



HAL
open science

Counterexamples to mean square almost periodicity of the solutions of some SDEs with almost periodic coefficients

Omar Mellah, Paul Raynaud de Fitte

► **To cite this version:**

Omar Mellah, Paul Raynaud de Fitte. Counterexamples to mean square almost periodicity of the solutions of some SDEs with almost periodic coefficients. *Electronic Journal of Differential Equations*, 2013, 2013 (91), pp.1-7. hal-00726462

HAL Id: hal-00726462

<https://hal.science/hal-00726462v1>

Submitted on 30 Aug 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COUNTEREXAMPLES TO MEAN SQUARE ALMOST PERIODICITY OF THE SOLUTIONS OF SOME SDES WITH ALMOST PERIODIC COEFFICIENTS

OMAR MELLAH AND PAUL RAYNAUD DE FITTE

ABSTRACT. We show that, contrarily to what is claimed in some papers, the nontrivial solutions of some stochastic differential equations with almost periodic coefficients are never mean square almost periodic (but they can be almost periodic in distribution).

1. INTRODUCTION

Almost periodicity for stochastic processes and in particular for solutions of stochastic differential equations is investigated in an increasing number of papers since the works of C. Tudor and his collaborators [11, 12, 1, 8], who proved almost periodicity in distribution of solutions of some SDEs with almost periodic coefficients. More recently, Bezandry and Diagana [3, 4, 5] claimed that some SDEs with almost periodic coefficients have solutions which satisfy the stronger property of mean square almost periodicity. These claims are repeated in some subsequent papers and a book by different authors.

The aim of this short note is to give counterexamples to the results of [3, 4, 5].

Notations and definitions We denote by $\text{law}(Y)$ the distribution of a random variable Y . If \mathbb{X} is a metrizable topological space, we denote by $\mathcal{M}^{1,+}(\mathbb{X})$ the set of Borel probability measures on \mathbb{X} , endowed with the topology of narrow (or weak) convergence, i.e. the coarsest topology such that the mappings $\mu \mapsto \mu(\varphi)$, $\mathcal{M}^{1,+}(\mathbb{X}) \rightarrow \mathbb{R}$ are continuous for all bounded continuous $\varphi : \mathbb{X} \rightarrow \mathbb{R}$.

Let (\mathbb{X}, d) be a metric space. A continuous mapping $f : \mathbb{R} \rightarrow \mathbb{X}$ is said to be *almost periodic* (in Bohr's sense) if, for every $\varepsilon > 0$, there exists a number $l(\varepsilon) > 0$ such that every interval I of length greater than $l(\varepsilon)$ contains an ε -almost period, that is, a number $\tau \in I$ such that $d(f(t+\tau), f(t)) \leq \varepsilon$ for all $t \in \mathbb{R}$. Equivalently, by a criterion of Bochner, f is almost periodic if and only if the set $\{x(t + \cdot), t \in \mathbb{R}\}$ is totally bounded in the space $C(\mathbb{R}, \mathbb{X})$ endowed with the topology of uniform convergence. Thanks to another criterion of Bochner [6], almost periodicity of f does not depend on the metric d nor on the uniform structure of (\mathbb{X}, d) , but only on f and the topology generated by d (see [2] for details). We refer to e.g. [7, 13] for beautiful expositions of almost periodic functions and their many properties.

Let $X = (X_t)_{t \in \mathbb{R}}$ be a continuous stochastic process with values in a separable Banach space \mathbb{E} :

2010 *Mathematics Subject Classification.* 34C27;60H25;34F05;60H10.

Key words and phrases. Square-mean almost periodic; almost periodic in distribution; Ornstein-Uhlenbeck; semilinear stochastic differential equation; stochastic evolution equation .

