On the Semiclassical Magnetic Laplacian and Connected Topics

Nicolas Raymond

C'est l'homme aux mille tours, Muse, qu'il faut me dire,
Celui qui tant erra quand, de Troade, il eut pillé la ville sainte,
Celui qui visita les cités de tant d'hommes et connut leur esprit,
Celui qui, sur les mers, passa par tant d'angoisses,
En luttant pour survivre et ramener ses gens.

ON THE SEMICLASSICAL MAGNETIC LAPLACIAN AND CONNECTED TOPICS

NICOLAS RAYMOND

ABSTRACT. The aim of this course is to introduce the reader to the general techniques appearing in the spectral theory of the semiclassical magnetic Laplacian. We explain how we can construct quasi-eigenpairs and how the investigation of the magnetic Laplacian can be reduced to the one of model operators. In particular, the localization estimates of Agmon and the Born-Oppenheimer approximation are discussed in this course. We also propose to analyze two recent examples and we finally provide some other perspectives (Birkhoff normal form and semiclassical waveguides).

1. INTRODUCTION

The aim of this course is to introduce the reader to the techniques appearing in the spectral theory of the semiclassical magnetic Laplacian. This fascinating subject has been extensively studied in the last fifteen years by many authors. The study of the magnetic Laplacian is the occasion to deal with the standard semiclassical and spectral methods. Therefore we will focus this lecture on the magnetic Laplacian, but we will also propose other applications. In particular, we will discuss connected perspectives such as the Birkhoff normal form and waveguides.

1.1. Motivation.

Before defining the operator that we analyze in this course, let us mention the different motivations.

The first motivation arises from the mathematical theory of superconductivity. A model for this theory (see [92]) is given by the Ginzburg-Landau functional:

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega |(-i\nabla + \kappa\sigma\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \, dx + \kappa^2 \int_\Omega |\sigma\nabla \times \mathbf{A} - \sigma\boldsymbol{\beta}|^2 \, dx,$$

where: $\Omega \subset \mathbb{R}^d$ is the place occupied by the superconductor, ψ is the so-called order parameter ($|\psi|^2$ is the density of Cooper pairs), $\mathbf{A}$ is a magnetic potential and $\boldsymbol{\beta}$ the applied magnetic field. The parameter κ is characteristic of the sample (the superconductors of type II are such that $\kappa >> 1$) and σ corresponds to the intensity of the applied magnetic field. Roughly speaking, the question is to know what the nature of minimizers is. Are they normal, that is $(\psi, \mathbf{A}) = (0, \boldsymbol{F})$ with $\nabla \times \boldsymbol{F} = \boldsymbol{\beta}$ (and $\nabla \cdot \boldsymbol{F} = 0$), or not ? We can mention the important result of Giorgi-Phillips [45] which states that, if the applied magnetic field does not vanish, then, for σ large enough, the normal state is the unique minimizer of $\mathcal{G}$ (with the divergence free condition). When analyzing the local minimality of $(0, \boldsymbol{F})$, we are led to compute the

Hessian of $\mathcal{G}$ at $(0, \boldsymbol{F})$ and to analyze the positivity of:

$$(-i\nabla + \kappa\sigma\mathbf{A})^2 - \kappa^2.$$

For further details, we refer to the book of Fournais and Helffer [40] and also [70, 71]. Therefore the theory of superconductivity leads to investigate an operator which is in the form $(-ih\nabla + \mathbf{A})^2$, where $h > 0$ is small (κ is assumed to be large). We will define it more in details in the next subsection.

The second motivation is to understand at which point there is an analogy between the electric Laplacian $-h^2\Delta + V(x)$ and the magnetic Laplacian $(-ih\nabla + \mathbf{A})^2$. For instance, it is well-known that we can perform WKB constructions for the electric Laplacian (see the book of Dimassi and Sjöstrand [30, Chapter 3]) and that such constructions do not seem to be possible in general for the magnetic case (see the course of Helffer [56, Section 6] and the references therein). In some generic situations, we can prove accurate asymptotic (in the semiclassical regime: $h \to 0$) expansions for the eigenvalues of the electric Laplacian and also provide a very fine (WKB) approximation of the attached eigenfunctions. For the magnetic situation, such accurate expansions are difficult to obtain. In fact, the more we know about the expansion of the eigenpairs, the more we can estimate the tunnel effect in the spirit of the electric tunnel effect of Helffer and Sjöstrand (see for instance [52, 53]) when there are symmetries. Estimating the magnetic tunnel effect is still a widely open question directly related to the approximation of the eigenfunctions (see [54] for electric tunneling in presence of magnetic field and [11] in the case with corners).

As we will see in this course we will focus on problems with magnetic fields. Nevertheless, the generality of the techniques and ideas will lead us to discuss other topics such as the Birkhoff normal form and the spectrum of waveguides. In fact, the reader can consider this course as an introduction to general semiclassical and spectral techniques through the example of the magnetic Laplacian.

1.2. **Definition of the Magnetic Laplacian.** Let us now define the operators which will be mainly analyzed in this course. We will assume that Ω is bounded and Lipschitzian and that $\mathbf{A} \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^d)$.

- *The magnetic operator.* Let us denote $\mathbf{A} = (A_1, \cdots, A_d)$. We consider the 1-form[1]:

$$\omega_{\mathbf{A}} = \sum_{k=1}^{d} A_k dx_k.$$

We introduce the exterior derivative of $\omega_{\mathbf{A}}$:

$$\sigma_\beta := d\omega_{\mathbf{A}} = \sum_{j<k} \beta_{j,k} dx_j \wedge dx_k.$$

In dimension 2, the only coefficient is $\beta_{12} = \beta = \partial_{x_1} A_2 - \partial_{x_2} A_1$. In dimension 3, the magnetic vector is defined as:

$$\boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) = (\beta_{23}, -\beta_{13}, \beta_{12}) = \nabla \times \mathbf{A}.$$

[1]We refer to [4, Chapter 7] to introduce the k-forms.

We will discuss in this course the spectral properties of the self-adjoint realizations of the magnetic operator:

$$\mathcal{L}_{h,\mathbf{A},\Omega} = \sum_{k=1}^{d} (-ih\partial_k + A_k)^2,$$

where $h > 0$ is a parameter (related to the Planck constant). We notice the fundamental property, called gauge invariance:

$$e^{-i\phi}(-i\nabla + \mathbf{A})e^{i\phi} = -i\nabla + \mathbf{A} + \nabla\phi$$

so that:

$$e^{-i\phi}(-i\nabla + \mathbf{A})^2 e^{i\phi} = (-i\nabla + \mathbf{A} + \nabla\phi)^2.$$

• *The Dirichlet realization.* Let us consider the following quadratic form which is defined for $u \in \mathcal{C}_0^\infty(\Omega)$ by:

$$Q_{h,\mathbf{A}}(u) = \int_\Omega |(-ih\nabla + \mathbf{A})u|^2 \, d\mathbf{x} \geq 0.$$

The standard Friedrichs procedure (see [89, p. 177]) allows to define a self-adjoint operator $\mathcal{L}_{h,\mathbf{A}}^{\mathsf{Dir}}$ whose (closed) quadratic form is:

$$Q_{h,\mathbf{A}}(u) = \int_\Omega |(-ih\nabla + \mathbf{A})u|^2 \, d\mathbf{x} \geq 0, \quad u \in H_0^1(\Omega)$$

and such that:

$$\langle \mathcal{L}_{h,\mathbf{A}}^{\mathsf{Dir}} u, v\rangle = Q_{h,\mathbf{A}}(u,v), \quad u, v \in \mathcal{C}_0^\infty(\Omega).$$

The domain of the Friedrichs extension is defined as:

$$\mathsf{Dom}(\mathcal{L}_{h,\mathbf{A}}^{\mathsf{Dir}}) = \left\{ u \in H_0^1(\Omega) : \mathcal{L}_{h,\mathbf{A}} u \in L^2(\Omega) \right\}.$$

When Ω is regular, we have the characterization:

$$\mathsf{Dom}(\mathcal{L}_{h,\mathbf{A}}^{\mathsf{Dir}}) = H_0^1(\Omega) \cap H^2(\Omega).$$

• *The Neumann realization.* We consider the other quadratic form defined by:

$$Q_{h,\mathbf{A}}(u) = \int_\Omega |(-ih\nabla + \mathbf{A})u|^2 \, d\mathbf{x}, \quad u \in \mathcal{C}^\infty(\overline{\Omega}).$$

This form admits a Friedrichs extension (a closure) defined by:

$$Q_{h,\mathbf{A}}(u) = \int_\Omega |(-ih\nabla + \mathbf{A})u|^2 \, d\mathbf{x}, \quad u \in H^1(\Omega).$$

By the Friedrichs theorem, we can define a self-adjoint operator $\mathcal{L}_{h,\mathbf{A}}^{\mathsf{Neu}}$ whose domain is given by:

$$\mathsf{Dom}(\mathcal{L}_{h,\mathbf{A}}^{\mathsf{Neu}}) = \left\{ u \in H^1(\Omega) : \mathcal{L}_{h,\mathbf{A}} u \in L^2(\Omega), (-ih\nabla + \mathbf{A})u \cdot \nu = 0, \text{ on } \partial\Omega \right\}.$$

When Ω is regular, this becomes:

$$\mathsf{Dom}(\mathcal{L}_{h,\mathbf{A}}^{\mathsf{Neu}}) = \left\{ u \in H^1(\Omega) : u \in H^2(\Omega), (-ih\nabla + \mathbf{A})u \cdot \nu = 0, \text{ on } \partial\Omega \right\}.$$

The main operators being now defined, let us recall a few elements of spectral theory.

1.3. **Elements of spectral theory.**

• *Spectrum of an unbounded operator.* Let A be an unbounded operator on an Hilbert space H with domain $\mathsf{Dom}(A)$. We recall the following characterizations of its spectrum $\sigma(A)$, its essential spectrum $\sigma_{\mathsf{ess}}(A)$ and its discrete spectrum $\sigma_{\mathsf{dis}}(A)$:

- Spectrum: $\lambda \in \sigma(A)$ if and only if $(A - \lambda\,\mathsf{Id})$ is not invertible from $\mathsf{Dom}(A)$ onto H,
- Essential spectrum: $\lambda \in \sigma_{\mathsf{ess}}(A)$ if and only if $(A - \lambda\,\mathsf{Id})$ is not Fredholm[2] from $\mathsf{Dom}(A)$ into H (see [89, Chapter VI] and [69, Chapter 3]),
- Discrete spectrum: $\sigma_{\mathsf{dis}}(A) := \sigma(A) \setminus \sigma_{\mathsf{ess}}(A)$.

We list now several fundamental properties of essential and discrete spectrum.

Lemma 1.1 (Weyl criterion). *We have $\lambda \in \sigma_{\mathsf{ess}}(A)$ if and only if there exists a sequence $(u_n) \in \mathsf{Dom}(A)$ such that $\|u_n\|_H = 1$, (u_n) has no subsequence converging in H and $(A - \lambda\,\mathsf{Id})u_n \underset{n\to+\infty}{\to} 0$ in H.*

From this lemma, one can deduce (see [69, Proposition 2.21 and Proposition 3.11]):

Lemma 1.2. *The discrete spectrum is formed by isolated eigenvalues of finite multiplicity.*

• *The example of the magnetic Laplacian.* Since Ω is bounded and Lipschitzian, the form domains $H^1_0(\Omega)$ and $H^1(\Omega)$ are compactly included in $L^2(\Omega)$ (by the Riesz-Fréchet-Kolmogorov criterion, see [17]) so that the corresponding Friedrichs extensions $\mathcal{L}^{\mathsf{Dir}}_{h,\mathbf{A}}$ and $\mathcal{L}^{\mathsf{Neu}}_{h,\mathbf{A}}$ have compact resolvents. Therefore these operators have discrete spectra and we can consider the non decreasing sequences of their eigenvalues $(\lambda^{\mathsf{Dir}}_n(h))_{n\in\mathbb{N}^*}$ and $(\lambda^{\mathsf{Neu}}_n(h))_{n\in\mathbb{N}^*}$.

Remark 1.3. Let us give a basic example of Fredholm operator. We consider $P = \mathcal{L}^{\mathsf{Dir}}_{h,\mathbf{A}}$ when Ω is bounded and regular. Let us take λ an eigenvalue of P ($\lambda \in \mathbb{R}$ since P is self-adjoint). As we said $\ker(P - \lambda)$ has finite dimension. Since P is self-adjoint, we can write:

$$\overline{\Im(P-\lambda)} = \ker(P-\lambda)^{\perp}.$$

This is easy to see that the image of $P - \lambda$ is closed. There exists $c > 0$ such that (exercise):

$$\|(P-\lambda)u\| \geq c\|u\|^2, \quad \forall u \in \ker(P-\lambda)^{\perp}.$$

Let us now assume that we have $(P - \lambda)u_n \to v \in L^2(\Omega)$, with $u_n \in \ker(P - \lambda)^{\perp}$. We immediately deduce that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and the conclusion follows. P is a Fredholm operator.

1.4. **The Harmonic Oscillator.**

Before going further we shall discuss the spectrum of the harmonic oscillator which we will encounter many times in this lecture. We introduce a useful notation:

$$D_x = -i\partial_x$$

and we are interested in the self-adjoint realization on $L^2(\mathbb{R})$ of:

$$H_{\mathsf{harm}} = D_x^2 + x^2.$$

[2] We recall that an operator is said to be *Fredholm* if its kernel is finite dimensional, its range is closed and with finite codimension.

