Vehicle Sharing System Pricing Regulation: A Fluid Approximation

Ariel Waserhole, Vincent Jost

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Abstract

This paper gives a fluid approximation for a Vehicle Sharing System Pricing problem (VSS-P).

1. The VSS-P is formulated as a Closed Queuing Network with finite buffers, time dependent service time variation and continuous controls on transition rates for the pricing.
2. Solving the model for general (dynamic) policies seams intractable. Therefore a fluid approximation giving piecewise static policy is formulated. It is conjectured asymptotically optimal when the problem is scaled by a factor tending to infinity.
3. A reusable benchmark and an experimental protocol is created for the general Vehicle Sharing System optimization problem.
4. Numerical experiments are run on toy cities comparing the classic protocol with reservation of the parking spot at destination and the static policy given by the fluid approximation.

1 Introduction

1.1 Context

Shoup [20] reports that, based on a sample of 22 US studies, cars looking for a parking spot contribute to 30% of the city traffic. Moreover most cars are not in use 90% of the time (use 2 hours per day) but still occupying parking spots! Could we have less vehicles for the same demand satisfaction?

Recently, the interest in Vehicle Sharing Systems (VSS) in cities has increased significantly. Indeed nowadays, urban policies intend to discourage citizens to use their personal car downtown by reducing the number of parking spots, street width, etc. VSS seem to be a promising solution to reduce traffic congestion (parking issues), noise, and air pollution (using bikes or electric cars). They offer personal mobility allowing users to pay only for the usage.

We are interested in short-term one-way VSS where vehicles can be taken and returned at different places paying by the minute. Associated with classical public transportation systems, short-term one-way VSS help to solve one of the most difficult public transportation problem: the last kilometer issue [9]. This is not the case for round-trip VSS, where vehicles have to be returned at the station where they were taken.

The first large scale short-term one-way VSS was the Bicycle Sharing System (BSS) Vélib' [3]. It was implemented in Paris in 2007 and now has more than 1200 stations and 20 000 vehicles selling around 110 000 trips per day. It has inspired several other cities all around the world, and now more than 300 cities have such a system, including Montréal, Bejing, Barcelona, Mexico City, Tel Aviv [9].

1.2 One-way Vehicle Sharing Systems: a management issue

However if the freedom increases for the user in the one way model, it implies a higher complexity in its management. In round trip type rental systems, the only stock that is relevant when managing yield and reservations is the number of available vehicles. In one-way systems, a new problem occurs because vehicles aren't the only key resources any more. In practice, parking stations have a maximum number of spots, and when the total number of vehicle is comparable to the total number of parking spots, available parking spots become a new key resource.

Since first BSS, problems of bikes and parking spots availability have appeared very often. Causes are various but we can highlight two important phenomenons: the gravitational effect which indicates that a station is constantly unbalanced (as Montmarte hill in Vélib'), and the tide phenomenon representing the oscillation of demand intensity along the day (as morning and evening flows between working and residential areas).

To improve the efficiency of the system in the literature different perspectives have been studied. At a strategic level, some authors considered the optimal capacity and locations of bike rental stations. Shu et al. [21] proposed a stochastic network flow model to support these decisions. They used their model to design a BSS in Singapore based on demand forecast derived from current usage of the mass transit system. Lin and Ta-Hui [15] considered a similar problem but formulated it as a deterministic mathematical model. Their model is aware of the bike path network and mode sharing with other means of public transportation.

At a tactical level, other authors have investigate the optimal number of vehicles given a set of station. George and Xia [13] have studied the fleet sizing problem with constant demand and no parking capacity. Fricker and Gast [10, 11] looked into the optimal sizing of a fleet in "toy" cities, where demand is constant over time and identical for every possible trip, and all stations have the same capacity $\mathcal{K}$. They showed that even with an optimal fleet sizing in the most "perfect" city, if there is no operational system management, there is at least a probability of

$\frac{1}{\mathcal{K}+1}$ that any given station is empty or full.

At an operational level, when the system is not able to meet the demand with a reasonable standard of quality, currently in BSS, trucks are used to balance the bikes among the stations. The problem is to schedule vehicle routes to visit some of the stations to perform pickup and delivery so as to minimize the number of users who cannot be served, i.e., the number of users who try to take a bike from an empty station or to return it in a full station. In the literature many papers deal already with this problem. A static version of the BSS balancing problem is treated in Chemla et al. [7] and a dynamic one in Contardo et al. [8].

1.3 Our scope: Self Regulation Systems

A new type of VSS have appeared lately: one-way Car Sharing Systems (CSS) with Autolib' in Paris [1] and Car2go [2] in more than 10 cities (Vancouver, San Diego, Lyon, Ulm...). With cars, operational balancing optimization through relocation seems inappropriate due to their size. This is the reason why we are looking here into a different operational management approach: Self Regulating Systems through pricing. In this study, assuming that demand is elastic, we use the prices to influence user choices and drive the system towards its most efficient dynamic.

2 Model: An intractable Markov Decision Process

2.1 Restriction to a simple protocol

In a real context, a user wants to use a vehicle to take a trip between an original (GPS) location a, and a final one b, during a specified moment. On a station based VSS, he tries to find the closest station to location a with a vehicle to take and the closest station to location b with a parking spot to return it. All along this process users decisions rely on several correlated inputs such as: trip total price, walking distance, public transportation competition, time frame...

A time elastic GPS to GPS demand forecast correlated to a user's decision protocol ruling his behaviour to take a trip between two specific stations at a specific time seems closer to reality but introduces of course a big complexity (use of utility function for instance).

Therefore we are here going to consider a simple station to station demand forecast with only real time reservation for a specified trip. It means that the user will engage himself to return the vehicle at a specified station and time. Finally it amounts in considering a demand for the following simplified protocol:

1. A user asks for a vehicle at station a (here and now), with destination b and rental duration $\mu_{a,b}^{-1}$;
2. The system offers a price (or rejects the user = infinite price);

3. The user accepts the price, takes the vehicle and a parking spot is reserved (or leaves the system).

2.2 A Vehicle Sharing System Stochastic Model

2.2.1 Markovian framework

We define in this section a framework to model a stochastic Vehicle Sharing Systems with the protocol defined in the previous section.

