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# An automatic correction of Ma's thinning algorithm based on $P$ -simple points

Christophe LOHOU and Julien DEHOS

<sup>1</sup> Laboratoire d'Algorithmique et Image de Clermont, Université d'Auvergne, France

<sup>2</sup> Laboratoire d'Informatique du Littoral, Université du Littoral Côte d'Opale, France

## Abstract

The notion of  $P$ -simple points has been introduced by Bertrand to conceive parallel thinning algorithms. In 'A 3D fully parallel thinning algorithm for generating medial faces', Ma has proposed an algorithm for which there exists objects whose topology is not preserved. In this paper, we propose a new application of  $P$ -simple points: to automatically correct Ma's algorithm.

**Keywords:** topology preservation; thinning algorithm correction; surface skeleton

## 1 Introduction

In the image processing framework, thinning is a technique which extracts skeletons from objects. Such a representation has the advantage of being compact while conserving important features of objects like topology and geometry. Thinning has many applications such as compression, analysis and recognition [Reniers and Telea(2007)]. In a binary image, objects are represented by black points. The complement is represented by white points. Thinning consists of changing black points to white. It is generally done in successive iterations of deletions. Removing a point should preserve certain properties like topology. This leads to the essential notion of a simple point: a simple point is a point which can be removed without changing topology [Kong and Rosenfeld(1989)].

The literature gives several kinds of 3D thinning algorithms. Parallel thinning algorithms proceed by removing black points in parallel. Implemented on appropriate computers, such algorithms can efficiently achieve thinning. The problem is that the notion of a simple point is not powerful enough to guarantee topology preservation in this context. For example, consider an object made of two neighboring points. Both points are simple but removing them both does change topology. This problem cannot occur by sequential processing because the deletion of the first point makes the second point become non-simple. To solve this problem, Bertrand has proposed the notion of  $P$ -simple points [Bertrand(1995a)]: a point of a set  $P$  is  $P$ -simple if it is simple when any subset of  $P$  is removed. Parallel thinning algorithms removing  $P$ -simple points have been proposed [Bertrand(1995b), Lohou and Bertrand(2004), Lohou and Bertrand(2005), Lohou and Bertrand(2007)]. Such algorithms have remarkable topology-preserving property.

In this paper we propose another application of  $P$ -simple points: ensuring topology preservation of an existing parallel thinning algorithm. In [Ma(1995)], Ma proposes a 3D parallel thinning algorithm. This algorithm is one of the few thinning algorithms able to preserve surfaces and is used as a reference [Palágyi(2008), Lohou and Bertrand(2007)]. In [Lohou(2008)], one of the authors gives an object whose topology is not preserved by Ma's algorithm. This proves that the algorithm fails to preserve topology, which it should do. Here, we propose a topology preserving algorithm by using  $P$ -simple points to repair Ma's algorithm. As far as we know, this is the first time  $P$ -simple points are used to automatically correct an image operator.

This paper is organized as follows. Section 2 gives some basic notions of Digital Topology and presents  $P$ -simple points. Section 3 shows how to use  $P$ -simple points in a thinning algorithm. In section 4, we propose such an algorithm to correct Ma's. Section 5 gives some results and section 6 concludes.

## 2 Basic notions of Digital Topology

### 2.1 Neighborhoods and connected components

A point  $x \in \mathbb{Z}^3$  is defined by  $(x_1, x_2, x_3)$  with  $x_i \in \mathbb{Z}$ . We consider the three neighborhoods:  $N_{26}(x) = \{x' \in \mathbb{Z}^3 : \max[|x_1 - x'_1|, |x_2 - x'_2|, |x_3 - x'_3|] \leq 1\}$ ,  $N_6(x) = \{x' \in \mathbb{Z}^3 : |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| \leq 1\}$ , and  $N_{18}(x) = \{x' \in \mathbb{Z}^3 : |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| \leq 2\} \cap N_{26}(x)$ . We define  $N_n^*(x) = N_n(x) \setminus \{x\}$ . We call respectively 6-, 18-, 26-*neighbors* of  $x$  the points of

