# Vehicle Sharing System Pricing Regulation: A Fluid Approximation

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**Abstract**

This paper gives a fluid approximation for a Vehicle Sharing System Pricing problem (VSS-P).

1. The VSS-P is formulated as a Closed Queuing Network with finite buffers, time dependent service time variation and continuous controls on transition rates for the pricing.
2. Solving the model for general (dynamic) policies seams intractable. Therefore a fluid approximation giving piecewise static policy is formulated. It is conjectured asymptotically optimal when the problem is scaled by a factor tending to infinity.
3. A reusable benchmark and an experimental protocol is created for the general Vehicle Sharing System optimization problem.
4. Numerical experiments are run on toy cities comparing the classic protocol with reservation of the parking spot at destination and the static policy given by the fluid approximation.

# 1 Introduction

## 1.1 Context

Shoup (2005) reports that, based on a sample of 22 US studies, cars looking for a parking spot contribute to 30% of the city traffic. Moreover cars are used less than 2 hours per day on average but still occupy a parking spot the rest of the time! Could we have less vehicles and satisfy the same demand level?

Recently, the interest in Vehicle Sharing Systems (VSS) in cities has increased significantly. Indeed, urban policies intend to discourage citizens to use their personal car downtown by reducing the number of parking spots, street width, etc. VSS seem to be a promising solution to reduce jointly traffic and parking congestion, noise, and air pollution (proposing bikes or electric cars). They offer personal mobility allowing users to pay only for the usage.

We are interested in short-term one-way VSS where vehicles can be taken and returned at different places (paying by the minute). Associated with classical public transportation systems, short-term one-way VSS help to solve one of the most difficult public transportation problem: the last kilometer issue (DeMaio, 2009). This is not the case for round-trip VSS where vehicles have to be returned at the station where they were taken.

The first large-scale short-term one-way VSS was the bicycle VSS Vélib'. It was implemented in Paris in 2007 and now has more than 1200 stations and 20 000 bikes selling around 110 000 trips per day. It has inspired several other cities all around the world; Now more than 300 cities have such a system, including Montréal, Bejing, Barcelona, Mexico City, Tel Aviv (DeMaio, 2009).

## 1.2 One-way Vehicle Sharing Systems: a management issue

However if freedom increases for the user in the one way model, it implies a higher complexity in its management. In round trip type rental systems, the only stock that is relevant when managing yield and reservations is the number of available vehicles. In one-way systems, a new problem occurs since vehicles aren't the only key resource any more. In practice, parking stations have a maximum number of spots, and when the total number of vehicle is comparable to the total number of parking spots, available parking spots become a new key resource.

Since first bicycle VSS, problems of bikes and parking spots availability have appeared very often. Reasons are various but we can highlight two important phenomenon: the gravitational effect which indicates that a station is constantly unbalanced (as Montmarte hill in Vélib'), and the tide phenomenon representing the oscillation of demand intensity along the day (as morning and evening flows between working and residential areas).

To improve the efficiency of the system, in the literature, different perspectives are studied. At a strategic level, some authors consider the optimal capacity and locations of bike rental stations. Shu et al. (2010) propose a stochastic network flow model to support these decisions. They use their model to design a bicycle VSS in Singapore based on demand forecast derived from current usage of the mass transit system. Lin and Ta-Hui (2011) consider a similar problem but formulate it as a deterministic mathematical model. Their model is aware of the bike path network and mode sharing with other means of public transportation.

At a tactical level, other authors investigate the optimal number of vehicles given a set of stations. George and Xia (2011) study the fleet sizing problem with constant demand and no parking capacity. Fricker and Gast (2012); Fricker et al. (2012) look into the optimal sizing of a fleet in"toy" cities, where demand is constant over time and identical for every possible trip, and all stations have the same capacity $\mathcal{K}$. They show that even with an optimal fleet sizing in the most "perfect" city, if there is no operational system management, there is at least a probability of $\frac{2}{\mathcal{K}+1}$ that any given station is empty or full.

At an operational level, in order to be able to meet the demand with a reasonable standard of quality, in most bicycle VSS trucks are used to balance the bikes among the stations. The problem is to schedule vehicle routes to visit some of the stations to perform pickup and delivery so as to minimize the number of users who cannot be served, *i.e.*, the number of users who try to take a bike from an empty station or to return it to a full station. In the literature many papers deal already with this problem. A static version of the bicycle VSS balancing problem is treated in Chemla et al. (2011) and a dynamic one in Contardo et al. (2012).

A new type of VSS has appeared lately: one-way Car VSS with Autolib' in Paris and Car2go in more than 10 cities (Vancouver, San Diego, Lyon, Ulm...). With cars, operational balancing optimization through relocation seems inappropriate due to their size. We have to find another way to optimize the system.