Definition 1 (Vehicle Sharing Systems Markovian Model) *In a city there is is a fleet of N vehicles along a set $\mathcal{M}$ ($|\mathcal{M}|= M$) of stations with capacity $\mathcal{K}_i$, $i \in \mathcal{M}$. There is an elastic demand between each station $\mathcal{D} = \mathcal{M} \times \mathcal{M}$. This demand is piecewise constant on time steps $\mathcal{T}$, it follows a Poisson distribution of parameter $\lambda^t_{a,b}(p^t_{a,b})$ to go from station $a \in \mathcal{M}$ to station $b \in \mathcal{M}$ at the period $t \in \mathcal{T}$ and is function of the proposed price $p^t_{a,b}$.*

All durations follow an exponential distribution: The transportation time to go from station a to station b at time step t has for mean $1/\mu^t_{a,b}$; The time step duration $t \in \mathcal{T}$ has for mean $1/\tau^t$ and the total horizon length has for mean $T = \sum_{t\in\mathcal{T}} 1/\tau^t$.

2.2.2 Closed queuing network model

For a given demand $\lambda^t_{a,b}$ for every trips $(a, b) \in \mathcal{D}$ and every time steps $t \in \mathcal{T}$ (*i.e.* for a given price $p^t_{a,b}$). We can model this stochastic Vehicle Sharing System by a Closed Queuing Network with finite buffers and service time variation, see figure 1.

Each demand $(a, b) \in \mathcal{D}$ is represented by a server $(a - b)$ which has a time dependent service rate equal to the average number of clients willing to take a trip from station a to station b: $\lambda^t_{a,b}$. Demands with same station of origin a ($(a, b) \in \mathcal{D}$, $\forall b \in \mathcal{M}$) are sharing the same finite buffer of size $\mathcal{K}_a$: the capacity of station a. When a vehicle (a job) is picked up to take the trip (a, b) (is processed by server (a, b)) before going to station (server) b it has to pass by a transportation state (server) $(a - b)$ with a service time proportional to the number of vehicles $n_{a,b}$ in transit (in the buffer): $n_{a,b} \times \mu^t_{a,b}$.

2.2.3 Continuous-Time Markov Chain (CT-MC) formulation

If a price $p^t_{a,b}$ is set for all trips $(a, b) \in \mathcal{D}$ at all time steps $t \in \mathcal{T}$, we can model the closed queuing network by a Continuous Time-Markov Chain on a set of states $\mathcal{S}$:

$$\mathcal{S} = \left\{ \Big(n_a : \ a \in \mathcal{M}, \ n_{a,b} : \ (a, b) \in \mathcal{D}, \ t \in \mathcal{T} \Big) \ / \sum_{i\in\mathcal{M}\cup\mathcal{D}} n_i = N \ \& \ n_a + \sum_{b\in\mathcal{M}} n_{b,a} \le \mathcal{K}_a, \ \forall a \in \mathcal{M}, \ \forall t \in \mathcal{T} \right\}$$

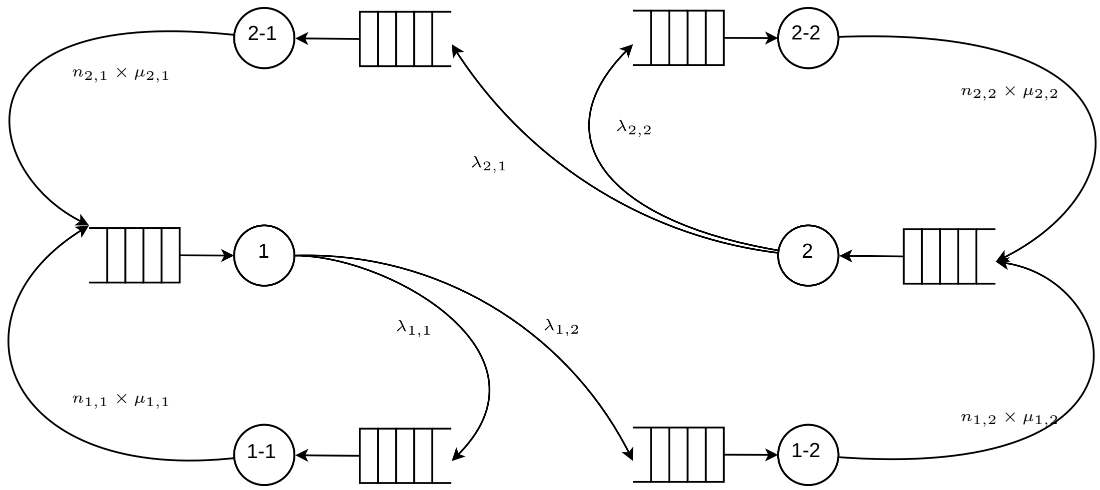

Figure 1: A closed queuing network model with servers for demands and transportation times.

A state $s = (n_a : \ a \in \mathcal{M}, \ n_{a,b} : \ (a,b) \in \mathcal{D}, \ t \in \mathcal{T})$ represents the repartition of the vehicles in the city space (in station or in transit) at a given time. At time step t, n_a is the number of vehicles in station $a \in \mathcal{M}$, $n_{a,b}$ the number of vehicles in transit between stations a and b serving a trip demand $(a,b) \in \mathcal{D}$.

The transition rates between states are either:

- The taking of a vehicle at a station a to go to a station b which gives a transition rate $\lambda^t_{a,b}(p^t_{a,b})$ between states $(\dots, \ n_a, \dots, \ n_{a,b} \dots, \ t)$ and states $(\dots, \ n_a - 1, \dots, \ n_{a,b} + 1, \dots, \ t)$ with $n_a > 0$ and $n_b + \sum_{c \in \mathcal{M}} n_{c,b} < \mathcal{K}_b$;
- The arrival of a vehicle at a station b from a station a which gives a transition rate $n_{a,b} \times \mu^t_{a,b}$ between states $(\dots, \ n_b, \dots, \ n_{a,b} \dots, \ t)$ and states $(\dots, \ n_b + 1, \dots, \ n_{a,b} - 1, \dots, \ t)$ with $n_{a,b} \geq 1$;
- The changing of piecewise constant demand time step which gives a transition rate τ^t between states $(\dots, \ t)$ and states $(\dots, \ t + 1 \ \mathrm{mod} |\mathcal{T}|)$.

We can note that there is an exponential number of states. Even with only one time-step, without transportation time but with infinite station capacities there is $\binom{N+M-1}{N}$ states. For instance for a small system with $N = 150$ vehicles and $M = 50$ stations it gives roughly 10^{47} states! This number grows linearly with the number of time steps and exponentially with the number of different transportation times considered.

