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Examples of rigid and flexible Seifert fibred cone-manifolds

Alexander Kolpakov*

Abstract

The present paper gives an example of a rigid spherical cone-manifold and that of a flexible one which are both Seifert fibred.

Keywords cone-manifold, rigidity, flexibility, Seifert fibration **MSC (2010)** 53A35, 57R18, 57M25

1 Introduction

The theory of three-dimensional orbifolds and cone-manifolds attracts attention of many mathematicians since the original work of Thurston [29]. An introduction to the theory of orbifolds could be found in [29, Ch.13]. For a basic introduction to the geometry of three-dimensional cone-manifolds and cone-surfaces we refer the reader to [6]. The main motivation for studying three-dimensional cone-manifolds comes from Thurston's approach to geometrization of three-orbifolds: three-dimensional cone-manifolds provide a way to deform geometric orbifold structures. The orbifold theorem has been proven in full generality by M. Boileau, B. Leeb and J. Porti, see [1, 2]. One of the main questions in the theory of three-dimensional cone-manifolds is the rigidity problem. First, the rigidity property was discovered for hyperbolic manifolds (so-called Mostow-Prasad rigidity, see [19, 24]). After that, the global rigidity property for hyperbolic three-dimensional cone-manifolds with singular locus a link and cone angles less than π was proven by S. Kojima [16]. The key result that implies global rigidity is due to Hodgson and Kerckhoff [13], who showed the local rigidity of hyperbolic cone manifolds with singularity of link or knot type and cone angles less than 2π . The de Rham rigidity for spherical orbifolds was established in [26, 27]. Detailed analysis of the rigidity property for three-dimensional cone-manifolds

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was carried out in [31, 32] for hyperbolic and spherical cone-manifolds with singularity a trivalent graph and cone angles less than π .

Recently, the local rigidity for hyperbolic cone-manifolds with cone angles less than 2π was proven in [18, 33]. However, examples of infinitesimally flexible hyperbolic cone-manifolds had already been given in [5]. For other examples of flexible cone-manifolds one may refer to [15, 21, 28].

The theorem of [32] concerning the global rigidity for spherical threedimensional cone-manifolds was proven under the condition of being not Seifert fibred. Recall that due to [22] a cone-manifold is Seifert fibred if its underlying space carries a Seifert fibration such that components of the singular stratum are leafs of the fibration. In particular, if its singular stratum is represented by a link, then the complement is a Seifert fibred three-manifold. All Seifert fibred link complements in the three-sphere are described by [4]. In the present paper, we give an explicit example of a rigid spherical cone-manifold and a flexible one which are both Seifert fibred. The singular locus for each of these cone-manifolds is a link and the underlying space is the three-sphere \mathbb{S}^3 . The rigid cone-manifold given in the paper has cone-angles of both kinds, less or greater than π . The flexible one has cone-angles strictly greater than π . Deformation of its geometric structure comes essentially from those of the base cone-surface. However, hyperbolic orbifolds, which are Seifert fibred over a disc, are rigid. Their geometric structure degenerates to the minimal-perimeter hyperbolic polygon, as shown in [23]. These are uniquely determined by cone angles.

The paper is organised as follows: first, we recall some common facts concerning spherical geometry. In the second section, the geometry of the Hopf fibration is considered and a number of lemmas are proven. After that, we construct two explicit examples of Seifert fibred cone-manifolds. The first one is a globally rigid cone-manifold and its moduli space is parametrised by its cone angles only. The second one is a flexible Seifert fibred conemanifold. This means that we can deform its metric while keeping its cone angles fixed. Rigorously speaking, the following assertion is proven: the given cone-manifold has a one-parameter family of distinct spherical cone metrics with the same cone angles.

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2 Spherical geometry

Below we present several common facts concerning spherical geometry in dimension two and three.

Let us identify a point p = (w, x, y, z) of the three-dimensional sphere

$$\mathbb{S}^3 = \{(w, x, y, z) \in \mathbb{R}^4 | w^2 + x^2 + y^2 + z^2 = 1\}$$

with an $SU_2(\mathbb{C})$ matrix of the form

$$P = \left(\begin{array}{cc} w + ix & y + iz \\ -y + iz & w - ix \end{array}\right).$$

Then, replace the group $\operatorname{Isom}^+ \mathbb{S}^3 \cong SO_4(\mathbb{R})$ of orientation preserving isometries with its two-fold covering $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$. Finally, define the action of $\langle A, B \rangle \in SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ on $P \in SU_2(\mathbb{C})$ by

$$\langle A, B \rangle : P \longmapsto A^t P \overline{B}.$$

Thus, we define the action of $SO_4(\mathbb{R}) \cong SU_2(\mathbb{C}) \times SU_2(\mathbb{C})/\{\pm id\}$ on the three-sphere \mathbb{S}^3 .

By assuming w = 0, we obtain the two-dimensional sphere

$$\mathbb{S}^2 = \{ (x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1 \}.$$

Let us identify a point (x, y, z) of \mathbb{S}^2 with the matrix

$$Q = \left(\begin{array}{cc} ix & y + iz \\ -y + iz & -ix \end{array}\right),$$

which represents a pure imaginary unit quaternion $Q \in \mathbf{H}$. Instead of $\operatorname{Isom}^+ \mathbb{S}^2 \cong SO_3(\mathbb{R})$ we use its two-fold covering $SU_2(\mathbb{C})$ acting by

$$A:q\longmapsto A^tq\overline{A}$$

for every $A \in SU_2(\mathbb{C})$ and every $q \in \mathbb{S}^2$.

Equip each \mathbb{S}^3 and \mathbb{S}^2 with an intrinsic metric of constant sectional curvature +1. We call the distance between two points P and Q of \mathbb{S}^n (n = 2, 3) a real number d(P, Q) uniquely defined by the conditions

$$0 \le d(P,Q) \le \pi,$$
$$\cos d(P,Q) = \frac{1}{2} \operatorname{tr} P^t \overline{Q}$$

The next step is to describe spherical geodesic lines in \mathbb{S}^n . Let us recall the following theorem [25, Theorem 2.1.5].

Theorem 1 A function $\lambda : \mathbb{R} \to \mathbb{S}^n$ is a geodesic line if and only if there are orthogonal vectors x, y in \mathbb{S}^n such that

$$\lambda(t) = (\cos t)x + (\sin t)y.$$

Taking into account the preceding discussion, we may reformulate the statement above.

Lemma 1 Every geodesic line (a great circle) in \mathbb{S}^3 (respectively, \mathbb{S}^2) could be represented in the form

$$C(t) = P\cos t + Q\sin t,$$

where $P, Q \in SU_2(\mathbb{C})$ (respectively $P, Q \in \mathbf{H}$) satisfy orthogonality condition

$$\cos d(P,Q) = 0.$$

By virtue of this lemma, one may regard P as the starting point of the curve C(t) and Q as the velocity vector at P, since C(0) = P, $\dot{C}(0) = \frac{d}{dt}C(t)|_{t=0} = Q$ and $d(C(0), \dot{C}(0)) = \frac{\pi}{2}$ (the latter holds up to a change of the parameter sign).