In terms of the philosophy of the last section, this operator is defined as the Friedrichs extension associated with the closed quadratic form defined by:

$$Q_{\mathsf{harm}}(\psi) = \|\psi'\|^2 + \|x\psi\|^2, \quad \psi \in B^1(\mathbb{R}),$$

where

$$B^1(\mathbb{R}) = \{\psi \in L^2(\mathbb{R}) : \psi' \in L^2(\mathbb{R}), x\psi \in L^2(\mathbb{R})\}.$$

The domain of the operator can be characterized (thanks to the difference quotients method, see [17, Theorem IX. 25]) as:

$$\mathsf{Dom}(H_{\mathsf{harm}}) = \{\psi \in L^2(\mathbb{R}) : \psi'' \in L^2(\mathbb{R}), x^2\psi \in L^2(\mathbb{R})\}.$$

The self-adjoint operator H_{harm} has compact resolvent since $B^1(\mathbb{R})$ is compactly included in $L^2(\mathbb{R})$. Its spectrum is a sequence of eigenvalues which tends to $+\infty$. Let us explain how we can get the spectrum of H_{harm}. We let:

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad a^* = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right).$$

We have:

$$[a, a^*] = aa^* - a^*a = 1.$$

We let:

$$f_0(x) = e^{-x^2/2}.$$

We investigate the spectrum of a^*a. We have: $af_0 = 0$. We let $f_n = (a^*)^n f_0$. This is easy to prove that $a^*af_n = nf_n$ and that $af_n = nf_{n-1}$. The (f_n) form a Hilbertian basis of $L^2(\mathbb{R})$. These functions are called Hermite's functions. The eigenvalues of H_{harm} are the numbers $2n + 1, n \in \mathbb{N}$. They are simple and associated with the normalized Hermite's functions.

Exercise. [3] We wish to study the $2D$ harmonic oscillator: $-\Delta + |x|^2$.

(1) Write the operator in terms of radial coordinates.
(2) Explain how the spectral analysis can be reduced to the study of:

$$-\partial_\rho^2 - \rho^{-1}\partial_\rho + \rho^{-2}m^2 + \rho^2,$$

on $L^2(\rho d\rho)$ with $m \in \mathbb{Z}$.
(3) Perform the change of variable $t = \rho^2$.
(4) For which α is $t \mapsto t^\alpha e^{-t/2}$ an eigenfunction ?
(5) Conjugate the operator by $t^{-m/2}e^{t/2}$. On which space is the new operator $\mathcal{L}_m$ acting ? Describe the new scalar product.
(6) Find eigenvalues of $\mathcal{L}_m$ by noticing that $\mathbb{R}_N[X]$ is stable by $\mathcal{L}_m$.
(7) Conclude.

[3]This exercise is an example of exact WKB expansions. We will recognize Laguerre's polynomials.

1.5. **The Case with Constant Magnetic Field.** Let us now come back to the magnetic Laplacian, in dimension 2. We consider the case with constant magnetic field. By the gauge invariance, the magnetic Laplacian can be written in the form:

$$h^2 D_{x_1}^2 + (hD_{x_2} - x_1)^2.$$

We want to determine its spectrum. If we use the Fourier transform with respect to x_2, we observe that it is unitarily equivalent to:

$$h^2 D_{x_1}^2 + (h\xi_2 - x_1)^2.$$

If we use the translation $x_1 = \tilde{x}_1 + h\xi_2$, we get the unitarily equivalent operator:

$$h^2 D_{\tilde{x}_1}^2 + \tilde{x}_1^2.$$

The spectrum is essential. The elements of the spectrum are given by $(2n+1)h, n \in \mathbb{N}$. These "eigenvalues" have infinite mutliplicity.

Remark 1.4. We observe that the investigation of the Laplacian with constant magnetic field leads to use transformations in the phase space $\mathbb{R}^4$. Indeed the symbol (in terms of the Weyl quantization) of the magnetic Laplacian is:

$$\xi_1^2 + (\xi_2 - x_1)^2$$

and is transformed into:

$$\tilde{\xi}_1^2 + \tilde{x}_1^2.$$

Such transformations are called "symplectic" and corresponds to transformations which preserve the structure of the Hamilton-Jacobi equations (classical mechanics).

1.6. **What do we know in general ?** We can try to give a panorama of the numerous results concerning the semiclassical spectral analysis of the magnetic Laplacian. For that purpose, we divide the exposition into two parts.

1.6.1. *Constant magnetic field.*

• *Dimension 2.* In 2D the constant magnetic field case is treated when Ω is a disk (with Neumann condition) by Bauman, Phillips and Tang in [6] (see also [7, 29] and [8] for the Dirichlet case). In particular, they prove a two terms expansion in the form:

$$\lambda_1(h) = \Theta_0 h - \frac{C_1}{R} h^{3/2} + o(h^{3/2}),$$

where $\Theta_0 \in (0,1)$ and $C_1 > 0$ are universal constants which will be defined later. This result is generalized to smooth and bounded domains by Helffer and Morame in [59] where it is proved that:

$$\lambda_1(h) = \Theta_0 h - C_1 \kappa_{max} h^{3/2} + o(h^{3/2}),$$

where κ_{max} is the maximal curvature of the boundary. Let us emphasize that, in these papers, the authors are only concerned by the first terms of the asymptotic expansion of $\lambda_1(h)$. In the case of smooth domains the complete asymptotic expansion of all the eigenvalues is done by Fournais and Helffer in [39].

When the boundary is not smooth, we may mention the papers of Jadallah and Pan [65, 79]. In the semiclassical regime, we refer to the papers of Bonnaillie-Noël, Dauge and Fournais [9, 13, 12]. For a numerical investigation, the reader may consider the paper [11].

• *Dimension 3.* In 3D the constant magnetic field case (with intensity 1) is treated by Helffer and Morame in [61] under generic assumptions on the (smooth) boundary of Ω:

$$\lambda_1(h) = \Theta_0 h + \hat{\gamma}_0 h^{4/3} + o(h^{4/3}),$$

where the constant $\hat{\gamma}_0$ is related to the magnetic curvature of a curve in the boundary along which the magnetic field is tangent to the boundary. The case of the ball is analyzed in details by Fournais and Persson in [41]. When the boundary is not smooth, the problem is studied in the thesis of N. Popoff [82] and a complete expansion of all the eigenvalues is performed in [83].

1.6.2. *Variable magnetic field.*

• *Dimension 2.* For the case with a non vanishing variable magnetic field, we refer to [70, 84] for the first terms of the lowest eigenvalue. For a complete expansion, we can refer to [86]. For the Dirichlet case, we can refer to [57] and to the paper in preparation by Faure, Raymond and Vũ Ngọc [38].

When the magnetic field vanishes, the first analysis of the lowest eigenvalue is due to Montgomery in [75] soon followed by Helffer and Morame in [58] (see also [80]). The most recent investigations in this case are the papers [49, 51] and [31]. In particular, in [31], a complete expansion is proved and solves the conjecture of Helffer [56, Section 5.2].

• *Dimension 3.* When the magnetic field is variable (with Neumann condition on a smooth boundary), the first term of $\lambda_1(h)$ is given by Lu and Pan in [71]. The next terms in the expansion are investigated in [85]. A toy model is also analyzed in [87] where a complete expansion of the eigenpairs is established. The generalization of [87] to general magnetic fields and general smooth boundaries is still an open and difficult problem. The case with Dirichlet boundary condition is partially studied by Helffer and Kordyukov in [50] and by Raymond and Vũ Ngọc (in progress).

1.7. **Comments on the philosophy of the proofs related to the magnetic Laplacian.** We can now make some general comments about the results on the semiclassical spectral analysis of the magnetic Laplacian. It is quite natural that the more we know about the eigenvalues, the more we learn about the eigenfunctions and conversely. As we have noticed, many results only concern the lowest eigenvalue $\lambda_1(h)$ and a few terms in its expansion. Therefore, in general, the corresponding expansion of the eigenfunction is not known. Even the main term of this expansion is not well understood. To understand the eigenpairs, we will be led to introduce approximate eigenvalues and eigenfunctions ; we will observe that formal power series give hints about the structure of the spectrum. In fact, in almost all the papers that we have mentioned, these power series expansions are used as a fundamental step to guess the behavior of the true eigenpairs. The most difficult part of the analysis is to prove that such a formal investigation completely describes the spectrum. We will see that the transition from "a few terms of $\lambda_1(h)$" to "a complete expansion of $\lambda_n(h)$" is not only a technical problem, but reflects the deepest properties of the magnetic Laplacian. In particular, we will have to establish accurate localization (and micro-localization) properties of the true eigenfunctions (in the spirit of Agmon, see [1]) as it is the case in [39] where the authors have to combine a very fine analysis using pseudo-differential calculus (to catch the a priori behavior of the eigenfunctions with respect to a phase variable) and the Grushin reduction machinery (see [47]). Fortunately, in this course we will see how we can avoid the introduction of the pseudo-differential calculus. The basic idea to analyze the spectrum of an operator is to compare it to a simplest one. For that purpose, we will use many change of variables, functions and gauge to simplify the "principal symbol" of the magnetic Laplacian: All these changes correspond to unitary conjugations which will be completely explicit and known as Fourier Integral Operators (see the classical references [91, 30, 73] and maybe also the initial paper of Egorov [34]). After these reductions (which can be compared to the Birkhoff normal form, see [93, 21, 94]), we will be reduced to an operator which has the "Born-Oppenheimer form", a notion coming from the original paper [16] and generalized by Martinez (see for instance [72, 67]). This "principal part" of the magnetic Laplacian will allow us to deduce the complete asymptotic expansion of the eigenpairs. Here this is interesting to underline that the operator which approximates the magnetic Laplacian (and which can be studied through the Born-Oppenheimer approximation) is nothing but the beginning of a Birkhoff normal form.

1.8. **Organization.** In Section 2 we recall the fundamental theorems in spectral theory and provide examples of applications. In Section 3 we discuss important model operators related to the magnetic Laplacian and recall formulas of Kato's theory. In Section 4 we explain how the magnetic Laplacian can be reduced to the models. In Section 5 we introduce the estimates of Agmon which describe some localization properties of the eigenfunctions. In Section 6, we discuss elementary aspects of the Born-Oppenheimer theory in relation with the model operators. In Section 7, we provide a complete example of application of the philosophy developed in the previous sections ; in particular we analyze the case when the magnetic field vanishes in $2D$. In Section 8, we describe a example in $3D$ with a non smooth boundary. In Section 9, we investigate the magnetic Laplacian in terms of symplectic geometry and present the Birkhoff normal form procedure. In Section 10, we analyze the Dirichlet spectrum of an isosceles triangle whose aperture goes to zero. In Section 11, we investigate the spectrum of broken waveguides.

• *Acknowledgments.* The author is grateful to his colleagues and collaborators which have directly or indirectly influenced the contents and the presentation of this course: V. Bonnaillie-Noël, M. Dauge, N. Dombrowski, F. Faure, S. Fournais, B. Helffer, D. Krejčiřík, Y. Lafranche, F. Nier, T. Ourmières, M. Persson, N. Popoff and S. Vũ Ngọc. He is also grateful to H. Najar for his invitation to give a lecture in Tunisia.

2. Spectral Theorem and Quasimodes

This section is devoted to recall basic tools in spectral analysis.

2.1. Min-max principle.

2.1. Min-max principle. We give a standard method to estimate the discrete spectrum and the bottom of the essential spectrum of a self-adjoint operator A on an Hilbert space H. We recall first the definition of the Rayleigh quotients of a self-adjoint operator A.

Definition 2.1. The Rayleigh quotients associated with the self-adjoint operator A on H of domain $\mathsf{Dom}(A)$ are defined for all positive natural number j by

$$\lambda_j(A) = \inf_{\substack{u_1,\ldots,u_j \in \mathsf{Dom}(A)\\ \text{independent}}} \sup_{u\in[u_1,\ldots,u_j]} \frac{\langle Au, u\rangle_H}{\langle u, u\rangle_H}.$$

Here $[u_1, \ldots, u_j]$ denotes the subspace generated by the j independent vectors $u_1, \ldots, u_j$.

The following statement gives the relation between Rayleigh quotients and eigenvalues.

Theorem 2.2. *Let A be a self-adjoint operator of domain $\mathsf{Dom}(A)$. We assume that A is semi-bounded from below. We set $\gamma = \min \sigma_{\mathsf{ess}}(A)$. Then the Rayleigh quotients λ_j of A form a non-decreasing sequence and there holds*

(1) *If $\lambda_j(A) < \gamma$, it is an eigenvalue of A,*
(2) *If $\lambda_j(A) \geq \gamma$, then $\lambda_j = \gamma$,*
(3) *The j-th eigenvalue $< \gamma$ of A (if exists) coincides with $\lambda_j(A)$.*

A consequence of this theorem which is often used is the following:

Proposition 2.3. *Suppose that there exists $a \in \mathbb{R}$ and an n-dimensional space $V \subset \mathsf{Dom}A$ such that:*

$$\langle A\psi, \psi\rangle \leq a\|\psi\|^2.$$

Then, we have:

$$\lambda_n(A) \leq a.$$

Remark 2.4. For the proof we refer to [69, Proposition 6.17 and 13.1] or to [90, Chapter XIII].

Let us give a characterization of the bottom of the essential spectrum (see [81] and also [40]).

Theorem 2.5. *Let V be real-valued, semi-bounded potential and $\mathbf{A} \in \mathcal{C}^1(\mathbb{R}^n)$ a magnetic potential. Let $P_{\mathbf{A},V}$ be the corresponding self-adjoint, semi-bounded Schrödinger operator. The, the bottom of the essential spectrum is given by:*

$$\inf \sigma_{\mathsf{ess}}(P_{V,\mathbf{A}}) = \Sigma(P_{V,\mathbf{A}}),$$

where:

$$\Sigma(P_{V,\mathbf{A}}) = \sup_{K\subset\mathbb{R}^n} \left[\inf_{\|\phi\|=1} \langle P_{V,\mathbf{A}}\phi, \phi\rangle \mid \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus K)\right].$$

Let us notice that generalizations including the presence of a boundary are possible.

2.2. **The Spectral Theorem.** We state a theorem which will be one of the fundamental tools in this course.

Theorem 2.6. *Let us assume that* $(A, \mathsf{Dom}(A))$ *is a self-adjoint operator. Then, if* $\lambda \notin \sigma(A)$*, we have:*

$$\|(A-\lambda)^{-1}\| \leq \frac{1}{d(\lambda, \sigma(A))}.$$

Remark 2.7. This theorem is known as the spectral theorem and a proof can be found in [90] and [66, Section VI.5]. An immediate consequence of this theorem is that, for all $\psi \in \mathsf{Dom}(A)$:

$$\|\psi\| d(\lambda, \sigma(A)) \leq \|(A-\lambda)\psi\|.$$

In particular, if we find $\psi \in \mathsf{Dom}(A)$ such that $\|\psi\| = 1$ and $\|(A-\lambda)\psi\| \leq \varepsilon$, we get: $d(\lambda, \sigma(A)) \leq \varepsilon$.