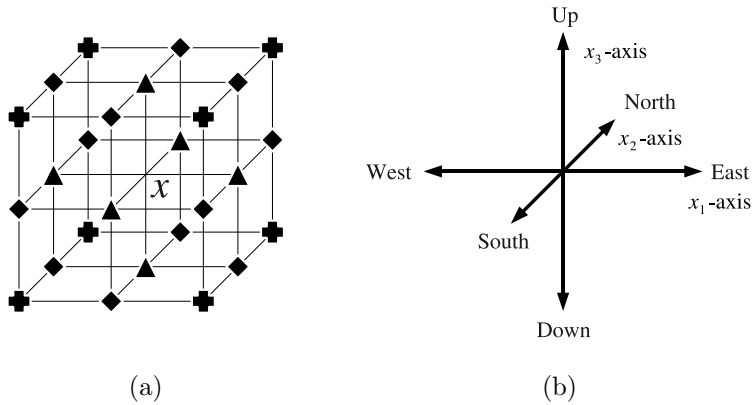


Figure 1: (a): The 6-neighbors (triangles), 18-neighbors (diamonds) and 26-neighbors (crosses) of  $x$ . (b): Axis and orientations.

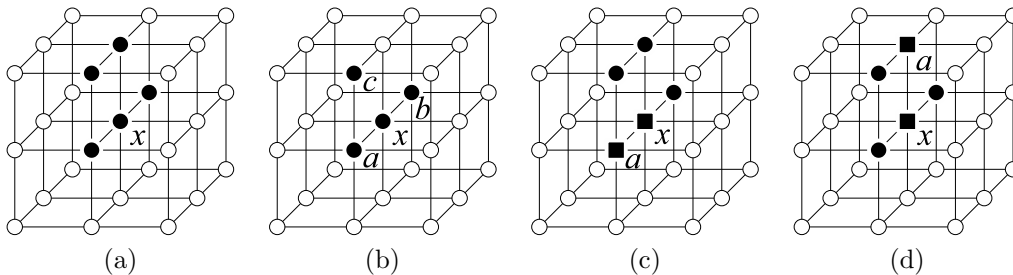


Figure 2: Simple points and  $P$ -simple points. Black points represent the object  $X$ . White points and hidden points represent the complement  $\overline{X}$ . (a):  $x$  is 26-simple for  $X$ . (b):  $x$  is not 26-simple for  $X$ . In (c) and (d), black squares represent  $P$  and black discs represent  $X \setminus P$ . (c):  $x$  is  $P$ -simple for  $X$ . (d):  $x$  is not  $P$ -simple for  $X$ .

$N_6^*(x)$ ,  $N_{18}^*(x) \setminus N_6^*(x)$ ,  $N_{26}^*(x) \setminus N_{18}^*(x)$ . Such points are represented in Fig. 1 (a). Let  $X \subseteq \mathbb{Z}^3$ . The points belonging to  $X$  (resp.  $\overline{X}$ , the complement of  $X$  in  $\mathbb{Z}^3$ ) are called *black points* (resp. *white points*). Axis and orientations used in this paper are given in Fig. 1 (b).

