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Examples of rigid and flexible Seifert fibred cone-manifolds

Alexander Kolpakov*

Abstract

The present paper gives an example of a rigid spherical cone-manifold and that of a flexible one which are both Seifert fibred.

Keywords cone-manifold, rigidity, flexibility, Seifert fibration
MSC (2010) 53A35, 57R18, 57M25

1 Introduction

The theory of three-dimensional orbifolds and cone-manifolds attracts attention of many mathematicians since the original work of Thurston [29]. An introduction to the theory of orbifolds could be found in [29, Ch.13]. For a basic introduction to the geometry of three-dimensional cone-manifolds and cone-surfaces we refer the reader to [6]. The main motivation for studying three-dimensional cone-manifolds comes from Thurston's approach to geometrization of three-orbifolds: three-dimensional cone-manifolds provide a way to deform geometric orbifold structures. The orbifold theorem has been proven in full generality by M. Boileau, B. Leeb and J. Porti, see [1, 2].

One of the main questions in the theory of three-dimensional cone-manifolds is the rigidity problem. First, the rigidity property was discovered for hyperbolic manifolds (so-called Mostow-Prasad rigidity, see [19, 24]). After that, the global rigidity property for hyperbolic three-dimensional cone-manifolds with singular locus a link and cone angles less than π was proven by S. Kojima [16]. The key result that implies global rigidity is due to Hodgson and Kerckhoff [13], who showed the local rigidity of hyperbolic cone manifolds with singularity of link or knot type and cone angles less than 2π . The de Rham rigidity for spherical orbifolds was established in [26, 27]. Detailed analysis of the rigidity property for three-dimensional cone-manifolds

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was carried out in [31, 32] for hyperbolic and spherical cone-manifolds with singularity a trivalent graph and cone angles less than π .

Recently, the local rigidity for hyperbolic cone-manifolds with cone angles less than 2π was proven in [18, 33]. However, examples of infinitesimally flexible hyperbolic cone-manifolds had already been given in [5]. For other examples of flexible cone-manifolds one may refer to [15, 21, 28].

The theorem of [32] concerning the global rigidity for spherical three-dimensional cone-manifolds was proven under the condition of being *not Seifert fibred*. Recall that due to [22] a cone-manifold is *Seifert fibred* if its underlying space carries a Seifert fibration such that components of the singular stratum are leaves of the fibration. In particular, if its singular stratum is represented by a link, then the complement is a Seifert fibred three-manifold. All Seifert fibred link complements in the three-sphere are described by [4]. In the present paper, we give an explicit example of a rigid spherical cone-manifold and a flexible one which are both Seifert fibred. The singular locus for each of these cone-manifolds is a link and the underlying space is the three-sphere \mathbb{S}^3 . The rigid cone-manifold given in the paper has cone-angles of both kinds, less or greater than π . The flexible one has cone-angles strictly greater than π . Deformation of its geometric structure comes essentially from those of the base cone-surface. However, hyperbolic orbifolds, which are Seifert fibred over a disc, are rigid. Their geometric structure degenerates to the minimal-perimeter hyperbolic polygon, as shown in [23]. These are uniquely determined by cone angles.

The paper is organised as follows: first, we recall some common facts concerning spherical geometry. In the second section, the geometry of the Hopf fibration is considered and a number of lemmas are proven. After that, we construct two explicit examples of Seifert fibred cone-manifolds. The first one is a globally rigid cone-manifold and its moduli space is parametrised by its cone angles only. The second one is a flexible Seifert fibred cone-manifold. This means that we can deform its metric while keeping its cone angles fixed. Rigorously speaking, the following assertion is proven: the given cone-manifold has a one-parameter family of distinct spherical cone metrics with the same cone angles.

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2 Spherical geometry

Below we present several common facts concerning spherical geometry in dimension two and three.

Let us identify a point $p = (w, x, y, z)$ of the three-dimensional sphere

$$\mathbb{S}^3 = \{(w, x, y, z) \in \mathbb{R}^4 | w^2 + x^2 + y^2 + z^2 = 1\}$$

with an $SU_2(\mathbb{C})$ matrix of the form

$$P = \begin{pmatrix} w + ix & y + iz \\ -y + iz & w - ix \end{pmatrix}.$$

Then, replace the group $\text{Isom}^+ \mathbb{S}^3 \cong SO_4(\mathbb{R})$ of orientation preserving isometries with its two-fold covering $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$. Finally, define the action of $\langle A, B \rangle \in SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ on $P \in SU_2(\mathbb{C})$ by

$$\langle A, B \rangle : P \mapsto A^t P \bar{B}.$$

Thus, we define the action of $SO_4(\mathbb{R}) \cong SU_2(\mathbb{C}) \times SU_2(\mathbb{C}) / \{\pm \text{id}\}$ on the three-sphere \mathbb{S}^3 .

By assuming $w = 0$, we obtain the two-dimensional sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

Let us identify a point (x, y, z) of \mathbb{S}^2 with the matrix

$$Q = \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix},$$

which represents a pure imaginary unit quaternion $Q \in \mathbf{H}$.

Instead of $\text{Isom}^+ \mathbb{S}^2 \cong SO_3(\mathbb{R})$ we use its two-fold covering $SU_2(\mathbb{C})$ acting by

$$A : q \mapsto A^t q \bar{A}$$

for every $A \in SU_2(\mathbb{C})$ and every $q \in \mathbb{S}^2$.

Equip each \mathbb{S}^3 and \mathbb{S}^2 with an intrinsic metric of constant sectional curvature $+1$. We call the distance between two points P and Q of \mathbb{S}^n ($n = 2, 3$) a real number $d(P, Q)$ uniquely defined by the conditions

$$0 \leq d(P, Q) \leq \pi,$$

$$\cos d(P, Q) = \frac{1}{2} \text{tr } P^t \bar{Q}.$$

The next step is to describe spherical geodesic lines in \mathbb{S}^n . Let us recall the following theorem [25, Theorem 2.1.5].

Theorem 1 *A function $\lambda : \mathbb{R} \rightarrow \mathbb{S}^n$ is a geodesic line if and only if there are orthogonal vectors x, y in \mathbb{S}^n such that*

$$\lambda(t) = (\cos t)x + (\sin t)y.$$

Taking into account the preceding discussion, we may reformulate the statement above.

Lemma 1 *Every geodesic line (a great circle) in \mathbb{S}^3 (respectively, \mathbb{S}^2) could be represented in the form*

$$C(t) = P \cos t + Q \sin t,$$

where $P, Q \in SU_2(\mathbb{C})$ (respectively $P, Q \in \mathbf{H}$) satisfy orthogonality condition

$$\cos d(P, Q) = 0.$$

By virtue of this lemma, one may regard P as the starting point of the curve $C(t)$ and Q as the velocity vector at P , since $C(0) = P$, $\dot{C}(0) = \frac{d}{dt} C(t)|_{t=0} = Q$ and $d(C(0), \dot{C}(0)) = \frac{\pi}{2}$ (the latter holds up to a change of the parameter sign).