- We say that X is *mean square almost periodic* if X_t is square integrable for each t and the mapping $t \mapsto X_t, \mathbb{R} \rightarrow L^2(\mathbb{E})$ is almost periodic.
- We say that X is *almost periodic in distribution* (in Bohr's sense) if the mapping $t \mapsto \text{law}(X_{t+\cdot}), \mathbb{R} \mapsto \mathcal{M}^{1,+}(\mathcal{C}(\mathbb{R}, \mathbb{E}))$ is almost periodic, where $\mathcal{C}(\mathbb{R}, \mathbb{E})$ is endowed with the topology of uniform convergence on compact subsets.

It is shown in [2] that, if X is mean square almost periodic, then X is almost periodic in distribution. The counterexamples of this paper also show that the converse implication is false (actually, it is proved in [2] that the converse implication is true under a tightness condition).

2. TWO EXPLICIT COUNTEREXAMPLES

The following very simple counterexample, inspired by [2, Counterexample 2.16], was suggested to us by Adam Jakubowski. It contradicts [3, Theorem 3.2], [4, Theorem 3.3], and [5, Theorem 4.2].

Example 2.1. (stationary Ornstein-Uhlenbeck process) Let $W = (W_t)_{t \in \mathbb{R}}$ be a standard Brownian motion on the real line. Let $\alpha, \sigma > 0$, and let X be the stationary Ornstein-Uhlenbeck process (see [10]) defined by

$$X_t = \sqrt{2\alpha\sigma} \int_{-\infty}^t e^{-\alpha(t-s)} dW_s. \quad (2.1)$$

Then X is the only L^2 -bounded solution of the following SDE, which is a particular case of Equation (3.1) in [3]:

$$dX_t = -\alpha X_t dt + \sqrt{2\alpha\sigma} dW_t.$$

The process X is Gaussian with mean 0, and we have, for all $t \in \mathbb{R}$ and $\tau \geq 0$,

$$\text{Cov}(X_t, X_{t+\tau}) = \sigma^2 e^{-\alpha\tau}.$$

Assume that X is mean square almost periodic, and let (t_n) be any increasing sequence of real numbers which converges to ∞ . By Bochner's characterization, we can extract a sequence (still denoted by (t_n) for simplicity) such that (X_{t_n}) converges in L^2 to a random variable Y . Necessarily Y is Gaussian with law $\mathcal{N}(0, 2\alpha\sigma^2)$, and Y is \mathcal{G} -measurable, where $\mathcal{G} = \sigma(X_{t_n}; n \geq 0)$. Moreover (X_{t_n}, Y) is Gaussian for every n , and we have, for any integer n ,

$$\text{Cov}(X_{t_n}, Y) = \lim_{m \rightarrow \infty} \text{Cov}(X_{t_n}, X_{t_n+m}) = 0.$$

because $(X_t^2)_{t \in \mathbb{R}}$ is uniformly integrable. This proves that Y is independent of X_{t_n} for every n , thus Y is independent of \mathcal{G} . Thus Y is constant, a contradiction.

Thus (2.1) has no mean square almost periodic solution.

A similar reasoning applies to the next counterexample, which also contradicts [3, Theorem 3.2], [4, Theorem 3.3], and [5, Theorem 4.2]:

Example 2.2. Again, $W = (W_t)_{t \in \mathbb{R}}$ is a standard Brownian motion on the real line. Let X be defined by

$$X_t = e^{-t+\sin(t)} \int_{-\infty}^t e^{s-\sin(s)} \sqrt{1-\cos(s)} dW_s.$$

Then X satisfies the SDE with periodic coefficients

$$dX_t = (-1 + \cos(t))X_t dt + \sqrt{1 - \cos(t)} dW_t.$$

The process X is Gaussian, with $\mathbb{E} X_t = 0$ and

$$\begin{aligned} \text{Cov}(X_t, X_{t+\tau}) &= e^{-t-\tau+\sin(t+\tau)} e^{-t+\sin(t)} \int_{-\infty}^t e^{2(s-\sin(s))} (1 - \cos(s)) ds \\ &= \frac{1}{2} e^{-\tau+\sin(t+\tau)-\sin(t)} \rightarrow 0 \text{ when } \tau \rightarrow +\infty \end{aligned}$$

in particular $\mathbb{E} X_t^2 = \frac{1}{2} e^{2\sin(t)} \geq \frac{1}{2} e^{-2}$ thus the same reasoning as in Example 2.1 shows that X is not mean-square almost periodic, because if X_{t_n} converges in L^2 to Y , with $t_n \rightarrow \infty$, then $Y = 0$ and $\mathbb{E} Y^2 \geq e^{-2}/2$.