Two points  $x$  and  $y$  are said to be  $n$ -adjacent if  $y \in N_n^*(x)$  ( $n = 6, 18, 26$ ). An  $n$ -path is a sequence of points  $x_0, \dots, x_k$ , with  $x_i$   $n$ -adjacent to  $x_{i-1}$  and  $1 \leq i \leq k$ . If  $x_0 = x_k$ , the path is *closed*. Let  $X \subseteq \mathbb{Z}^3$ . Two points  $x \in X$  and  $y \in X$  are  $n$ -connected if they can be linked by an  $n$ -path included in  $X$ . The equivalence classes relative to this relation are the  $n$ -connected components of  $X$ . The presence of an  $n$ -hole is detected whenever there is a closed  $n$ -path in  $X$  that cannot be deformed, in  $X$ , into a single point (see [Kong(1989)], for further details). In order to have a correspondence between the topology of  $X$  and that of  $\overline{X}$ , we have to consider two different kinds of adjacency for  $X$  and for  $\overline{X}$  [Kong and Rosenfeld(1989)]: if we use an  $n$ -adjacency for  $X$ , we have to use another  $\overline{n}$ -adjacency for  $\overline{X}$ . In this paper, we only consider  $(n, \overline{n}) = (26, 6)$ .

## 2.2 Simple points and topological numbers

Let  $X \subseteq \mathbb{Z}^3$ . A point  $x \in X$  is said to be  $n$ -simple for  $X$  if its deletion does not “change the topology” of the image, in the sense that there is a bijection between the components and the holes of  $X$  and  $\overline{X}$  and the ones of  $X \setminus \{x\}$  and  $\overline{X} \cup \{x\}$  (see [Kong(1989)] for a precise definition). In the following, we write  $x$  is simple instead of  $x$  is simple for  $X$ . In Fig. 2 (a), we may verify that  $x$  is simple. In Fig. 2 (b),  $X$  is a single connected component whereas  $X \setminus \{x\}$  is made of two connected components ( $\{a, c\}$  and  $\{b\}$ ), therefore the topology of  $X$  is not preserved by the deletion of  $x$ , i.e.  $x$  is not simple.

The set composed of all  $n$ -connected components of  $X$  is denoted by  $\mathcal{C}_n(X)$ . The set of all  $n$ -connected components of  $X$  and  $n$ -adjacent to a point  $x$  is denoted by  $\mathcal{C}_n^x(X)$ . Let  $\#X$  denote the number of elements which belong to  $X$ . The *topological numbers* relative to  $X$  and  $x$  are the two numbers [Bertrand(1994)]:  $T_6(x, X) = \#\mathcal{C}_6^x(N_{18}^*(x) \cap X)$  and  $T_{26}(x, X) = \#\mathcal{C}_{26}^x(N_{26}^*(x) \cap X)$  (in fact,  $T_{26}(x, X) = \#\mathcal{C}_{26}(N_{26}^*(x) \cap X)$ , since any 26-connected component of black points in  $N_{26}^*(x)$  is inevitably 26-adjacent to  $x$ ). These numbers lead to a very concise characterization of 3D simple points [Malandain and Bertrand(1992), Bertrand and Malandain(1994)]:  $x \in X$  is 26-simple if and only if  $T_{26}(x, X) = 1$  and  $T_6(x, \overline{X}) = 1$ . For example, in Fig. 2 (a), the points of  $N_{26}^*(x) \cap X$  (resp.  $N_{18}^*(x) \cap \overline{X}$ ) make a single 26-connected (resp. 6-connected and 6-adjacent to  $x$ ) component. Thus  $(T_{26}(x, X), T_6(x, \overline{X})) = (1, 1)$ ; therefore  $x$  is simple. In Fig. 2 (b),  $N_{26}^*(x) \cap X$  is constituted by two

26-connected components ( $\{a, c\}$  and  $\{b\}$ ).  $N_{18}^*(x) \cap \overline{X}$  is made of a single 6-connected component 6-adjacent to  $x$ . Thus, we have,  $(T_{26}(x, X), T_6(x, \overline{X})) = (2, 1)$ ; therefore  $x$  is not simple.

### 2.3 $P$ -simple points

This section recalls the definition of  $P$ -simple points and several properties of such points [Bertrand(1995a)]. In the following, we consider a subset  $X$  of  $\mathbb{Z}^3$ , a subset  $P$  of  $X$  and a point  $x$  of  $P$ .