Given two geodesic lines $C_1(t)$ and $C_2(t)$, define their common perpendicular $C_{12}(t)$ as a geodesic line such that there exist $0 \leq t_1, t_2 \leq 2\pi$, $0 \leq \delta \leq \pi$ with the following properties:

$$\begin{aligned} C_{12}(0) &= C_1(t_1), \quad C_{12}(\delta) = C_2(t_2), \\ d(\dot{C}_{12}(0), \dot{C}_1(t_1)) &= d(\dot{C}_{12}(\delta), \dot{C}_2(t_2)) = \frac{\pi}{2}. \end{aligned}$$

We call δ the distance between the geodesics $C_1(t)$ and $C_2(t)$. Note, that for an arbitrary pair of geodesics their common perpendicular should not be unique.

For an additional explanation of spherical geometry we refer the reader to [25] and [31, Chapter 6.4.2].

3 Links arising from the Hopf fibration

The present section is devoted to the construction of a family of links \mathcal{H}_n ($n \geq 2$) which we shall use later. These links have a nice property – each of them is formed by $n \geq 2$ fibres of the Hopf fibration. Recall that the Hopf map $h : \mathbb{S}^3 \xrightarrow{\mathbb{S}^1} \mathbb{S}^2$ has geometric nature [14, p. 654]. Our aim is to prove a number of lemmas concerning the geometry of the Hopf fibration in more detail.

3.1 Links \mathcal{H}_n as fibres of the Hopf fibration

The Hopf map h is defined as follows [14]: for every point $(w, x, y, z) \in \mathbb{S}^3$ let its image on \mathbb{S}^2 be

$$h(w, x, y, z) = (2(xz + wy), 2(yz - wx), 1 - 2(x^2 + y^2)).$$

The fibre $h^{-1}(a, b, c)$ over the point $(a, b, c) \in \mathbb{S}^2$ is a geodesic line in \mathbb{S}^3 of the form

$$C(t) = \frac{1}{\sqrt{2(1+c)}} ((1+c, -b, a, 0) \cos t + (0, a, b, 1+c) \sin t).$$

The exceptional point $(0, 0, -1)$ has the fibre $(0, \cos t, -\sin t, 0)$.

The line $C(t)$ is a great circle of \mathbb{S}^3 and can be rewritten in the matrix form

$$C(t) = P(a, b, c) \cos t + Q(a, b, c) \sin t,$$

where

$$P(a, b, c) = \frac{1}{\sqrt{2(1+c)}} \begin{pmatrix} (1+c) - ib & a \\ -a & (1+c) + ib \end{pmatrix},$$

$$Q(a, b, c) = P(a, b, c) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We call

$$F(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos t + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \sin t$$

the generic fibre $h^{-1}(0, 0, 1)$. Moreover, every fibre $h^{-1}(a, b, c)$ can be described as a circle $C(t) = P(a, b, c) F(t)$. Note, that $P(a, b, c)$ is an $SU_2(\mathbb{C})$ matrix. Thus $C(t)$ could be obtained from $F(t)$ by means of the isometry $\langle P(a, b, c)^t, \text{id} \rangle$. For the exceptional point $(0, 0, -1) \in \mathbb{S}^2$, we set

$$P(0, 0, -1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is known, that every pair of distinct fibres of the Hopf fibration represents simply linked circles in \mathbb{S}^3 (the Hopf link). Thus, n fibres form a link \mathcal{H}_n whose every two components form the Hopf link. One can obtain it by drawing n straight vertical lines on a cylinder and identifying its ends by a rotation through the angle of 2π . Hence \mathcal{H}_n is an (n, n) torus link.

Another remark is that the \mathcal{H}_n link could be arranged around a point in order to reveal its n -th order symmetry, as depicted in Fig. 1. This fact allows us to consider n -fold branched coverings of the corresponding cone-manifolds with singular locus \mathcal{H}_n that appear in Section 4.

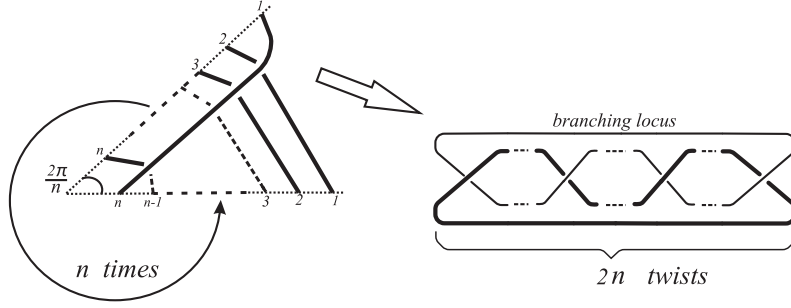


Figure 1: n -fold branched covering of $(2, 2n)$ torus link by \mathcal{H}_n

3.2 Geometry of the Hopf fibration

Here and below we use the polar coordinate system (ψ, θ) on \mathbb{S}^2 instead of the Cartesian one. Suppose

$$a = \cos \psi \sin \theta, \quad b = \sin \psi \sin \theta, \quad c = \cos \theta,$$

$$0 \leq \psi \leq 2\pi, \quad 0 \leq \theta \leq \pi$$

and let

$$M(\psi, \theta) = P(a, b, c) = \begin{pmatrix} \cos \frac{\theta}{2} - i \sin \psi \sin \frac{\theta}{2} & \cos \psi \sin \frac{\theta}{2} \\ -\cos \psi \sin \frac{\theta}{2} & \cos \frac{\theta}{2} + i \sin \psi \sin \frac{\theta}{2} \end{pmatrix}.$$

A rotation of \mathbb{S}^3 about the generic fibre $F(t)$ through angle ω has the form $\langle R(\omega), R(\omega) \rangle$, where

$$R(\omega) = \begin{pmatrix} \cos \frac{\omega}{2} & i \sin \frac{\omega}{2} \\ i \sin \frac{\omega}{2} & \cos \frac{\omega}{2} \end{pmatrix}.$$

The image of $F(t)$ under the Hopf map h is $(0, 0)$ w.r.t. the polar coordinates. The following lemma shows how to obtain a rotation about the pre-image $h^{-1}(\psi, \theta)$ of an arbitrary point (ψ, θ) .

Lemma 2 *A rotation through angle ω about an axis $C(t)$ in \mathbb{S}^3 which is the pre-image of a point $(\psi, \theta) \in \mathbb{S}^2$ with respect to the Hopf map is*

$$\overline{M(\psi, \theta)} R(\omega) M(\psi, \theta)^t, R(\omega).$$

Proof. Since we have that $C(t) = M(\psi, \theta)F(t)$ and $R(\omega)^t F(t) \overline{R(\omega)} = F(t)$ for every $0 \leq t \leq 2\pi$, then

$$\begin{aligned} \left(\overline{M(\psi, \theta)R(\omega)M(\psi, \theta)^t} \right)^t C(t) \overline{R(\omega)} &= M(\psi, \theta)R(\omega)^t F(t) \overline{R(\omega)} = \\ &= M(\psi, \theta)F(t) = C(t) \end{aligned}$$

by a straightforward computation. Here we use the fact that $M(\psi, \theta) \in SU_2(\mathbb{C})$, and so $\overline{M(\psi, \theta)^t} M(\psi, \theta) = \text{id}$. \square

Another remarkable property of the Hopf fibration is discussed below.