## 1.3 Regulation through pricing

The origin of Revenue Management (RM) lies in airline industry. It started in the 1970s and 1980s with the deregulation of the market in the United States. In the early 1990s RM techniques were then applied to improve the efficiency of round trip Vehicle Rental Systems (VRS), see Carroll and Grimes (1995) and Geraghty and Johnson (1997). One way rental is now offered in many VRS, however as one can see in practice for car VRS that it is always much more expensive than round trip rental. We haven't found in the literature authors tackling the one way VRS RM problem. We can only cite Haensela et al. (2011) that model a network of only round trip car VRS but with the possibility of transferring cars between rental sites for a fixed cost. For trucks rental on the contrary, companies such as Rentn'Drop in France or Budget Truck Rental in the United States are specialized in the one way rental offering dynamic pricing. This problem is tackled by Guerriero et al. (2012) that consider the optimal managing of a fleet of trucks rented by a logistic operator, to serve customers. The logistic operator has to decide whether to accept or reject a booking request and which type of truck should be used to address it.

Anyway results for one way VRS are not directly applicable to VSS, because they differ on several points: 1) Renting are by the day in VRS and by the minute in VSS with a possible high intensity; 2) One way rental is the core in VSS, for instance only 5% of round trip rental in Bixi (Morency et al., 2011), and it is classically the opposite in car VRS. 3) There is usually no booking in advance in VSS, it is a first come first serve rule, whereas usually trips are planned several days in advance in VRS.

In this paper we are looking at VSS and optimization through pricing. Assuming that demand is elastic, we want to use prices to influence user choices in order to drive the system towards its most efficient dynamic.

This work is part of a preliminary study using operation research to 1) Establish the interest of VSS pricing regulation system 2) Give good and possibly simple pricing policies for the operational management.

# 2 Model: An intractable Markov Decision Process

## 2.1 Restriction to a simple protocol

In a real context, a user wants to use a vehicle to take a trip between an original (GPS) location $a$, and a final one $b$, during a specified moment. On a station based VSS, he tries to find the closest station to location $a$ with a vehicle to take and the closest station to location $b$ with a parking spot to return it. All along this process users decisions rely on several correlated inputs such as: trip total price, walking distance, public transportation competition, time frame...

A time elastic GPS to GPS demand forecast correlated to a user's decision protocol ruling his behaviour to take a trip between two specific stations at a specific time seems closer to reality but introduces of course a big complexity (use of utility function for instance).

Therefore we are here going to consider a simple station to station demand forecast with only real time reservation for a specified trip. It means that the user will engage himself to return the vehicle at a specified station and time. Finally it amounts in considering a demand for the following simplified protocol:

1. A user asks for a vehicle at station $a$ (here and now), with destination $b$ and rental duration $\mu_{a,b}^{-1}$;
2. The system offers a price (or rejects the user = infinite price);
3. The user accepts the price, takes the vehicle and a parking spot is reserved (or leaves the system).

## 2.2 A Vehicle Sharing System Stochastic Model

### 2.2.1 Markovian framework

We define in this section a framework to model a stochastic Vehicle Sharing Systems with the protocol defined in the previous section.

**Definition 1 (Vehicle Sharing Systems Markovian Model)** *In a city there is is a fleet of $N$ vehicles along a set $\mathcal{M}$ ($|\mathcal{M}| = M$) of stations with capacity $\mathcal{K}_i$, $i \in \mathcal{M}$. There is an elastic demand between each station $\mathcal{D} = \mathcal{M} \times \mathcal{M}$. This demand is piecewise constant on time steps $\mathcal{T}$, it follows a Poisson distribution of parameter $\lambda_{a,b}^t(p_{a,b}^t)$ to go from station $a \in \mathcal{M}$ to station $b \in \mathcal{M}$ at the period $t \in \mathcal{T}$ and is function of the proposed price $p_{a,b}^t$.*

*All durations follow an exponential distribution: The transportation time to go from station $a$ to station $b$ at time step $t$ has for mean $1/\mu_{a,b}^t$; The time step duration $t \in \mathcal{T}$ has for mean $1/\tau^t$ and the total horizon length has for mean $T = \sum_{t\in\mathcal{T}} 1/\tau^t$.*

### 2.2.2 Closed queuing network model

For a given demand $\lambda^t_{a,b}$ for every trips $(a,b) \in \mathcal{D}$ and every time steps $t \in \mathcal{T}$ (*i.e.* for a given price $p^t_{a,b}$). We can model this stochastic Vehicle Sharing System by a Closed Queueing Network with finite buffers and service time variation, see figure 1.

Each demand $(a,b) \in \mathcal{D}$ is represented by a server $(a-b)$ which has a time dependent service rate equal to the average number of clients willing to take a trip from station $a$ to station $b$: $\lambda^t_{a,b}$. Demands with same station of origin $a$ ($(a,b) \in \mathcal{D},\ \forall b \in \mathcal{M}$) are sharing the same finite buffer of size $\mathcal{K}_a$: the capacity of station $a$. When a vehicle (a job) is picked up to take the trip $(a,b)$ (is processed by server $(a,b)$) before going to station (server) $b$ it has to pass by a transportation state (server) $(a-b)$ with a service time proportional to the number of vehicles $n_{a,b}$ in transit (in the buffer): $n_{a,b} \times \mu^t_{a,b}$.