By [8, Theorem 4.1], the process X is periodic in distribution.

The argument in the previous counterexamples can be slightly generalized for non necessarily Gaussian processes as follows:

Lemma 2.3. *Let X be a continuous square integrable stochastic process with values in a Banach space \mathbb{E} . Assume that $(\|X_t\|^2)_{t \in \mathbb{R}}$ is uniformly integrable and that there exists a sequence (t_n) of real numbers, $t_n \rightarrow \infty$, such that for any $x^* \in \mathbb{E}^*$ and any integer $n \geq 0$,*

$$\lim_{m \rightarrow \infty} \text{Cov}(\langle x^*, X_{t_n} \rangle, \langle x^*, X_{t_m} \rangle) = 0, \quad (2.2)$$

$$\lim_{m \rightarrow \infty} \text{Var}(\|X_{t_m}\|) > 0. \quad (2.3)$$

Then X is not mean square almost periodic.

Proof. Assume that X is mean square almost periodic. Then, for some subsequence (t'_n) of (t_n) , $X_{t'_n}$ converges in L^2 to some random vector Y . By (2.3) and the uniform integrability hypothesis, Y is not constant. On the other hand, by (2.2) and the uniform integrability hypothesis, we have

$$\text{Cov}(\langle x^*, X_{t'_n} \rangle, \langle x^*, Y \rangle) = 0$$

for every $x^* \in \mathbb{E}^*$ and every integer n . Then

$$\text{Var} \langle x^*, Y \rangle = \lim_n \text{Cov}(\langle x^*, X_{t'_n} \rangle, \langle x^*, Y \rangle) = 0,$$

thus Y is constant, a contradiction. \blacksquare

3. GENERALIZATION

We present a generalization of Counterexamples 2.1 and 2.2 in a Hilbert space setting. Other generalizations in the same setting are possible.

From now on, \mathbb{H} and \mathbb{U} are separable Hilbert spaces, Q is a symmetric nonnegative operator on \mathbb{U} with finite trace, and $(W_t)_{t \in \mathbb{R}}$ is a Q -Brownian motion with values in \mathbb{U} . We denote $\mathbb{U}_0 = Q^{1/2}\mathbb{U}$ and $L_2^0 = L_2(\mathbb{U}_0, \mathbb{H})$ the space of Hilbert-Schmidt operators from \mathbb{U}_0 to \mathbb{H} , endowed with the Hilbertian norm

$$\|\Psi\|_{L_2^0}^2 = \left\| \Psi Q^{1/2} \right\|_{L_2}^2 = \text{Tr}(\Psi Q \Psi^*).$$

It is well known that, if Φ is a predictable stochastic process with values in L_2^0 such that $\int_0^t \|\Phi_s\|_{L_2^0}^2 ds < +\infty$, then we have the Ito isometry

$$\mathbb{E} \left(\left\| \int_0^t \Phi_s dW_s \right\|^2 \right) = \int_0^t \|\Phi_s\|_{L_2^0}^2 ds.$$

Recall (see e.g. [9, Definitions 1.4.1 and 1.4.2]) that a linear operator $A(t)$ on \mathbb{H} with domain $D(A(t))$ generates an *evolution semigroup* $(U(t, s))_{t \geq s}$ on \mathbb{H} , if $(U(t, s))_{t \geq s}$ is a family of bounded linear operators on \mathbb{H} such that