Given two geodesic lines $C_1(t)$ and $C_2(t)$, define their common perpendicular $C_{12}(t)$ as a geodesic line such that there exist $0 \le t_1, t_2 \le 2\pi, 0 \le \delta \le \pi$ with the following properties:

$$C_{12}(0) = C_1(t_1), \ C_{12}(\delta) = C_2(t_2),$$
$$d(\dot{C}_{12}(0), \dot{C}_1(t_1)) = d(\dot{C}_{12}(\delta), \dot{C}_2(t_2)) = \frac{\pi}{2}.$$

We call δ the distance between the geodesics $C_1(t)$ and $C_2(t)$. Note, that for an arbitrary pair of geodesics their common perpendicular should not be unique.

For an additional explanation of spherical geometry we refer the reader to [25] and [31, Chapter 6.4.2].

3 Links arising from the Hopf fibration

The present section is devoted to the construction of a family of links \mathcal{H}_n $(n \geq 2)$ which we shall use later. These links have a nice property – each of them is formed by $n \geq 2$ fibres of the Hopf fibration. Recall that the Hopf map $h : \mathbb{S}^3 \xrightarrow{\mathbb{S}^1} \mathbb{S}^2$ has geometric nature [14, p. 654]. Our aim is to prove a number of lemmas concerning the geometry of the Hopf fibration in more detail.

3.1 Links \mathcal{H}_n as fibres of the Hopf fibration

The Hopf map h is defined as follows [14]: for every point $(w, x, y, z) \in \mathbb{S}^3$ let its image on \mathbb{S}^2 be

$$h(w, x, y, z) = \left(2(xz + wy), 2(yz - wx), 1 - 2(x^2 + y^2)\right).$$

The fibre $h^{-1}(a,b,c)$ over the point $(a,b,c)\in\mathbb{S}^2$ is a geodesic line in \mathbb{S}^3 of the form

$$C(t) = \frac{1}{\sqrt{2(1+c)}} \left((1+c, -b, a, 0) \cos t + (0, a, b, 1+c) \sin t \right).$$

The exceptional point (0, 0, -1) has the fibre $(0, \cos t, -\sin t, 0)$. The line C(t) is a great circle of \mathbb{S}^3 and can be rewritten in the matrix form

$$C(t) = P(a, b, c) \cos t + Q(a, b, c) \sin t,$$

where

$$P(a,b,c) = \frac{1}{\sqrt{2(1+c)}} \begin{pmatrix} (1+c) - ib & a\\ -a & (1+c) + ib \end{pmatrix},$$
$$Q(a,b,c) = P(a,b,c) \begin{pmatrix} 0 & i\\ i & 0 \end{pmatrix}.$$

We call

$$F(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos t + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \sin t$$

the generic fibre $h^{-1}(0,0,1)$. Moreover, every fibre $h^{-1}(a,b,c)$ can be described as a circle C(t) = P(a,b,c) F(t). Note, that P(a,b,c) is an $SU_2(\mathbb{C})$ matrix. Thus C(t) could be obtained from F(t) by means of the isometry $\langle P(a,b,c)^t, \mathrm{id} \rangle$. For the exceptional point $(0,0,-1) \in \mathbb{S}^2$, we set

$$P(0,0,-1) = \left(\begin{array}{cc} 0 & 1\\ -1 & 0 \end{array}\right).$$

It is known, that every pair of distinct fibres of the Hopf fibration represents simply linked circles in \mathbb{S}^3 (the Hopf link). Thus, *n* fibres form a link \mathcal{H}_n whose every two components form the Hopf link. One can obtain it by drawing *n* straight vertical lines on a cylinder and identifying its ends by a rotation through the angle of 2π . Hence \mathcal{H}_n is an (n, n) torus link.

Another remark is that the \mathcal{H}_n link could be arranged around a point in order to reveal its *n*-th order symmetry, as depicted in Fig. 1. This fact allows us to consider *n*-fold branched coverings of the corresponding conemanifolds with singular locus \mathcal{H}_n that appear in Section 4.



Figure 1: *n*-fold branched covering of (2, 2n) torus link by \mathcal{H}_n

3.2 Geometry of the Hopf fibration

Here and below we use the polar coordinate system (ψ,θ) on \mathbb{S}^2 instead of the Cartesian one. Suppose

$$a = \cos \psi \sin \theta, \ b = \sin \psi \sin \theta, \ c = \cos \theta,$$
$$0 \le \psi \le 2\pi, \ 0 \le \theta \le \pi$$

and let

$$M(\psi,\theta) = P(a,b,c) = \begin{pmatrix} \cos\frac{\theta}{2} - i\sin\psi\sin\frac{\theta}{2} & \cos\psi\sin\frac{\theta}{2} \\ -\cos\psi\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + i\sin\psi\sin\frac{\theta}{2} \end{pmatrix}.$$

A rotation of \mathbb{S}^3 about the generic fibre F(t) through angle ω has the form $\langle R(\omega), R(\omega) \rangle$, where

$$R(\omega) = \begin{pmatrix} \cos\frac{\omega}{2} & i\sin\frac{\omega}{2} \\ i\sin\frac{\omega}{2} & \cos\frac{\omega}{2} \end{pmatrix}.$$

The image of F(t) under the Hopf map h is (0,0) w.r.t. the polar coordinates. The following lemma shows how to obtain a rotation about the pre-image $h^{-1}(\psi,\theta)$ of an arbitrary point (ψ,θ) .

Lemma 2 A rotation through angle ω about an axis C(t) in \mathbb{S}^3 which is the pre-image of a point $(\psi, \theta) \in \mathbb{S}^2$ with respect to the Hopf map is

$$\langle M(\psi,\theta)R(\omega)M(\psi,\theta)^t,R(\omega)\rangle$$

Proof. Since we have that $C(t) = M(\psi, \theta)F(t)$ and $R(\omega)^t F(t)\overline{R(\omega)} = F(t)$ for every $0 \le t \le 2\pi$, then

$$\left(\overline{M(\psi,\theta)}R(\omega)M(\psi,\theta)^t\right)^t C(t)\overline{R(\omega)} = M(\psi,\theta)R(\omega)^t F(t)\overline{R(\omega)} =$$
$$= M(\psi,\theta)F(t) = C(t)$$

by a straightforward computation. Here we use the fact that $M(\psi, \theta) \in SU_2(\mathbb{C})$, and so $\overline{M(\psi, \theta)^t}M(\psi, \theta) = \mathrm{id.} \square$

Another remarkable property of the Hopf fibration is discussed below.