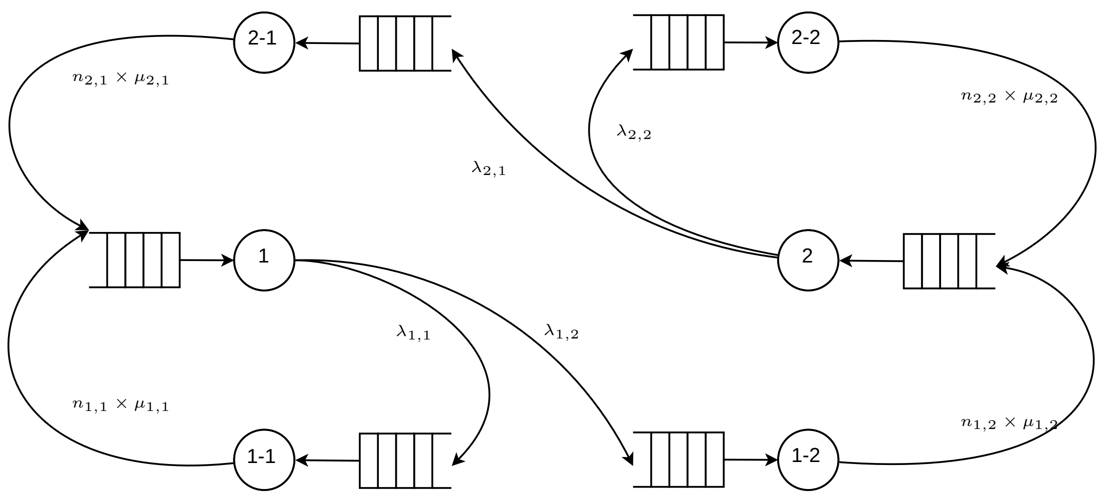


Figure 1: A closed queuing network model with servers for demands and transportation times.

### 2.2.3 Continuous-Time Markov Chain (CT-MC) formulation

If a price $p^t_{a,b}$ is set for all trips $(a,b) \in \mathcal{D}$ at all time steps $t \in \mathcal{T}$, we can model the closed queuing network by a Continuous Time-Markov Chain on a set of states $\mathcal{S}$:

$$\mathcal{S} = \left\{ \Big( n_a :\ a \in \mathcal{M},\ n_{a,b} :\ (a,b) \in \mathcal{D},\ t \in \mathcal{T} \Big) \ / \sum_{i \in \mathcal{M} \cup \mathcal{D}} n_i = N \ \&\ n_a + \sum_{b \in \mathcal{M}} n_{b,a} \leq \mathcal{K}_a,\ \forall a \in \mathcal{M},\ \forall t \in \mathcal{T} \right\}$$

A state $s = (n_a :\ a \in \mathcal{M},\ n_{a,b} :\ (a,b) \in \mathcal{D},\ t \in \mathcal{T})$ represents the repartition of the vehicles in the city space (in station or in transit) at a given time. At time step $t$, $n_a$ is the number of vehicles in station $a \in \mathcal{M}$, $n_{a,b}$ the number of vehicles in transit between stations $a$ and $b$ serving a trip demand $(a,b) \in \mathcal{D}$.

The transition rates between states are either:

- The taking of a vehicle at a station $a$ to go to a station $b$ which gives a transition rate $\lambda^t_{a,b}(p^t_{a,b})$ between states $(\dots,\ n_a, \dots,\ n_{a,b} \dots,\ t)$ and states $(\dots,\ n_a - 1, \dots,\ n_{a,b} + 1, \dots,\ t)$ with $n_a > 0$ and $n_b + \sum_{c \in \mathcal{M}} n_{c,b} < \mathcal{K}_b$;

- The arrival of a vehicle at a station $b$ from a station $a$ which gives a transition rate $n_{a,b} \times \mu^t_{a,b}$ between states $(\dots,\ n_b, \dots,\ n_{a,b} \dots,\ t)$ and states $(\dots,\ n_b + 1, \dots,\ n_{a,b} - 1, \dots,\ t)$ with $n_{a,b} \geq 1$;

- The changing of piecewise constant demand time step which gives a transition rate $\tau^t$ between states $(\dots,\ t)$ and states $(\dots,\ t + 1 \bmod |\mathcal{T}|)$.

We can note that there is an exponential number of states. Even with only one time-step, without transportation time but with infinite station capacities there is $\binom{N+M-1}{N}$ states. For instance for a small system with $N = 150$ vehicles and $M = 50$ stations it gives roughly $10^{47}$ states! This number grows linearly with the number of time steps and exponentially with the number of different transportation times considered.