2.3 Model optimization

2.3.1 Vehicle Sharing System Pricing Problem

In the previous section we modelled the system by a closed queuing network with finite buffer and service time variation that can be described explicitly through a Continuous-Time Markov Chain. Through this model we want now to optimize the Vehicle Sharing System in order to maximize the average revenue. To do so we have as leverage the possibility to change the price to take a trip which will, assuming an elastic demand, influence the demand for such trip. We call this problem the Stochastic Vehicle Sharing System Pricing Problem.

Definition 2 (VSS Pricing Problem) *The Stochastic Vehicle Sharing System Pricing Problem amounts in setting price for every trip in order to maximize the gain of the Vehicle Sharing System Markovian Model.*

Prices can be Discrete, i.e. selected in a set of possibilities, or Continuous i.e. chosen in a range. Pricing policies can be Dynamic, i.e. dependent on system's state (vehicle repartition and period of the day), or Static i.e. independent on system's state, set in advance and function of the trip and the time of the day.

2.3.2 An intractable Markov Decision Process (MDP) Resolution

We can solve this queuing network model through the well known Markov Decision Process framework based on the Continuous-Time Markov Chain given in the previous section. To do so define a set $\mathcal{Q}$ of possible discrete prices for each trip at each time step. A trip $(a, b) \in \mathcal{D}$ at time step $t \in \mathcal{T}$ at price $p^{t,q}_{a,b},\ q \in \mathcal{Q}$ have a demand (transition rate) $\lambda^t_{a,b}(p^{t,q}_{a,b}) = \lambda^{t,q}_{a,b}$.

Solving this MDP computes the best Dynamic System State Dependent Trip Discrete Pricing policy, *i.e.* the price for a trip depends of the current state of the system (vehicle repartition). MDP are known to be polynomially solvable on the number of states $|\mathcal{S}|$ and the number of actions $|\mathcal{A}|$ available in each state. For example through Value Iteration, Policy Iteration algorithm or Linear Programming techniques, *c.f.* book of Puterman [19].

In this particular case, we are dealing with a pricing problem and the action space $\mathcal{A}(s)$ in each state $s \in \mathcal{S}$ is the Cartesian product of the available prices for each trip *i.e.* $\mathcal{A}(s) = \mathcal{Q}^M$. However to not suffer from this exponential explosion we can model this problem as a Action Decomposable Markov Decision Process, *c.f.* Waserhole et al. [22]. It is a general method based on the Event Based Dynamic Programming, *c.f.* Koole [14], to reduce the complexity of the action space to $\mathcal{A}(s) = \mathcal{Q} \times M$. It allows to use Value Iteration, Policy Iteration algorithm or Linear Programming solution techniques.

Lemma 1 (Dynamic Pricing MDP) *The previous MDP gives the best Dynamic System State Dependent Discrete Pricing Policy for the stochastic VSS Pricing problem.*

Anyway the state space is still exponential so it does not give a polynomial algorithm. We have therefore to look into approximations or simplifications to tackle the problem.

2.3.3 State of the art on this model

In the literature simple forms of this closed queuing network model with the relationship to the underlying CT-MC have already been studied for VSS. George and Xia [13] have considered a VSS with only one time step, one price and infinite station capacities. With these assumptions they have been able to deduce a compact form to compute the system performance through the help of the BCMP network theory [4]. In the end considering a cost to maintain a vehicle and a gain to rent it, they have successfully deduced the optimal sizing of the fleet.

Fricker and Gast [10] have considered simple cities that they called homogeneous. They have a unique fixed station capacity $\mathcal{K}_a = \mathcal{K}$, a constant (one time step) arrival rate and uniform routing matrix, $\lambda_{a,b}^t = \frac{\lambda}{M}$ and a unique exponential travel time of mean ${\mu_{a,b}^t}^{-1} = \mu^{-1}$. Through the use of a Mean Field Approximation, they have been able to give some asymptotic results when the number of stations tends to infinity: $M \to \infty$. If there is no operational regulation system the optimal sizing is to have $\frac{\mathcal{K}}{2} + \frac{\lambda}{\mu}$ vehicles per station which correspond in filling half of the station plus the average number of vehicles in transit $(\frac{\lambda}{\mu})$. They show that even with an optimal sizing there is a probability $\frac{1}{\mathcal{K}+1}$ to be empty or full for each station which is considered pretty bad. In another paper Fricker et al. [11] have extended some analytical results and just verified experimentally some others on inhomogeneous cities modelled by clusters.

Fricker and Gast [10] have also studied in homogeneous cities a heuristic called "The power of two choices" using incentives that can be seen as a Dynamic Station State Pricing. When a user is showing at a station and takes a vehicle, he gives randomly two possible stations of destination and the system is directing him to the least loaded one. They show that this policy allows to drastically reduce the probability to be empty or full for each station to $2^{-\frac{\mathcal{K}}{2}}$.

Finally none of these models includes time dependent demands, pricing or full heterogeneity which is the subject of this paper with the fluid approximation.

3 A fluid approximation

3.1 A plumbing problem

We use the term plumbing because we are considering continuous vehicles that can in fact be seen as a flow passing through pipes. The stations are tanks with capacity the size of the station, and the demands between stations are pipes with width the amount of these demands. Finally the length of the pipes represents the duration of the trip reservation. Figure 2 gives an example for two stations.

We are modelling a system which has no direct interaction with the user. The decisions are static and have to be taken before, once for all. They amount in setting the width of a pipe by changing the price to pass flow in it: the higher the price is, the smaller the pipe (demand) will be.

However a free system does not mean that it is possible to do everything! Indeed first, if a pipe (a demand) exists and there is some flow (vehicle) available in the tank (station), according to gravity first come first serve law, the flow has to pass through the pipe until no flow is available.

Secondly if there is not enough flow to fulfil all pipes (demands), according to the same law there should be some equity between them. Hence the proportion of filling up of all pipes should be equal.

However there is some restrictions to this rule. If it the arrival tank of a pipe is full, it might be impossible to fulfil this pipe. In this case another equity rule should be applied to all pipes discharging into this tank.

In other words for each pipe, if its discharging tank is full it has the same proportion of filling up as the other pipes discharging in this tank, otherwise it has the same proportion of filling up as the other pipes coming from its source tank.