Lemma 3 Every two fibres $C_1(t)$ and $C_2(t)$ of the Hopf fibration are equidistant geodesic lines (great circles) in \mathbb{S}^3 . If $C_i(t)$, $i \in \{1, 2\}$ are pre-images of the points $\widehat{C}_i \in \mathbb{S}^2$, then the length δ of the common perpendicular for $C_1(t)$ and $C_2(t)$ equals $\frac{1}{2}d(\widehat{C}_1, \widehat{C}_2)$.

Proof. The proof follows from the fact that the Hopf fibration is a Riemannian submersion between \mathbb{S}^3 and $\mathbb{S}_{\frac{1}{2}}^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = \frac{1}{4}\}$ with their standard Riemannian metrics of sectional curvature +1 and +4 respectively, see Proposition 1.1 and Proposition 1.2 of [9]. \Box

Every rotation about a fibre of the Hopf fibration induces a rotation about a point of its base.

Lemma 4 Given a rotation $\langle A, B \rangle \in SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ about a fibre C(t)of the Hopf fibration, the transformation $A \in SU_2(\mathbb{C})$ induces a rotation of \mathbb{S}^2 about the point to which C(t) projects under the Hopf map.

Proof. Rotation about the fibre $C(t) = M(\psi, \theta)F(t)$ which projects to the point $(\psi, \theta) \in \mathbb{S}^2$ has the form

$$\langle A, B \rangle = \langle \overline{M(\psi, \theta)} R(\omega) M(\psi, \theta)^t, R(\omega) \rangle.$$

Observe that the rotation $\langle R(\omega), R(\omega) \rangle$ fixes the geodesic F(t) in \mathbb{S}^3 and $R(\omega)$ fixes the point $\widehat{F} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ in \mathbb{S}^2 . Thus $A \in SU_2(\mathbb{C})$ fixes the point $\widehat{C} = M(\psi, \theta) \widehat{F} \overline{M(\psi, \theta)^t}$. By a straightforward computation, we obtain that

$$\widehat{C} = \left(\begin{array}{cc} i\cos\psi\sin\theta & \sin\theta\sin\psi + i\cos\theta \\ -\sin\theta\sin\psi + i\cos\theta & -i\cos\psi\sin\theta \end{array} \right).$$

The point $\widehat{C} \in \mathbb{S}^2$ corresponds to (ψ, θ) w.r.t. the polar coordinates. \Box

4 Examples of rigidity and flexibility

In this section we work out two principal examples of Seifert fibred conemanifolds: the first represents a rigid cone-manifold, the second one is flexible.

4.1 Case of rigidity: the cone-manifold $\mathcal{H}_3(\alpha, \beta, \gamma)$

Let $\mathcal{H}_3(\alpha, \beta, \gamma)$ denote a three-dimensional cone-manifold with underlying space the sphere \mathbb{S}^3 and singular locus formed by the link \mathcal{H}_3 with cone angles α, β and γ along its components. The remaining discussion is devoted to the proof of

Theorem 2 The cone-manifold $\mathcal{H}_3(\alpha, \beta, \gamma)$ admits a spherical structure if the following inequalities are satisfied:

$$\begin{aligned} &2\pi-\gamma < \alpha+\beta < 2\pi+\gamma,\\ &-2\pi+\gamma < \alpha-\beta < 2\pi-\gamma. \end{aligned}$$

The spherical structure on $\mathcal{H}_3(\alpha, \beta, \gamma)$ is unique (i.e. $\mathcal{H}_3(\alpha, \beta, \gamma)$ is globally rigid).

The lengths ℓ_{α} , ℓ_{β} , ℓ_{γ} of its singular strata are pairwise equal and the following formula holds:

$$\ell_{\alpha} = \ell_{\beta} = \ell_{\gamma} = \frac{\alpha + \beta + \gamma}{2} - \pi.$$

The volume of $\mathcal{H}_3(\alpha, \beta, \gamma)$ equals

$$\operatorname{Vol} \mathcal{H}_3(\alpha, \beta, \gamma) = \frac{1}{2} \left(\frac{\alpha + \beta + \gamma}{2} - \pi \right)^2.$$

Proof. First, we construct a holonomy map for $\mathcal{H}_3(\alpha, \beta, \gamma)$. By applying Wirtinger's algorithm, one obtains the following fundamental group presentation for the link \mathcal{H}_3 :

$$\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3) = \langle a, b, c, h | acb = bac = cba = h, h \in Z(\Gamma) \rangle,$$

that is a central extension by h of the thrice-punctured sphere group

$$\Gamma_0 = \pi_1(\mathbb{S}^2 \setminus \{3 \text{ points}\}) = \langle a, b, c | acb = bac = cba = \mathrm{id} \rangle.$$

Consider a holonomy map

$$\rho: \Gamma \longmapsto \operatorname{Isom}^+ \mathbb{S}^3 \cong SO_4(\mathbb{R}).$$



Figure 2: The link \mathcal{H}_3

Let $\tilde{\rho}$ denote its lift to $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, which is a two-fold covering of $SO_4(\mathbb{R})$ (see [7]):

$$\widetilde{\rho} = \langle \widetilde{\rho}_1, \widetilde{\rho}_2 \rangle : \Gamma \longmapsto SU_2(\mathbb{C}) \times SU_2(\mathbb{C}).$$

Let us note, that if holonomy images of any two generators of Γ commute, then the whole homomorphic image $\tilde{\rho}(\Gamma)$ is abelian. Thus, for a representation $\tilde{\rho}$ we have that the following three cases, up to a suitable conjugation, are possible:

- (i) $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2) : \Gamma \to SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, both $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are non-abelian,
- (ii) $\tilde{\rho}: \Gamma \to \mathbb{S}^1 \times \mathbb{S}^1$, an abelian representation,
- (iii) $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2) : \Gamma \to SU_2(\mathbb{C}) \times \mathbb{S}^1$, where $\tilde{\rho}_1$ is non-abelian.

For case (i), let us first suppose that $\tilde{\rho}(h)$ is non-trivial. Since the holonomy images of the meridians a, b and c have to commute with the holonomy image of h, they are simultaneously diagonalisable. We arrive at case (ii). If $\tilde{\rho}(h)$ is trivial, then we have two non-abelian representations $\tilde{\rho}_i : \Gamma_0 \to$ $SU_2(\mathbb{C})$. Since the holonomy images of the meridians correspond to rotations along geodesic lines in \mathbb{S}^3 , it follows by [2, Lemma 9.2] that $\operatorname{tr} \tilde{\rho}_1(x) = \operatorname{tr} \tilde{\rho}_2(x)$ for $x \in \{a, b, c\}$. The base space of the fibred cone-manifold $\mathcal{H}_3(\alpha, \beta, \gamma)$ is a turnover $\mathbb{S}^2(\alpha, \beta, \gamma)$, with α, β, γ cone angles. Then, by [10, Lemma 4.1], up to a conjugation, $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_1)$. The representation $\rho : \Gamma \to SO(4)$ is conjugate into SO(3) and the holonomy images of the meridians have a common fixed point in \mathbb{S}^3 . Thus, their axis intersect, which does not correspond to a non-degenerate spherical structure on the cone-manifold $\mathcal{H}_3(\alpha,\beta,\gamma)$.