## 2.3 Model optimization

### 2.3.1 Vehicle Sharing System Pricing Problem

In the previous section we modelled the system by a closed queuing network with finite buffer and service time variation that can be described explicitly through a Continuous-Time Markov Chain. Through this model we want now to optimize the Vehicle Sharing System in order to maximize the average revenue. To do so we have as leverage the possibility to change the price to take a trip which will, assuming an elastic demand, influence the demand for such trip. We call this problem the Stochastic Vehicle Sharing System Pricing Problem.

**Definition 2 (VSS Pricing Problem)** *The Stochastic Vehicle Sharing System Pricing Problem amounts in setting price for every trip in order to maximize the gain of the Vehicle Sharing System Markovian Model.*

*Prices can be Discrete, i.e. selected in a set of possibilities, or Continuous i.e. chosen in a range. Pricing policies can be Dynamic, i.e. dependent on system's state (vehicle repartition and period of the day), or Static i.e. independent on system's state, set in advance and function of the trip and the time of the day.*

### 2.3.2 An intractable Markov Decision Process (MDP) Resolution

We can solve this queuing network model through the well known Markov Decision Process framework based on the Continuous-Time Markov Chain given in the previous section. To do so define a set $\mathcal{Q}$ of possible discrete prices for each trip at each time step. A trip $(a, b) \in \mathcal{D}$ at time step $t \in \mathcal{T}$ at price $p^{t,q}_{a,b},\ q \in \mathcal{Q}$ have a demand (transition rate) $\lambda^t_{a,b}(p^{t,q}_{a,b}) = \lambda^{t,q}_{a,b}$.

Solving this MDP computes the best Dynamic System State Dependent Trip Discrete Pricing policy, *i.e.* the price for a trip depends of the current state of the system (vehicle repartition). MDP are known to be polynomially solvable on the number of states $|\mathcal{S}|$ and the number of actions $|\mathcal{A}|$ available in each state. For example through Value Iteration, Policy Iteration algorithm or Linear Programming techniques, *c.f.* book of Puterman (1994).

In this particular case, we are dealing with a pricing problem and the action space $\mathcal{A}(s)$ in each state $s \in \mathcal{S}$ is the Cartesian product of the available prices for each trip *i.e.* $\mathcal{A}(s) = \mathcal{Q}^M$. However to not suffer from this exponential explosion we can model this problem as a Action Decomposable Markov Decision Process, *c.f.* Waserhole et al. (2012). It is a general method based on the Event Based Dynamic Programming, *c.f.* Koole (1998), to reduce the complexity of the action space to $\mathcal{A}(s) = \mathcal{Q} \times M$. It allows to use Value Iteration, Policy Iteration algorithm or Linear Programming solution techniques.

**Lemma 1 (Dynamic Pricing MDP)** *The previous MDP gives the best Dynamic System State Dependent Discrete Pricing Policy for the stochastic VSS Pricing problem.*

Anyway the state space is still exponential so it does not give a polynomial algorithm. We have therefore to look into approximations or simplifications to tackle the problem.

### 2.3.3 State of the art on this model

In the literature only simple forms of this closed queuing network model with the relationship to the underlying CT-MC have already been studied for VSS. George and Xia (2011) consider a VSS with only one time step, one price and infinite station capacities. Under these assumptions they establish a compact form to compute the system performance using the BCMP network theory (Baskett et al., 1975). They solve an optimal fleet sizing problem considering a cost to maintain a vehicle and a gain to rent it.

Fricker and Gast (2012) consider simple cities that they call homogeneous. These cites have a unique fixed station capacity $\mathcal{K}_a = \mathcal{K}$, a constant (one time step) arrival rate and uniform routing matrix: $\lambda_{a,b}^t = \frac{\lambda}{M}$; they also have a unique travel time following an exponential distribution of mean ${\mu_{a,b}^t}^{-1} = \mu^{-1}$. With a Mean Field Approximation, they obtain some asymptotic results when the number of stations tends to infinity ($M \to \infty$): If there is no operational regulation system, the optimal sizing is to have $\frac{\mathcal{K}}{2} + \frac{\lambda}{\mu}$ vehicles per station which corresponds in filling half of the station plus the average number of vehicles in transit ($\frac{\lambda}{\mu}$). Moreover they show that even with an optimal sizing each station has still a probability $\frac{1}{\mathcal{K}+1}$ to be empty or full which is considered pretty bad. In another paper Fricker et al. (2012) extend to inhomogeneous cities modelled by clusters some analytical results and verified experimentally some others.