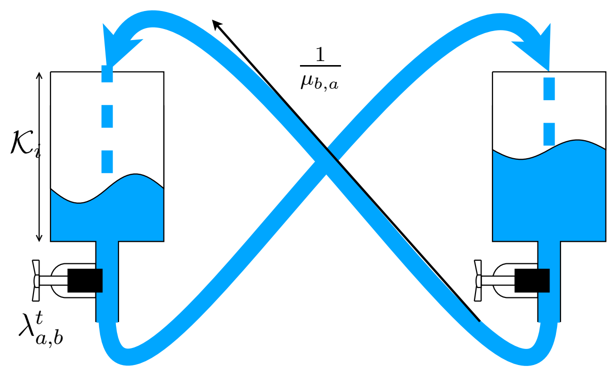

Figure 2: A Plumbing Problem.

3.2 More formally

The fluid model is constructed by replacing stochastic demands by a continuous flow with the corresponding deterministic rate. It gives a deterministic and continuous dynamics and evolves as a continuous process. Optimizing the fluid model to give heuristics on the stochastic one is a well know technique derived as a limit under a strong-law-of-large numbers type of scaling as the potential demand and the capacity grow proportionally large; see Gallego and van Ryzin [12]. Lots of application of this principle are available in the literature to deal with revenue management problem, see [16] for instance. However to the best of our knowledge there is no direct classic approach available for a general case including our application. Nevertheless there is some work on theorizing the fluid approximation scheme: Meyn [18] described some approaches to the synthesis of optimal policies for multiclass queueing network models based upon the close connection between stability of queueing networks and their associated fluid limit models; Bäuerle [6] generalized open multi class queueing networks and routeing problems.

What goes out of all studies is that the fluid model might not be easily constructed and even if found, the convergence will not be trivial to prove. Sometime some little modification, called tracking policy, have to be made on the solution to be asymptotic optimal, but in any case the fluid approximation is known to give a good approximation and also an upper bound on optimization, see Bäuerle [5].

In the next section we propose a continuous prices (controls) fluid approximation giving a static continuous prices policy directly usable in our Markovian Model. We conjecture that the value of this flow approximation gives an upper bound on any type of policy (even dynamic ones).

N.B. We propose a discrete control (prices) fluid formulation in appendix but this formulation is rather complicated and does not straight forwardly lead to a polynomial optimization. Since we have no reason to prefer continuous or discrete prices, we therefore focus on continuous ones.

3.3 SCLP model

In this section we are building a Mathematical Programming Model for the fluid approximation of the Stochastic VSS Pricing Problem with continuous prices.

For each trip $(a, b) \in \mathcal{D}$, at each time step $t \in \mathcal{T}$, we assume the existence of a continuous surjective function, strictly decreasing and covering all values in $[0,\ \Lambda^t_{a,b}]$, computing the demand for a given price: $demand(price)$. Where $\Lambda^t_{a,b}$ is the maximum demand from station a at time t to station b at time $t + \mu^{-1}_{a,b}$.

3.3.1 A Continuous Convex Program

To build the flow model we define variables:

$y_{a,b}(t)$	the flow leaving station a at time step t to go to station b at time step $t + \mu^{-1}_{a,b}$ at price $price(y_{a,b}(t))$;
$s_a(t)$	the available stock at station a;
$r_a(t)$	the number of parking spots reserved at station a.

We can now build a Continuous Convex Program (CCP) giving the best fluid policy with the trick that we set the prices to have a demand exactly equal to the flow passing: $y^t_{a,b} = \lambda^t_{a,b}$. This avoid "arrival equity" issues, which occur for discrete prices.

Fluid CCP

$$\max \sum_{(a,b)\in D} \int_0^T y_{a,b}(t) \times price(y_{a,b}(t))\ dt \qquad \text{(Gain)} \tag{1a}$$

$$\text{s.t.} \sum_{a\in\mathcal{M}} s_a(0) = N \qquad \text{(Flow initialization)} \tag{1b}$$

$$s_a(0) = s_a(T) + r_a(T) \qquad \forall a \in \mathcal{M} \qquad \text{(Flow stabilization)} \tag{1c}$$

$$s_a(t) = s_a(0) + \int_0^t \sum_{(b,a)\in\mathcal{D}} y_{b,a}(\theta - \mu_{b,a}^{-1}) - y_{a,b}(\theta)\ d\theta \qquad \forall a \in \mathcal{M},\ \forall t \in [0,T] \qquad \text{(Flow conservation)} \tag{1d}$$

$$0 \leq y_{a,b}(t) \leq \Lambda_{a,b}^t \qquad \forall a,b \in \mathcal{M},\ \forall t \in [0,T] \qquad \text{(Max demand)} \tag{1e}$$

$$r_a(t) = \sum_{b\in\mathcal{M}} \int_0^{\mu_{b,a}^{-1}} y_{b,a}(t-\theta)\ d\theta \qquad \forall a \in \mathcal{M},\ \forall t \in [0,T] \qquad \text{(Reserved Park Spot)} \tag{1f}$$

$$s_a(t) + r_a(t) \leq \mathcal{K}_a \qquad \forall a \in \mathcal{M},\ \forall t \in [0,T] \qquad \text{(Station capacity)} \tag{1g}$$

$$s_a(t) \geq 0 \qquad \forall a \in \mathcal{M},\ \forall t \in [0,T] \tag{1h}$$

Equations (1b) initialize the flow with the N vehicles available. Equations (1c) constrain the solution to be stable, *i.e.* cyclic over the horizon. Equations (1d) is continuous version of the classic flow conservation. Equations (1e) constrain the flow on a demand edge not to be greater than the maximum demand. Equations (1f) set the reserved parking spot variable. Equations (1g) constrain the maximum capacity on a station and the parking spot reservation: For a station the number of reserved parking spots plus the number of vehicles already parked should not exceed its capacity.

Note that this model assume that there is an "off period" between the cycling horizons where all vehicles are parked at a station. However if it is not the case only some small changes have to be made in flow equations.

3.3.2 A SCLP approximation

If we consider a concave gain, we can approximate the Continuous Convex Program by a Separated Continuous Linear Program (SCLP) polynomially solvable, see Weiss [23] who gives an extension of the simplex to solve it.

We keep all constraints of the previous Continuous Convex Program since they are linear and there is only few of them. We now simply make a linear approximation of the objective concave function adding a series of linear constraints:

$$gain_{a,b}(t) \leq y_{a,b}(t) \times price(\lambda_{a,b}(t)) \quad \rightsquigarrow \quad gain_{a,b}(t) \leq \left| \begin{array}{c} a_1(a,b,t) \times y_{a,b}(t) + b_1(a,b,t) \\ \dots \\ a_k(a,b,t) \times y_{a,b}(t) + b_k(a,b,t) \end{array} \right|$$

That gives us finally a SCLP program with the following objective:

$$\max \sum_{(a,b)\in\mathcal{D}} \int_0^T gain_{a,b}(t)\ dt.$$

3.4 Discussion

The main advantage of this model is that it considers time dependent demands and that hence will give a macro management of the tide phenomenon. It gives static policies but nevertheless it can also help designing Dynamic ones, see Maglaras and Meissner [17]. N.B. A simple way is to make a multiple launch heuristic.