For case (ii), up to a suitable conjugation, the representation $\tilde{\rho}$ preserves the Hopf fibration. Thus, by Lemma 4, it descends to an abelian representation of Γ_0 , which cannot be a holonomy of a non-degenerate spherical structure on the base of the fibration.

Finally, case (iii) is left. By [2, Lemma 9.2], one has

$$\widetilde{\rho}(a) = \langle m_a^t R(\alpha) \overline{m_a}, R(\alpha) \rangle,$$

$$\widetilde{\rho}(b) = \langle m_b^t R(\beta) \overline{m_b}, R(\beta) \rangle,$$

$$\widetilde{\rho}(c) = \langle m_c^t R(\gamma) \overline{m_c}, R(\gamma) \rangle$$

for $m_a, m_b, m_c \in SU_2(\mathbb{C})$.

Note, that every matrix $m \in SU_2(\mathbb{C})$ is of the form $m = R(\tau)M(\psi,\theta)$ for suitable $0 \leq \psi \leq \pi, 0 \leq \theta, \tau \leq 2\pi$. Then we obtain that the image of every meridian in $\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3)$ has the form

$$\langle m^t R(\omega)\overline{m}, R(\omega) \rangle = \langle M^t(\psi, \theta) R^t(\tau) R(\omega) R(\tau) M(\psi, \tau), R(\omega) \rangle = \langle M^t(\psi, \theta) R(\omega) \overline{M(\psi, \theta)}, R(\omega) \rangle,$$

since $R(\omega)$ and $R(\tau)$ commute. Hence, Lemma 2 implies that every meridian is mapped by $\tilde{\rho}$ to a rotation about an appropriate fibre of the Hopf fibration. By Propositions 2.1 and 2.2 of [9], the holonomy preserves the fibration structure.

Let $A = \tilde{\rho}(a), B = \tilde{\rho}(b), C = \tilde{\rho}(c)$ be holonomy images of the generators a, b, c for $\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3)$.

After a suitable conjugation in $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, we obtain

$$A = \langle A_l, A_r \rangle = \langle R(\alpha), R(\alpha) \rangle ,$$
$$B = \langle B_l, B_r \rangle = \left\langle \overline{M(0, \phi)} R(\beta) M(0, \phi)^t, R(\beta) \right\rangle ,$$
$$C = \langle C_l, C_r \rangle = \left\langle \overline{M(\psi, \theta)} R(\gamma) M(\psi, \theta)^t, R(\gamma) \right\rangle .$$

In order for the holonomy map $\tilde{\rho}$ to be a homomorphism, the following relations should hold:

$$A_l C_l B_l = B_l A_l C_l = C_l B_l A_l,$$
$$A_r C_r B_r = B_r A_r C_r = C_r B_r A_r.$$

The latter of them are satisfied by the construction of $\tilde{\rho}: \Gamma \to SU_2(\mathbb{C}) \times \mathbb{S}^1$. Let us consider the former relations. By Lemma 4, the elements A_l , B_l and C_l are rotations of \mathbb{S}^2 about the points $\hat{F}_a = (0,0)$, $\hat{F}_b = (0,\phi)$ and $\hat{F}_c = (\psi, \theta)$, respectively. Since \hat{F}_a , \hat{F}_b , \hat{F}_c form a triangle on \mathbb{S}^2 and the base space of $\mathcal{H}_3(\alpha, \beta, \gamma)$ is a turnover with α , β , γ cone angles, one may expect the following

Lemma 5 The points $\widehat{F}_a = (0,0)$, $\widehat{F}_b = (0,\phi)$ and $\widehat{F}_c = (\psi,\theta)$ form a triangle with angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$ at the corresponding vertices.

Proof. By a straightforward computation, we obtain that

$$\begin{aligned} A_l C_l B_l - B_l A_l C_l &= \begin{pmatrix} iR_1 & R_2 + iR_3 \\ -R_2 + iR_3 & -iR_1 \end{pmatrix}, \\ C_l B_l A_l - B_l A_l C_l &= \begin{pmatrix} iR_4 & R_5 + iR_3 \\ -R_5 + iR_3 & -iR_4 \end{pmatrix}, \end{aligned}$$

where

$$R_1 = 2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sin\theta\cos\phi\sin\left(\frac{\alpha}{2} - \psi\right),\,$$

$$R_2 = 2\sin\frac{\beta}{2}\left(\cos\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\phi + \sin\frac{\gamma}{2}\left(-\cos\phi\cos\left(\frac{\alpha}{2} - \psi\right)\sin\theta + \cos\frac{\alpha}{2}\cos\theta\sin\phi\right)\right),$$

$$R_3 = -2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sin\theta\sin\phi\sin\left(\frac{\alpha}{2} - \psi\right),\,$$

$$R_4 = 2\sin\frac{\gamma}{2} \left(\cos\theta\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\phi - \left(\cos\frac{\beta}{2}\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\phi\right)\sin\theta\sin\psi\right),$$

$$R_5 = 2\sin\frac{\gamma}{2} \left(\cos\frac{\beta}{2}\cos\psi\sin\frac{\alpha}{2}\sin\theta + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}(\cos\phi\cos\psi\sin\theta - \cos\theta\sin\phi)\right)$$

In order to determine the parameters ϕ , ψ and θ , one can proceed as follows: these are determined by the system of equations $R_k = 0, k \in \{1, \dots, 5\}$ under the restrictions $0 < \alpha, \beta, \gamma < 2\pi$ and $0 < \psi \leq 2\pi, 0 < \theta \leq \pi$. Thus, the common solutions to R_1 and R_3 are $\psi = \frac{\alpha}{2}$ and $\psi = \frac{\alpha}{2} \pm \pi$. We claim that the cone angles in the base space of $\mathcal{H}_3(\alpha, \beta, \gamma)$ and along its fibres are the same, and choose $\psi = \frac{\alpha}{2}$.