Fricker and Gast (2012) also study in homogeneous cities a heuristic called “The power of two choices” using incentives that can be seen as a Dynamic Station State Pricing. When a user is showing at a station and is taking a vehicle, he gives randomly two possible destination stations and the system is directing him to the least loaded one. They show that this policy allows to drastically reduce the probability to be empty or full for each station to $2^{-\frac{K}{2}}$.

Finally none of these models includes time dependent demands, pricing or full heterogeneity which is the subject of this paper with the fluid approximation.

# 3 A fluid approximation

## 3.1 A plumbing problem

We use the term plumbing because we are considering continuous vehicles that can in fact be seen as a flow passing through pipes. The stations are tanks with capacity the size of the station, and the demands between stations are pipes with width the amount of these demands. Finally the length of the pipes represents the duration of the trip reservation. Figure 2 gives an example for two stations.

We are modelling a system which has no direct interaction with the user. The decisions are static and have to be taken before, once for all. They amount in setting the width of a pipe by changing the price to pass flow in it: the higher the price is, the smaller the pipe (demand) will be.

However a free system does not mean that it is possible to do everything! Indeed first, if a pipe (a demand) exists and there is some flow (vehicle) available in the tank (station), according to gravity first come first serve law, the flow has to pass through the pipe until no flow is available.

Secondly if there is not enough flow to fulfil all pipes (demands), according to the same law there should be some equity between them. Hence the proportion of filling up of all pipes should be equal.

However there is some restrictions to this rule. If it the arrival tank of a pipe is full, it might be impossible to fulfil this pipe. In this case another equity rule should be applied to all pipes discharging into this tank.

In other words for each pipe, if its discharging tank is full it has the same proportion of filling up as the other pipes discharging in this tank, otherwise it has the same proportion of filling up as the other pipes coming from its source tank.

## 3.2 More formally

The fluid model is constructed by replacing stochastic demands by a continuous flow with the corresponding deterministic rate. It gives a deterministic and continuous dynamics and evolves as a continuous process. Optimizing the fluid model to give heuristics on the stochastic model is a well know technique. It is derived as a limit under a strong-law-of-large numbers, type of scaling, as the potential demand and the capacity grow proportionally large; see Gallego and van Ryzin

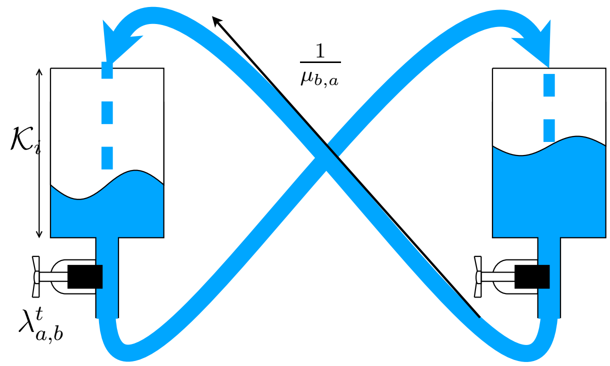


Figure 2: A Plumbing Problem.

(1994). Lots of application of this principle are available in the literature to deal with revenue management problems, see Maglaras (2006) for instance. However to the best of our knowledge there is no direct classic approach available for a general case including our application. Nevertheless there is some work on theorizing the fluid approximation scheme: Meyn (1997) describes some approaches to the synthesis of optimal policies for multiclass queueing network models based upon the close connection between stability of queueing networks and their associated fluid limit models; Bäuerle (2002) generalizes open multi class queueing networks and routing problems.

What goes out of all studies is that the fluid model might not be easily constructed and, even if found, the convergence is not be trivial to prove. Sometimes little modifications, called tracking policy, have to be made on the solution to be asymptotic optimal. In any case the fluid approximation is known to give a good approximation and also an upper bound on optimization, see Bäuerle (2000).

In the next section we propose a continuous prices (controls) fluid approximation giving a static continuous prices policy directly usable in our Markovian Model. We conjecture that the value of this flow approximation gives an upper bound on any type of policy (even dynamic ones).

N.B. We propose a discrete control (prices) fluid formulation in appendix but this formulation is rather complicated and does not straight forwardly lead to a polynomial optimization. Since we have no reason to prefer continuous or discrete prices, we therefore focus on continuous ones.

## 3.3 SCLP model

In this section we are building a Mathematical Programming Model for the fluid approximation of the Stochastic VSS Pricing Problem with continuous prices.

For each trip $(a, b) \in \mathcal{D}$, at each time step $t \in \mathcal{T}$, we assume the existence of a continuous surjective function, strictly decreasing and covering all values in $[0,\ \Lambda^t_{a,b}]$, computing the demand for a given price: $demand(price)$. Where $\Lambda^t_{a,b}$ is the maximum demand from station $a$ at time $t$

to station $b$ at time $t+\mu_{a,b}^{-1}$.