We assumed continuous surjective demand function and a concave gain. For instance $price(\lambda) = \lambda^{-\alpha}$ with $\alpha \in [0,1]$. Note that if we tolerate randomized pricing policy, our solution technique still works for general demand functions.

In the end, a weakness of this approach is that we have no control on the static policy time step. Indeed the optimal solution might lead to change the price every 5 minutes which is in practice not suitable.

Moreover since it is a deterministic approximation, this model doesn't take into account the stochastic aspect of the demand. For stations with small capacities it can be a problem since the variance of the demand could often lead to the problematic states: empty or full.

3.5 Questions & Conjectures

To the best of our understanding the Convex Continuous Program (1) is a fluid approximation of the stochastic VSS problem. It is classic in the literature, as in Maglaras [16], to formally prove it linking the fluid model to an asymptotic limit of a scaled problem.

Definition 3 (s-Scaled VSS Pricing) *The s-Scaled VSS Pricing problem is the VSS Pricing problem when demand rates, station capacities and number of vehicles are multiplied by a coefficient s while the gain to serve each trip and variance in transportation time are divided also by s.*

Remark 1 *The reduction of the variance in the transportation time can be modelled by replacing the trip length transition rate μ by a succession of s trips with transition rates $s \times \mu$.*

However in our case when s tends to infinity there still exists some degenerated instances where the gap between the s-scaled problem and the fluid approximation cannot be reduce. For instance for 2 vehicles and 2 stations of capacity 1, a s-scaled problem would never be able to sell any trip contrary to the fluid model. For the instances dealt with this study, simulation results indicate that the s-scaled problem converges not too far from the fluid model as s tends to infinity.

One would expect that the uncertainty in sales in the stochastic problem results in lower expected revenues. It was indeed shown in many applications, as in Gallego and van Ryzin [12].

Conjecture 1 (Fluid CCP UB) *The Convex Continuous Program* (1) *optimal solution gives an Upper Bound on the Dynamic VSS Pricing problem.*

We think that one could prove the convergence of dynamic s-scaled problem toward the fluid model in some subclass of instances, such as system with infinite station capacity. Indeed in general we conjecture that as s grows, the variance of the demand diminishes, and hence the optimal value of the dynamic s-scaled problem increases. When s tends to infinity the optimal solution tends to be static and could be the solution of the fluid CCP formulation in some sub-classes of instances. The fluid CCP formulation would hence gives an Upper Bound on the s-scaled problem and notably the 1-scaled problem.

Conjecture 2 (s-scaled increasing) *The value of the solution of the dynamic s-scaled problem is increasing with s.*

4 Simulation

To be able to consider real data we would need to see exactly what was the original demand and not only which trips were finally served as is available in some exploitation data. Currently we do not dispose of any data and generating them with time dependent elastic demand, structured tides, etc. is a complex task. To have a clear experimental protocol we therefore restrain to an analysis of toy cities.

4.1 Experimental protocol

A city formed with stations on a grid We consider a Vehicle Sharing System implemented in a city where stations are positioned on a grid of width w and length l and travel time unity $t_{\min} = 15$ (closest distance between two points of the grid). A number $M = l \times w$ of stations are positioned at regular interval on this grid and the distance to go from one to another is computed thanks to the Manhattan distance in time.

There is either a unique station capacity $\mathcal{K} = 10$ with a number of $N = 60\% \times M \times \mathcal{K}$ of vehicles, or infinite station capacities but with the same number of vehicles as in the system with station capacity.

Demand pattern A day lasts 12 hours (say from 6h00 to 18h00). At the end of each day, all vehicles must return to a station. We take as a base case a fully homogeneous city, *i.e.* demands for any trip have the same probability to happen: $\lambda^t_{a,b} = \lambda,\ \forall(a,b) \in \mathcal{D}, \forall t \in \mathcal{T}$. We then introduce tide and gravitational phenomenon to look at their influence.

Instances We work on homogeneous cities, it means cities where all trip requests have the same probability to happen, *i.e.* $\lambda^t_{a,b} = \lambda,\ \forall(a,b) \in \mathcal{D},\ \forall t \in \mathcal{T}$. We only consider one way trips: $\lambda^t_{a,a} = 0,\ \forall a \in \mathcal{M},\ \forall t \in \mathcal{T}$. Instance "$M\ w \times l\ I\lambda_s\ [G\Gamma]\ [T\Theta]$" have to be read as follow: It is an homogeneous city with M stations spread on a grid of size w times l with a demand intensity λ_s per minute per station ($\lambda_s = (M-1) \times \lambda$) and with possibly a gravitational effect of intensity Γ or a tide effect of intensity Θ.

Optimizing the Number of trip sold In this study, to avoid to consider a complex demand elasticity function we only optimize the number of trips sold by the system. Therefore we only have to consider that there exists a continuous surjective demand function, with a maximum possible demand λ, i.e. there exists a price to obtain any demands between 0 and the maximum demand λ.

We are investigating if it possible to improve a classic VSS (typical BSS) with a fixed price set as the lowest possible (CLASSIC), with the fluid heuristic (FLUID) . Note that in our model the maximum demand for a trip corresponds to this lowest price even if we think that in reality, with pricing strategy, it would be interesting to reduce this price (even maybe paying users) to attract more users on certain trips.

Simulation We test the fluid approximation heuristic through a simulation on 250 days with similar demand patterns (using a 10 days warm up). We use a reservation protocol, *i.e.* users have to book a parking spot at destination in order to take a vehicle, and compare FLUID and CLASSIC pricing policy on the same scenario (same realization of demands),

Regarding the number of trips sold we look at: The average number of trip requests (Nb trips); The average number of trips sold by CLASSIC policy; The average number of trips sold by FLUID policy and its relative gain: (FLUID-CLASSIC)/CLASSIC; And the optimization efficiency with the upper bound (UB) given by the value of the fluid model, its number of trips sold and relative gain (UB-CLASSIC)/CLASSIC.

Regarding the vehicle utilization, we look at the proportion of time a vehicle is in use for CLASSIC, FLUID policy (or expected for UB): $Time\ (Sold\ or\ Expected)/(N \times (12h + \max_{(a,b)\in\mathcal{D}} \mu_{a,b}^{-1})$, and the relative gain of FLUID and UB: (FLUID or UB-CLASSIC)/CLASSIC.