Taking into account that $0 < \alpha, \beta, \gamma < 2\pi$ (this implies that the sine functions of half cone angles are non-zero), turn the set of relations R_k , $k \in \{1, \ldots, 5\}$ into a new one:

$$\widetilde{R}_1 = -\cos\phi\sin\frac{\gamma}{2}\sin\theta + \left(\sin\frac{\alpha}{2}\cos\frac{\gamma}{2} + \cos\frac{\alpha}{2}\sin\frac{\gamma}{2}\cos\theta\right)\sin\phi,$$
$$\widetilde{R}_2 = -\cos\theta\sin\frac{\beta}{2}\sin\phi + \left(\sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\phi\right)\sin\theta.$$

Note, that the conditions of Theorem 2 concerning cone angles are exactly the existence conditions for a spherical triangle with angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$. For the latter, the following trigonometric identities (spherical cosine and sine rules) are satisfied [25, Theorems 2.5.2 and 2.5.4]:

$$\cos \phi = \frac{\cos \frac{\gamma}{2} + \cos \frac{\alpha}{2} \cos \frac{\beta}{2}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}},$$
$$\cos \theta = \frac{\cos \frac{\beta}{2} + \cos \frac{\alpha}{2} \cos \frac{\gamma}{2}}{\sin \frac{\alpha}{2} \sin \frac{\gamma}{2}},$$
$$\frac{\sin \phi}{\sin \frac{\gamma}{2}} = \frac{\sin \theta}{\sin \frac{\beta}{2}}.$$

These identities state that the points \widehat{F}_a , \widehat{F}_b and \widehat{F}_c form a triangle on \mathbb{S}^2 with angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$ at the corresponding vertices. Its double provides the base turnover with cone angles α , β and γ for the fibred cone-manifold $\mathcal{H}_3(\alpha, \beta, \gamma)$.

On substituting the expressions for $\cos \phi$ and $\cos \psi$ above in the relations \widetilde{R}_k , $k \in \{1, 2\}$ and taking into account the sine rule, one obtains that $\widetilde{R}_k = 0$, $k \in \{1, 2\}$. The lemma is proven. \Box

Let \mathcal{S} denote the domain of cone angles indicated in the statement of the theorem:

$$\mathcal{S} = \left\{ \overrightarrow{\alpha} = (\alpha, \beta, \gamma) \middle| \begin{array}{c} 2\pi - \gamma < \alpha + \beta < 2\pi + \gamma \\ -2\pi + \gamma < \alpha - \beta < 2\pi - \gamma \end{array} \right\}.$$

Let \mathcal{S}^* denote the subset of \mathcal{S} , such that for every triple of cone angles $\overrightarrow{\alpha} = (\alpha, \beta, \gamma) \in \mathcal{S}^*$ there exists a spherical structure on $\mathcal{H}_3(\overrightarrow{\alpha})$. Our next step is to show that \mathcal{S}^* coincides with \mathcal{S} .

The set S^* is non-empty. From [8], it follows that $\mathcal{H}_3(\pi, \pi, \pi)$ has a spherical structure. The orbifold $\mathcal{H}_3(\pi, \pi, \pi)$ is Seifert fibred and its base is a turnover with cone angles equal to π . Thus, the point $(\pi, \pi, \pi) \in S$ belongs to S^* .

The set S^* is open, because a deformation of the holonomy induces a deformation of the structure [20].

In order to prove that the set S^* is closed, we consider a sequence $\overrightarrow{\alpha}_n = (\alpha_n, \beta_n, \gamma_n)$ in S^* converging to $\overrightarrow{\alpha}_{\infty} = (\alpha_{\infty}, \beta_{\infty}, \gamma_{\infty})$ in S. Since every spherical cone-manifold with cone angles $\leq 2\pi$ is an Alexandrov space with curvature ≥ 1 [3], we obtain that the diameter of $\mathcal{H}_3(\overrightarrow{\alpha}_n)$ is bounded above: diam $\mathcal{H}_3(\overrightarrow{\alpha}_n) \leq \pi$.

Let dist $\mathcal{H}_3(\overrightarrow{\alpha}_n)$ denote the minimum of the mutual distances between the axis of rotations A, B and C. Since $\overrightarrow{\alpha}_{\infty} \in S$, we have by Lemma 5 that the turnover $\mathbb{S}^2(\overrightarrow{\alpha}_{\infty})$ is non-degenerate. By making use of Lemma 3, one obtains that (restricting to a subsequence, if needed) for every $\overrightarrow{\alpha}_n \in S$, $n = 1, 2, \ldots$ the function dist $\mathcal{H}_3(\overrightarrow{\alpha}_n)$ is uniformly bounded below away from zero:

dist
$$\mathcal{H}_3(\overrightarrow{\alpha}_n) \ge d_0 > 0, \ n = 1, 2, \dots$$

Then we use the following facts [3]:

- 1. The Gromov-Hausdorff limit of Alexandrov spaces with curvature ≥ 1 , dimension = 3 and bounded diameter is an Alexandrov space with curvature ≥ 1 and dimension ≤ 3 ,
- 2. Dimension of an Alexandrov space with curvature ≥ 1 holds the same at every point (the word "dimension" means Hausdorff or topological dimension, which are equal in the case of curvature ≥ 1).

Since dist $\mathcal{H}_3(\overrightarrow{\alpha}_n) \ge d_0 > 0$, the sequence $\mathcal{H}_3(\overrightarrow{\alpha}_n)$ does not collapse. Thus, the cone-manifold $\mathcal{H}_3(\overrightarrow{\alpha}_\infty)$ has a non-degenerate spherical structure and $\overrightarrow{\alpha}_\infty \in \mathcal{S}^*$.

The subset $S^* \subset S$ is non-empty, as well as both closed and open. This implies $S^* = S$.

Finally, we claim the following fact concerning the geometric characteristics of $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold:

Lemma 6 Let ℓ_{α} , ℓ_{β} , ℓ_{γ} denote the lengths of the singular strata for $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold with cone angles α , β and γ . Then

$$\ell_{\alpha} = \ell_{\beta} = \ell_{\gamma} = \frac{\alpha + \beta + \gamma}{2} - \pi.$$

The volume of $\mathcal{H}_3(\alpha, \beta, \gamma)$ is

Vol
$$\mathcal{H}_3(\alpha, \beta, \gamma) = \frac{1}{2} \left(\frac{\alpha + \beta + \gamma}{2} - \pi \right)^2$$
.

Proof. Let us calculate the geometric parameters explicitly, using the holonomy map defined above. First, we introduce two notions suitable for the further discussion. Given an element $M = \langle M_l, M_r \rangle \in SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, one may assume that the pair of matrices $\langle M_l, M_r \rangle$ is conjugated, by means of a certain element $\langle C_l, C_r \rangle \in SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, to the pair of diagonal matrices

$$\left\langle \left(\begin{array}{cc} e^{i\gamma} & 0\\ 0 & e^{-i\gamma} \end{array}\right), \left(\begin{array}{cc} e^{i\varphi} & 0\\ 0 & e^{-i\varphi} \end{array}\right) \right\rangle$$

with $0 \leq \gamma, \varphi \leq \pi$.