### 3.3.1 A Continuous Convex Program

To build the flow model we define variables:

| | |
|---|---|
| $y_{a,b}(t)$ | the flow leaving station $a$ at time step $t$ to go to station $b$ at time step $t+\mu_{a,b}^{-1}$ at price $price(y_{a,b}(t))$; |
| $s_a(t)$ | the available stock at station $a$; |
| $r_a(t)$ | the number of parking spots reserved at station $a$. |

We can now build a Continuous Convex Program (CCP) giving the best fluid policy with the trick that we set the prices to have a demand exactly equal to the flow passing: $y_{a,b}^t=\lambda_{a,b}^t$. This avoid "arrival equity" issues, which occur for discrete prices.

**Fluid CCP**

$$\max \sum_{(a,b)\in D} \int_0^T y_{a,b}(t)\times price(y_{a,b}(t))\ dt \qquad \text{(Gain)} \tag{1a}$$

$$\text{s.t.} \sum_{a\in\mathcal{M}} s_a(0)=N \qquad \text{(Flow initialization)} \tag{1b}$$

$$s_a(0)=s_a(T)+r_a(T) \qquad \forall a\in\mathcal{M} \qquad \text{(Flow stabilization)} \tag{1c}$$

$$s_a(t)=s_a(0)+\int_0^t \sum_{(b,a)\in\mathcal{D}} y_{b,a}(\theta-\mu_{b,a}^{-1})-y_{a,b}(\theta)\ d\theta \qquad \forall a\in\mathcal{M},\ \forall t\in[0,T] \qquad \text{(Flow conservation)} \tag{1d}$$

$$0\le y_{a,b}(t)\le \Lambda_{a,b}^t \qquad \forall a,b\in\mathcal{M},\ \forall t\in[0,T] \qquad \text{(Max demand)} \tag{1e}$$

$$r_a(t)=\sum_{b\in\mathcal{M}} \int_0^{\mu_{b,a}^{-1}} y_{b,a}(t-\theta)\ d\theta \qquad \forall a\in\mathcal{M},\ \forall t\in[0,T] \qquad \text{(Reserved Park Spot)} \tag{1f}$$

$$s_a(t)+r_a(t)\le \mathcal{K}_a \qquad \forall a\in\mathcal{M},\ \forall t\in[0,T] \qquad \text{(Station capacity)} \tag{1g}$$

$$s_a(t)\ge 0 \qquad \forall a\in\mathcal{M},\ \forall t\in[0,T] \tag{1h}$$

Equations (1b) initialize the flow with the $N$ vehicles available. Equations (1c) constrain the solution to be stable, *i.e.* cyclic over the horizon. Equations (1d) is continuous version of the classic flow conservation. Equations (1e) constrain the flow on a demand edge not to be greater than the maximum demand. Equations (1f) set the reserved parking spot variable. Equations (1g) constrain the maximum capacity on a station and the parking spot reservation: For a station the number of reserved parking spots plus the number of vehicles already parked should not exceed its capacity.

Note that this model assume that there is an "off period" between the cycling horizons where all vehicles are parked at a station. However if it is not the case only some small changes have to be made in flow equations.

### 3.3.2 A SCLP approximation

If we consider a concave gain, we can approximate the Continuous Convex Program by a Separated Continuous Linear Program (SCLP) polynomially solvable, see Weiss (2008) who gives an extension of the simplex to solve it.

We keep all constraints of the previous Continuous Convex Program since they are linear and there is only few of them. We now simply make a linear approximation of the objective concave function adding a series of linear constraints:

$$gain_{a,b}(t) \leq y_{a,b}(t) \times price(\lambda_{a,b}(t)) \quad \rightsquigarrow \quad gain_{a,b}(t) \leq \left| \begin{array}{c} a_1(a,b,t) \times y_{a,b}(t) + b_1(a,b,t) \\ \cdots \\ a_k(a,b,t) \times y_{a,b}(t) + b_k(a,b,t) \end{array} \right|$$

That gives us finally a SCLP program with the following objective:

$$\max \sum_{(a,b)\in\mathcal{D}} \int_0^T gain_{a,b}(t)\ dt.$$

## 3.4 Discussion

The main advantage of this model is that it considers time dependent demands and that hence will give a macro management of the tide phenomenon. It gives static policies but nevertheless it can also help designing Dynamic ones, see Maglaras and Meissner (2006). N.B. A simple way is to make a multiple launch heuristic.

We assumed continuous surjective demand function and a concave gain. For instance $price(\lambda) = \lambda^{-\alpha}$ with $\alpha \in [0,1]$. Note that if we tolerate randomized pricing policy, our solution technique still works for general demand functions.

In the end, a weakness of this approach is that we have no control on the static policy time step. Indeed the optimal solution might lead to change the price every 5 minutes which is in practice not suitable.

Moreover since it is a deterministic approximation, this model doesn't take into account the stochastic aspect of the demand. For stations with small capacities it can be a problem since the variance of the demand could often lead to the problematic states: empty or full.