2.3. **Quasimodes for the $1D$ Electric Laplacian.** We illustrate the application of the spectral theorem in the case of the electric Laplacian $\mathcal{L}_{h,V} = -h^2\Delta + V(x)$. We assume that $V \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, that $V(x) \to +\infty$ when $|x| \to +\infty$ and that it admits a unique and non degenerate minimum at 0. This example is also the occasion to understand more in details how we construct quasi-eigenpairs in general. From a heuristic point of view, we guess that the lowest eigenvalues correspond to functions localized near the minimum of the potential (intuition coming from the classical mechanics). Therefore we can use a Taylor expansion of V near 0:

$$V(x) = \frac{V''(0)}{2}x^2 + O(|x|^3).$$

We can then try to compare $-h^2\Delta + V(x)$ with $-h^2\Delta + \frac{V''(0)}{2}x^2$. For an homogeneity reason, we try the rescaling $x = h^{1/2}y$. The electric operator becomes:

$$\tilde{\mathcal{L}}_{h,V} = -h\Delta_y + V(h^{1/2}y).$$

Let us use the Taylor formula:

$$V(h^{1/2}y) \sim \frac{V''(0)}{2}hy^2 + \sum_{j\geq 3} h^{j/2}\frac{V^{(j)}(0)}{j!}y^j.$$

This provides the formal expansion:

$$\tilde{\mathcal{L}}_{h,V} \sim h\left(H_0 + \sum_{j\geq 1} h^{j/2}H_j\right),$$

where

$$H_0 = -\partial_y^2 + \frac{V''(0)}{2}y^2.$$

We look for a quasimode in the form:

$$u \sim \sum_{j\geq 0} u_j(y)h^{j/2}$$

and an eigenvalue:

$$\mu \sim h \sum_{j\geq 0} \mu_j h^{j/2}.$$

Let us investigate the system of PDE that we get when solving in the formal series:

$$\tilde{\mathcal{L}}_{h,V} u \sim \mu u.$$

- *Term of order h.* We get the equation:

$$H_0 u_0 = \mu_0 u_0.$$

Therefore we can take for (μ_0, u_0) a L^2-normalized eigenpair of the harmonic oscillator.

- *Term of order h^2.* We solve:

$$(H_0 - \mu_0) u_1 = (\mu_1 - H_1) u_0.$$

We want to determine μ_1 and u_1. We can verify that $H_0 - \mu_0$ is a Fredholm operator so that a necessary and sufficient condition to solve this equation is given by:

$$\langle (\mu_1 - H_1) u_0, u_0 \rangle = 0.$$

Lemma 2.8. *Let us consider the equation:*

$$(H_0 - \mu_0) u = f, \tag{2.1}$$

with $f \in \mathcal{S}(\mathbb{R})$ such that $\langle f, u_0 \rangle = 0$. The (2.1) *admits a unique solution which is orthogonal to u_0 and this solution is in the Schwartz class.*

Proof. Let us just sketch the proof to enlighten the general idea. We know that we can find $u \in \mathsf{Dom}(H_0)$ and that u is determined modulo u_0 which is in the Schwartz class. Therefore, we have: $y^2 u \in L^2(\mathbb{R})$ and $u \in H^2(\mathbb{R})$. Let us introduce a smooth cutoff function $\chi_R(y) = \chi(R^{-1} y)$. $\chi_R y^2 u$ is in the form domain of H_0 as well as in the domain of H_0 so that we can write:

$$\langle H_0(\chi_R y^2 u), \chi_R y^2 u \rangle = \langle [H_0, \chi_R y^2] u, \chi_R y^2 u \rangle + \langle \chi_R y^2 u(\mu_0 u + f), \chi_R y^2 u \rangle.$$

The commutator can easily be estimated and, by dominate convergence, we find the existence of $C > 0$ such that for R large enough we have:

$$\|\chi_R y^3 u\|^2 \leq C.$$

The Fatou lemma involves:

$$y^3 u \in L^2(\mathbb{R}).$$

This is then a standard iteration procedure which gives that $\partial_y^l (y^k u) \in L^2(\mathbb{R})$. The Sobolev injection ($H^s(\mathbb{R}) \hookrightarrow \mathcal{C}^{s-\frac{1}{2}}(\mathbb{R})$ for $s > \frac{1}{2}$) gives the conclusion. □

This determines a unique value of $\mu_1 = \langle H_1 u_0, u_0 \rangle$. For this value we can find a unique $u_1 \in \mathcal{S}(\mathbb{R})$ orthogonal to u_0.

- *Iteration.* This is easy to see that this procedure can be continued at any order.

• *Application of the spectral theorem.* Let us consider the (μ_j, u_j) that we have constructed and let us introduce:

$$U_{J,h} = \sum_{j=0}^{J} u_j(y) h^{j/2}, \quad \mu_{J,h} = h \sum_{j=0}^{J} \mu_j h^{j/2}.$$

We estimate:

$$\|(\tilde{\mathcal{L}}_{h,V} - \mu_{J,h}) U_{J,h}\|.$$

By using the Taylor formula and the definition of the μ_j and u_j, we have:

$$\|(\tilde{\mathcal{L}}_{h,V} - \mu_{J,h}) U_{J,h}\| \leq C_J h^{(J+1)/2},$$

since $h^{(J+1)/2} \|y^{(J+1)/2} U_{J,h}\| \leq C_J h^{(J+1)/2}$ due to the fact that $u_j \in \mathcal{S}(\mathbb{R})$. The spectral theorem implies:

$$d\left(\mu_{J,h}, \sigma_{\mathsf{dis}}(\tilde{\mathcal{L}}_{h,V})\right) \leq C_J h^{(J+1)/2}.$$

2.4. **Magnetic Example.** Let us now give an explicit example of construction of quasimodes for the magnetic Laplacian in $\mathbb{R}^2$. We investigate the operator:

$$\mathcal{L}_{h,\mathbf{A}} = (hD_1 + A_1)^2 + (hD_2 + A_2)^2,$$

with domain:

$$\mathsf{Dom}\mathcal{L}_{h,\mathbf{A}} = \{\psi \in L^2(\mathbb{R}^2) : \left((hD_1 + A_1)^2 + (hD_1 + A_2)^2\right)\psi \in L^2(\mathbb{R}^2)\}.$$

2.4.1. *Compact resolvent ?* Let us state an easy lemma.

Lemma 2.9. *We have:*

$$Q_{h,\boldsymbol{A}}(\psi) \geq \left| \int_{\mathbb{R}^2} h\beta(x) |\psi|^2 \, d\mathbf{x} \right|, \quad \forall \psi \in \mathcal{C}_0^\infty(\mathbb{R}^2).$$

Proof. We notice that:

$$[hD_1 + A_1, hD_2 + A_2] = -ih\beta.$$

We find:

$$\langle [hD_1 + A_1, hD_2 + A_2]\psi, \psi \rangle = -ih \int_{\mathbb{R}^2} \beta |\psi|^2 \, d\mathbf{x}.$$

By integration by parts, we deduce:

$$|\langle [hD_1 + A_1, hD_2 + A_2]\psi, \psi \rangle| \leq 2\|(hD_1 + A_1)\psi\| \|(hD_2 + A_2)\psi\| \leq Q_{h,\mathbf{A}}(\psi).$$

□

Proposition 2.10. *Suppose that* $\boldsymbol{A} \in \mathcal{C}^\infty(\mathbb{R}^2)$ *and that* $\beta = \nabla \times \boldsymbol{A} \geq 0$ *and* $\beta(x) \underset{|x \to +\infty|}{\to} +\infty$. *Then,* $\mathcal{L}_{h,\boldsymbol{A}}$ *has compact resolvent.*

Proof. This is an application of the Riesz-Fréchet-Kolmogorov theorem, see [17, Theorem IV.25] (the form domain has compact injection in $L^2(\mathbb{R}^2)$). □

2.4.2. *Quasimodes.* Let us give a simple example inspired by [57]. Let us choose $\mathbf{A}$ such that $\beta = 1 + x^2 + y^2$. We take $A_1 = 0$ and $A_2 = x + \frac{x^3}{3} + y^2 x$. We study:

$$\mathcal{L}_{h,\mathbf{A}} = h^2 D_x^2 + \left(hD_y + x + \frac{x^3}{3} + y^2 x\right)^2 .$$

Let us try the rescaling $x = h^{1/2}u$, $y = h^{1/2}v$. We get a new operator:

$$\tilde{\mathcal{L}}_{h,\mathbf{A}} = hD_u^2 + h\left(D_v + u + h\frac{u^3}{3} + hv^2 u\right)^2 .$$

Let us conjugate by the partial Fourier transform with respect to v ; we get the unitarily equivalent operator:

$$\hat{\mathcal{L}}_{h,\mathbf{A}} = hD_u^2 + h\left(\xi + u + h\frac{u^3}{3} + huD_\xi^2\right)^2 .$$

Let us now use the transvection: $u = \check{u} - \check{\xi}, \xi = \check{\xi}$. We have:

$$D_u = D_{\check{u}}, \quad D_\xi = D_{\check{u}} + D_{\check{\xi}}.$$

We are reduced to the study of:

$$\check{\mathcal{L}}_{h,\mathbf{A}} = hD_{\check{u}}^2 + h\left(\check{u} + h\frac{(\check{u}-\check{\xi})^3}{3} + h(\check{u}-\check{\xi})(D_\xi + D_{\check{u}})^2\right)^2$$

We can expand $\check{\mathcal{L}}_{h,\mathbf{A}}$ in formal power series:

$$\check{\mathcal{L}}_{h,\mathbf{A}} = hP_0 + h^2 P_1 + \cdots ,$$

where $P_0 = D_{\check{u}}^2 + \check{u}^2$ and $P_1 = \frac{2}{3}\check{u}(\check{u}-\check{\xi})^3 + (\check{u}-\check{\xi})(D_{\check{\xi}} + D_{\check{u}})^2\check{u} + \check{u}(\check{u}-\check{\xi})(D_{\check{\xi}} + D_{\check{u}})^2$.

Let us look for quasi-eigenpairs in the form

$$\lambda \sim h\lambda_0 + h^2\lambda_1 + \cdots , \quad \psi \sim \psi_0 + h\psi_1 + \cdots$$

- *Term of order h.* We solve the equation:

$$P_0\psi_0 = \lambda_0\psi_0.$$

We take $\lambda_0 = 1$ and $\psi_0(\check{u}, \check{\xi}) = g_0(\check{u})f_0(\check{\xi})$ where g_0 is the first normalized eigenfunction of the harmonic oscillator. f_0 is a function to be determined.

- *Term of order h^2.* The second equation of the formal system is:

$$(P_0 - \lambda_0)\psi_1 = (\lambda_1 - P_1)\psi_0.$$

The Fredholm condition gives, for all $\check{\xi}$:

$$\langle(\lambda_1 - P_1)\psi_0, g_0\rangle_{L^2(\mathbb{R}_{\check{u}})} = 0.$$

Let us analyze the different terms which appear in this differential equation. There should be a term in $\check{\xi}^3$. Its coefficient is:

$$\int \check{u} g_0(\check{u})^2 \, d\check{u} = 0.$$

For the same parity reason, there is no term in $\check{\xi}$. Let us now analyze the term in $D_{\check{\xi}}$. Its coefficient is:

$$\langle (D_{\check{u}}\check{u} + \check{u}D_{\check{u}})g_0, \check{u}g_0\rangle = 0,$$

for a parity reason. In the same way, there is no term in $\check{\xi}D_{\check{\xi}}^2$. The coefficient of $\check{\xi}D_{\check{\xi}}$ is:

$$2\int (\check{u}D_{\check{u}} - D_{\check{u}}\check{u})g_0 g_0 \, d\check{u} = 0.$$

The compatibility equation is in the form:

$$(aD_{\check{\xi}}^2 + b\check{\xi}^2 + c)f_0 = \lambda_1 f_0.$$

It turns out that (exercise):

$$a = b = 2\int \check{u}^2 g_0^2 \, d\check{u} = 1.$$

In the same way c can be explicitly found. This leads to a family of choices for (λ_1, f_0): We can take $\lambda_1 = c + (2m+1)$ and $f_0 = g_m$ the corresponding Hermite function.

This construction provides us a family of quasimodes (which are in the Schwartz class). By the spectral theorem, we infer that, for each $m \in \mathbb{N}$, there exists $C_m > 0$ such that:

$$d\left(h + (2m+1+c)h^2, \sigma_{\mathsf{dis}}(P_{h,\mathbf{A}})\right) \leq C_m h^3.$$

Remark 2.11. One could continue the expansion at any order and one could also consider the other possible values of λ_0 (next eigenvalues of the harmonic oscillator).

Remark 2.12. The fact that the construction can be continued as much as the appearance of the harmonic oscillator is a clue that our initial scaling is actually the good one. We can also guess that the lowest eigenfunctions are concentrated near zero at the scale $h^{1/2}$ if the quasimodes approximate the true eigenfunctions.

3. Magnetic Model Operators

As we mentioned in the introduction, the analysis of the magnetic Laplacian leads to the study of numerous model operators. We saw in the last example that the harmonic oscillator is such a model.

3.1. **De Gennes Operator.** The analysis of the $2D$ magnetic Laplacian with Neumann condition on $\mathbb{R}^2_+$ makes the so-called de Gennes operator to appear. We refer to [26] where this model is studied in details (see also [40]). This operator is defined as follows. For $\xi \in \mathbb{R}$, we consider the Neumann realization $H(\xi)$ in $L^2(\mathbb{R}_+)$ associated with the operator

$$-\frac{d^2}{dt^2} + (t-\xi)^2, \quad \mathsf{Dom}(H(\xi)) = \{u \in B^2(\mathbb{R}_+) : u'(0) = 0\}. \tag{3.1}$$

The operator $H(\xi)$ has compact resolvent by standard arguments. By the Cauchy-Lipschitz theorem, all the eigenvalues are simple.