**Definition 1** *The point  $x$  is  $P$ -simple for  $X$  if for each subset  $S$  of  $P \setminus \{x\}$ ,  $x$  is 26-simple for  $X \setminus S$ .*

For example, in Fig. 2 (c),  $P = \{x, a\}$ . We may verify that  $x$  is simple for  $X \setminus \{a\}$ ; therefore  $x$  is  $P$ -simple. In Fig. 2 (d),  $P = \{x, a\}$ . With  $S = \{a\}$ , we may verify that  $x$  is not simple for  $X \setminus S$ ; therefore  $x$  is not  $P$ -simple.

We have the remarkable property that any algorithm removing subsets composed solely of  $P$ -simple points is guaranteed to keep the topology unchanged [Bertrand(1995a)].

The following proposition permits to locally characterize  $P$ -simple points [Bertrand(1995c)].

**Proposition 1** *The point  $x$  is  $P$ -simple if and only if:*

$$\left\{ \begin{array}{l} T_{26}(x, X \setminus P) = 1, \\ T_6(x, \overline{X}) = 1, \\ \forall y \in N_{26}^*(x) \cap P, \exists z \in X \setminus P \text{ such that } z \text{ is 26-adjacent to } x \text{ and to } y, \\ \forall y \in N_6^*(x) \cap P, \exists z \in \overline{X} \text{ and } \exists t \in \overline{X} \text{ such that } \{x, y, z, t\} \text{ is a unit square.} \end{array} \right.$$

For example, in Fig. 2 (c),  $x$  is  $P$ -simple because it concurs with the four conditions of Proposition 1. In Fig. 2 (d),  $x$  is not  $P$ -simple because the first condition is not satisfied ( $T_{26}(x, X \setminus P) = 2$ ). In both examples, Proposition 1 makes it possible to consider only a few points around  $x$  to determine whether  $x$  is  $P$ -simple or not.

## 3 Thinning algorithms based on $P$ -simple points

Let us consider an algorithm which removes  $P$ -simple points in parallel, in successive iterations. Since some points are deleted during an iteration, this algorithm thins objects. A skeleton is obtained when no more points can be removed. As shown in the previous section, removing subsets composed of solely  $P$ -simple points guarantees to keep the topology unchanged. Consequently, topology preservation, which is one of the most important features of a thinning algorithm, is automatically proved for such algorithms.

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**Algorithm 1** : P-SimpleThinningAlgorithm( $X, C$ )  $\rightarrow X$

---

```

1: repeat
2:   {parallel labelling of points which belong to  $P$ }
3:    $P \leftarrow \emptyset$ 
4:   for each point  $x$  in  $X$ , in parallel, do
5:     if  $x$  verifies the condition  $C$  then
6:       put  $x$  in  $P$ 
7:     end if
8:   end for
9:   {parallel deletion of  $P$ -simple points}
10:  for each point  $x$  in  $P$ , in parallel, do
11:    if  $x$  is  $P$ -simple for  $X$  then
12:      delete  $x$  in  $X$ 
13:    end if
14:  end for
15: until no points are deletable

```

---

Algorithm 1 gives such an algorithm; where  $C$  denotes the condition a point  $x$  of  $X$  should satisfy to belong to  $P$  (the set of candidate points to deletion). Each iteration is composed of two passes. Thanks to  $P$ -simple points, the algorithm can examine points in parallel. Furthermore, using the local characterization of  $P$ -simple points (Proposition 1), the algorithm can consider limited neighborhoods. It is thus well-suited for parallel machines.

Algorithm 1 does not explain how to determine  $P$  (the set of candidate points for deletion). In [Bertrand(1995a)], Bertrand proposes that  $P$  contains the simple points of  $X$  to obtain a certain kind of skeleton.