Lemma 3 *Every two fibres $C_1(t)$ and $C_2(t)$ of the Hopf fibration are equidistant geodesic lines (great circles) in \mathbb{S}^3 .*

If $C_i(t)$, $i \in \{1, 2\}$ are pre-images of the points $\widehat{C}_i \in \mathbb{S}^2$, then the length δ of the common perpendicular for $C_1(t)$ and $C_2(t)$ equals $\frac{1}{2}d(\widehat{C}_1, \widehat{C}_2)$.

Proof. The proof follows from the fact that the Hopf fibration is a Riemannian submersion between \mathbb{S}^3 and $\mathbb{S}^2_{\frac{1}{2}} = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = \frac{1}{4}\}$ with their standard Riemannian metrics of sectional curvature $+1$ and $+4$ respectively, see Proposition 1.1 and Proposition 1.2 of [9]. \square

Every rotation about a fibre of the Hopf fibration induces a rotation about a point of its base.

Lemma 4 *Given a rotation $\langle A, B \rangle \in SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ about a fibre $C(t)$ of the Hopf fibration, the transformation $A \in SU_2(\mathbb{C})$ induces a rotation of \mathbb{S}^2 about the point to which $C(t)$ projects under the Hopf map.*

Proof. Rotation about the fibre $C(t) = M(\psi, \theta)F(t)$ which projects to the point $(\psi, \theta) \in \mathbb{S}^2$ has the form

$$\langle A, B \rangle = \langle \overline{M(\psi, \theta)R(\omega)M(\psi, \theta)^t}, R(\omega) \rangle.$$

Observe that the rotation $\langle R(\omega), R(\omega) \rangle$ fixes the geodesic $F(t)$ in \mathbb{S}^3 and $R(\omega)$ fixes the point $\widehat{F} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ in \mathbb{S}^2 . Thus $A \in SU_2(\mathbb{C})$ fixes the point $\widehat{C} = M(\psi, \theta)\widehat{F}\overline{M(\psi, \theta)^t}$. By a straightforward computation, we obtain that

$$\widehat{C} = \begin{pmatrix} i \cos \psi \sin \theta & \sin \theta \sin \psi + i \cos \theta \\ -\sin \theta \sin \psi + i \cos \theta & -i \cos \psi \sin \theta \end{pmatrix}.$$

The point $\widehat{C} \in \mathbb{S}^2$ corresponds to (ψ, θ) w.r.t. the polar coordinates. \square

4 Examples of rigidity and flexibility

In this section we work out two principal examples of Seifert fibred cone-manifolds: the first represents a rigid cone-manifold, the second one is flexible.

4.1 Case of rigidity: the cone-manifold $\mathcal{H}_3(\alpha, \beta, \gamma)$

Let $\mathcal{H}_3(\alpha, \beta, \gamma)$ denote a three-dimensional cone-manifold with underlying space the sphere \mathbb{S}^3 and singular locus formed by the link \mathcal{H}_3 with cone angles α , β and γ along its components. The remaining discussion is devoted to the proof of

Theorem 2 *The cone-manifold $\mathcal{H}_3(\alpha, \beta, \gamma)$ admits a spherical structure if the following inequalities are satisfied:*

$$2\pi - \gamma < \alpha + \beta < 2\pi + \gamma,$$

$$-2\pi + \gamma < \alpha - \beta < 2\pi - \gamma.$$

The spherical structure on $\mathcal{H}_3(\alpha, \beta, \gamma)$ is unique (i.e. $\mathcal{H}_3(\alpha, \beta, \gamma)$ is globally rigid).

The lengths ℓ_α , ℓ_β , ℓ_γ of its singular strata are pairwise equal and the following formula holds:

$$\ell_\alpha = \ell_\beta = \ell_\gamma = \frac{\alpha + \beta + \gamma}{2} - \pi.$$

The volume of $\mathcal{H}_3(\alpha, \beta, \gamma)$ equals

$$\text{Vol } \mathcal{H}_3(\alpha, \beta, \gamma) = \frac{1}{2} \left(\frac{\alpha + \beta + \gamma}{2} - \pi \right)^2.$$

Proof. First, we construct a holonomy map for $\mathcal{H}_3(\alpha, \beta, \gamma)$. By applying Wirtinger's algorithm, one obtains the following fundamental group presentation for the link \mathcal{H}_3 :

$$\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3) = \langle a, b, c, h \mid acb = bac = cba = h, h \in Z(\Gamma) \rangle,$$

that is a central extension by h of the thrice-punctured sphere group

$$\Gamma_0 = \pi_1(\mathbb{S}^2 \setminus \{3 \text{ points}\}) = \langle a, b, c \mid acb = bac = cba = \text{id} \rangle.$$

Consider a holonomy map

$$\rho : \Gamma \longmapsto \text{Isom}^+ \mathbb{S}^3 \cong SO_4(\mathbb{R}).$$

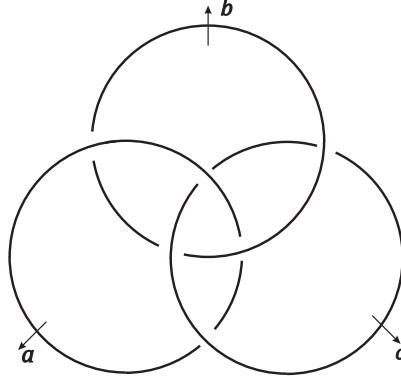


Figure 2: The link \mathcal{H}_3

Let $\tilde{\rho}$ denote its lift to $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, which is a two-fold covering of $SO_4(\mathbb{R})$ (see [7]):

$$\tilde{\rho} = \langle \tilde{\rho}_1, \tilde{\rho}_2 \rangle : \Gamma \mapsto SU_2(\mathbb{C}) \times SU_2(\mathbb{C}).$$

Let us note, that if holonomy images of any two generators of Γ commute, then the whole homomorphic image $\tilde{\rho}(\Gamma)$ is abelian. Thus, for a representation $\tilde{\rho}$ we have that the following three cases, up to a suitable conjugation, are possible:

- (i) $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2) : \Gamma \rightarrow SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, both $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are non-abelian,
- (ii) $\tilde{\rho} : \Gamma \rightarrow \mathbb{S}^1 \times \mathbb{S}^1$, an abelian representation,
- (iii) $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_2) : \Gamma \rightarrow SU_2(\mathbb{C}) \times \mathbb{S}^1$, where $\tilde{\rho}_1$ is non-abelian.

For case (i), let us first suppose that $\tilde{\rho}(h)$ is non-trivial. Since the holonomy images of the meridians a , b and c have to commute with the holonomy image of h , they are simultaneously diagonalisable. We arrive at case (ii).