Algorithm implementation To compute easily the SCLP program, we use a discrete time approximation (with 5 minutes time step) that gives us a Linear Program solvable by a classic solver. It is a classic way to approximate Continuous Time Linear Program to discretize time into fixed length time step. It is obviously not optimal, however since in our case we are looking at a general behaviour having an error of 1% doesn't really matter, and as we see table 1 taking 5 minutes time seems to ensure us to have reasonable results.

Note that there is monotonic increases in the FLUID and UB Gap when looking at time step divisors. However even if the general tendency is in the increase there are examples of smaller time step being less efficient than a bigger one (for instance 2 and 5 minutes time step in table 1).

In our instances, transportation times are multiple of 15 minutes (Manhattan distances). It as to be scale to fit the demand and the station capacities.

Another point is that time step of discretization rules fluid time step policy, yet we have piecewise constant demand therefore to be the most accurate we should have time step of discretization divisor of it.

N.B. When a demand is changing within a time step, a constant constant demand equals to the average demand is taken.

Time step	180	90	60	45	30	20	15	12
FLUID Rel. Gain.	-82.8%	-55.6 %	-33.9%	-10.3%	-1.8%	-2.2%	6.4%	2.5%
UB Rel. Gain.	-70.5 %	-38.4%	-6.2%	23.4 %	38.6%	36.1%	57.3%	45.3%
Time step	10	8	6	5	4	3	2	
FLUID Rel. Gain.	5.4%	5.9%	6.2%	7%	6.1%	7.3%	6.8%	
UB Rel. Gain.	52.3%	52.9%	54.5%	58.4%	54.4%	58.6%	57.2%	

Table 1: Time discretization (in min.) convergence for instance "024 4x6 I3 T3" with station capacity 10.

4.2 Influence of the demand intensity

We first look at the influence of the demand intensity in an homogeneous city. In table 2 and 3, we study the behaviour of the Fluid Approximation heuristic (optimizing the number of trips sold) in an homogeneous city with 24 stations where the demand intensity ranges from 0.1 to 0.9 demands per station per minute.

In table 2 we see that the performance of the optimization is strongly related to demand intensity: The higher the demand intensity is, the higher the improvement of the fluid heuristic will be on the FREE system. However as can be seen in table 3, that looks at the average vehicle proportion at use, if the Fluid Heuristic increases the number of trips sold it also decreases the use of the vehicles. It can be explained since in order to optimize the number of trips sold in a homogeneous city, the system favours short distance trips, especially if demand is intense.

		$\mathcal{K} = 10$		
Instance	Nb trips	CLASSIC	FLUID	UB
024 4x6 I1	1656	1215	1215.1 (0.0%)	1655.9 (36.2%)
024 4x6 I3	4968	2000	2047.1 (2.3%)	3042.0 (52.0%)
024 4x6 I6	9932	2169	2622.2 (20.8%)	3912.0 (80.2%)
024 4x6 I9	14906	2214	2984.3 (34.7%)	4482.0 (102.3%)

Table 2: Influence of the demand intensity in homogeneous cities. Legend $A(R)$: A = Absolute number of trips sold; R = Relative gain compared to the CLASSIC policy.

	$\mathcal{K} = 10$		
Instance	CLASSIC	FLUID	UB
024 4x6 I1	45.4%	45.4% (0.0%)	68.4% (50.5%)
024 4x6 I3	74.4%	54.5% (-26.7%)	88.7% (19.1%)
024 4x6 I6	80.8%	54.6% (-32.3%)	88.7% (9.7%)
024 4x6 I9	82.6%	54.4% (-34.0%)	88.7% (7.3%)

Table 3: Influence of the demand intensity in homogeneous cities: Legend $A(R)$: A= Average absolute proportion of vehicles in use; R: Relative gain compared to the CLASSIC policy.

So how to choose the correct demand intensity? In Vélib' Paris [3] there is approximately 150 000 trips sold per day for about 1400 stations. If we now consider that the majority of the trips are made during 18 hours of the day it gives approximately an arrival intensity of 0.1 client per station per minute ($\approx$ 1750 trips sold per day for 24 stations). In Vélib' this represents the final satisfied demand, that means in our simulation what was sold by the CLASSIC policy (regardless of the relocation process), hence from simulation table 2, we can consider in the following of this study that an intensity of 0.3 clients per station per minute ($\approx$ 2000 trips sold per day) is in the good order of magnitude.

4.3 Influence of gravitation

We study the influence of the introduction of gravitation in an homogeneous city with demand intensity λ. We split the city into two equal subgrids: $\mathcal{L}$ (left) and $\mathcal{R}$ (right) and increase by a factor Γ the demands for trips going from a station $l \in \mathcal{L}$ to a station $r \in \mathcal{R}$ while we decrease the opposite demand by the same factor Γ, *i.e.* $\lambda_{a,b} = \Gamma \times \lambda$ and $\lambda_{b,a} = \Gamma^{-1} \times \lambda$ for $(a, b) \in \mathcal{L} \times \mathcal{R}$ and $\lambda_{a,b} = \lambda$ otherwise.

This process makes stations $\mathcal{L}$ lacking of vehicles and $\mathcal{R}$ exceeding. Yet another consequence is that the overall number of trip requests increases. This is why to decorrelate this phenomenon, that we show previous section was in itself a source of optimization gap, we normalize the overall demand in order to keep in average the same number of requests as in the full homogeneous city.

Table 4 gives some results of simulation with different gravitation factors on a city with 24 stations of capacity 10 and demand intensity per station $\lambda_s = 0.3$. Although FLUID is not improving a full homogeneous city, as soon as some gravitation appears the gain of using this heuristic increases significantly, it even sells more than twice the number of trips than CLASSIC for a gravitation factor of parameter $\Gamma = 6$.

		$\mathcal{K} = 10$		
Instance	Nb trips	CLASSIC	FLUID	UB
024 4x6 I3	4968	2000	2047.1 (2.3%)	3042.0 (52.0%)
024 4x6 I3 G3	4963	1256	1649.3 (31.3%)	2403.8 (91.3%)
024 4x6 I3 G6	4970	466	1047.3 (124.3%)	1345.4 (188.1%)
024 4x6 I3 G9	4968	317	747.5 (135.5%)	933.0 (194.0%)

Table 4: Influence of gravitations in homogeneous cities.