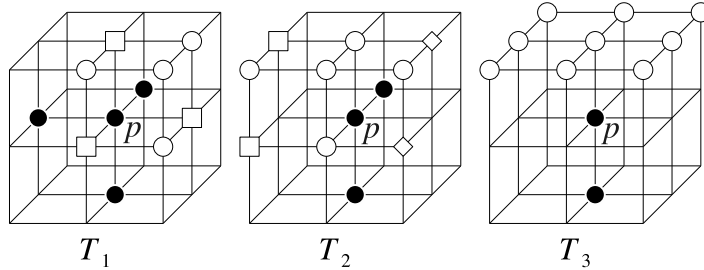


Figure 3: The templates of Ma’s algorithm. A point  $p$  satisfies a template if each disc is a black point, each circle is a white point, at least one square is a white point and at least one diamond is a white point.

## 4 A new thinning algorithm correcting Ma’s algorithm

In the previous section, we have recalled how  $P$ -simple points can be used to conceive thinning algorithms. We now suggest a new application of  $P$ -simple points: to correct an existing thinning algorithm which fails to preserve topology. Indeed, a lot of parallel thinning algorithms (including Ma’s algorithm) are based on “local” templates. Basically, they consist of removing in parallel and in successive iterations, all the points matching one of the given templates. A skeleton is obtained when no more points can be removed. However, when a templates-based thinning algorithm is proposed, its topology preservation should be proven afterward. This is very difficult because it implies considering a great number of configurations in a greater neighborhood than the  $N_{26}$  neighborhood of the considered point. Thus, giving an analytical proof is difficult and computing the correctness of the algorithm cannot be done in reasonable time using actual computers. That’s why an error can be made easily, when a templates-based thinning algorithm is conceived. This is what has occurred with Ma’s algorithm. A correction could be made by changing the initial templates but, once again, proof has to be given, which leads us to the previously mentioned difficulties. In this section, we explain how to use  $P$ -simple points to correct, in a simple and effective manner, a templates-based thinning algorithm which fails to preserve topology.

### 4.1 Study of Ma’s algorithm

This algorithm has been proposed in [Ma(1995)]. It is a 3D parallel thinning algorithm based on templates requiring access of up to 30 points to decide whether a point is deletable or not.

In [Ma(1995)], Ma gives the following definitions and rules:

- A black point is called *border point* if it is 6-adjacent to a white point.
- Definition 1.1: Let  $p$  be a black point of a 3D image. Then  $p$  is called an *edge point* if  $p$  has exactly one black 26-neighbor in  $N_{26}^*(p)$  or in any of the three orthogonal  $3 \times 3$  planes containing  $p$ ;  $p$  is called a *non-edge point* if it is not an edge point.
- Suppose  $p$  is a point in a 3D image. Let  $s(p)$ ,  $w(p)$  and  $d(p)$  be the south, west and down neighbors of  $p$ , respectively.
- Rule 2.1: Suppose all four corners of a unit lattice square are to be deleted. Then the corner with the smallest sum of coordinates is preserved if and only if it is non-simple after the other three corners are deleted.
- Let  $\Omega$  be the set of all rotations and reflections of all three configurations (shown in Fig. 3). For each element  $T$  of  $\Omega$ , a black point  $p$  is said to *satisfy*  $T$  if [it satisfies the corresponding template and]<sup>1</sup> all of the following conditions are satisfied:
 
$$\left\{ \begin{array}{l} p \text{ is a non-edge point,} \\ \text{if } p \text{ is a north border point, then } s(s(p)) \text{ is a black point,} \\ \text{if } p \text{ is a east border point, then } w(w(p)) \text{ is a black point,} \\ \text{if } p \text{ is a up border point, then } d(d(p)) \text{ is a black point.} \end{array} \right.$$

A black point  $p$  is said to *satisfy*  $\Omega$  if  $p$  satisfies any element in  $\Omega$ .

Using these definitions, Ma gives the Algorithm 2, denoted by MA95.