If $\tilde{\rho}(h)$ is trivial, then we have two non-abelian representations $\tilde{\rho}_i : \Gamma_0 \rightarrow SU_2(\mathbb{C})$. Since the holonomy images of the meridians correspond to rotations along geodesic lines in \mathbb{S}^3 , it follows by [2, Lemma 9.2] that $\text{tr}\tilde{\rho}_1(x) = \text{tr}\tilde{\rho}_2(x)$ for $x \in \{a, b, c\}$. The base space of the fibred cone-manifold $\mathcal{H}_3(\alpha, \beta, \gamma)$ is a turnover $\mathbb{S}^2(\alpha, \beta, \gamma)$, with α, β, γ cone angles. Then, by [10, Lemma 4.1], up to a conjugation, $\tilde{\rho} = (\tilde{\rho}_1, \tilde{\rho}_1)$. The representation $\rho : \Gamma \rightarrow SO(4)$ is conjugate into $SO(3)$ and the holonomy images of the meridians have a common fixed point in \mathbb{S}^3 . Thus, their axis intersect, which does not

correspond to a non-degenerate spherical structure on the cone-manifold $\mathcal{H}_3(\alpha, \beta, \gamma)$.

For case (ii), up to a suitable conjugation, the representation $\tilde{\rho}$ preserves the Hopf fibration. Thus, by Lemma 4, it descends to an abelian representation of Γ_0 , which cannot be a holonomy of a non-degenerate spherical structure on the base of the fibration.

Finally, case (iii) is left. By [2, Lemma 9.2], one has

$$\begin{aligned}\tilde{\rho}(a) &= \langle m_a^t R(\alpha) \overline{m_a}, R(\alpha) \rangle, \\ \tilde{\rho}(b) &= \langle m_b^t R(\beta) \overline{m_b}, R(\beta) \rangle, \\ \tilde{\rho}(c) &= \langle m_c^t R(\gamma) \overline{m_c}, R(\gamma) \rangle\end{aligned}$$

for $m_a, m_b, m_c \in SU_2(\mathbb{C})$.

Note, that every matrix $m \in SU_2(\mathbb{C})$ is of the form $m = R(\tau)M(\psi, \theta)$ for suitable $0 \leq \psi \leq \pi$, $0 \leq \theta, \tau \leq 2\pi$. Then we obtain that the image of every meridian in $\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3)$ has the form

$$\begin{aligned}\langle m^t R(\omega) \overline{m}, R(\omega) \rangle &= \langle M^t(\psi, \theta) R^t(\tau) R(\omega) \overline{R(\tau)} \overline{M(\psi, \tau)}, R(\omega) \rangle = \\ &= \langle M^t(\psi, \theta) R(\omega) \overline{M(\psi, \theta)}, R(\omega) \rangle,\end{aligned}$$

since $R(\omega)$ and $R(\tau)$ commute. Hence, Lemma 2 implies that every meridian is mapped by $\tilde{\rho}$ to a rotation about an appropriate fibre of the Hopf fibration. By Propositions 2.1 and 2.2 of [9], the holonomy preserves the fibration structure.

Let $A = \tilde{\rho}(a)$, $B = \tilde{\rho}(b)$, $C = \tilde{\rho}(c)$ be holonomy images of the generators a, b, c for $\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3)$.

After a suitable conjugation in $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, we obtain

$$\begin{aligned}A &= \langle A_l, A_r \rangle = \langle R(\alpha), R(\alpha) \rangle, \\ B &= \langle B_l, B_r \rangle = \left\langle \overline{M(0, \phi)} R(\beta) M(0, \phi)^t, R(\beta) \right\rangle, \\ C &= \langle C_l, C_r \rangle = \left\langle \overline{M(\psi, \theta)} R(\gamma) M(\psi, \theta)^t, R(\gamma) \right\rangle.\end{aligned}$$

In order for the holonomy map $\tilde{\rho}$ to be a homomorphism, the following relations should hold:

$$\begin{aligned}A_l C_l B_l &= B_l A_l C_l = C_l B_l A_l, \\ A_r C_r B_r &= B_r A_r C_r = C_r B_r A_r.\end{aligned}$$

The latter of them are satisfied by the construction of $\tilde{\rho} : \Gamma \rightarrow SU_2(\mathbb{C}) \times \mathbb{S}^1$. Let us consider the former relations. By Lemma 4, the elements A_l , B_l and C_l are rotations of \mathbb{S}^2 about the points $\widehat{F}_a = (0, 0)$, $\widehat{F}_b = (0, \phi)$ and $\widehat{F}_c = (\psi, \theta)$, respectively. Since \widehat{F}_a , \widehat{F}_b , \widehat{F}_c form a triangle on \mathbb{S}^2 and the base space of $\mathcal{H}_3(\alpha, \beta, \gamma)$ is a turnover with α , β , γ cone angles, one may expect the following

Lemma 5 *The points $\widehat{F}_a = (0, 0)$, $\widehat{F}_b = (0, \phi)$ and $\widehat{F}_c = (\psi, \theta)$ form a triangle with angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$ at the corresponding vertices.*

Proof. By a straightforward computation, we obtain that

$$A_l C_l B_l - B_l A_l C_l = \begin{pmatrix} iR_1 & R_2 + iR_3 \\ -R_2 + iR_3 & -iR_1 \end{pmatrix},$$

$$C_l B_l A_l - B_l A_l C_l = \begin{pmatrix} iR_4 & R_5 + iR_3 \\ -R_5 + iR_3 & -iR_4 \end{pmatrix},$$

where

$$R_1 = 2 \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sin \theta \cos \phi \sin \left(\frac{\alpha}{2} - \psi \right),$$

$$R_2 = 2 \sin \frac{\beta}{2} \left(\cos \frac{\gamma}{2} \sin \frac{\alpha}{2} \sin \phi + \sin \frac{\gamma}{2} \left(-\cos \phi \cos \left(\frac{\alpha}{2} - \psi \right) \sin \theta + \cos \frac{\alpha}{2} \cos \theta \sin \phi \right) \right),$$

$$R_3 = -2 \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sin \theta \sin \phi \sin \left(\frac{\alpha}{2} - \psi \right),$$

$$R_4 = 2 \sin \frac{\gamma}{2} \left(\cos \theta \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \phi - \left(\cos \frac{\beta}{2} \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \phi \right) \sin \theta \sin \psi \right),$$

$$R_5 = 2 \sin \frac{\gamma}{2} \left(\cos \frac{\beta}{2} \cos \psi \sin \frac{\alpha}{2} \sin \theta + \cos \frac{\alpha}{2} \sin \frac{\beta}{2} (\cos \phi \cos \psi \sin \theta - \cos \theta \sin \phi) \right).$$

In order to determine the parameters ϕ , ψ and θ , one can proceed as follows: these are determined by the system of equations $R_k = 0$, $k \in \{1, \dots, 5\}$

under the restrictions $0 < \alpha, \beta, \gamma < 2\pi$ and $0 < \psi \leq 2\pi$, $0 < \theta \leq \pi$. Thus, the common solutions to R_1 and R_3 are $\psi = \frac{\alpha}{2}$ and $\psi = \frac{\alpha}{2} \pm \pi$. We claim that the cone angles in the base space of $\mathcal{H}_3(\alpha, \beta, \gamma)$ and along its fibres are the same, and choose $\psi = \frac{\alpha}{2}$.