4.4 Tides in homogeneous cities

We study the influence of tides in an homogeneous city with demand intensity λ. We divide the day into three periods, morning from 6h to 9h middle of the day from 9h to 15h and evening from 15h to 18h. The city is split into two equals sub grid: $\mathcal{L}$ and $\mathcal{R}$. We have two tides of equal intensity Θ: First in the morning there is Θ times more demands for trips going from a station $l \in \mathcal{L}$ to a station $r \in \mathcal{R}$ and Θ less in the opposite direction, *i.e.* $\lambda_{l,r}^{[6,9]} = \Theta \times \lambda$ and $\lambda_{r,l}^{[6,9]} = \Theta^{-1} \times \lambda$; Secondly in the evening there is an opposed tide from $r \in \mathcal{R}$ to $l \in \mathcal{L}$, *i.e.* $\lambda_{r,l}^{[15,18]} = \Theta \times \lambda$ and $\lambda_{l,r}^{[15,18]} = \Theta^{-1} \times \lambda$. Otherwise $\lambda_{r,l}^{t} = \lambda$. Once again to decorrelate the tide phenomenon from the simple increase of demands, we normalize the overall demand in order to keep in average the same number of demands as in the full homogeneous city.

Tables 5 and 6 give simulation results for different tide factors on a city with 24 stations and demand intensity per station $\lambda_s = 0.3$. In cities with a uniform station capacity (table 5), the optimization has more impact than cities with infinite station capacity (table 6). It might be explained by the fact that the system with infinite station capacity can absorb the tides. This shows the interest of a good station capacity sizing.

		$\mathcal{K} = 10$		
Instance	Nb trips	CLASSIC	FLUID	UB
024 4x6 I3	4968	2000	2047.1 (2.3%)	3042.0 (52.0%)
024 4x6 I3 T3	4970	1742	1873.2 (7.4%)	2780.9 (59.5%)
024 4x6 I3 T6	4972	1347	1543.6 (14.5%)	2278.0 (69.0%)
024 4x6 I3 T9	4970	1156	1332.7 (15.2%)	1951.2 (68.7%)

Table 5: Influence of tide in homogeneous cities with uniform station capacities.

		$\mathcal{K} = \infty$		
Instance	Nb trips	CLASSIC	FLUID	UB
024 4x6 I3	4967	1998	2074.3 (3.7%)	3042.0 (52.1%)
024 4x6 I3 T3	4975	1809	1893.6 (4.6%)	2780.9 (53.7%)
024 4x6 I3 T6	4968	1524	1577.7 (3.4%)	2278.0 (49.3%)
024 4x6 I3 T9	4968	1323	1356.9 (2.4%)	1951.2 (47.3%)

Table 6: Influence of tide in homogeneous cities with infinite station capacities.

		$\mathcal{K} = 10$			
Scale	Nb trips.	CLASSIC	FLUID	UB	Rel Diff. FLUID/UB
1	864	448	477 (6.3%)	632(41.1%)	-24.6%
2	867	488	533 (9.2%)	632(29.6%)	-15.7%
5	861	517	572 (10.5%)	632(22.2%)	-9.6%
10	864	533	594 (11.4%)	632(18.7%)	-6.1%
20	863	541	605 (11.8%)	632(16.8%)	-4.2%
50	863	549	616 (12.2%)	632(15.1%)	-2.5%
100	863	552	620 (12.3%)	632(14.5%)	-1.9%
200	863	554	622 (12.2%)	632(14.0%)	-1.5%
500	864	561	625 (11.2%)	632(12.7%)	-1.2%
1000	864	568	625 (10.0%)	632(11.4%)	-1.1%

Table 7: Asymptotic convergence of fluid approximation heuristic and s-scaled problem for instance "004 2x2 I3 T3"

4.5 Fluid approximation asymptotic convergence

We study the convergence of the s-scaled problem toward the fluid approximation. We scale the problem by multiplying the demand rates, station capacities and number of vehicles by a coefficient s while the gain to serve each trip and average transportation time variance are, in the mean time, divided by s.

Table 7 shows the convergence of the optimization gap and the upper bound for the instance "004 2x2 I3 T3" with station capacities, (this phenomenon can be observed on any instances with or without capacities). We can see that the less variance the system has (as the scale increases), the more demands the system is receiving as average, the more each policy sells trips and the more the Fluid Model was accurate in its expectation. Note that the absolute number of trips sold by the Upper Bound is constant, since the fluid optimization doesn't take into account the variance of the demands anyway.

N.B. for a reason of implementation simplicity, in this simulation we have taken a version of s-scaled problem with deterministic transportation times.

4.6 Simulation conclusion

To sum up simulation observations we can say that: There is a strong link between demand intensity and optimization efficiency in homogeneous cities. When optimizing the number of trips sold, this relation is in fact natural since there is no need to select which trips to serve when there is few demands compared to the size of the fleet. We show also that optimizing the number of trips sold could have for collateral damage to reduce vehicle time usage advantaging short trips.

When we add tides and gravitations in homogeneous cities, the Fluid Approximation Heuristic is shown efficient. In all simulations the Upper Bound given by the fluid approximation is rather far from the final optimization gain given by the corresponding policy. It is probably due to the high variance of the demands. Indeed when the system is scaled to reduce the variance the gap between the real and the expected optimization also reduces. In the end we conjecture that the fluid approximation would be efficient in systems with high traffic and appropriate sizing. However for relatively low traffic systems, the fluid approximation policy seams worse than the classic policy.

This is raising a question regarding optimization gap in VSS. Since the Upper Bound given is conjectured for any policies, static or dynamic, is it possible to find better static policies or optimizing low traffic VSS should make use of dynamic policies?

Maybe however, the gap comes from the non-tightness of the fluid Upper Bound itself, in which case sharper (but still tractable) approximations models would be useful.