This algorithm works by iterations. Each iteration is composed of two passes (find black points satisfying  $\Omega$ ; delete points not preserved by Rule 2.1). During a pass, points can be considered in parallel.

<sup>1</sup>not mentioned in the original article

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**Algorithm 2** : MA95

---

```
1: repeat
2:   parallel delete black point that satisfies  $\Omega$  and is not preserved by Rule 2.1
3: until no points are deleted
```

---

In [Lohou(2008)], it is shown that Ma’s algorithm does not always preserve topology. Now, we propose an algorithm which corrects this problem.

## 4.2 Our proposal

In section 3, we proposed a generic parallel thinning algorithm (Algorithm 1). First, this algorithm detects, in parallel, the points which verify a condition  $C$ . These points compose the set  $P$ . Then, the algorithm deletes  $P$ -simple points in parallel among the points of  $P$ .

To correct Ma’s algorithm, we propose to remove  $P$ -simple points in parallel in successive iterations where  $P$  is precisely the set of points which would be deleted by Ma’s algorithm. This is equivalent to Algorithm 1 where the condition  $C$  is “the point satisfies  $\Omega$  and is not preserved by Rule 2.1”.

Checking Rule 2.1 for a point  $x$  requires us to determine if some neighbors of  $x$  are to be deleted. This can be done by checking whether they satisfy  $\Omega$  or not. This method means accessing a larger neighborhood of  $x$  (see [Lohou and Bertrand(2004)] for further details). To avoid this, we can first find the points that satisfy  $\Omega$  and, in another pass, determine which ones are preserved by Rule 2.1. This algorithm, denoted by LD09, is given in Algorithm 3.

---

**Algorithm 3** : LD09( $X$ )  $\rightarrow X$ 

---

```
1: repeat
2:   {parallel labelling of points which belong to  $P'$ }
3:    $P' \leftarrow \emptyset$ 
4:   for each point  $x$  in  $X$ , in parallel, do
5:     if  $x$  satisfies  $\Omega$  then
6:       put  $x$  in  $P'$ 
7:     end if
8:   end for
9:   {parallel application of Rule 2.1}
10:   $P \leftarrow \emptyset$ 
11:  for each point  $x$  in  $P'$ , in parallel, do
12:    if  $x$  is not preserved by Rule 2.1 in  $P'$  then
13:      put  $x$  in  $P$ 
14:    end if
15:  end for
16:  {parallel deletion of  $P$ -simple points}
17:  for each point  $x$  in  $P$ , in parallel, do
18:    if  $x$  is  $P$ -simple for  $X$  then
19:      delete  $x$  in  $X$ 
20:    end if
21:  end for
22: until no points are deletable
```

---

LD09 is indeed a correction of MA95. First, it removes  $P$ -simple points only, which ensures topology preservation. And finally, it can delete only the same points as MA95. This use of  $P$ -simple points implies that LD09 preserves topology. On the contrary, a correction based only on a modification of the initial templates would have required a difficult and error-prone proof of the topology preservation.

We may notice that other algorithms can be derived from LD09. For instance, Rule 2.1 may be checked in the same time as the  $P$ -simplicity, which makes it possible to bypass the specific pass. Another algorithm can be defined by letting  $P$  be the set of points which match the templates and then removing  $P$ -simple points which are non-edge points and non-border points. We are currently working on these points and a few other ones [DEHOS and LOHOU(2009)].

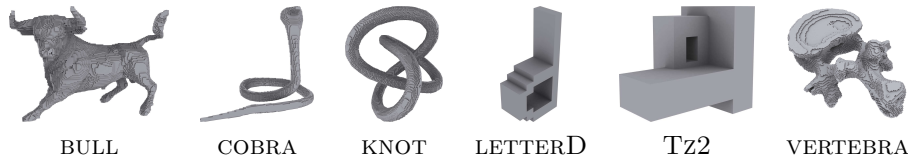


Figure 4: Six of the objects used for our tests.