Taking into account that $0 < \alpha, \beta, \gamma < 2\pi$ (this implies that the sine functions of half cone angles are non-zero), turn the set of relations R_k , $k \in \{1, \dots, 5\}$ into a new one:

$$\tilde{R}_1 = -\cos \phi \sin \frac{\gamma}{2} \sin \theta + \left(\sin \frac{\alpha}{2} \cos \frac{\gamma}{2} + \cos \frac{\alpha}{2} \sin \frac{\gamma}{2} \cos \theta \right) \sin \phi,$$

$$\tilde{R}_2 = -\cos \theta \sin \frac{\beta}{2} \sin \phi + \left(\sin \frac{\alpha}{2} \cos \frac{\beta}{2} + \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \phi \right) \sin \theta.$$

Note, that the conditions of Theorem 2 concerning cone angles are exactly the existence conditions for a spherical triangle with angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$. For the latter, the following trigonometric identities (spherical cosine and sine rules) are satisfied [25, Theorems 2.5.2 and 2.5.4]:

$$\begin{aligned} \cos \phi &= \frac{\cos \frac{\gamma}{2} + \cos \frac{\alpha}{2} \cos \frac{\beta}{2}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}}, \\ \cos \theta &= \frac{\cos \frac{\beta}{2} + \cos \frac{\alpha}{2} \cos \frac{\gamma}{2}}{\sin \frac{\alpha}{2} \sin \frac{\gamma}{2}}, \\ \frac{\sin \phi}{\sin \frac{\gamma}{2}} &= \frac{\sin \theta}{\sin \frac{\beta}{2}}. \end{aligned}$$

These identities state that the points \widehat{F}_a , \widehat{F}_b and \widehat{F}_c form a triangle on \mathbb{S}^2 with angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$ and $\frac{\gamma}{2}$ at the corresponding vertices. Its double provides the base turnover with cone angles α , β and γ for the fibred cone-manifold $\mathcal{H}_3(\alpha, \beta, \gamma)$.

On substituting the expressions for $\cos \phi$ and $\cos \psi$ above in the relations \tilde{R}_k , $k \in \{1, 2\}$ and taking into account the sine rule, one obtains that $\tilde{R}_k = 0$, $k \in \{1, 2\}$. The lemma is proven. \square

Let \mathcal{S} denote the domain of cone angles indicated in the statement of the theorem:

$$\mathcal{S} = \left\{ \vec{\alpha} = (\alpha, \beta, \gamma) \mid \begin{array}{l} 2\pi - \gamma < \alpha + \beta < 2\pi + \gamma \\ -2\pi + \gamma < \alpha - \beta < 2\pi - \gamma \end{array} \right\}.$$

Let \mathcal{S}^* denote the subset of \mathcal{S} , such that for every triple of cone angles $\vec{\alpha} = (\alpha, \beta, \gamma) \in \mathcal{S}^*$ there exists a spherical structure on $\mathcal{H}_3(\vec{\alpha})$. Our next step is to show that \mathcal{S}^* coincides with \mathcal{S} .

The set \mathcal{S}^* is non-empty. From [8], it follows that $\mathcal{H}_3(\pi, \pi, \pi)$ has a spherical structure. The orbifold $\mathcal{H}_3(\pi, \pi, \pi)$ is Seifert fibred and its base is a turnover with cone angles equal to π . Thus, the point $(\pi, \pi, \pi) \in \mathcal{S}$ belongs to \mathcal{S}^* . The set \mathcal{S}^* is open, because a deformation of the holonomy induces a deformation of the structure [20].

In order to prove that the set \mathcal{S}^* is closed, we consider a sequence $\vec{\alpha}_n = (\alpha_n, \beta_n, \gamma_n)$ in \mathcal{S}^* converging to $\vec{\alpha}_\infty = (\alpha_\infty, \beta_\infty, \gamma_\infty)$ in \mathcal{S} . Since every spherical cone-manifold with cone angles $\leq 2\pi$ is an Alexandrov space with curvature ≥ 1 [3], we obtain that the diameter of $\mathcal{H}_3(\vec{\alpha}_n)$ is bounded above: $\text{diam } \mathcal{H}_3(\vec{\alpha}_n) \leq \pi$.

Let $\text{dist } \mathcal{H}_3(\vec{\alpha}_n)$ denote the minimum of the mutual distances between the axis of rotations A , B and C . Since $\vec{\alpha}_\infty \in \mathcal{S}$, we have by Lemma 5 that the turnover $\mathbb{S}^2(\vec{\alpha}_\infty)$ is non-degenerate. By making use of Lemma 3, one obtains that (restricting to a subsequence, if needed) for every $\vec{\alpha}_n \in \mathcal{S}$, $n = 1, 2, \dots$ the function $\text{dist } \mathcal{H}_3(\vec{\alpha}_n)$ is uniformly bounded below away from zero:

$$\text{dist } \mathcal{H}_3(\vec{\alpha}_n) \geq d_0 > 0, \quad n = 1, 2, \dots$$

Then we use the following facts [3]:

1. The Gromov-Hausdorff limit of Alexandrov spaces with curvature ≥ 1 , dimension = 3 and bounded diameter is an Alexandrov space with curvature ≥ 1 and dimension ≤ 3 ,
2. Dimension of an Alexandrov space with curvature ≥ 1 holds the same at every point (the word ‘‘dimension’’ means Hausdorff or topological dimension, which are equal in the case of curvature ≥ 1).

Since $\text{dist } \mathcal{H}_3(\vec{\alpha}_n) \geq d_0 > 0$, the sequence $\mathcal{H}_3(\vec{\alpha}_n)$ does not collapse. Thus, the cone-manifold $\mathcal{H}_3(\vec{\alpha}_\infty)$ has a non-degenerate spherical structure and $\vec{\alpha}_\infty \in \mathcal{S}^*$.

The subset $\mathcal{S}^* \subset \mathcal{S}$ is non-empty, as well as both closed and open. This implies $\mathcal{S}^* = \mathcal{S}$.