Then call the translation length of M the quantity $\delta(M) := \varphi - \gamma$ and call the "jump" of M the quantity $\nu(M) := \varphi + \gamma$, see [11] and [31, Ch.6.4.2]. We suppose that $\varphi > \gamma$, otherwise changing γ , φ for $2\pi - \gamma$ and $\pi - \varphi$ makes the considered tuple to have the desired form.

Recall that the representation of $\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3)$ is

$$\Gamma = \langle a, b, c, h | acb = bac = cba = h, h \in Z(\Gamma) \rangle,$$

where a, b, c are meridians and h is a longitudinal loop that represents a fibre. Denote by H the image of h under the holonomy map $\tilde{\rho}$. Then we obtain

$$\ell_{\alpha} = \ell_{\beta} = \ell_{\gamma} = \delta(H).$$

Since $A = \tilde{\rho}(a)$ and $H = \tilde{\rho}(h)$ commute, there exists an element $C = \langle C_l, C_r \rangle$ of $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ such that

$$\begin{split} CAC^{-1} &= \left\langle \left(\begin{array}{cc} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{array} \right), \left(\begin{array}{cc} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{array} \right) \right\rangle, \\ CHC^{-1} &= \left\langle \left(\begin{array}{cc} e^{i\gamma(H)} & 0 \\ 0 & e^{-i\gamma(H)} \end{array} \right), \left(\begin{array}{cc} e^{i\varphi(H)} & 0 \\ 0 & e^{-i\varphi(H)} \end{array} \right) \right\rangle. \end{split}$$

By a straightforward computation similar to that in Lemma 5, one obtains

$$2\cos\gamma(H) = \operatorname{tr} H_l = \operatorname{tr} A_l C_l B_l = \operatorname{tr}(-\operatorname{id}) = 2\cos\pi$$

and

$$2\cos\varphi(H) = \operatorname{tr} H_r = \operatorname{tr} A_r C_r B_r = 2\cos\frac{\alpha + \beta + \gamma}{2}.$$

From the foregoing discussion, the singular stratum's length is

$$\ell_{\alpha} = \delta(H) = \frac{\alpha + \beta + \gamma}{2} - \pi.$$

An analogous equality holds for ℓ_{β} and ℓ_{γ} . By the Schläfli formula [12], the following relation holds:

2 dVol
$$\mathcal{H}_3(\alpha, \beta, \gamma) = \ell_\alpha d\alpha + \ell_\beta d\beta + \ell_\gamma d\gamma.$$

Solving this differential equality, we obtain that

$$\operatorname{Vol} \mathcal{H}_3(\alpha, \beta, \gamma) = \frac{1}{2} \left(\frac{\alpha + \beta + \gamma}{2} - \pi \right)^2 + \operatorname{Vol}_0,$$

where Vol_0 is an arbitrary constant. Since the geometric structure on the base space of the fibration (consequently, on the whole $\mathcal{H}_3(\alpha, \beta, \gamma)$ conemanifold) degenerates when $\alpha + \beta + \gamma \longrightarrow 2\pi$, the equality $\operatorname{Vol}_0 = 0$ follows from the volume function continuity. \Box

Consider a holonomy $\tilde{\rho} = \langle \tilde{\rho}_1, \tilde{\rho}_2 \rangle : \Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3) \to SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ for $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold. As we already know from the preceding discussion, one has $\tilde{\rho} : \Gamma \to SU_2(\mathbb{C}) \times \mathbb{S}^1$ essentially, and $\tilde{\rho}_1$ determines $\tilde{\rho}_2$ up to a conjugation by means of the equality $\operatorname{tr} \tilde{\rho}_1(m) = \operatorname{tr} \tilde{\rho}_2(m)$ for meridians in Γ . So any deformation of $\tilde{\rho}$ is a deformation of $\tilde{\rho}_1$. In the case of $\mathcal{H}_3(\alpha, \beta, \gamma)$, the map $\tilde{\rho}_1$ is a non-abelian representation of the base turnover group. Spherical turnover is rigid, that means $\tilde{\rho}_1$ is determined only by the corresponding cone angles. Thus $\mathcal{H}_3(\alpha, \beta, \gamma)$ is locally rigid.

The global rigidity follows from the fact that every $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold could be deformed to the orbifold $\mathcal{H}_3(\pi, \pi, \pi)$ by a continuous path through locally rigid structures. This assertion holds since \mathcal{S}^* contains the point (π, π, π) and \mathcal{S}^* is convex. The global rigidity of $\mathcal{H}_3(\pi, \pi, \pi)$ spherical orbifold follows from [26, 27] and implies the global rigidity of $\mathcal{H}_3(\alpha, \beta, \gamma)$ by means of deforming the orbifold structure backwards to the considered conemanifold one. \Box

4.2 Case of flexibility: the cone-manifold $\mathcal{H}_4(\alpha)$

Let $\mathcal{H}_4(\alpha)$ denote a three-dimensional cone-manifold with underlying space the sphere \mathbb{S}^3 and singular locus formed by the link \mathcal{H}_4 with cone angle α along all its components.

The following theorem provides an example of a flexible cone-manifold which is Seifert fibred.



Figure 3: The link \mathcal{H}_4

Theorem 3 The cone-manifold $\mathcal{H}_4(\alpha)$ admits a spherical structure if

 $\pi < \alpha < 2\pi$.

This structure is not unique (i.e. $\mathcal{H}_4(\alpha)$ is not globally, nor locally rigid). The deformation space contains an open interval, that provides a oneparameter family of distinct spherical cone-metrics on \mathbb{S}^3 . The length of each singular stratum is

$$\ell = 2(\alpha - \pi).$$

The volume of $\mathcal{H}_4(\alpha)$ equals

$$\operatorname{Vol} \mathcal{H}_4(\alpha) = 2(\alpha - \pi)^2.$$

Proof. The following lemma precedes the proof of the theorem.