Object	Black points	Deleted points (first iteration)		Deleted points (total)		Iterations	
		MA95	LD09	MA95	LD09	MA95	LD09
BULL	424 346	31 344	31 290	399 207	395 099	42	42
COBRA	89 890	18 924	18 922	80 626	79 962	10	11
KNOT	231 239	35 572	35 542	212 503	210 972	14	14
LETTERD	2 079	1 281	1 281	1 665	1 665	3	3
Tz2	4 299	1 739	1 739	3 879	3 873	6	7
VERTEBRA	180 161	23 042	23 034	162 715	161 215	24	25

Table 1: Comparison between MA95 and LD09 applied to the 6 objects depicted in Fig. 4.

## 5 Results

We have compared Ma’s algorithm and ours thanks to 5 small synthetic objects, 5 large objects obtained by voxelisation and 1 object obtained by medical imagery (see Table 1. Some of these objects are depicted in Fig. 4. We have found that 3 objects have the same number of deleted points with both algorithms (for instance, LETTERD and Tz2), 8 objects have more deleted points with Ma’s algorithm and no object has more deleted points with our algorithm. Furthermore, 6 objects have the same number of iterations with both algorithms (for instance, BULL), one object has more iterations with Ma’s algorithm and 4 objects have more iterations with our algorithm (for instance, VERTEBRA).

If we apply these algorithms to the same object, we can get different results after the first iteration, because LD09 removes some but not always all of the points removed by MA95. Thus, the next iteration happens on different objects therefore the final results are difficult to compare. By its conception, our algorithm cannot delete more points of a given object than Ma’s algorithm, during one iteration (columns 3 and 4 of Table 1. According to our tests, our algorithm generally deletes less points and performs more iterations than Ma’s. However, we cannot prove anything from this. For example, as mentioned before, one object of our database is thinned by less iterations with our algorithm than with Ma’s.

Fig. 5 shows the result of both algorithms applied to an object of the database. The initial object (a) is a dolphin obtained by voxelisation. The skeletons given by Ma’s algorithm (b) and ours (c) are quite similar. However, we can see on their respective close-ups (d) and (e), that Ma’s algorithm does not preserve the topology whereas ours does. For example, in (d), the stamp 1 depicts a voxel which is disconnected from the skeleton by Ma’s algorithm. In the same way, the stamp 2 shows a hole created by the algorithm. We can see on (e) that our algorithm does not have these problems. From another viewpoint (f), we can notice that, like Ma’s skeleton (g), our skeleton (h) presents medial faces. Finally, if the skeletons have common points (i), they also have points which are unique to themselves (j, k).

## 6 Conclusion

Up until now,  $P$ -simple points have been used to develop new topology-preserving thinning algorithms. In this paper, thanks to  $P$ -simple points, we have proposed an automatic correction of a templates-based thinning algorithm which fails to preserve topology. Like this latter, our algorithm allows us to extract medial-face skeletons. However, our algorithm ensures topology preservation. To our knowledge, it is the first time  $P$ -simple points are used for such an application. Our algorithm could be seen as a correcting patch for current applications using Ma’s algorithm.