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A Mathematical formulation of the fluid model for discrete prices

Data:

$\mathcal{M}$	the set of stations
$\mathcal{K}_a$	capacity of station a
$\mathcal{D}$	the set of possible trips ($= \mathcal{M} \times \mathcal{M}$)
$t_{a,b}(t)$	the transportation time between station a and $b = \mu_{a,b}^{-1}$
$\lambda_{a,b}(t)$	the transition rate of demands from station a at time t to station b at time $t + t_{a,b}$ at price $p(\lambda_{a,b}(t))$
N	the number of cars available

Variables:

$p_a^+(t)$	the proportion of requests accepted among those willing to leave station a at t
$p_a^-(t)$	the proportion of requests accepted among those willing to have departure at time t, with a as destination and that have not been refused a departure at their departure station.
$y_{a,b}(t)$	the flow leaving station a at time step t (and arriving at station b at time step $t + t_{a,b}$)
$y_{a,b}^{dep}(t)$	the flow accepted by station a (but not yet accepted by station b)
$y_{a,b}^{ref}(t)$	the flow refused by station b returning to station a (one has $y_{a,b}^{dep}(t) = y_{a,b}^{ref}(t) + y_{a,b}(t)$)
$s_a(t)$	the available stock at station a
$r_a(t)$	the number of parking spots reserved at station a (flow in transit toward a)

Figures 3 and 4 gives an instance with its corresponding variables.

$$
\begin{aligned}
\max \quad & \sum_{(a,b)\in D} \int_0^T y_{a,b}(t) \times price(a,b,t))\ dt && && \text{(Gain)}\\
\text{s.t.} \quad & \sum_{a\in\mathcal{M}} s_a(0) = N && && \text{(Flow initialization)}\\
& s_a(0) = s_a(T) && \forall a \in \mathcal{M} && \text{(Flow stabilization)}\\
& \delta_a^+(t) = \sum_b y_{a,b}(t) && \forall a \in \mathcal{M},\ \forall t \in [0,T] && \text{(Output Flow)}\\
& \delta_a^-(t) = \sum_b y_{b,a}(t - t_{b,a}) + \sum_b y_{a,b}^{ref}(t) && \forall a \in \mathcal{M},\ \forall t \in [0,T] && \text{(Input Flow)}\\
& y_{a,b}^{dep}(t) = y_{a,b}^{ref}(t) + y_{a,b}(t) && \forall (a,b) \in \mathcal{D},\ \forall t \in [0,T] && \text{(Flow conservation)}\\
& s_a(t) = s_a(0) + \int_0^t \delta_a^-(\theta) - \delta_a^+(\theta)\ d\theta && \forall a \in \mathcal{M},\ \forall t \in [0,T] && \text{(Flow conservation)}\\
& \frac{\delta s_a(t)}{\delta\theta} = \delta_a^-(t) - \delta_a^+(t) && \forall a \in \mathcal{M},\ \forall t \in [0,T] && \text{(Flow conservation V)}\\
& p_a^+(t) = \begin{cases} 1 & \text{if } s_a(t) > 0,\\ \min\left\{1,\ \frac{\delta_a^-(t)}{\sum_b \lambda_{a,b}(t)}\right\} & \text{otherwise.} \end{cases} && \forall a \in \mathcal{M},\ \forall t \in [0,T] && \text{(Departure equity)}\\
& p_a^-(t) = \begin{cases} 1 & \text{if } s_a(t) + r_a(t) < \mathcal{K}_a,\\ \min\left\{1,\ \frac{\delta_a^+(t)}{\sum_b y_{a,b}^{dep}(t)}\right\} & \text{otherwise.} \end{cases} && \forall a \in \mathcal{M},\ \forall t \in [0,T] && \text{(Arrival equity)}\\
& y_{a,b}^{dep}(t) = p_a^+(t) \times \lambda_{a,b}(t) && \forall (a,b) \in \mathcal{D},\ \forall t \in [0,T] && \\
& y_{a,b}(t) = p_b^-(t) \times y_{a,b}^{dep}(t) && \forall (a,b) \in \mathcal{D},\ \forall t \in [0,T] && \text{(Pushing flow with equity)}\\
& r_a(t) = \sum_b \int_0^{t_{b,a}} y_{b,a}(t-\theta)\ d\theta && \forall a \in \mathcal{M},\ \forall t \in [0,T] && \text{(Reserved Park Spot)}\\
& s_a(t) + r_a(t) \le \mathcal{K}_a && \forall a \in \mathcal{M},\ \forall t \in [0,T] && \text{(Station Capacity)}\\
& \lambda_a(t) = \Lambda(price(a,b,t)) && \forall a \in \mathcal{M},\ \forall t \in [0,T] && \text{(Demand elasticity)}
\end{aligned}
$$

Remark 2 *Without the flow stabilization constraint, it would be easy to compute the value of a solution with one price. A simple iterative algorithm on the horizon would work. With flow stabilization constraint it is not clear that cycling on that iterative algorithm would lead to a stationary solution.*

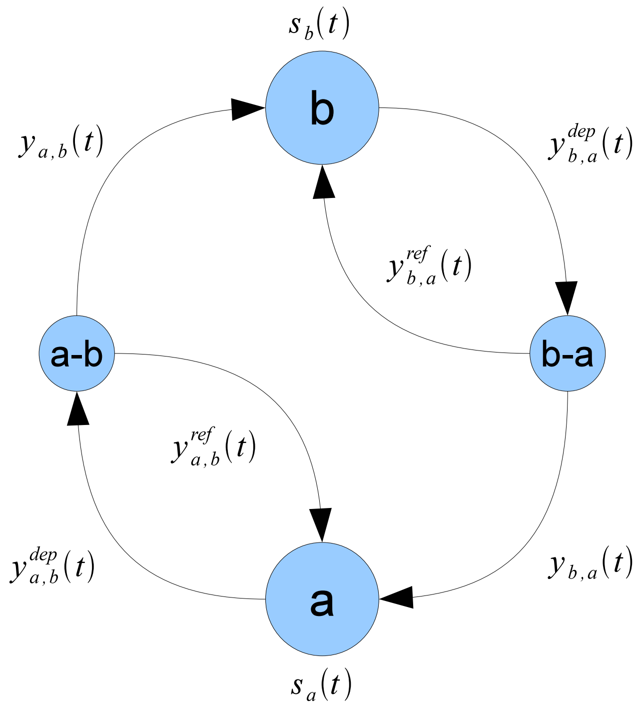

Figure 3: Variables for 2 stations.

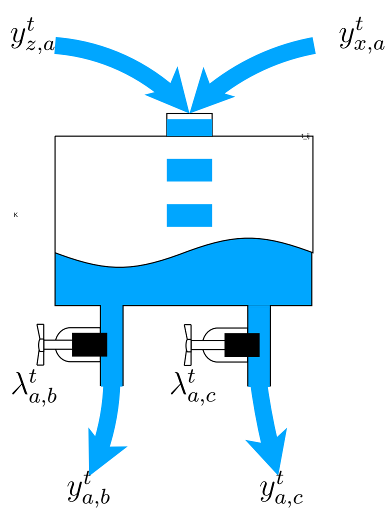

Figure 4: An equity issue.