Lemma 7 Given a quadrangle Q on \mathbb{S}^2 with three right angles and one angle $\frac{\alpha}{2}$ (see Fig. 4), the following statements hold:

- 1. The quadrangle Q exists if $\pi < \alpha < 2\pi$,
- $2. \sin \ell_1 \sin \ell_2 = -\cos \frac{\alpha}{2},$
- 3. $\cos \phi = \frac{\cos \ell_1 \cos \ell_2}{\sin \frac{\alpha}{2}}$,

- 4. $\cos \psi = \tan \ell_1 \cot \phi$,
- 5. $0 \le \ell_1, \, \ell_2, \, \phi, \, \psi \le \frac{\pi}{2}.$

Proof. We refer the reader to $[30, \S 3.2]$ for a detailed proof of the statements above. \Box



Figure 4: The quadrangle Q

Given a quadrangle Q from Lemma 7 (so-called Saccheri's quadrangle) one can construct another one, depicted in Fig.5, by reflecting Q in its sides incident to the vertex O. We may regard O to be the point $(0,0) \in \mathbb{S}^2$. Thus, the fibres over the corresponding vertices are

$$F_a(t) = M(\psi, \phi) F(t),$$

$$F_b(t) = M(\pi - \psi, \phi) F(t),$$

$$F_c(t) = M(\pi + \psi, \phi) F(t),$$

$$F_d(t) = M(2\pi - \psi, \phi) F(t).$$

Let $A = \langle A_l, A_r \rangle$, $B = \langle B_l, B_r \rangle$, $C = \langle C_l, C_r \rangle$, $D = \langle D_l, D_r \rangle$ denote the respective rotations through angle α about the axis F_a , F_b , F_c and F_d . From Lemma 2, one obtains

$$A_{l} = \overline{M(\psi, \phi)} R(\alpha) M(\psi, \phi)^{t}, A_{r} = R(\alpha);$$

$$B_{l} = \overline{M(\pi - \psi, \phi)} R(\alpha) M(\pi - \psi, \phi)^{t}, B_{r} = R(\alpha);$$

$$C_{l} = \overline{M(\pi + \psi, \phi)} R(\alpha) M(\pi + \psi, \phi)^{t}, B_{r} = R(\alpha);$$

$$D_{l} = \overline{M(2\pi - \psi, \phi)} R(\alpha) M(2\pi - \psi, \phi)^{t}, D_{r} = R(\alpha);$$

We assume that ℓ_1 , ℓ_2 , ϕ and ψ satisfy the identities of Lemma 7.



Figure 5: The base quadrangle P for $\mathcal{H}_4(\alpha)$

The fundamental group of $\pi_1(\mathbb{S}^3 \setminus \mathcal{H}_4)$ has the presentation

$$\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_4) = \langle a, b, c, d, h | adcb = badc = cbad = dcba = h, h \in Z(\Gamma) \rangle.$$

Let us construct a lift of the holonomy map $\tilde{\rho}: \Gamma \to SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ as follows:

$$\widetilde{\rho}(a) = A, \ \widetilde{\rho}(b) = B, \ \widetilde{\rho}(c) = C, \ \widetilde{\rho}(d) = D.$$

Here we choose $\tilde{\rho}: \Gamma \to SU_2(\mathbb{C}) \times \mathbb{S}^1$ by the same reason as in Theorem 2. In order to show that the map $\tilde{\rho}$ is a homomorphism, one has to check whether the following relations are satisfied:

$$A_l D_l C_l B_l = B_l A_l D_l C_l = C_l B_l A_l D_l = D_l C_l B_l A_l,$$
$$A_r D_r C_r B_r = B_r A_r D_r C_r = C_r B_r A_r D_r = D_r C_r B_r A_r$$

The latter relations hold in view of the fact that the matrices A_r , B_r , C_r and D_r pairwise commute. Then, we show that the following equality holds:

$$A_l D_l C_l B_l = \mathrm{id}.$$

To do this, split the quadrangle P into two triangles by drawing a geodesic line from B to D. Since A_l , B_l , C_l and D_l are rotations about the vertices of the quadrangle depicted in Fig. 6, let us decompose the rotations $B_l = B'_l B''_l$ and $D_l = D'_l D''_l$ into the products of rotations B'_l , B''_l through angles β_1 ,



Figure 6: Section of P by the line joining vertices B and D

 β_2 and the rotations D'_l , D''_l through angles δ_1 and δ_2 , respectively. The following equalities hold: $\beta_1 + \beta_2 = \frac{\alpha}{2}$ and $\delta_1 + \delta_2 = \frac{\alpha}{2}$. Thus, the triples D''_l , C_l , B'_l and A_l , D'_l , B''_l consist of rotations about the vertices of two disjoint triangles depicted in Fig. 6. Similar to the computation of Lemma 6, we have $D''_l C_l B'_l = -\mathrm{id}$

and

$$A_l D_l' B_l'' = -\mathrm{id}.$$

From the identities above, it follows that

$$A_l D_l C_l B_l = A_l D'_l D''_l C_l B'_l B''_l = -A_l D'_l B''_l = \mathrm{id}.$$

The statement holds under a cyclic permutation of the factors. Thus,

$$A_l D_l C_l B_l = B_l A_l D_l C_l = C_l B_l A_l D_l = D_l C_l B_l A_l = \mathrm{id}.$$

Below we shall consider the side-length ℓ_1 as a parameter. Let $\ell_1 := \tau$. Then by Lemma 7 one has that $\sin \ell_2 = -\frac{\cos \frac{\alpha}{2}}{\sin \tau}$ and $\ell_2 := \ell_2(\tau)$ is a well-defined continuous function of τ . The quadrangle P depends on the parameter τ continuously while keeping the angles in its vertices equal to $\frac{\alpha}{2}$. Let $\mathcal{H}_4(\alpha; \tau)$ denote a three-dimensional cone-manifold with underlying

space the sphere \mathbb{S}^3 and singular locus the link \mathcal{H}_4 with cone angle α along its components. Furthermore, its holonomy map is determined by the quadrangle P described above (see Fig. 5) depending on the parameter τ . This means that the double of P forms a "pillowcase" cone-surface with all cone angles equal to α , which is the base space for the fibred cone-manifold $\mathcal{H}_4(\alpha; \tau)$. Let $\mathbb{L}_n(\alpha,\beta)$ be a cone-manifold with underlying space the sphere \mathbb{S}^3 and singular locus a torus link of the type (2,2n) with cone angles α and β along its components. Torus links of the type (2,2n) are two-bridge links. The corresponding cone-manifolds were previously considered in [17, 22]. Since the cone-manifold $\mathcal{H}_4(\alpha)$ forms a 4-fold branched covering of the conemanifold $\mathbb{L}_4(\alpha, \frac{\pi}{2})$, from [17, Theorem 2] we obtain that $\mathcal{H}_4(\alpha)$ has a spherical structure if $\pi < \alpha < 2\pi$. The length of each singular stratum equals to $\ell = 2(\alpha - \pi)$ and the volume is Vol $\mathcal{H}_4(\alpha) = 2(\alpha - \pi)^2$.

Under the assumption that $\ell_1 = \ell_2$, the base quadrangle depicted in Fig. 5 appears to have a four order symmetry. Moreover, by making use of Lemma 7, one may derive the following equalities: $\psi = \frac{\pi}{4}$, $\cos \phi = \cot \frac{\alpha}{4}$. The general formulas for the holonomy of $\mathcal{H}_4(\alpha)$ cone-manifold derived above subject to the condition $\ell_1 = \ell_2$ (equivalently, the cone-manifold $\mathcal{H}_4(\alpha)$ has a four order symmetry) give the holonomy map induced by the covering. Thus $\mathcal{H}_4(\alpha) \cong \mathcal{H}_4(\alpha; \arccos(\sqrt{2}\cos \frac{\alpha}{4}))$ is a spherical cone-manifold.