Finally, we claim the following fact concerning the geometric characteristics of $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold:

Lemma 6 *Let $l_\alpha, l_\beta, l_\gamma$ denote the lengths of the singular strata for $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold with cone angles α, β and γ . Then*

$$l_\alpha = l_\beta = l_\gamma = \frac{\alpha + \beta + \gamma}{2} - \pi.$$

The volume of $\mathcal{H}_3(\alpha, \beta, \gamma)$ is

$$\text{Vol } \mathcal{H}_3(\alpha, \beta, \gamma) = \frac{1}{2} \left(\frac{\alpha + \beta + \gamma}{2} - \pi \right)^2.$$

Proof. Let us calculate the geometric parameters explicitly, using the holonomy map defined above. First, we introduce two notions suitable for the further discussion. Given an element $M = \langle M_l, M_r \rangle \in SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, one may assume that the pair of matrices $\langle M_l, M_r \rangle$ is conjugated, by means of a certain element $\langle C_l, C_r \rangle \in SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$, to the pair of diagonal matrices

$$\left\langle \left(\begin{array}{cc} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{array} \right), \left(\begin{array}{cc} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{array} \right) \right\rangle$$

with $0 \leq \gamma, \varphi \leq \pi$.

Then call *the translation length* of M the quantity $\delta(M) := \varphi - \gamma$ and call *the “jump”* of M the quantity $\nu(M) := \varphi + \gamma$, see [11] and [31, Ch.6.4.2]. We suppose that $\varphi > \gamma$, otherwise changing γ, φ for $2\pi - \gamma$ and $\pi - \varphi$ makes the considered tuple to have the desired form.

Recall that the representation of $\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3)$ is

$$\Gamma = \langle a, b, c, h \mid acb = bac = cba = h, h \in Z(\Gamma) \rangle,$$

where a, b, c are meridians and h is a longitudinal loop that represents a fibre. Denote by H the image of h under the holonomy map $\tilde{\rho}$. Then we obtain

$$\ell_\alpha = \ell_\beta = \ell_\gamma = \delta(H).$$

Since $A = \tilde{\rho}(a)$ and $H = \tilde{\rho}(h)$ commute, there exists an element $C = \langle C_l, C_r \rangle$ of $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ such that

$$\begin{aligned} CAC^{-1} &= \left\langle \left(\begin{array}{cc} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{array} \right), \left(\begin{array}{cc} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{array} \right) \right\rangle, \\ CHC^{-1} &= \left\langle \left(\begin{array}{cc} e^{i\gamma(H)} & 0 \\ 0 & e^{-i\gamma(H)} \end{array} \right), \left(\begin{array}{cc} e^{i\varphi(H)} & 0 \\ 0 & e^{-i\varphi(H)} \end{array} \right) \right\rangle. \end{aligned}$$

By a straightforward computation similar to that in Lemma 5, one obtains

$$2 \cos \gamma(H) = \text{tr } H_l = \text{tr } A_l C_l B_l = \text{tr}(-\text{id}) = 2 \cos \pi$$

and

$$2 \cos \varphi(H) = \text{tr } H_r = \text{tr } A_r C_r B_r = 2 \cos \frac{\alpha + \beta + \gamma}{2}.$$

From the foregoing discussion, the singular stratum's length is

$$\ell_\alpha = \delta(H) = \frac{\alpha + \beta + \gamma}{2} - \pi.$$

An analogous equality holds for ℓ_β and ℓ_γ .

By the Schläfli formula [12], the following relation holds:

$$2 \, \text{dVol } \mathcal{H}_3(\alpha, \beta, \gamma) = \ell_\alpha \text{d}\alpha + \ell_\beta \text{d}\beta + \ell_\gamma \text{d}\gamma.$$

Solving this differential equality, we obtain that

$$\text{Vol } \mathcal{H}_3(\alpha, \beta, \gamma) = \frac{1}{2} \left(\frac{\alpha + \beta + \gamma}{2} - \pi \right)^2 + \text{Vol}_0,$$

where Vol_0 is an arbitrary constant. Since the geometric structure on the base space of the fibration (consequently, on the whole $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold) degenerates when $\alpha + \beta + \gamma \rightarrow 2\pi$, the equality $\text{Vol}_0 = 0$ follows from the volume function continuity. \square

Consider a holonomy $\tilde{\rho} = \langle \tilde{\rho}_1, \tilde{\rho}_2 \rangle : \Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_3) \rightarrow SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$ for $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold. As we already know from the preceding discussion, one has $\tilde{\rho} : \Gamma \rightarrow SU_2(\mathbb{C}) \times \mathbb{S}^1$ essentially, and $\tilde{\rho}_1$ determines $\tilde{\rho}_2$ up to a conjugation by means of the equality $\text{tr } \tilde{\rho}_1(m) = \text{tr } \tilde{\rho}_2(m)$ for meridians in Γ . So any deformation of $\tilde{\rho}$ is a deformation of $\tilde{\rho}_1$. In the case of $\mathcal{H}_3(\alpha, \beta, \gamma)$, the map $\tilde{\rho}_1$ is a non-abelian representation of the base turnover group. Spherical turnover is rigid, that means $\tilde{\rho}_1$ is determined only by the corresponding cone angles. Thus $\mathcal{H}_3(\alpha, \beta, \gamma)$ is locally rigid.

The global rigidity follows from the fact that every $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold could be deformed to the orbifold $\mathcal{H}_3(\pi, \pi, \pi)$ by a continuous path through locally rigid structures. This assertion holds since \mathcal{S}^* contains the point (π, π, π) and \mathcal{S}^* is convex. The global rigidity of $\mathcal{H}_3(\pi, \pi, \pi)$ spherical orbifold follows from [26, 27] and implies the global rigidity of $\mathcal{H}_3(\alpha, \beta, \gamma)$ by means of deforming the orbifold structure backwards to the considered cone-manifold one. \square

4.2 Case of flexibility: the cone-manifold $\mathcal{H}_4(\alpha)$

Let $\mathcal{H}_4(\alpha)$ denote a three-dimensional cone-manifold with underlying space the sphere \mathbb{S}^3 and singular locus formed by the link \mathcal{H}_4 with cone angle α along all its components.

The following theorem provides an example of a flexible cone-manifold which is Seifert fibred.

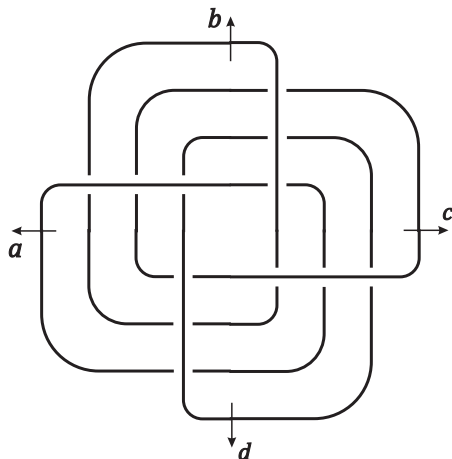


Figure 3: The link \mathcal{H}_4

Theorem 3 *The cone-manifold $\mathcal{H}_4(\alpha)$ admits a spherical structure if*

$$\pi < \alpha < 2\pi.$$

This structure is not unique (i.e. $\mathcal{H}_4(\alpha)$ is not globally, nor locally rigid). The deformation space contains an open interval, that provides a one-parameter family of distinct spherical cone-metrics on \mathbb{S}^3 .

The length of each singular stratum is

$$\ell = 2(\alpha - \pi).$$

The volume of $\mathcal{H}_4(\alpha)$ equals

$$\text{Vol } \mathcal{H}_4(\alpha) = 2(\alpha - \pi)^2.$$

Proof. The following lemma precedes the proof of the theorem.