Notation 3.1. The lowest eigenvalue of $H(\xi)$ is denoted $\mu(\xi)$; the associated L^2-normalized and positive eigenstate is denoted by $u_\xi = u(\cdot, \xi)$.

We easily get that u_ξ is in the Schwartz class.

Lemma 3.2. *The function $\xi \mapsto \mu(\xi)$ is smooth and so is $\xi \mapsto u(\cdot, \xi)$.*

Proof. The family $(H(\xi))_{\xi\in\mathbb{R}}$ is analytic of type (A), see [66, p. 375]. □

Lemma 3.3. *$\xi \mapsto \mu(\xi)$ admits a unique minimum and it is non degenerate.*

Proof. This an easy application of the min-max principle which proves that

$$\lim_{\xi\to-\infty} \mu(\xi) = +\infty.$$

Let us now show that:

$$\lim_{\xi\to+\infty} \mu(\xi) = 1.$$

The de Gennes operator is equivalent to the operator $-\partial_t^2 + t^2$ on $(-\xi, +\infty)$ with Neumann condition at $-\xi$. Let us begin with upper bound. An easy and explicit computation gives:

$$\mu(\xi) \leq \langle(-\partial_t^2 + t^2)e^{-t^2/2}, e^{-t^2/2}\rangle_{L^2((-\xi,+\infty))} \underset{\xi\to+\infty}{\to} 1.$$

Let us investigate the converse inequality. Let us prove some concentration of u_ξ near 0 when ξ increases (the reader can compare this with the estimates of Agmon of Section 5). We have:

$$\int_0^{+\infty} (t-\xi)^2 |u_\xi(t)|^2 \, dt \leq \mu(\xi).$$

If $\lambda(\xi)$ is the lowest Dirichlet eigenvalue, we have:

$$\mu(\xi) \leq \lambda(\xi).$$

By monotonicity of the Dirichlet eigenvalue with respect to the domain, we have, for $\xi > 0$:

$$\lambda(\xi) \leq \lambda(0) = 3.$$

It follows that:

$$\int_0^1 |u_\xi(t)|^2\, dt \leq \frac{3}{(\xi-1)^2}, \quad \xi \geq 2.$$

Let us introduce the test function: $\chi(t)u_\xi(t)$ with χ supported in $(0,+\infty)$ and being 1 for $t \geq 1$. We have:

$$\langle (-\partial_t^2+(t-\xi)^2)\chi(t)u_\xi(t), \chi(t)u_\xi(t)\rangle_{L^2(\mathbb{R})} \geq \|\chi(\cdot+\xi)u_\xi(\cdot+\xi)\|^2_{L^2(\mathbb{R})} = \|\chi u_\xi\|^2_{L^2(\mathbb{R})} = 1+O(|\xi|^{-2}).$$

Moreover, we get:

$$\langle (-\partial_t^2+(t-\xi)^2)\chi(t)u_\xi(t), \chi(t)u_\xi(t)\rangle_{L^2(\mathbb{R})} = \langle (-\partial_t^2+(t-\xi)^2)\chi(t)u_\xi(t), \chi(t)u_\xi(t)\rangle_{L^2(\mathbb{R}_+)}.$$

We have:

$$\langle (-\partial_t^2+(t-\xi)^2)\chi(t)u_\xi(t), \chi(t)u_\xi(t)\rangle_{L^2(\mathbb{R}_+)} = \mu(\xi)\|\chi u_\xi\|^2 + \|\chi' u_\xi\|^2$$

which can be controlled by the concentration result. We infer that, for ξ large enough:

$$\mu(\xi) \geq 1 - C|\xi|^{-1}.$$

From these limits, we deduce the existence of a minimum strictly less than 1.

We now use the Feynman-Hellmann formula which will be established later. We have:

$$\mu'(\xi) = -2\int_{t>0} (t-\xi)|u_\xi(t)|^2\, dt.$$

For $\xi < 0$, we get an increasing function. Moreover, we see that $\mu(0) = 1$. The minima are obtained for $\xi > 0$.

We can write that:

$$\mu'(\xi) = 2\int_{t>0} (t-\xi)^2 u_\xi u_\xi'\, dt + \xi^2 u_\xi(0)^2.$$

This implies:

$$\mu'(\xi) = (\xi^2 - \mu(\xi))u_\xi(0)^2.$$

Let ξ_c a critical point for μ. We get:

$$\mu''(\xi_c) = 2\xi_c u_{\xi_c}(0)^2.$$

The critical points are all non degenerate. They correspond to local minima.We conclude that there is only one critical point and that is the minimum. We denote it ξ_0 and we have $\mu(\xi_0) = \xi_0^2$. □

We let:

$$\Theta_0 = \mu(\xi_0), \tag{3.2}$$

$$C_1 = \frac{u_{\xi_0}^2(0)}{3}. \tag{3.3}$$

Exercise. We propose to prove be elementary means that $\xi \mapsto \mu(\xi)$ and $\xi \mapsto u(\cdot,\xi)$ are smooth. Let us fix $\xi_1 \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \sigma(H(\xi_1))$.

(1) Prove that, for ξ close enough to ξ_1, $H(\xi) - z$ is invertible. For that purpose, one could show that: $t(H(\xi_1) - z)^{-1}$ is bounded with a uniform bound with respect to z.

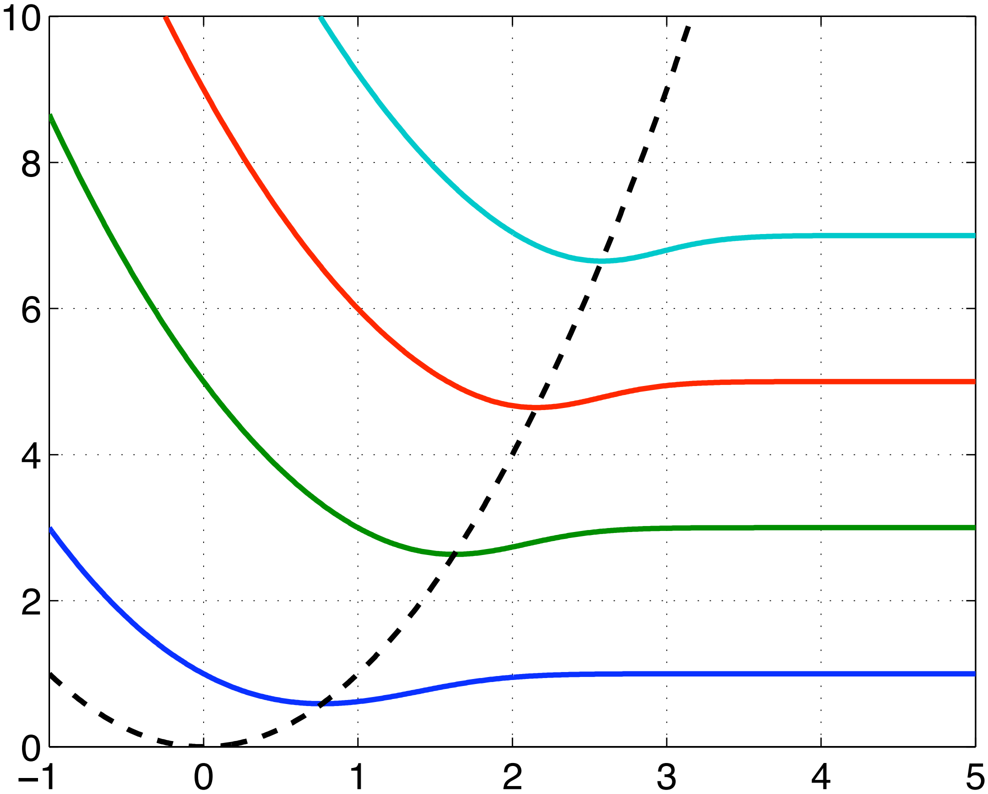


FIGURE 1. $\xi \mapsto \mu_k(\xi)$, for $k = 1, 2, 3, 4$

(2) Prove that $\xi \mapsto (H(\xi) - z)^{-1}$ is analytic as soon as ξ is close to ξ_1.
(3) Establish the resolvent formula:

$$(H(\xi_1) - z)^{-1} - (H(\xi) - z)^{-1} = (\xi_1 - \xi)(H(\xi) - z)^{-1}(2t - \xi - \xi_1)(H(\xi_1) - z)^{-1}.$$

(4) By using the fact that $H(\xi)$ has compact resolvent and is self-adjoint, prove that:

$$P_\Gamma(\xi) = \frac{1}{2i\pi} \int_\Gamma (H(\xi) - z)^{-1} \, dz$$

is the projection on the space generated by the eigenfunctions associated with eigenvalues enclosed by the smooth contour Γ.
(5) Prove that:

$$\|P_\Gamma(\xi) - P_\Gamma(\xi_1)\| \leq C|\xi - \xi_1|,$$

when ξ is close to ξ_1. See [66, I.8].
(6) Deduce that near each $\mu_n(\xi_1)$ there exists an element $\mu_p(\xi)$ and conversely.
(7) Deduce that $\xi \mapsto \mu_k(\xi)$ are continuous functions near ξ_1.

(8) Conclude that, if Γ is a coutour small enough around $\mu_n(\xi_1)$, then, for ξ close enough to ξ_1, it only contains $\mu_n(\xi)$. Finally, prove that the corresponding normalized eigenfunction is analytic with respect to ξ and so is the eigenvalue.

3.2. **Montgomery Operator.** Let us now discuss another important model. This one was introduced by Montgomery in [75] to study the case of vanishing magnetic fields in $2D$ (see also [80] and [61, Section 2.4]). This model was revisited by Helffer in [48] and generalized by Helffer and Persson in [62].

The Montgomery operator with parameters $\eta \in \mathbb{R}$ and $\delta > 0$ is the self-adjoint realization on $\mathbb{R}$ of:

$$M_{\eta,\delta} = D_t^2 + \left(-\eta + \frac{\delta}{2}t^2\right)^2. \tag{3.4}$$

The Montgomery operator has clearly compact resolvent.

Notation 3.4. The lowest eigenvalue of $M_{\eta,\delta}$ is denoted by $\nu_\delta(\eta)$

In fact, ν_δ is related to ν_1. Indeed, we can perform a rescaling $t = \delta^{-1/3}\tau$ so that $H_{\eta,\delta}$ is unitarily equivalent to:

$$\delta^{2/3}\left(D_\tau^2 + (-\eta\delta^{-1/3} + \frac{1}{2}\tau^2)^2\right) = \delta^{2/3} M_{\eta\delta^{-1/3},1}.$$

It is known (see [48, 62]) that, for all $\delta > 0$:

$$\eta \mapsto \nu_\delta(\eta) \text{ admits a unique and non-degenerate minimum at a point } \eta_0. \tag{3.5}$$

We may write:

$$\inf_{\eta\in\mathbb{R}} \nu_\delta(\eta) = \delta^{2/3}\nu_1(\eta_0). \tag{3.6}$$

For fixed $\delta > 0$, the family $(M_{\eta,\delta})_{\eta\in\mathbb{R}}$ is an analytic family of type (B) so that the eigenpair $(\nu_1(\eta), u_\eta)$ has an analytic dependence on η (see [66]).

Numerical computations of η_0 and ν_{η_0} are performed by V. Bonnaillie-Noël (see [62, Table 1]) and give $\eta_0 \approx 0.35$ and $\nu_1(\eta_0) \approx 0,57$. It is also proved that:

$$\lim_{|\eta|\to+\infty} \nu_1(\eta) = +\infty.$$

3.3. **Popoff Operator.** The next model operator that we will encounter has been introduced more recently by Popoff in [82] in order to study the Neumann Laplacian on an edge in a constant magnetic field. Let us defined the corner with fixed angle $\alpha \in (0, \pi)$:

$$C_\alpha = \{(t, z) \in \mathbb{R}^2 : |z| \leq t \tan\left(\frac{\alpha}{2}\right)\}.$$

The edge of angle α is defined by:

$$\mathcal{E}_\alpha = \mathbb{R} \times C_\alpha.$$

We are interested in the Neumann realization on $L^2(\mathcal{E}_\alpha, dsdtdz)$ of the following operator:

$$\mathcal{L}_\alpha = D_t^2 + D_z^2 + (D_s - t)^2.$$

Using the Fourier transform with respect to s, we have the decomposition (into a direct integral, see [90, p. 281-284]):

$$\mathcal{L}_\alpha = \int^{\oplus} L_{\alpha,\eta}\, d\eta,$$

where we have introduced the following Neumann realization on $L^2(C_\alpha, dtdz)$:

$$L_{\alpha,\eta} = D_t^2 + D_z^2 + (\eta - t)^2,$$

where $\eta \in \mathbb{R}$ is a parameter.

Notation 3.5. For each $\alpha \in (0,\pi)$, we denote by $\nu(\alpha,\eta)$ the lowest eigenvalue (which is simple) of $L_{\alpha,\eta}$ and we denote by $u_{\alpha,\eta}$ the corresponding eigenfunction.

Notation 3.6. $\nu(\alpha)$ denotes the bottom of the spectrum of $\mathcal{L}_\alpha$.

We have:

$$\nu(\alpha) = \inf_{\eta\in\mathbb{R}} \nu(\alpha,\eta).$$

• *Properties related to $L_{\alpha,\eta}$ and $\mathcal{L}_\alpha$.* Let us gather a few elementary properties:

Lemma 3.7. *We have:*

(1) *The function $(0,\pi) \ni \alpha \mapsto \nu(\alpha)$ is non increasing.*
(2) *For all $\eta \in \mathbb{R}$, the function $(0,\pi) \ni \alpha \mapsto \nu(\alpha,\eta)$ is decreasing.*
(3) *The function $(0,\pi) \times \mathbb{R} \ni (\alpha,\eta) \mapsto \nu(\alpha,\eta)$ is analytic.*

We will admit that (open question):

Assumption 3.8. For all $\alpha \in (0,\pi)$, $\eta \mapsto \nu(\alpha,\eta)$ has a unique critical point denoted by $\eta_0(\alpha)$ and it is non degenerate.