- (i) $U(t, r)U(r, s) = U(t, s)$ for all $t, r, s \in \mathbb{R}$ such that $s \leq r \leq t$, and, for every $t \in \mathbb{R}$, $U(t, t) = I$ the identity operator on \mathbb{H} ,
- (ii) for every $x \in \mathbb{H}$, the mapping $(t, s) \mapsto U(t, s)x$ from $\{(t, s); t \geq s\}$ to \mathbb{H} is continuous,
- (iii) for every $T > 0$, there exists $K_T < \infty$ such that $\|U(t, s)\| \leq K_T$ for $0 \leq s \leq t \leq T$,
- (iv) for all $t, s \in \mathbb{R}$ such that $s \leq t$, the domain $D(A(t))$ is dense in \mathbb{H} , $U(t, s)D(A(s)) \subset D(A(t))$, and

$$\frac{\partial}{\partial t} U(t, s)x = A(t)U(t, s)x \text{ for } t > s \text{ and } x \in D(A(s)).$$

The following theorem contains Counterexamples 2.1 and 2.2. For example, Counterexample 2.2 can be seen as a particular case of Equation (3.1) below, with $A(t) = -1 + \cos(t)$ which generates the evolution semigroup $U(t, s) = e^{-(t-s) + \sin(t) - \sin(s)}$.

Theorem 3.1. (linear evolution equations with almost periodic noise) *Let us consider the stochastic evolution equation*

$$dX_t = A(t)X_t dt + g(t) dW_t \quad (3.1)$$

where $A(\cdot)$ generates an evolution semigroup $(U(t, s))_{t \geq s}$ on \mathbb{H} . We assume that

- (a) (see Hypothesis 1 in [8]) the Yosida approximations $A_n(t) = nA(t)(nI - A(t))^{-1}$ of $A(t)$, $t \in \mathbb{R}$, generate corresponding evolution operators $(U_n(t, s))_{t \geq s}$ such that, for every $x \in \mathbb{H}$ and for all $t, s \in \mathbb{R}$ such that $s \leq t$,

$$\lim_{n \rightarrow \infty} U_n(t, s)x = U(t, s)x$$

- (b) A is uniformly dissipative (see Hypothesis 3 in [8]), i.e. there exists $\beta > 0$ such that

$$\langle A(t)x, x \rangle \leq -\beta \|x\|^2, \quad t \in \mathbb{R}, \quad x \in D(A(t)),$$

- (c) U is exponentially stable (see Hypothesis H0 in [4]), i.e.

$$\|U(t, s)\| \leq M e^{-\delta(t-s)}, \quad t \geq s \quad (3.2)$$

- (d) $g : \mathbb{R} \rightarrow L_2^0$ is almost periodic and satisfies

$$0 < \int_{-\infty}^{+\infty} \|U(t, s)g(s)\|_{L_2^0}^2 ds < +\infty. \quad (3.3)$$

Then (3.1) has no mean square almost periodic solution. However, if the family $(X_t)_{t \in \mathbb{R}}$ is tight, the only L^2 -bounded solution of (3.1) is almost periodic in distribution.

Note that, if A and g are T -periodic, then by [8, Theorem 4.1] the L^2 -bounded solution is T -periodic in distribution, that is, the mapping $t \mapsto \text{law}(X_{t+})$, $\mathbb{R} \mapsto \mathcal{M}^{1,+}(\mathbb{C}(\mathbb{R}, \mathbb{E}))$, is periodic.

Proof. The only L^2 -bounded (mild) solution to (3.1) is given by

$$X_t = \int_{-\infty}^t U(t, s)g(s) dW_s, \quad (3.4)$$

see the proof of [8, Theorem 3.3]. Note that X is Gaussian because the integrand in (3.4) is deterministic. By [9, Theorem 1.4.5], X has a continuous version (actually Theorem 1.4.5 of [9] is given for processes defined on the half line \mathbb{R}^+ , but we can repeat the argument on any interval $[-R, \infty)$). By [8, Theorem 4.3], if the family $(X_t)_{t \in \mathbb{R}}$ is tight, X is almost periodic in distribution.