Lemma 7 *Given a quadrangle Q on \mathbb{S}^2 with three right angles and one angle $\frac{\alpha}{2}$ (see Fig. 4), the following statements hold:*

1. *The quadrangle Q exists if $\pi < \alpha < 2\pi$,*
2. $\sin \ell_1 \sin \ell_2 = -\cos \frac{\alpha}{2}$,
3. $\cos \phi = \frac{\cos \ell_1 \cos \ell_2}{\sin \frac{\alpha}{2}}$,

$$4. \cos \psi = \tan \ell_1 \cot \phi,$$

$$5. 0 \leq \ell_1, \ell_2, \phi, \psi \leq \frac{\pi}{2}.$$

Proof. We refer the reader to [30, § 3.2] for a detailed proof of the statements above. \square

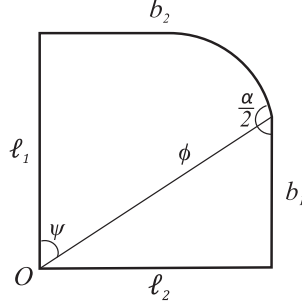


Figure 4: The quadrangle Q

Given a quadrangle Q from Lemma 7 (so-called Saccheri's quadrangle) one can construct another one, depicted in Fig.5, by reflecting Q in its sides incident to the vertex O . We may regard O to be the point $(0, 0) \in \mathbb{S}^2$. Thus, the fibres over the corresponding vertices are

$$F_a(t) = M(\psi, \phi) F(t),$$

$$F_b(t) = M(\pi - \psi, \phi) F(t),$$

$$F_c(t) = M(\pi + \psi, \phi) F(t),$$

$$F_d(t) = M(2\pi - \psi, \phi) F(t).$$

Let $A = \langle A_l, A_r \rangle$, $B = \langle B_l, B_r \rangle$, $C = \langle C_l, C_r \rangle$, $D = \langle D_l, D_r \rangle$ denote the respective rotations through angle α about the axis F_a , F_b , F_c and F_d . From Lemma 2, one obtains

$$A_l = \overline{M(\psi, \phi)} R(\alpha) M(\psi, \phi)^t, \quad A_r = R(\alpha);$$

$$B_l = \overline{M(\pi - \psi, \phi)} R(\alpha) M(\pi - \psi, \phi)^t, \quad B_r = R(\alpha);$$

$$C_l = \overline{M(\pi + \psi, \phi)} R(\alpha) M(\pi + \psi, \phi)^t, \quad C_r = R(\alpha);$$

$$D_l = \overline{M(2\pi - \psi, \phi)} R(\alpha) M(2\pi - \psi, \phi)^t, \quad D_r = R(\alpha).$$

We assume that ℓ_1 , ℓ_2 , ϕ and ψ satisfy the identities of Lemma 7.

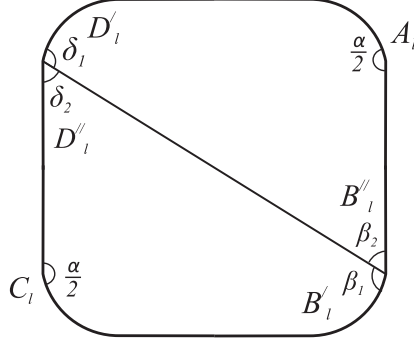


Figure 6: Section of P by the line joining vertices B and D

β_2 and the rotations D_l' , D_l'' through angles δ_1 and δ_2 , respectively. The following equalities hold: $\beta_1 + \beta_2 = \frac{\alpha}{2}$ and $\delta_1 + \delta_2 = \frac{\alpha}{2}$. Thus, the triples D_l'', C_l, B_l' and A_l, D_l', B_l'' consist of rotations about the vertices of two disjoint triangles depicted in Fig. 6. Similar to the computation of Lemma 6, we have

$$D_l'' C_l B_l' = -\text{id}$$

and

$$A_l D_l' B_l'' = -\text{id}.$$

From the identities above, it follows that

$$A_l D_l C_l B_l = A_l D_l' D_l'' C_l B_l' B_l'' = -A_l D_l' B_l'' = \text{id}.$$

The statement holds under a cyclic permutation of the factors. Thus,

$$A_l D_l C_l B_l = B_l A_l D_l C_l = C_l B_l A_l D_l = D_l C_l B_l A_l = \text{id}.$$

Below we shall consider the side-length ℓ_1 as a parameter. Let $\ell_1 := \tau$. Then by Lemma 7 one has that $\sin \ell_2 = -\frac{\cos \frac{\alpha}{2}}{\sin \tau}$ and $\ell_2 := \ell_2(\tau)$ is a well-defined continuous function of τ . The quadrangle P depends on the parameter τ continuously while keeping the angles in its vertices equal to $\frac{\alpha}{2}$.

Let $\mathcal{H}_4(\alpha; \tau)$ denote a three-dimensional cone-manifold with underlying space the sphere \mathbb{S}^3 and singular locus the link \mathcal{H}_4 with cone angle α along its components. Furthermore, its holonomy map is determined by the quadrangle P described above (see Fig. 5) depending on the parameter τ . This means that the double of P forms a ‘‘pillowcase’’ cone-surface with all cone angles equal to α , which is the base space for the fibred cone-manifold $\mathcal{H}_4(\alpha; \tau)$.

Let $\mathbb{L}_n(\alpha, \beta)$ be a cone-manifold with underlying space the sphere \mathbb{S}^3 and singular locus a torus link of the type $(2, 2n)$ with cone angles α and β along its components. Torus links of the type $(2, 2n)$ are two-bridge links. The corresponding cone-manifolds were previously considered in [17, 22]. Since the cone-manifold $\mathcal{H}_4(\alpha)$ forms a 4-fold branched covering of the cone-manifold $\mathbb{L}_4(\alpha, \frac{\pi}{2})$, from [17, Theorem 2] we obtain that $\mathcal{H}_4(\alpha)$ has a spherical structure if $\pi < \alpha < 2\pi$. The length of each singular stratum equals to $\ell = 2(\alpha - \pi)$ and the volume is $\text{Vol } \mathcal{H}_4(\alpha) = 2(\alpha - \pi)^2$.

Under the assumption that $\ell_1 = \ell_2$, the base quadrangle depicted in Fig. 5 appears to have a four order symmetry. Moreover, by making use of Lemma 7, one may derive the following equalities: $\psi = \frac{\pi}{4}$, $\cos \phi = \cot \frac{\alpha}{4}$. The general formulas for the holonomy of $\mathcal{H}_4(\alpha)$ cone-manifold derived above subject to the condition $\ell_1 = \ell_2$ (equivalently, the cone-manifold $\mathcal{H}_4(\alpha)$ has a four order symmetry) give the holonomy map induced by the covering. Thus $\mathcal{H}_4(\alpha) \cong \mathcal{H}_4(\alpha; \arccos(\sqrt{2} \cos \frac{\alpha}{4}))$ is a spherical cone-manifold.