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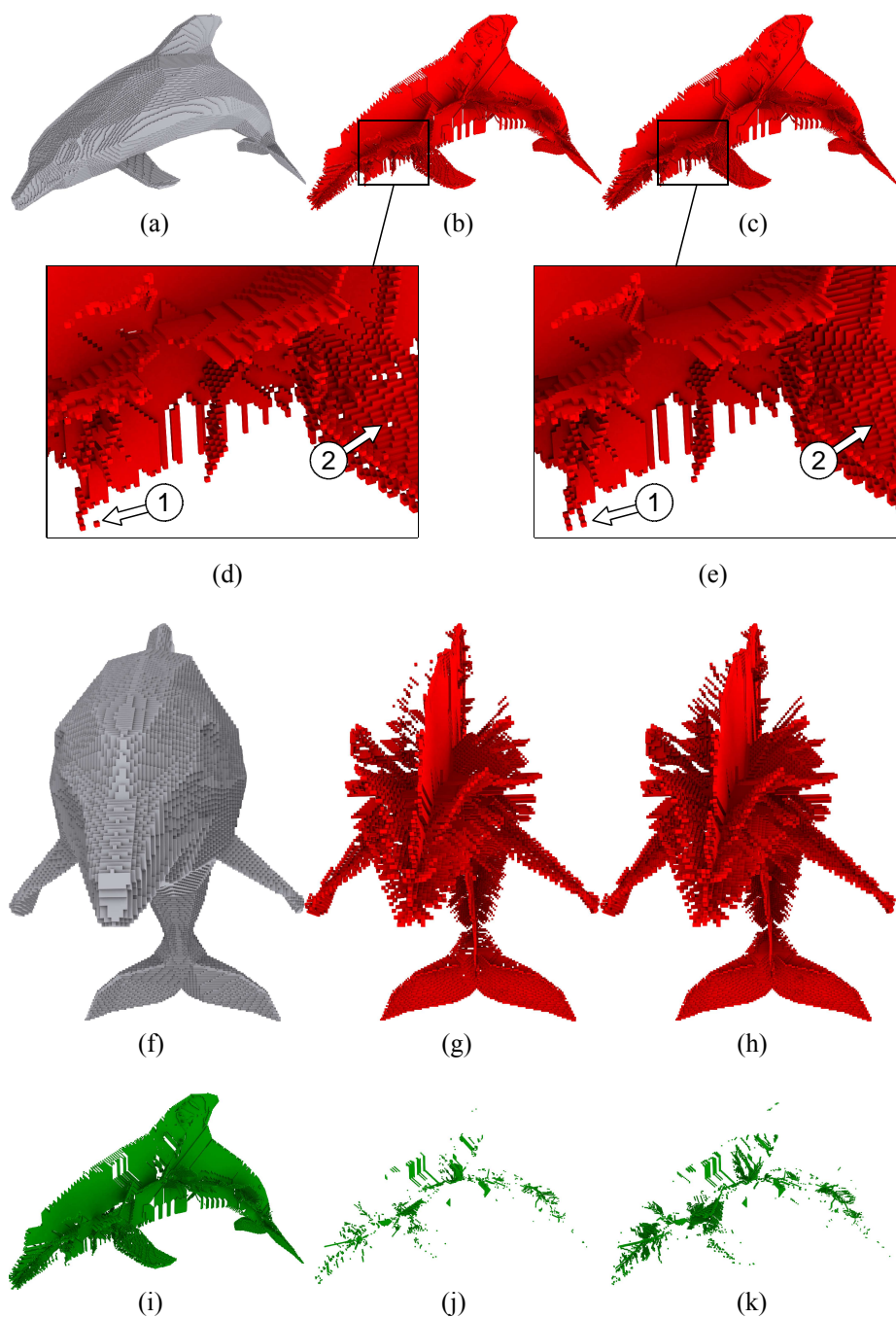


Figure 5: Result of both algorithms applied to a same object. (a): Initial object (1 202 772 black points). (b): Skeleton obtained with Ma's algorithm (1 153 121 deleted points in 60 iterations, 49 651 remaining points). (c): Skeleton obtained with our algorithm (1 150 230 deleted points in 61 iterations, 52 542 remaining points). (d): Close-up of (b). (e): Close-up of (c). (f): Another viewpoint of (a). (g): (b) from the same viewpoint as (f). (h): (c) from the same viewpoint as (f). (i): Intersection of (b) and (c) (45 967 points). (j): Difference between (b) and (c) (3 684 points). (k): Difference between (c) and (b) (6 575 points).



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