Under this assumption and using the analytic implicit function theorem, we deduce:

Lemma 3.9. *Under Assumption 3.8, the function $(0,\pi) \ni \alpha \mapsto \eta_0(\alpha)$ is analytic and so is $(0,\pi) \ni \alpha \mapsto \nu(\alpha)$. Moreover the function $(0,\pi) \ni \alpha \mapsto \nu(\alpha)$ is decreasing.*

3.4. **Helffer-Lu-Pan Operator.** Let us present a last model operator appearing in $3D$ in the case of smooth Neumann boundary (see [71, 60, 10]).

We denote by $\mathbf{x} = (s,t)$ the coordinates in $\mathbb{R}^2$ and by Ω the half-plane:

$$\Omega = \mathbb{R}^2_+ = \{x = (s,t) \in \mathbb{R}^2,\ t > 0\}.$$

We introduce the self-adjoint Neumann realization on the half-plane Ω of the Schrödinger operator $\mathcal{L}_\theta$ with potential V_θ:

$$\mathcal{L}_\theta = -\Delta + V_\theta = D_s^2 + D_t^2 + V_\theta,$$

where V_θ is defined for any $\theta \in (0, \frac{\pi}{2})$ by

$$V_\theta \colon \mathbf{x} = (s,t) \in \Omega \longmapsto (t\cos\theta - s\sin\theta)^2.$$

We can notice that V_θ reaches its minimum 0 all along the line $t\cos\theta = s\sin\theta$, which makes the angle θ with $\partial\Omega$. We denote by $\mathsf{Dom}(\mathcal{L}_\theta)$ the domain of $\mathcal{L}_\theta$ and we consider the associated quadratic form Q_θ defined by:

$$Q_\theta(u) = \int_\Omega \left(|\nabla u|^2 + V_\theta |u|^2\right) d\mathbf{x},$$

whose domain $\mathsf{Dom}(q_\theta)$ is:

$$\mathsf{Dom}(Q_\theta) = \{u \in L^2(\Omega),\ \nabla u \in L^2(\Omega),\ \sqrt{V_\theta}\, u \in L^2(\Omega)\}.$$

Let $\sigma_n(\theta)$ denote the n-th Rayleigh quotient of $\mathcal{L}_\theta$. Let us recall some fundamental spectral properties of $\mathcal{L}_\theta$ when $\theta \in \left(0, \frac{\pi}{2}\right)$.

It is proved in [60] that $\sigma_{\mathsf{ess}}(\mathcal{L}_\theta) = [1, +\infty)$ and that $\theta \mapsto \sigma_n(\theta)$ is non decreasing. Moreover, the function $(0, \frac{\pi}{2}) \ni \theta \mapsto \sigma_1(\theta)$ is increasing, and corresponds to a simple eigenvalue < 1 associated with a positive eigenfunction (see [71, Lemma 3.6]). As a consequence $\theta \mapsto \sigma_1(\theta)$ is analytic (see [66, Chapter 7]).

• *A few numerical simulations.* Let us provide a few numerical experiments (coming from [10]) related to the Helffer-Lu-Pan operator. Below we give the result of numerical simulations giving the first eigenvalues of $\mathcal{L}_\theta$ as a function of θ. The next figure describes the first

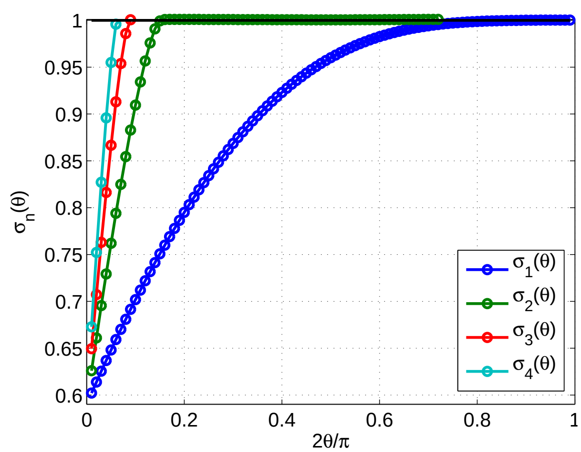
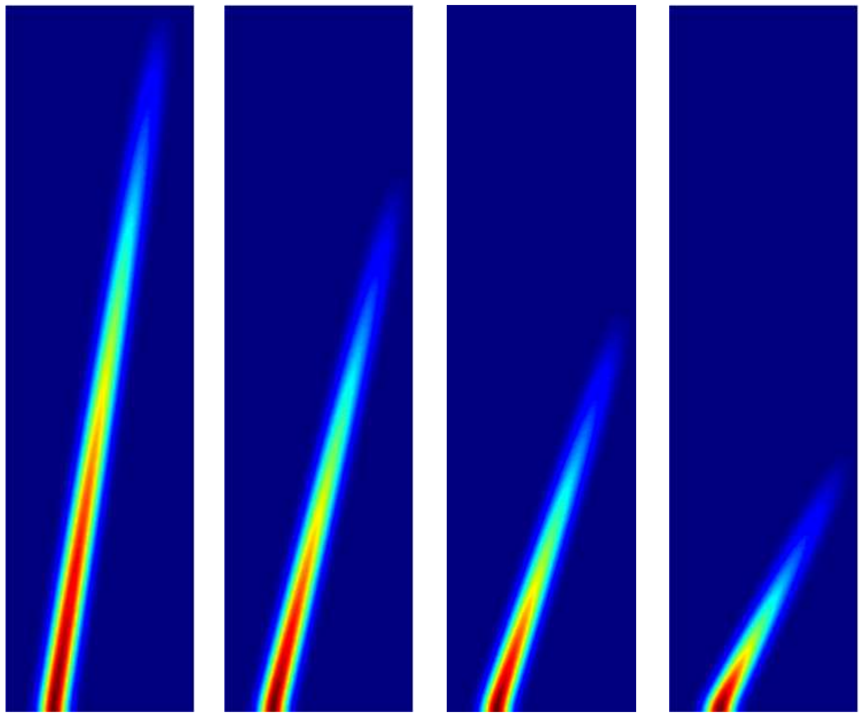


FIGURE 2. $\sigma_n(\theta; 100, 100, 100)$ for $n = 1, \ldots, 4$ (ordinates) versus $\vartheta = 2\theta/\pi$ (abscissa). Sampling: $\vartheta = k/100,\ 1 \le k \le 99$.

eigenmode for different values of θ.



$\breve{\sigma}_1(\theta)$ 1.0001656284 0.99987798948 0.99910390126 0.99445407220

FIGURE 3. First eigenmode of $\mathcal{L}_\theta$ for $\theta = \vartheta\pi/2$ with $\vartheta = 0.9, 0.85, 0.8$ and 0.7.

3.5. **Kato Theory: Feynman-Hellmann Formulas.** As we can notice, all the operators that we have introduced depend on parameters and are analytic of type (B) in terms of Kato's theory. Moreover, we also observe that the lowest eigenvalues of the previous model operators are simple, we systematically deduce that they analytically depend on the parameters.

In order to illustrate the Feynman-Hellmann formulas, let us examine a few examples.

• *De Gennes operator.* Let us prove propositions which are often used in the study of the magnetic Laplacian.

For $\rho > 0$ and $\xi \in \mathbb{R}$, let us introduce the Neumann realization on $\mathbb{R}_+$ of:

$$H_{\rho,\xi} = -\rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2.$$

By scaling, we observe that $H_{\rho,\xi}$ is unitarily equivalent to H_ξ and that $H_{1,\xi} = H_\xi$ (the corresponding eigenfunction is $u_{1,\xi} = u_\xi$).

Remark 3.10. The introduction of the scaling parameter ρ is related to the Virial theorem (see [95]) which was used by physicists in the theory of superconductivity (see [32] and also [3, 20]). We also refer to the papers [85] and [86] where it is used many times.

The form domain of $H_{\rho,\xi}$ is $B^1(\mathbb{R}_+)$ and is independent from ρ and ξ so that the family $(H_{\rho,\xi})_{\rho>0,\xi\in\mathbb{R}}$ is an analytic family of type (B). The lowest eigenvalue of $H_{\rho,\xi}$ is $\mu(\xi)$ and we will denote by $u_{\rho,\xi}$ the corresponding normalized eigenfunction:

$$u_{\rho,\xi}(\tau) = \rho^{1/4} u_{\xi}(\rho^{1/2}\tau).$$

Since u_ξ satisfies the Neumann condition, we observe that $\partial_\rho^m \partial_\xi^n u_{\rho,\xi}$ also satisfies it. In order to lighten the notation and when it is not ambiguous we will write H for $H_{\rho,\xi}$, u for $u_{\rho,\xi}$ and μ for $\mu(\xi)$.

The main idea is now to take derivatives of:

$$Hu = \mu u \tag{3.7}$$

with respect to ρ and ξ. Taking the derivative with respect to ρ and ξ, we get the proposition:

Proposition 3.11. *We have:*

$$(H-\mu)\partial_\xi u = 2(\rho^{1/2}\tau - \xi)u + \mu'(\xi)u \tag{3.8}$$

and

$$(H-\mu)\partial_\rho u = \left(-\rho^{-2}\partial_\tau^2 - \xi\rho^{-1}(\rho^{1/2}\tau-\xi) - \rho^{-1}\tau(\rho^{1/2}\tau-\xi)^2\right)u. \tag{3.9}$$

Moreover, we get:

$$(H-\mu)(Su) = Xu, \tag{3.10}$$

where

$$X = -\frac{\xi}{2}\mu'(\xi) + \rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2$$

and

$$S = -\frac{\xi}{2}\partial_\xi - \rho\partial_\rho.$$

Proof. Taking the derivatives with respect to ξ and ρ of (3.7), we get:

$$(H-\mu)\partial_\xi u = \mu'(\xi)u - \partial_\xi H u$$

and

$$(H-\mu)\partial_\rho u = -\partial_\rho H.$$

We have: $\partial_\xi H = -2(\rho^{1/2}\tau - \xi)$ and $\partial_\rho H = \rho^{-2}\partial_\tau^2 + \rho^{-1/2}\tau(\rho^{1/2}\tau - \xi)$. □

Taking $\rho = 1$ and $\xi = \xi_0$ in (3.8), we deduce, with the Fredholm alternative:

Corollary 3.12. *We have:*

$$(H_{\xi_0} - \mu(\xi_0))v_{\xi_0} = 2(t-\xi_0)u_{\xi_0},$$

with:

$$v_{\xi_0} = (\partial_\xi u_\xi)_{|\xi=\xi_0}.$$

Moreover, we have:

$$\int_{\tau>0} (\tau - \xi_0)u_{\xi_0}^2 \, d\tau = 0.$$

Corollary 3.13. *We have, for all $\rho > 0$:*

$$\int_{\tau>0} (\rho^{1/2}\tau - \xi_0)u_{\rho,\xi_0}^2 \, d\tau = 0$$

and:

$$\int_{\tau>0} (\tau - \xi_0) \left(\partial_\rho u\right)_{\rho=1,\xi=\xi_0} u \, d\tau = -\frac{\xi_0}{4}.$$

Corollary 3.14. *We have:*

$$(H_{\xi_0} - \mu(\xi_0))S_0 u = \left(\partial_\tau^2 + (\tau - \xi_0)^2\right) u_{\xi_0},$$

where:

$$S_0 u = -\left(\partial_\rho u_{\rho,\xi}\right)_{|\rho=1,\xi=\xi_0} - \frac{\xi_0}{2} v_{\xi_0}.$$

Moreover, we have:

$$\|\partial_\tau u_{\xi_0}\|^2 = \|(\tau - \xi_0)u_{\xi_0}\|^2 = \frac{\Theta_0}{2}.$$

The next proposition deals with the second derivative of (3.7) with respect to ξ.

Proposition 3.15. *We have:*

$$(H_\xi - \mu(\xi))w_{\xi_0} = 4(\tau - \xi_0)v_{\xi_0} + (\mu''(\xi_0) - 2)u_{\xi_0},$$

with

$$w_{\xi_0} = \left(\partial_\xi^2 u_\xi\right)_{|\xi=\xi_0}.$$

Moreover, we have:

$$\int_{\tau>0} (\tau - \xi_0)v_{\xi_0} u_{\xi_0} \, d\tau = \frac{2 - \mu''(\xi_0)}{4}.$$

Proof. Taking the derivative of (3.8) with respect to ξ (with $\rho = 1$), we get:

$$(H_\xi - \mu(\xi))\partial_\xi^2 u_\xi = 2\mu'(\xi)\partial_\xi u_\xi + 4(\tau - \xi)\partial_\xi u_\xi + (\mu''(\xi) - 2)u_\xi.$$

It remains to take $\xi = \xi_0$ and to write the Fredholm alternative. □

• *Helffer-Lu-Pan operator.* The following result is obtained in [10].