Let $p > 2$. Applying Burkholder-Davis-Gundy inequalities to the process $t \mapsto \int_{-\infty}^t U(t_0, s)g(s) dW_s$ for fixed t_0 , and then setting $t = t_0$ yields, for some constant c_p ,

$$\mathbb{E} \|X_t\|^p \leq c_p \left(\int_{-\infty}^t \|U(t, s)g(s)\|_{L_2^0}^2 ds \right)^{p/2} \leq c_p \left(\int_{-\infty}^{+\infty} \|U(t, s)g(s)\|_{L_2^0}^2 ds \right)^{p/2} < +\infty$$

(see e.g. [9, Theorems 1.2.1, 1.2.3-(e) and Proposition 1.3.3-(f)]). Thus (X_t) is bounded in L^p , which proves that $(\|X_t\|^2)_{t \in \mathbb{R}}$ is uniformly integrable.

We have $\mathbb{E}(X_t) = 0$ for all $t \in \mathbb{R}$. Let $x \in \mathbb{H}$, $t \in \mathbb{R}$ and $\tau \geq 0$, and let us compute the covariance $\text{Cov}(\langle x, X_t \rangle, \langle x, X_{t+\tau} \rangle)$: We get

$$\begin{aligned} & \text{Cov}(\langle x, X_t \rangle, \langle x, X_{t+\tau} \rangle) \\ &= \mathbb{E} \left(\left\langle x, \int_{-\infty}^t U(t, s)g(s) dW_s \right\rangle \right. \\ & \quad \left. \times \left\langle x, \left(\int_{-\infty}^t U(t+\tau, s)g(s) dW_s + \int_t^{t+\tau} U(t+\tau, s)g(s) dW_s \right) \right\rangle \right) \\ &= \mathbb{E} \left(\left\langle x, \int_{-\infty}^t U(t, s)g(s) dW_s \right\rangle \left\langle U(t+\tau, t)^* x, \int_{-\infty}^t U(t, s)g(s) dW_s \right\rangle \right). \end{aligned}$$

We deduce, using (3.2) and (3.3),

$$\lim_{\tau \rightarrow +\infty} |\text{Cov}(\langle x, X_t \rangle, \langle x, X_{t+\tau} \rangle)| \leq \lim_{\tau \rightarrow +\infty} \|U(t+\tau, t)\| \|x\|^2 \mathbb{E} \int_{-\infty}^t \|U(t, s)g(s)\|_{L_2^0}^2 ds = 0.$$

On the other hand, we have, using (3.3),

$$\text{Var}(\|X_t\|) = \mathbb{E} \left(\int_{-\infty}^t \|U(t, s)g(s)\|_{L_2^0}^2 ds \right) \rightarrow \int_{-\infty}^{+\infty} \|U(t, s)g(s)\|_{L_2^0}^2 ds > 0$$

We conclude by Lemma 2.3 that X is not mean square almost periodic. \blacksquare

4. CONCLUSION

A close look at the proofs of [3, 4, 5] shows the same error in each of those papers, which besides are clever at other places. Let us use the notations of the Hilbert setting of Section 3, and assume that all processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The error lies in the proof of the (untrue) assertion that, if $G : \mathbb{R} \times L^2(\mathbb{P}; \mathbb{H}) \rightarrow L^2(\mathbb{P}; L_2^0)$ is almost periodic in the first variable, uniformly with respect to the second on compact subsets of $L^2(\mathbb{P}; \mathbb{H})$, then the stochastic convolution

$$\Psi(Y)_t := \int_{-\infty}^t U(t, s)G(s, Y_s) dW_s$$

is mean square almost periodic for any continuous square integrable stochastic process Y . If this were true, then with $G(t, Y) = g(t)$ an almost periodic function, and assuming the hypothesis of Theorem 3.1, the process $X = \psi(1)$ of Equation (3.4), which is solution of (3.1), would be mean square almost periodic, but we know from Theorem 3.1 that this is not the case. The error consists in a wrong identification between integrals of the form $\int_{-\infty}^t Z_s dW_s$ and $\int_{-\infty}^t Z_s d\widetilde{W}_s$, where \widetilde{W} has the same distribution as W .