We claim that one can vary the parameter τ in certain ranges while keeping spherical structure on $\mathcal{H}_4(\alpha; \tau)$ non-degenerate.

Lemma 8 *If τ varies over $(\frac{\alpha-\pi}{2}, \frac{\pi}{2})$, the cone-manifold $\mathcal{H}_4(\alpha; \tau)$ has a non-degenerate spherical structure.*

Proof. The proof has much in common with the proof of the spherical structure existence on $\mathcal{H}_3(\alpha, \beta, \gamma)$ cone-manifold given in Theorem 2. Let us express the identities of Lemma 7 in terms of the parameter $\ell_1 := \tau$. We obtain

$$\begin{aligned} \cos \phi &= \cos \tau \sqrt{1 - \cot^2 \frac{\alpha}{2} \cot^2 \tau}, \\ \cos \psi &= \sqrt{\frac{1 - \cot^2 \frac{\alpha}{2} \cot^2 \tau}{1 + \cot^2 \frac{\alpha}{2} \cot^4 \tau}}, \\ \sin \ell_2 &= -\frac{\cos \frac{\alpha}{2}}{\sin \tau}. \end{aligned}$$

Since Lemma 7 states that $0 \leq \phi, \psi, \ell_2 \leq \frac{\pi}{2}$, the functions $\phi := \phi(\tau)$, $\psi := \psi(\tau)$, $\ell_2 := \ell_2(\tau)$ are well-defined and depend continuously on τ .

Moreover, the following relations hold:

$$\begin{aligned} \cos b_1 &= \frac{\cos \phi}{\cos \ell_2} = \cos \tau \sqrt{\frac{\sin^2 \tau - \cot^2 \frac{\alpha}{2} \cos^2 \tau}{\sin^2 \tau - \cos^2 \frac{\alpha}{2}}}, \\ \cos b_2 &= \frac{\cos \phi}{\cos \tau} = \sqrt{1 - \cot^2 \frac{\alpha}{2} \cot^2 \tau}. \end{aligned}$$

If one sets the centre O of the quadrangle P to $(0,0) \in \mathbb{S}^2$, the whole quadrangle is situated in the upper hemisphere provided $\phi < \frac{\pi}{2}$. From the fact that $\cos b_1 \geq \cos \phi$ and $\cos b_2 \geq \cos \phi$, it follows $b_1, b_2 \leq \phi$. Thus $b_1, b_2 \leq \frac{\pi}{2}$ and the functions $b_1 := b_1(\tau)$, $b_2 := b_2(\tau)$ are well-defined and continuous with respect to τ .

Observe that if the condition $\frac{\alpha-\pi}{2} < \tau < \frac{\pi}{2}$ is satisfied, then the required inequality $\phi < \frac{\pi}{2}$ holds.

Let \mathcal{S}_α^* denote the subset of $\mathcal{S}_\alpha = \{\tau | \frac{\alpha-\pi}{2} < \tau < \frac{\pi}{2}\}$ that consists of the points $\tau \in \mathcal{S}_\alpha$ such that the cone-manifold $\mathcal{H}_4(\alpha; \tau)$ has a non-degenerate spherical structure. We show $\mathcal{S}_\alpha^* = \mathcal{S}_\alpha$ by means of the fact that \mathcal{S}_α^* is both open and closed non-empty subset of \mathcal{S}_α .

As noticed above, $\tau = \arccos(\sqrt{2} \cos \frac{\alpha}{4})$ belongs to \mathcal{S}_α^* . Hence the set \mathcal{S}_α^* is non-empty.

The set \mathcal{S}_α^* is open by the fact that a deformation of the holonomy implies a deformation of the structure [20]. To prove that \mathcal{S}_α^* is closed, consider a sequence τ_n converging in \mathcal{S}_α^* to $\tau_\infty \in \mathcal{S}_\alpha$.

The lengths of common perpendiculars between the axis of rotations A, B, C and D defined above equal respectively b_1, b_2 and ϕ .

Since τ_∞ corresponds to a non-degenerated quadrangle, every cone-manifold $\mathcal{H}_4(\alpha; \tau_n)$ has the quantities $b_1(\tau_n)$, $b_2(\tau_n)$ and $\phi(\tau_n)$ uniformly bounded below away from zero. By the arguments similar to those of Theorem 2, we obtain that $\mathcal{H}_4(\alpha; \tau_\infty)$ is a non-degenerate spherical cone-manifold. Thus τ_∞ belongs to \mathcal{S}_α^* . Hence \mathcal{S}_α^* is closed.

Finally, we obtain that $\mathcal{S}_\alpha^* = \mathcal{S}_\alpha$. Thus, while τ varies over $(\frac{\alpha-\pi}{2}, \frac{\pi}{2})$ the cone-manifold $\mathcal{H}_4(\alpha; \tau)$ does not collapse. \square

The following lemma shows that the interval $(\frac{\alpha-\pi}{2}, \frac{\pi}{2})$ represents a part of the deformation space for possible spherical structures on $\mathcal{H}_4(\alpha; \tau)$.

Lemma 9 *The cone-manifolds $\mathcal{H}_4(\alpha; \tau_1)$ and $\mathcal{H}_4(\alpha; \tau_2)$ with $\pi < \alpha < 2\pi$ and $\frac{\alpha-\pi}{2} < \tau_1, \tau_2 < \frac{\pi}{2}$ are not isometric if $\tau_1 \neq \tau_2$.*

Proof. If the cone-manifolds $\mathcal{H}_4(\alpha; \tau_1)$ and $\mathcal{H}_4(\alpha; \tau_2)$ were isometric, then their holonomy maps $\tilde{\rho}_i$, $i = 1, 2$ would be conjugated representations of $\Gamma = \pi_1(\mathbb{S}^3 \setminus \mathcal{H}_4)$ into $SU_2(\mathbb{C}) \times SU_2(\mathbb{C})$. Then the mutual distances between the axis of rotations A_i, B_i, C_i and D_i , $i = 1, 2$, coming from the holonomy maps $\tilde{\rho}_1$ and $\tilde{\rho}_2$ would be equal for the corresponding pairs. From Lemma 3, it follows that the common perpendicular length for the given fibres C_1 and C_2 is half the distance between the images of C_1 and C_2 under the Hopf map. By applying Lemmas 3 and 8 to the base quadrangle P of $\mathcal{H}_4(\alpha; \tau_i)$, $i = 1, 2$ one makes sure that the inequality $\tau_1 \neq \tau_2$ implies the inequality for the lengths of corresponding common perpendiculars. \square

Note, that by the Schläfli formula the volume of $\mathcal{H}_4(\alpha)$ remains the same under any deformation preserving cone angles. Then the formulas for the volume and the singular stratum length follow from the covering properties of $\mathcal{H}_4(\alpha) \xrightarrow{4:1} \mathbb{L}_4(\alpha, \frac{\pi}{2})$ and Theorem 2 of [17]. Thus, Theorem 3 is proven. \square

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