Proposition 3.16. *For all $\theta \in \left(0, \frac{\pi}{2}\right)$, we have:*

$$\sigma_1(\theta)\cos\theta - \sigma_1'(\theta)\sin\theta > 0.$$

Moreover, we have:

$$\lim_{\substack{\theta \to \frac{\pi}{2} \\ \theta < \frac{\pi}{2}}} \sigma_1'(\theta) = 0.$$

Proof. For $\gamma \geq 0$, we introduce the operator (see [87]):

$$\mathcal{L}(\theta, \gamma) = D_s^2 + D_t^2 + (t(\cos\theta + \gamma) - s\sin\theta)^2$$

and we denote by $\sigma_1(\theta, \gamma)$ the bottom of its spectrum. Let $\rho > 0$ and $\alpha \in (0, \frac{\pi}{2})$ satisfy

$$\cos\theta + \gamma = \rho\cos\alpha \quad \text{and} \quad \sin\theta = \rho\sin\alpha.$$

We perform the rescaling $t = \rho^{-1/2}\hat{t}$, $s = \rho^{-1/2}\hat{s}$ and obtain that $\mathcal{L}(\theta, \gamma)$ is unitarily equivalent to:

$$\rho(D_{\hat{s}}^2 + D_{\hat{t}}^2 + (\hat{t}\cos\alpha - \hat{s}\sin\alpha)^2) = \rho\mathcal{L}_\alpha.$$

In particular, we observe that $\sigma_1(\theta, \gamma) = \rho\sigma_1(\alpha)$ is a simple eigenvalue: there holds

$$\sigma_1(\theta, \gamma) = \sqrt{(\cos\theta + \gamma)^2 + \sin^2\theta}\ \ \sigma_1\left(\arctan\left(\frac{\sin\theta}{\cos\theta + \gamma}\right)\right). \tag{3.11}$$

Performing the rescaling $\tilde{t} = (\cos\theta + \gamma)t$, we get the operator $\tilde{\mathcal{L}}(\theta, \gamma)$ which is unitarily equivalent to $\mathcal{L}(\theta, \gamma)$:

$$\tilde{\mathcal{L}}(\theta, \gamma) = D_s^2 + (\cos\theta + \gamma)^2 D_{\tilde{t}}^2 + (\tilde{t} - s\sin\theta)^2.$$

We observe that the domain of $\tilde{\mathcal{L}}(\theta, \gamma)$ does not depend on $\gamma \geq 0$. Denoting by $\tilde{u}_{\theta,\gamma}$ the L^2-normalized and positive eigenfunction of $\tilde{\mathcal{L}}(\theta, \gamma)$ associated with $\sigma_1(\theta, \gamma)$, we write:

$$\tilde{\mathcal{L}}(\theta, \gamma)\tilde{u}_{\theta,\gamma} = \sigma_1(\theta, \gamma)\tilde{u}_{\theta,\gamma}.$$

Taking the derivative with respect to γ, multiplying by $\tilde{u}_{\theta,\gamma}$ and integrating, we get the Feynman-Hellmann formula:

$$\partial_\gamma\sigma_1(\theta, \gamma) = 2(\cos\theta + \gamma)\int_\Omega |D_t\tilde{u}_{\theta,\gamma}|^2 ds dt \geq 0.$$

We deduce that, if $\partial_\gamma\sigma_1(\theta, \gamma) = 0$, then $D_t\tilde{u}_{\theta,\gamma} = 0$ and $\tilde{u}_{\theta,\gamma}$ only depends on s, which is a contradiction with $\tilde{u}_{\theta,\gamma} \in L^2(\Omega)$. Consequently, we have $\partial_\gamma\sigma_1(\theta, \gamma) > 0$ for any $\gamma \geq 0$. An easy computation using formula (3.11) provides:

$$\partial_\gamma\sigma_1(\theta, 0) = \sigma_1(\theta)\cos\theta - \sigma_1'(\theta)\sin\theta.$$

The function σ_1 is analytic and increasing. Thus we deduce:

$$\forall\theta \in \left(0, \frac{\pi}{2}\right), \qquad 0 \leq \sigma_1'(\theta) < \frac{\cos\theta}{\sin\theta}\sigma_1(\theta).$$

We get:

$$0 \leq \liminf_{\substack{\theta\to\frac{\pi}{2} \\ \theta<\frac{\pi}{2}}} \sigma_1'(\theta) \leq \limsup_{\substack{\theta\to\frac{\pi}{2} \\ \theta<\frac{\pi}{2}}} \sigma_1'(\theta) \leq 0,$$

which ends the proof. $\square$

4. Reduction to Local Models

We explain in this section how we can perform a reduction of the magnetic Laplacian to local models.

4.1. **Partition of Unity and Localization Formula.**

The presentation is inspired by [24]. We introduce the following partition of unity:

$$\sum_j \chi_{j,R}^2 = 1,$$

where the $\chi_{j,R}$ is a smooth cutoff function supported in a ball of center x_j and radius $R > 0$. Moreover, we can find such a partition of unity so that:

$$\sum_j \|\nabla \chi_{j,R}\|^2 \leq CR^{-2}.$$

The following formula is usually called "IMS formula" and allows to localize the electromagnetic Laplacian.

Proposition 4.1. *Let $\psi \in \mathsf{Dom}(q_{h,\boldsymbol{A},V})$. We have:*

$$Q_{h,\boldsymbol{A},V}(\psi) = \sum_j Q_{h,\boldsymbol{A},V}(\chi_{j,R}\psi) - h^2 \sum_j \|\nabla \chi_{j,R}\psi\|^2.$$

Proof. The proof is easy and instructive. By a density argument, it is enough to prove this for $\psi \in \mathsf{Dom}(P_{h,\mathbf{A},V})$. We can write:

$$Q_{h,\mathbf{A},V}(\chi_{j,R}\psi) = \langle P_{h,\mathbf{A},V}\chi_{j,R}\psi, \chi_{j,R}\psi\rangle.$$

We let $P = hD_k + A_k$ and $\chi = \chi_{j,R}$. It is enough to estimate:

$$\begin{aligned}\langle P\psi, P\chi^2\psi\rangle &= \langle \chi P\psi, [P,\chi]\psi\rangle + \langle \chi P\psi, P\chi\psi\rangle \\ &= \langle \chi P\psi, [P,\chi]\psi\rangle + \langle P\chi\psi, P\chi\psi\rangle + \langle [\chi, P]\psi, P\chi\psi\rangle \\ &= \langle P\chi\psi, P\chi\psi\rangle - \|[P,\chi]\psi\|^2 + \langle \chi P\psi, [P,\chi]\psi\rangle - \langle [P,\chi]\psi, \chi P\psi\rangle.\end{aligned}$$

Taking the real part, we find:

$$\langle P\psi, P\chi^2\psi\rangle = \|P\chi\psi\|^2 - \|[P,\chi]\psi\|^2.$$

We have: $[P,\chi] = -ih\partial_k\chi$. It remains to take the sum and the conclusion follows. □

4.2. **Magnetic Example.**

As we are going to see on an example, this localization formula is very convenient to prove lower bounds for the spectrum. Let us continue the study of:

$$\mathcal{L}_{h,\mathbf{A}}^{\mathsf{ex}} = h^2 D_x^2 + \left(hD_y + x + \frac{x^3}{3} + y^2 x\right)^2.$$

Proposition 4.2. *For all $n \in \mathbb{N}^*$, there exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$:*

$$\lambda_n(h) \geq h - Ch^{5/4}.$$

Proof. We introduce a partition of unity with radius $R > 0$ denoted by $(\chi_{j,R})_j$. Let us consider an eigenpair (λ, ψ). We have:

$$Q_{h,\mathbf{A}}(\psi) = \sum_j Q_{h,\mathbf{A}}(\chi_{j,R}\psi) - h^2 \sum_j \|\nabla \chi_{j,R}\psi\|^2$$

so that:

$$Q_{h,\mathbf{A}}(\psi) \geq \sum_j Q_{h,\mathbf{A}}(\chi_{j,R}\psi) - CR^{-2}h^2\|\psi\|^2$$

and:

$$\lambda\|\psi\|^2 \geq \sum_j Q_{h,\mathbf{A}}(\chi_{j,R}\psi) - CR^{-2}h^2\|\psi\|^2.$$

It remains to provide a lower bound for $Q_{h,\mathbf{A}}(\chi_{j,R}\psi)$. We choose $R = h^\rho$ with $\rho > 0$, to be chosen. We approximate the magnetic field in each ball by the constant magnetic field β_j:

$$|\beta - \beta_j| \leq C\|x - x_j\|.$$

In a suitable gauge, we have:

$$\|\mathbf{A} - \mathbf{A}_j^{\mathsf{lin}}\| \leq C\|x - x_j\|^2,$$

where $C > 0$ does not depend on j. Then, we have, for all $\varepsilon \in (0, 1)$:

$$Q_{h,\mathbf{A}}(\chi_{j,R}\psi) \geq (1-\varepsilon)Q_{h,\mathbf{A}_j^{\mathsf{lin}}}(\chi_{j,R}\psi) - C^2\varepsilon^{-1}R^4\|\chi_{j,R}\psi\|^2.$$

From the min-max principle, we deduce:

$$Q_{h,\mathbf{A}}(\chi_{j,R}\psi) \geq \left((1-\varepsilon)\beta_j h - C^2\varepsilon^{-1}h^{4\rho}\right)\|\chi_{j,R}\psi\|^2.$$

Optimizing ε, we take: $\varepsilon = h^{2\rho-1/2}$ and it follows:

$$Q_{h,\mathbf{A}}(\chi_{j,R}\psi) \geq \left(\beta_j h - Ch^{2\rho+1/2}\right)\|\chi_{j,R}\psi\|^2.$$

We now choose ρ such that $2\rho + 1/2 = 2 - 2\rho$. We are led to take: $\rho = \frac{3}{8}$ and the conclusion follows. $\square$

5. AGMON ESTIMATES

This section is devoted to the Agmon estimates in the semiclassical framework. We refer to the classical references [1, 2, 55, 52, 53].

5.1. Agmon identity for the electro-magnetic Laplacian.

Proposition 5.1. *Let Ω be a bounded open domain in $\mathbb{R}^m$ with Lipschitzian boundary. Let $V \in \mathcal{C}^0(\overline{\Omega}, \mathbb{R})$, $\boldsymbol{A} \in \mathcal{C}^0(\overline{\Omega}, \mathbb{R}^m)$ and Φ a real valued Lipschitzian function on $\overline{\Omega}$. Then, for $u \in \mathsf{Dom}(\mathcal{L}_{h,\boldsymbol{A},V})$ (with Dirichlet or magnetic Neumann condition), we have:*

$$\int_\Omega \|(-ih\nabla + \boldsymbol{A})e^{\Phi}u\|^2 \, d\mathbf{x} + \int_\Omega \left(V - h^2\|\nabla\Phi\|^2 e^{2\Phi}\right)|u|^2 \, d\mathbf{x} = \Re\langle \mathcal{L}_{h,\boldsymbol{A},V}u, e^{2\Phi}u\rangle.$$

Proof. We give the proof when Φ is smooth. Let us use the Green-Riemann formula:

$$\sum_{k=1}^m \langle(-ih\partial_k + A_k)^2 u, e^{2\Phi}u\rangle = \sum_{k=1}^m \langle(-ih\partial_k + A_k)u, (-ih\partial_k + A_k)e^{2\Phi}u\rangle,$$

where the boundary term has disappeared thanks to the boundary condition. In order to lighten the notation, we let $P = -ih\partial_k + A_k$.

$$\begin{aligned}\langle Pu, Pe^{2\Phi}u\rangle &= \langle e^{\Phi}Pu, [P, e^{\Phi}]u\rangle + \langle e^{\Phi}Pu, Pe^{\Phi}u\rangle \\ &= \langle e^{\Phi}Pu, [P, e^{\Phi}]u\rangle + \langle Pe^{\Phi}u, Pe^{\Phi}u\rangle + \langle [e^{\Phi}, P]u, Pe^{\Phi}u\rangle \\ &= \langle Pe^{\Phi}u, Pe^{\Phi}u\rangle - \|[P, e^{\Phi}]u\|^2 + \langle e^{\Phi}Pu, [P, e^{\Phi}]u\rangle - \langle [P, e^{\Phi}]u, e^{\Phi}Pu\rangle.\end{aligned}$$

We deduce:

$$\Re\left(\langle Pu, Pe^{2\Phi}u\rangle\right) = \langle Pe^{\Phi}u, Pe^{\Phi}u\rangle - \|[P, e^{\Phi}u]\|^2.$$

This is then enough to conclude.

□

5.2. Example of application.

Let us continue to study our favorite example (see Subsection 4.2).

Proposition 5.2. *There exist $C > 0, h_0 > 0$ such that, for $h \in (0, h_0)$ and (λ, ψ) an eigenpair of $\mathcal{L}^{\mathsf{ex}}_{h,\boldsymbol{A}}$ satisfying $\lambda \leq h + Ch^2$, we have:*

$$\int_{\mathbb{R}^2} e^{2h^{-1/8}|x|}|\psi|^2 \, d\mathbf{x} \leq C\|\psi\|^2.$$

Proof. We consider an eigenpair (λ, ψ) as in the proposition and we use the Agmon identity, jointly with the "IMS" formula (with balls of size $h^{3/8}$):

$$\sum_j Q_{h,\mathbf{A}}(\chi_{j,h}e^{\Phi}\psi) - h^2\|\nabla\chi_{j,h}e^{\Phi}\psi\|^2 - h^2\|\chi_{j,h}\nabla\Phi e^{\Phi}\psi\|^2 - \lambda\|\chi_{j,h}e^{\Phi}\psi\|^2 = 0.$$

This becomes:

$$\sum_j Q_{h,\mathbf{A}}(\chi_{j,h}e^{\Phi}\psi) - (h + Ch^{5/4})\|\chi_{j,h}e^{\Phi}\psi\|^2 - h^2\|\chi_{j,h}\nabla\Phi e^{\Phi}\psi\|^2 \leq 0.$$

We need to give a lower bound for $Q_{h,\mathbf{A}}(\chi_{j,h}e^{\Phi}\psi)$:

$$Q_{h,\mathbf{A}}(\chi_{j,h}e^{\Phi}\psi) \geq (\beta(\mathbf{x}_j)h - Ch^{5/4})\|e^{\Phi}\chi_{j,h}\psi\|^2.$$

This implies:

$$\sum_j ((\beta(\mathbf{x}_j)-1)h - Ch^{5/4})\|e^{\Phi}\chi_{j,h}\psi\|^2 - h^2\|\chi_{j,h}\nabla\Phi e^{\Phi}\psi\|^2 \leq 0.$$

We split the sum into two parts: the j such that $|\mathbf{x}_j| \geq C_0 h^{1/8}$ and the j such that $|\mathbf{x}_j| \leq C_0 h^{1/8}$, for some $C_0 > 0$ to be chosen. Moreover, we choose $\Phi(\mathbf{x}) = h^{-1/8}|\mathbf{x}|$.

Let us consider first j such that $|\mathbf{x}_j| \leq C_0 h^{1/8}$. Due to the non-degeneracy of the minimum of β, we get the existence of $c_0, \varepsilon_0 > 0$ such that, for all $C_0 > 0$:

$$\beta(\mathbf{x}_j) - 1 \geq \min(c_0 C_0^2 h^{5/4}, \varepsilon_0).$$

Then, we choose $C_0 > 0$ such that: $c_0 C_0^2 - C > 0$. Taking h small enough, we find the inequality:

$$\sum_{|\mathbf{x}_j| \geq C_0 h^{1/8}} \|e^{\Phi}\chi_{j,h}\psi\|^2 \leq \tilde{C} \sum_{|\mathbf{x}_j| \leq C_0 h^{1/8}} \|e^{\Phi}\chi_{j,h}\psi\|^2 \leq \hat{C}\|\psi\|^2.$$

Finally, we deduce:

$$\|e^{\Phi}\psi\| \leq C\|\psi\|.$$

□

- *Numerical simulations.* Let us give a few simulations of the eigenfunctions of $\mathcal{L}^{\mathsf{ex}}_{h,\mathbf{A}}$.