Actually, mean square almost periodicity appears to be a very strong property for solutions of SDEs. Our counter-examples suggest that there are “very few” examples of SDEs with non trivial mean square almost periodic solutions. The question of their characterization remains open.

REFERENCES

- [1] Ludwig Arnold and Constantin Tudor, *Stationary and almost periodic solutions of almost periodic affine stochastic differential equations*, Stochastics Stochastics Rep. **64** (1998), no. 3–4, 177–193. MR 1709282 (2000d:60098)
- [2] Fazia Bedouhene, Omar Mellah, and Paul Raynaud de Fitte, *Bochner-almost periodicity for stochastic processes*, Stoch. Anal. Appl. **30** (2012), no. 2, 322–342. MR 2891457
- [3] Paul H. Bezandry and Toka Diagana, *Existence of almost periodic solutions to some stochastic differential equations*, Appl. Anal. **86** (2007), no. 7, 819–827. MR 2355540 (2008i:60089)
- [4] ———, *Square-mean almost periodic solutions nonautonomous stochastic differential equations*, Electron. J. Differential Equations (2007), No. 117, 10 pp. (electronic). MR MR2349945 (2009e:34171)
- [5] ———, *Existence of quadratic-mean almost periodic solutions to some stochastic hyperbolic differential equations*, Electron. J. Differential Equations (2009), No. 111, 14. MR MR2539219
- [6] S. Bochner, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 2039–2043. MR MR0145283 (26 #2816)
- [7] C. Corduneanu, *Almost periodic functions*, Interscience Publishers [John Wiley & Sons], New York-London-Sydney, 1968, With the collaboration of N. Gheorghiu and V. Barbu, Translated from the Romanian by Gitta Bernstein and Eugene Tomer, Interscience Tracts in Pure and Applied Mathematics, No. 22. MR 0481915 (58 #2006)
- [8] G. Da Prato and C. Tudor, *Periodic and almost periodic solutions for semilinear stochastic equations*, Stochastic Anal. Appl. **13** (1995), no. 1, 13–33. MR MR1313204 (96e:60112)
- [9] Wilfried Grecksch and Constantin Tudor, *Stochastic evolution equations*, Mathematical Research, vol. 85, Akademie-Verlag, Berlin, 1995, A Hilbert space approach. MR 1353910 (96m:60130)
- [10] Georg Lindgren, *Lectures on stationary stochastic processes*, Tech. report, Lund University, Lund, Sweden, 2006, A course for PHD students in Mathematical Statistics and other fields available at <http://www.maths.lth.se/matstat/staff/georg/Publications/lecture2006.pdf>.
- [11] T. Morozan and C. Tudor, *Almost periodic solutions of affine Itô equations*, Stochastic Anal. Appl. **7** (1989), no. 4, 451–474. MR 1040479 (91k:60064)
- [12] C. Tudor, *Almost periodic solutions of affine stochastic evolution equations*, Stochastics Stochastics Rep. **38** (1992), no. 4, 251–266. MR MR1274905 (95e:60058)
- [13] S. Zaidman, *Almost-periodic functions in abstract spaces*, Research Notes in Mathematics, vol. 126, Pitman (Advanced Publishing Program), Boston, MA, 1985. MR 790316 (86j:42018)

DEPARTMENT OF MATHEMATICS FACULTY OF SCIENCES UNIVERSITY OF TIZI-OUZOU, ALGERIA
Current address: Laboratoire Raphaël Salem, UMR CNRS 6085, University of Rouen, France
E-mail address: omellah@yahoo.fr

LABORATOIRE RAPHAËL SALEM, UMR CNRS 6085, UNIVERSITY OF ROUEN, FRANCE
E-mail address: prf@univ-rouen.fr