## 3.5 Questions & Conjectures

To the best of our understanding the Convex Continuous Program (1) is a fluid approximation of the stochastic VSS problem. It is classic in the literature, as in Maglaras (2006), to formally prove it linking the fluid model to an asymptotic limit of a scaled problem.

**Definition 3 ($s$-Scaled VSS Pricing)** *The $s$-Scaled VSS Pricing problem is the VSS Pricing problem when demand rates, station capacities and number of vehicles are multiplied by a coefficient $s$ while the gain to serve each trip and variance in transportation time are divided also by $s$.*

**Remark 1** *The reduction of the variance in the transportation time can be modelled by replacing the trip length transition rate $\mu$ by a succession of $s$ trips with transition rates $s \times \mu$.*

However in our case when $s$ tends to infinity there still exists some degenerated instances where the gap between the $s$-scaled problem and the fluid approximation cannot be reduce. For instance for 2 vehicles and 2 stations of capacity 1, a $s$-scaled problem would never be able to sell any trip contrary to the fluid model. For the instances dealt with this study, simulation results indicate that the $s$-scaled problem converges not too far from the fluid model as $s$ tends to infinity.

One would expect that the uncertainty in sales in the stochastic problem results in lower expected revenues. It was indeed shown in many applications, as in Gallego and van Ryzin (1994).

**Conjecture 1 (Fluid CCP UB)** *The Convex Continuous Program* (1) *optimal solution gives an Upper Bound on the Dynamic VSS Pricing problem.*

We think that one could prove the convergence of dynamic $s$-scaled problem toward the fluid model in some subclass of instances, such as system with infinite station capacity. Indeed in general we conjecture that as $s$ grows, the variance of the demand diminishes, and hence the optimal value of the dynamic $s$-scaled problem increases. When $s$ tends to infinity the optimal solution tends to be static and could be the solution of the fluid CCP formulation in some sub-classes of instances. The fluid CCP formulation would hence gives an Upper Bound on the $s$-scaled problem and notably the 1-scaled problem.

**Conjecture 2 ($s$-scaled increasing)** *The value of the solution of the dynamic $s$-scaled problem is increasing with $s$.*

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# A Mathematical formulation of the fluid model for discrete prices

Data:

| | |
|---|---|
| $\mathcal{M}$ | the set of stations |
| $\mathcal{K}_a$ | capacity of station $a$ |
| $\mathcal{D}$ | the set of possible trips ($= \mathcal{M} \times \mathcal{M}$) |
| $t_{a,b}(t)$ | the transportation time between station $a$ and $b = \mu_{a,b}^{-1}$ |
| $\lambda_{a,b}(t)$ | the transition rate of demands from station $a$ at time $t$ to station $b$ at time $t + t_{a,b}$ at price $p(\lambda_{a,b}(t))$ |
| $N$ | the number of cars available |

Variables:

| | |
|---|---|
| $p_a^+(t)$ | the proportion of requests accepted among those willing to leave station $a$ at $t$ |
| $p_a^-(t)$ | the proportion of requests accepted among those willing to have departure at time $t$, with $a$ as destination and that have not been refused a departure at their departure station. |
| $y_{a,b}(t)$ | the flow leaving station $a$ at time step $t$ (and arriving at station $b$ at time step $t + t_{a,b}$) |
| $y_{a,b}^{dep}(t)$ | the flow accepted by station $a$ (but not yet accepted by station $b$) |
| $y_{a,b}^{ref}(t)$ | the flow refused by station $b$ returning to station $a$ (one has $y_{a,b}^{dep}(t) = y_{a,b}^{ref}(t) + y_{a,b}(t)$) |
| $s_a(t)$ | the available stock at station $a$ |
| $r_a(t)$ | the number of parking spots reserved at station $a$ (flow in transit toward $a$) |

Figures 3 and 4 gives an instance with its corresponding variables.