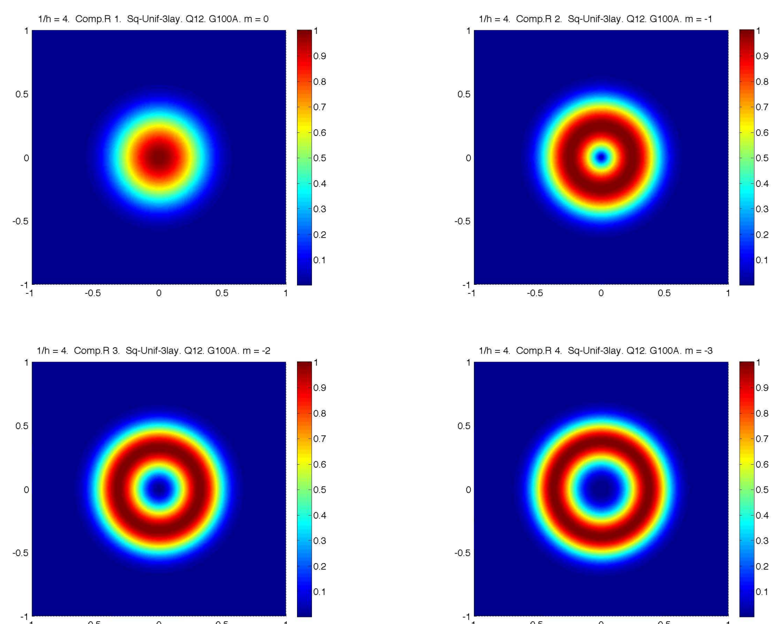


FIGURE 4. Eigenmodes

Another example of application of the estimates of Agmon is the theory of the Born-Oppenheimer approximation that we sketch in the next section.

6. BORN-OPPENHEIMER APPROXIMATION

This section presents the main idea behind the Born-Oppenheimer approximation (see [23, 72]). We do not strive for maximum generality.

6.1. Heuristics and framework.

Let us explain the question in which we are interested. We shall study operators in $L^2(\mathbb{R}\times\Omega)$ (with $\Omega\subset\mathbb{R}^d$) in the form:

$$\mathcal{H}(h)=h^2D_z^2+A(z),$$

where $A(z)=-\Delta_t+P(t,z)$ is a family of semi-bounded self-adjoint operators, with P polynomial. We will denote by Q_h the corresponding quadratic form.

We want to analyze the low lying eigenvalues of this operator. We will assume that the lowest eigenvalue $\nu(z)$ of $A(z)$ (which is simple) admits, as a function of z, a unique and non degenerate minimum at z_0.

• *Heuristics.* We now try to understand the heuristics. We hope that $\mathcal{H}(h)$ can be described by its "Born-oppenheimer" approximation:

$$\mathcal{H}^{\mathsf{BO}}(h)=h^2D_z^2+\nu(z),$$

with is a $1D$ electric Laplacian. Then, we guess that $\mathcal{H}^{\mathsf{BO}}(h)$ is well approximated by its Taylor expansion:

$$h^2D_z^2+\nu(z_0)+\frac{\nu''(z_0)}{2}(z-z_0)^2.$$

Therefore we imagine that the lowest eigenvalues of $\mathcal{H}(h)$ satisfy:

$$\lambda_n(h)=\nu(z_0)+h(2n-1)\left(\frac{\nu''(z_0)}{2}\right)^{1/2}+o(h).$$

In the next subsections we explain how to make this heuristics rigorous.

6.2. Recall of Feynman-Hellmann formulas.

We have:

$$A(z)v_z=\nu(z)v_z.$$

This is easy to prove that (the details are left as an exercise):

$$\langle A'(z_0)v_{z_0},v_{z_0}\rangle=0,$$

$$(A(z_0)-\nu(z_0))\left(\frac{d}{dz}v_z\right)_{|z=z_0}=-A'(z_0)v_{z_0}$$

and:

$$\left\langle A'(z_0)\left(\frac{d}{dz}v_z\right)_{|z=z_0}+\frac{A''(z_0)}{2}v_{z_0},v_{z_0}\right\rangle=\frac{\nu''(z_0)}{2}.$$

6.3. **Quasimodes.** As usual we begin with the construction of suitable quasimodes. Instead of $\mathcal{H}(h)$ we study:

$$\tilde{\mathcal{H}}(h) = hD_u^2 + A(z_0 + h^{1/2}u).$$

In terms of formal power series, we have:

$$\tilde{\mathcal{H}}(h) = A(z_0) + h^{1/2}uA'(z_0) + h\left(u^2\frac{A''(z_0)}{2} + D_u^2\right) + \cdots$$

We look for quasi-eigenpairs in the form:

$$\lambda \sim \lambda_0 + h^{1/2}\lambda_1 + h\lambda_2 + \cdots, \quad \psi \sim \psi_0 + h^{1/2}\psi_1 + h\psi_2 + \cdots$$

- *Term of order* h^0. We must solve:

$$A(z_0)\psi_0 = \lambda_0\psi_0.$$

Therefore, we choose $\lambda_0 = \nu(z_0)$ and $\psi_0(u,t) = v_{z_0}(t)f_0(u)$.

- *Term of order* $h^{1/2}$. We now meet the following equation:

$$(A(z_0) - \lambda_0)\psi_1 = (\lambda_1 - uA'(z_0))\psi_0.$$

The Feynman-Hellmann formula jointly with the Fredholm alternative implies that: $\lambda_1 = 0$ and that we can take:

$$\psi_1(u,t) = uf_0(u)\left(\frac{d}{dz}v_z\right)_{|z=z_0}(t) + uf_1(u)v_{z_0}.$$

- *Term of order* h^1. The crucial equation is given by:

$$(A(z_0) - \nu(z_0))\psi_2 = \lambda_2\psi_0 - uA'(z_0)\psi_1 - \left(u^2\frac{A''(z_0)}{2} + D_u^2\right)\psi_0.$$

The Fredholm alternative jointly with the Feynman-Hellmann formula provides:

$$\left(D_u^2 + \frac{\nu''(z_0)}{2}u^2\right)f_0 = \lambda_2 f_0.$$

This is an easy exercise to prove that this construction can be continued at any order.

6.4. **Essential spectrum.** Let us briefly discuss the properties related to the essential spectrum.

Assumption 6.1. Let us assume that $\liminf_{z\to\pm\infty}\nu(z) > \nu(z_0)$ and that for all z:

$$\inf_z \sigma_{\mathsf{ess}}(A(z)) > \nu(z_0).$$

We infer (exercise), as a consequence of the theorem of Persson (see Theorem 2.5):

Proposition 6.2. *Under Assumption 6.1, we have:*

$$\inf_{h>0}\inf \sigma_{\mathsf{ess}}(\mathcal{H}(h)) > \nu(z_0).$$

As a corollary, we get:

Proposition 6.3. *There exists $h_0 > 0, C > 0, \varepsilon_0 > 0$ such that, for $h \in (0, h_0)$, for all eigenpair (λ, ψ) such that $\lambda \leq \nu(z_0) + C_0 h$, we have:*

$$\int e^{2\varepsilon_0(|z|+|t|)}|\psi|^2\, dzdt \leq C\|\psi\|^2.$$

Proof. This is a consequence of Persson's theorem (see [81]). □

6.5. **Agmon Estimates.** We are now led to prove some localization behavior of the eigenfunctions associated with eigenvalues λ such that: $|\lambda - \nu(z_0)| \leq C_0 h$.

Proposition 6.4. *There exist $\varepsilon_0, h_0, C > 0$ such that for all eigenpair (λ, ψ) such that $|\lambda - \nu(z_0)| \leq C_0 h$, we have:*

$$\int e^{2\varepsilon_0 h^{-1/2}|z|}|\psi|^2\, d\mathbf{x} \leq C\|\psi\|^2.$$

and:

$$\left\| h\partial_z \left(e^{\varepsilon_0 h^{-1/2}|z|}\psi\right)\right\|^2 \leq Ch\|\psi\|^2.$$

Proof. Let us write an estimate of Agmon:

$$Q_h(e^{h^{-1/2}\varepsilon_0|z|}\psi) - h\varepsilon_0^2\|e^{h^{-1/2}\varepsilon_0|z|}\psi\|^2 = \lambda\|e^{h^{-1/2}\varepsilon_0|z|}\psi\|^2 \leq (\nu(z_0) + C_0 h)\|e^{h^{-1/2}\varepsilon_0|z|}\psi\|^2.$$

But we notice that:

$$Q_h(e^{h^{-1/2}\varepsilon_0|z|}\psi) \geq \int h^2 \left|\partial_z\left(e^{h^{-1/2}\varepsilon_0|z|}\psi\right)\right|^2 + \nu(z)\left|\left(e^{h^{-1/2}\varepsilon_0|z|}\psi\right)\right|^2\, d\mathbf{x}$$

This implies the inequality:

$$\int (\nu(z) - \nu(z_0) - C_0 h - \varepsilon_0^2 h)\left|\left(e^{h^{-1/2}\varepsilon_0|z|}\psi\right)\right|^2\, d\mathbf{x} \leq 0.$$

We leave the conclusion as an exercise. □

6.6. **Projection Method.** As we have observed, it can be more convenient to study $\tilde{\mathcal{H}}(h)$ instead of $\mathcal{H}(h)$. Let us introduce the Feshbach-Grushin projection (see [47]) on v_{z_0}:

$$\Pi_0\psi = \langle\psi, v_{z_0}\rangle_t v_{z_0}(t).$$

We want to estimate the projection of the eigenfunctions associated with eigenvalues λ such that: $|\lambda - \nu(z_0)| \leq C_0 h$. For that purpose, let us introduce the quadratic form:

$$q_0(\psi) = \int |\partial_t\psi|^2 + P(t, z_0)|\psi|^2\, dudt.$$

This quadratic form is associated with the operator: $\mathsf{Id}_u \otimes A(z_0)$ whereas Π_0 is the projection on its first eigenspace.

Proposition 6.5. *There exist $C, h_0 > 0$ such that, for $h \in (0, h_0)$, for all eigenpair (λ, ψ) of $\tilde{\mathcal{H}}(h)$ such that $\lambda \leq \nu(z_0) + C_0 h$:*

$$0 \leq q_0(\psi) - \nu(z_0)\|\psi\|^2 \leq Ch^{1/2}\|\psi\|^2.$$

Moreover, we have:

$$\|\psi - \Pi_0\psi\| + \|\partial_t(\psi - \Pi_0\psi)\| \leq Ch^{1/4}\|\psi\|.$$

Proof. The proof is rather easy. We write:

$$h\|\partial_u\psi\|^2 + \|\partial_t\psi\|^2 + \int P(t, z_0 + h^{1/2}u)|\psi|^2\, dzdt \leq (\lambda + C_0h)\|\psi\|^2. \tag{6.1}$$

Using the fact that P is a polynomial and the fact that, for $k, n \in \mathbb{N}$:

$$\int |t|^n|u|^k|\psi|^2\, dudt \leq C\|\psi\|^2,$$

we get the first estimate. For the second one, we notice that:

$$q_0(\psi) - \nu(z_0)\|\psi\|^2 = q_0(\psi - \Pi_0\psi) - \nu(z_0)\|\psi - \Pi_0\psi\|^2,$$

due to the fact that $\Pi_0\psi$ belongs to the kernel of $\mathsf{Id}_u \otimes A(z_0) - \nu(z_0)\mathsf{Id}$. We observe then that:

$$q_0(\psi - \Pi_0\psi) - \nu(z_0)\|\psi - \Pi_0\psi\|^2 \geq \int_u\int_t |\partial_t(\psi - \Pi_0\psi)|^2 + P(t, z_0)|(\psi - \Pi_0\psi)|^2\, dt\, du.$$

Since for each u, we have: $\langle\psi - \Pi_0\psi, v_{z_0}\rangle_t = 0$, we have the lower bound (min-max principle):

$$q_0(\psi - \Pi_0\psi) - \nu(z_0)\|\psi - \Pi_0\psi\|^2 \geq \int_u (\nu_2(z_0) - \nu(z_0))\int_t |\psi - \Pi_0\psi|^2\, dt\, du.$$

□

Proposition 6.6. *There exist $C, h_0 > 0$ such that, for $h \in (0, h_0)$, for all eigenpair (λ, ψ) of $\tilde{\mathcal{H}}(h)$ such that $\lambda \leq \nu(z_0) + C_0h$:*

$$0 \leq q_0(u\psi) - \nu(z_0)\|u\psi\|^2 \leq Ch^{1/2}\|\psi\|^2$$

and

$$0 \leq q_0(\partial_u\psi) - \nu(z_0)\|\partial_u\psi\|^2 \leq Ch^{1/4}\|\psi\|^2$$

Moreover, we have:

$$\|u\psi - u\Pi_0\psi\| + \|u\partial_t(\psi - u\Pi_0\psi)\| \leq Ch^{1/4}\|\psi\|$$

and

$$\|\partial_u(\psi - \Pi_0\psi)\| + \|\partial_u(\partial_t(\psi - \Pi_0\psi))\| \leq Ch^{1/8}\|\psi\|.$$

Proof. Using the "IMS" formula, we get:

$$q_h(u\psi) = \lambda\|u\psi\|^2 + h\|\psi\|^2 \leq (\nu(z_0) + C_0h)\|u\psi\|^2 + h\|\psi\|^2.$$

Using the estimates of Agmon, we find:

$$q_0(u\psi) - \nu(z_0)\|u\psi\|^2 \leq Ch^{1/2}\|\psi\|^2.$$

Let us analyze the estimate with ∂_u. We take the derivative with respect to u in the eigenvalue equation:

$$\left(hD_u^2 + D_t^2 + P(t, z_0 + h^{1/2}u)\right)\partial_u\psi = \lambda\partial_u\psi + [P(t, z_0 + h^{1/2}u), \partial_u]\psi. \tag{6.2}$$

Taking the scalar product with $\partial_u\psi$, we find (exercise):

$$q_h(\partial_u\psi) \leq (\nu(z_0) + C_0h)\|\partial_u\psi\|^2 + Ch^{1/2}\|\psi\|^2 \tag{6.3}$$

and:

$$q_0(\partial_u\psi)-\nu(z_0)\|\partial_u\psi\|^2\leq Ch^{1/4}\|\psi\|^2,$$

where we have used: $\|\partial_u^2\psi\|\leq Ch^{-1/4}\|\psi\|+C\|\partial_u\psi\|$ which is a consequence of (6.3) and $\|\partial_u\psi\|\leq C\|\psi\|$ which comes from (6.1). □

We can now use our approximation results to reduce the investigation to a $1D$ model operator.