We claim that one can vary the parameter τ in certain ranges while keeping spherical structure on $\mathcal{H}_4(\alpha; \tau)$ non-degenerate.

Lemma 8 If τ varies over $(\frac{\alpha-\pi}{2}, \frac{\pi}{2})$, the cone-manifold $\mathcal{H}_4(\alpha; \tau)$ has a nondegenerate spherical structure.

Proof. The proof has much in common with the proof of the spherical structure existence on $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold given in Theorem 2. Let us express the identities of Lemma 7 in terms of the parameter $\ell_1 := \tau$. We obtain

$$\cos \phi = \cos \tau \sqrt{1 - \cot^2 \frac{\alpha}{2} \cot^2 \tau},$$
$$\cos \psi = \sqrt{\frac{1 - \cot^2 \frac{\alpha}{2} \cot^2 \tau}{1 + \cot^2 \frac{\alpha}{2} \cot^4 \tau}},$$
$$\sin \ell_2 = -\frac{\cos \frac{\alpha}{2}}{\sin \tau}.$$

Since Lemma 7 states that $0 \leq \phi, \psi, \ell_2 \leq \frac{\pi}{2}$, the functions $\phi := \phi(\tau)$, $\psi := \psi(\tau), \ell_2 := \ell_2(\tau)$ are well-defined and depend continuously on τ . Moreover, the following relations hold:

$$\cos b_1 = \frac{\cos \phi}{\cos \ell_2} = \cos \tau \sqrt{\frac{\sin^2 \tau - \cot^2 \frac{\alpha}{2} \cos^2 \tau}{\sin^2 \tau - \cos^2 \frac{\alpha}{2}}}$$
$$\cos b_2 = \frac{\cos \phi}{\cos \tau} = \sqrt{1 - \cot^2 \frac{\alpha}{2} \cot^2 \tau}.$$

If one sets the centre O of the quadrangle P to $(0,0) \in \mathbb{S}^2$, the whole quadrangle is situated in the upper hemisphere provided $\phi < \frac{\pi}{2}$. From the fact that $\cos b_1 \ge \cos \phi$ and $\cos b_2 \ge \cos \phi$, it follows $b_1, b_2 \le \phi$. Thus $b_1, b_2 \le \frac{\pi}{2}$ and the functions $b_1 := b_1(\tau), b_2 := b_2(\tau)$ are well-defined and continuous with respect to τ .

Observe that if the condition $\frac{\alpha-\pi}{2} < \tau < \frac{\pi}{2}$ is satisfied, then the required inequality $\phi < \frac{\pi}{2}$ holds.

Let S^*_{α} denote the subset of $S_{\alpha} = \{\tau | \frac{\alpha - \pi}{2} < \tau < \frac{\pi}{2}\}$ that consists of the points $\tau \in S_{\alpha}$ such that the cone-manifold $\mathcal{H}_4(\alpha; \tau)$ has a non-degenerate spherical structure. We show $S^*_{\alpha} = S_{\alpha}$ by means of the fact that S^*_{α} is both open and closed non-empty subset of S_{α} .

As noticed above, $\tau = \arccos(\sqrt{2}\cos\frac{\alpha}{4})$ belongs to \mathcal{S}^*_{α} . Hence the set \mathcal{S}^*_{α} is non-empty.

The set \mathcal{S}^*_{α} is open by the fact that a deformation of the holonomy implies a deformation of the structure [20]. To prove that \mathcal{S}^*_{α} is closed, consider a sequence τ_n converging in \mathcal{S}^*_{α} to $\tau_{\infty} \in \mathcal{S}_{\alpha}$.

The lengths of common perpendiculars between the axis of rotations A, B, C and D defined above equal respectively b_1 , b_2 and ϕ .

Since τ_{∞} corresponds to a non-degenerated quadrangle, every cone-manifold $\mathcal{H}_4(\alpha;\tau_n)$ has the quantities $b_1(\tau_n)$, $b_2(\tau_n)$ and $\phi(\tau_n)$ uniformly bounded below away from zero. By the arguments similar to those of Theorem 2, we obtain that $\mathcal{H}_4(\alpha;\tau_{\infty})$ is a non-degenerate spherical cone-manifold. Thus τ_{∞} belongs to \mathcal{S}^*_{α} . Hence \mathcal{S}^*_{α} is closed.

Finally, we obtain that $S_{\alpha}^* = S_{\alpha}$. Thus, while τ varies over $\left(\frac{\alpha - \pi}{2}, \frac{\pi}{2}\right)$ the cone-manifold $\mathcal{H}_4(\alpha; \tau)$ does not collapse. \Box

The following lemma shows that the interval $(\frac{\alpha-\pi}{2}, \frac{\pi}{2})$ represents a part of the deformation space for possible spherical structures on $\mathcal{H}_4(\alpha; \tau)$.

Lemma 9 The cone-manifolds $\mathcal{H}_4(\alpha; \tau_1)$ and $\mathcal{H}_4(\alpha; \tau_2)$ with $\pi < \alpha < 2\pi$ and $\frac{\alpha-\pi}{2} < \tau_1, \tau_2 < \frac{\pi}{2}$ are not isometric if $\tau_1 \neq \tau_2$.

Proof. If the cone-manifolds $\mathcal{H}_4(\alpha; \tau_1)$ and $\mathcal{H}_4(\alpha; \tau_2)$ were isometric, then their holonomy maps $\tilde{\rho}_i$, i = 1, 2 would be conjugated representations of $\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_4)$ into $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$. Then the mutual distances between the axis of rotations A_i , B_i , C_i and D_i , i = 1, 2, coming from the holonomy maps $\tilde{\rho}_1$ and $\tilde{\rho}_2$ would be equal for the corresponding pairs. From Lemma 3, it follows that the common perpendicular length for the given fibres C_1 and C_2 is half the distance between the images of C_1 and C_2 under the Hopf map. By applying Lemmas 3 and 8 to the base quadrangle P of $\mathcal{H}_4(\alpha; \tau_i)$, i = 1, 2 one makes sure that the inequality $\tau_1 \neq \tau_2$ implies the inequality for the lengths of corresponding common perpendiculars. \Box Note, that by the Schläfli formula the volume of $\mathcal{H}_4(\alpha)$ remains the same under any deformation preserving cone angles. Then the formulas for the volume and the singular stratum length follow from the covering properties of $\mathcal{H}_4(\alpha) \xrightarrow{4:1} \mathbb{L}_4(\alpha, \frac{\pi}{2})$ and Theorem 2 of [17]. Thus, Theorem 3 is proven. \Box

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