$$
\begin{array}{llr}
\max \displaystyle\sum_{(a,b)\in D} \int_0^T y_{a,b}(t) \times price(a,b,t))\ dt & & \text{(Gain)} \\
\text{s.t.} \displaystyle\sum_{a\in\mathcal{M}} s_a(0) = N & & \text{(Flow initialization)} \\
s_a(0) = s_a(T) & \forall a \in \mathcal{M} & \text{(Flow stabilization)} \\
\delta_a^+(t) = \displaystyle\sum_b y_{a,b}(t) & \forall a \in \mathcal{M},\ \forall t \in [0,T] & \text{(Output Flow)} \\
\delta_a^-(t) = \displaystyle\sum_b y_{b,a}(t - t_{b,a}) + \sum_b y_{a,b}^{ref}(t) & \forall a \in \mathcal{M},\ \forall t \in [0,T] & \text{(Input Flow)} \\
y_{a,b}^{dep}(t) = y_{a,b}^{ref}(t) + y_{a,b}(t) & \forall (a,b) \in \mathcal{D},\ \forall t \in [0,T] & \text{(Flow conservation)} \\
s_a(t) = s_a(0) + \displaystyle\int_0^t \delta_a^-(\theta) - \delta_a^+(\theta)\ d\theta & \forall a \in \mathcal{M},\ \forall t \in [0,T] & \text{(Flow conservation)} \\
\dfrac{\delta s_a(t)}{\delta\theta} = \delta_a^-(t) - \delta_a^+(t) & \forall a \in \mathcal{M},\ \forall t \in [0,T] & \text{(Flow conservation V)} \\
p_a^+(t) = \begin{cases} 1 & \text{if } s_a(t) > 0, \\ \min\left\{1,\ \frac{\delta_a^-(t)}{\sum_b \lambda_{a,b}(t)}\right\} & \text{otherwise.} \end{cases} & \forall a \in \mathcal{M},\ \forall t \in [0,T] & \text{(Departure equity)} \\
p_a^-(t) = \begin{cases} 1 & \text{if } s_a(t) + r_a(t) < \mathcal{K}_a, \\ \min\left\{1,\ \frac{\delta_a^+(t)}{\sum_b y_{a,b}^{dep}(t)}\right\} & \text{otherwise.} \end{cases} & \forall a \in \mathcal{M},\ \forall t \in [0,T] & \text{(Arrival equity)} \\
y_{a,b}^{dep}(t) = p_a^+(t) \times \lambda_{a,b}(t) & \forall (a,b) \in \mathcal{D},\ \forall t \in [0,T] & \\
y_{a,b}(t) = p_b^-(t) \times y_{a,b}^{dep}(t) & \forall (a,b) \in \mathcal{D},\ \forall t \in [0,T] & \text{(Pushing flow with equity )} \\
r_a(t) = \displaystyle\sum_b \int_0^{t_{b,a}} y_{b,a}(t - \theta)\ d\theta & \forall a \in \mathcal{M},\ \forall t \in [0,T] & \text{(Reserved Park Spot)} \\
s_a(t) + r_a(t) \le \mathcal{K}_a & \forall a \in \mathcal{M},\ \forall t \in [0,T] & \text{(Station Capacity)} \\
\lambda_a(t) = \Lambda(price(a,b,t)) & \forall a \in \mathcal{M},\ \forall t \in [0,T] & \text{(Demand elasticity)}
\end{array}
$$

**Remark 2** *Without the flow stabilization constraint, it would be easy to compute the value of a solution with one price. A simple iterative algorithm on the horizon would work. With flow stabilization constraint it is not clear that cycling on that iterative algorithm would lead to a stationary solution.*

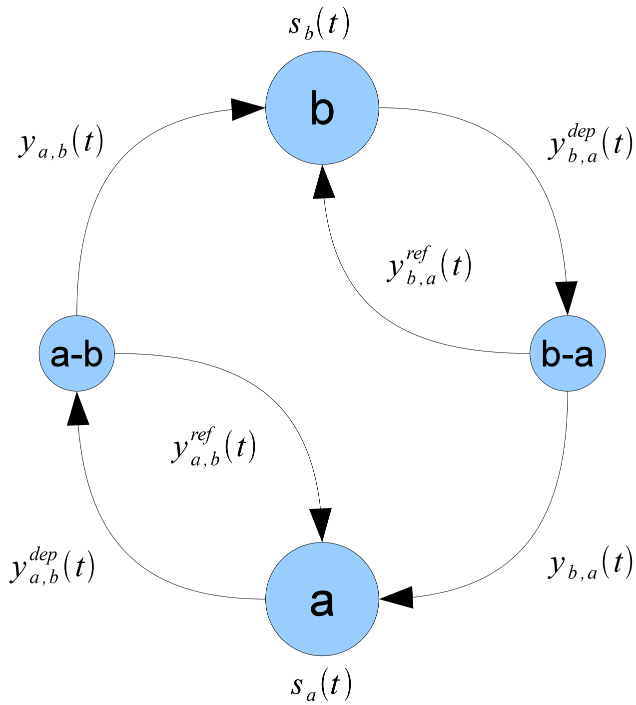


Figure 3: Variables for 2 stations.

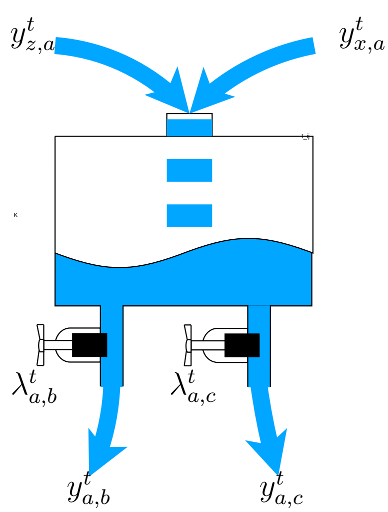


Figure 4: An equity issue.