6.7. **Accurate lower bound.** For all $N\geq 1$, let us consider the L^2-normalized eigenpairs $(\lambda_n(h),\psi_{n,h})_{1\leq n\leq N}$ such that $\langle\psi_{n,h},\psi_{m,h}\rangle=0$ when $n\neq m$. We consider the N dimensional space defined by:

$$\mathfrak{E}_N(h)=\underset{1\leq n\leq N}{\operatorname{span}}\psi_{n,h}.$$

It is rather easy to observe that, for $\psi\in\mathfrak{E}_N(h)$:

$$q_h(\psi)\leq\lambda_N(h)\|\psi\|^2.$$

We are going to prove a lower bound of q_h on $\mathfrak{E}_N(h)$. We notice that:

$$q_h(\psi)\geq\int h|\partial_u\psi|^2+\nu(z_0+h^{1/2}u)|\psi|^2\,du\,dt.$$

We have:

$$\begin{aligned}&\int h|\partial_u\psi|^2+\nu(z_0+h^{1/2}u)|\psi|^2\,du\,dt\\&=\int_{|uh^{1/2}|\leq\varepsilon_0}h|\partial_u\psi|^2+\nu(z_0+h^{1/2}u)|\psi|^2\,du\,dt+\int_{|uh^{1/2}|\geq\varepsilon_0}h|\partial_u\psi|^2+\nu(z_0+h^{1/2}u)|\psi|^2\,du\,dt.\end{aligned}$$

With the Taylor formula, we can write:

$$\begin{aligned}&\int_{|uh^{1/2}|\leq\varepsilon_0}h|\partial_u\psi|^2+\nu(z_0+h^{1/2}u)|\psi|^2\,du\,dt\\&\geq\int_{|uh^{1/2}|\leq\varepsilon_0}h|\partial_u\psi|^2+\nu(z_0)+h\frac{\nu''(z_0)}{2}u^2|\psi|^2\,du\,dt-Ch^{3/2}\int_{|uh^{1/2}|\leq\varepsilon_0}|u|^3|\psi|^2\,dudt.\end{aligned}$$

The estimates of Agmon give:

$$\begin{aligned}&\int_{|uh^{1/2}|\leq\varepsilon_0}h|\partial_u\psi|^2+\nu(z_0+h^{1/2}u)|\psi|^2\,du\,dt\\&\geq\int_{|uh^{1/2}|\leq\varepsilon_0}h|\partial_u\psi|^2+\nu(z_0)|\psi|^2+h\frac{\nu''(z_0)}{2}u^2|\psi|^2\,du\,dt-Ch^{3/2}\|\psi\|^2.\end{aligned}$$

Moreover, we have:

$$\int_{|uh^{1/2}|\geq\varepsilon_0}h|\partial_u\psi|^2+\nu(z_0|\psi|^2+h^{1/2}u)|\psi|^2\,du\,dt\geq(\nu(z_0)+\eta_0)\int_{|uh^{1/2}|\geq\varepsilon_0}|\psi|^2\,du\,dt=O(h^\infty)\|\psi\|^2.$$

We observe that:

$$\int_{|uh^{1/2}|\geq\varepsilon_0}h|\partial_u\psi|^2+\nu(z_0)|\psi|^2+h\frac{\nu''(z_0)}{2}u^2|\psi|^2\,du\,dt=O(h^\infty)\|\psi\|^2.$$

It follows that:

$$q_h(\psi) \geq \int h|\partial_u \psi|^2 + \nu(z_0)|\psi|^2 + h\frac{\nu''(z_0)}{2}u^2|\psi|^2 \, du \, dt - Ch^{3/2}\|\psi\|^2.$$

We can now use the approximation result and we infer (exercise):

$$\lambda_N(h)\|\psi\|^2 \geq q_h(\psi) \geq \nu(z_0)\|\psi\|^2 + \int h|\partial_u \Pi_0\psi|^2 + h\frac{\nu''(z_0)}{2}u^2|\Pi_0\psi|^2 \, du \, dt + o(h)\|\psi\|^2.$$

This becomes:

$$\int h|\partial_u \langle \psi, v_{z_0}\rangle|^2 + h\frac{\nu''(z_0)}{2}u^2|\langle \psi, v_{z_0}\rangle|^2 \, du \leq (\lambda_N(h) - \nu(z_0) + o(h))\|\langle \psi, v_{z_0}\rangle\|^2_{L^2(du)}.$$

By the min-max principle, we deduce:

$$\lambda_N(h) \geq \nu(z_0) + (2N-1)h\left(\frac{\nu''(z_0)}{2}\right)^{1/2} + o(h).$$

6.8. **Examples and exercises.** Let us now give examples which can be treated as exercises.

• *Helffer-Lu-Pan/de Gennes operator.* Our first example (which comes from [10] and [86]) is the Neumann realization of the operator acting on $L^2(\mathbb{R}^2_+, d\xi dt)$:

$$h^2D_\xi^2 + D_t^2 + (t-\xi)^2,$$

where $\mathbb{R}^2_+ = \{t > 0\}$.

• *Montgomery operator.* The second example (which is the core of [31]) is the self-adjoint realization on $L^2(d\xi dt)$ of:

$$h^2D_\xi^2 + D_t^2 + \left(\xi - t^2\right)^2.$$

• *Popoff operator.* Our last example (which comes from [83]) corresponds to the Neumann realization on $L^2(\mathcal{E}_\alpha, d\xi dz dt)$ of:

$$h^2D_\xi^2 + D_t^2 + D_z^2 + (t-\xi)^2.$$

The next two sections provide detailed examples of the philosophy explained in this course.

7. FROM THE MAGNETIC LAPLACIAN TO THE ELECTRIC LAPLACIAN: A REGULAR CASE IN 2D

7.1. **Motivation.** We consider a vector potential $\mathbf{A} \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and we consider the self-adjoint operator defined by:

$$\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla + \mathbf{A})^2.$$

In order $\mathcal{L}_{h,\mathbf{A}}$ to have compact resolvent, we will assume that:

$$\beta(x) \underset{|x|\to+\infty}{\to} +\infty. \tag{7.1}$$

As in [80, 49], we will investigate the case when β cancels along a closed and smooth curve Γ in $\mathbb{R}^2$. Let us notice that Assumption 7.1 could clearly be relaxed so that one could also consider a smooth, bounded and simply connected domain of $\mathbb{R}^2$ with Dirichlet or Neumann condition on the boundary as far as the magnetic field does not vanish near the boundary. Nevertheless we do not strive for maximum generality the present "generic" case giving enough information when the magnetic field "nicely" cancels (one could also make it to cancel at an higher order as in [49]). We let:

$$\Gamma = \{\gamma(s), s \in \mathbb{R}\}.$$

We assume that β is non positive inside Γ and non negative outside. We introduce the standard tubular coordinates (s,t) near Γ:

$$\Phi(s,t) = \gamma(s) + t\nu(s),$$

where $\nu(s)$ denotes the inward pointing normal to Γ at $\gamma(s)$. We let:

$$\tilde{\beta}(s,t) = \beta(\Phi(s,t))$$

so that:

$$\tilde{\beta}(s,0) = 0.$$

We consider the normal derivative of β on Γ, i.e. the function $\delta : s \mapsto \partial_t\tilde{\beta}(s,0)$. We will assume that:

(7.2) δ admits a unique, non-degenerate and positive minimum at x_0.

We let $\delta_0 = \delta(0)$ and assume without loss of generality that $x_0 = (0,0)$. Let us state the main result of this section:

Theorem 7.1. *We assume Assumptions 7.1 and 7.2. For all $n \geq 1$, there exist a sequence $(\theta_j^n)_{j\geq 0}$ and $h_0 > 0$ such that for $h \in (0, h_0)$, we have:*

$$\lambda_n(h) \sim h^{4/3} \sum_{j\geq 0} \theta_j^n h^{j/6}$$

where:

$$\theta_0^n = \delta_0^{2/3}\nu_1(\eta_0), \quad \theta_1^n = 0, \quad \theta_2^n = \delta_0^{2/3}C_0 + \delta_0^{2/3}(2n-1)\left(\frac{\alpha\nu(\eta_0)\nu''(\eta_0)}{3}\right)^{1/2},$$

where we have let:

$$\alpha = \frac{1}{2}\delta_0^{-1}\delta''(0) > 0 \tag{7.3}$$

and:

$$C_0 = \langle L u_{\eta_0}, u_{\eta_0}\rangle_{\hat\tau}, \tag{7.4}$$

where:

$$L = 2\kappa(0)\delta_0^{-4/3}\left(\frac{\hat\tau^2}{2}-\eta_0\right)\hat\tau^3 + 2\hat\tau\delta_0^{-1/3}k(0)\left(-\eta_0+\frac{\hat\tau^2}{2}\right)^2,$$

and:

$$\kappa(0) = \frac{1}{6}\partial_t^2\tilde\beta(0,0) - \frac{k(0)}{3}\delta_0.$$

7.2. **Normal Form.** We can write (exercise !) the operator near the cancellation line in the coordinates (s,t):

$$\widetilde{\mathcal{L}}_{h,\mathbf{A}} = h^2(1-tk(s))^{-1}D_t(1-tk(s))D_t + (1-tk(s))^{-1}\tilde P(1-tk(s))^{-1}\tilde P,$$

where

$$\tilde P = ih\partial_s + \tilde A(s,t)$$

with:

$$\tilde A(s,t) = \int_0^t (1-k(s)t')\tilde\beta(s,t')dt'.$$

In terms of the quadratic form, we can write:

$$\tilde Q_{h,\mathbf{A}}(\psi) = \int \left(|hD_t\psi|^2 + (1-tk(s))^{-2}|\tilde P\psi|^2\right) m(s,t)dsdt,$$

with:

$$m(s,t) = (1-tk(s)).$$

We consider the following operator on $L^2(dsdt)$ which is unitarily equivalent to $\widetilde{\mathcal{L}}_{h,\mathbf{A}}$ (see [64, Theorem 18.5.9 and below])[4]:

$$\mathcal{L}_{h,\mathbf{A}}^{\mathsf{new}} = m^{1/2}\widetilde{\mathcal{L}}_{h,\mathbf{A}}m^{-1/2} = P_1^2 + P_2^2 - \frac{h^2k(s)^2}{4m^2},$$

with $P_1 = m^{-1/2}(-hD_s + \tilde A(s,t))m^{-1/2}$ and $P_2 = hD_t$.

We wish to use a system of coordinates more adapted to the magnetic situation. Let us perform a Taylor expansion near $t=0$. We have:

$$\tilde\beta(s,t) = \delta(s)t + \partial_t^2\tilde\beta(s,0)\frac{t^2}{2} + O(t^3).$$

This provides:

$$\tilde A(s,t) = \frac{\delta(s)}{2}t^2 + \kappa(s)t^3 + O(t^4),$$

with:

$$\kappa(s) = \frac{1}{6}\partial_t^2\tilde\beta(s,0) - \frac{k(s)}{3}\delta(s)$$

This suggests, as for the model operator, to introduce the new magnetic coordinates in a fixed neighborhood of $(0,0)$:

$$\tau = \delta(s)^{1/3}t, \quad \sigma = s.$$

[4]Such a conjugation is standard in the universe of waveguides, see [33].

We can notice that it is a "scaling" depending on s. The change of coordinates for the derivatives is given by:

$$D_t = \delta(\sigma)^{1/3} D_\tau, \quad D_s = D_\sigma + \frac{1}{3}\delta'\delta^{-1}\tau D_\tau.$$

The space $L^2(dsdt)$ becomes $L^2(\delta(\sigma)^{-1/3} d\sigma d\tau)$. In the same way as previously, we shall conjugate $\mathcal{L}^{\mathsf{new}}_{h,\mathbf{A}}$. We introduce the self-adjoint operator on $L^2(d\sigma d\tau)$:

$$\check{\mathcal{L}}_{h,\mathbf{A}} = \delta^{-1/6}\mathcal{L}^{\mathsf{new}}_{h,\mathbf{A}}\delta^{1/6}.$$

We deduce:

$$\check{\mathcal{L}}_{h,\mathbf{A}} = h^2\delta(\sigma)^{2/3}D_\tau^2 + \check{P}^2,$$

where:

$$\check{P} = \delta^{-1/6}\check{m}^{-1/2}\left(-hD_\sigma + \check{A}(\sigma,\tau) - h\frac{1}{3}\delta'\delta^{-1}\tau D_\tau\right)\check{m}^{-1/2}\delta^{1/6},$$

with:

$$\check{A}(\sigma,\tau) = \tilde{A}(\sigma, \delta(\sigma)^{-1/3}\tau).$$

A straight forward computation provides:

$$\check{P} = \check{m}^{-1/2}\left(-hD_\sigma + \check{A}(\sigma,\tau) - h\frac{1}{6}\delta'\delta^{-1}(\tau D_\tau + D_\tau\tau)\right)\check{m}^{-1/2},$$

where we make the generator of dilations $\tau D_\tau + D_\tau \tau$ to appear (and which is related to the virial theorem, see [85, 86] where this theorem is often used). Up to a change of gauge, we can replace $\check{P}$ by:

$$\check{m}^{-1/2}\left(-hD_\sigma - \eta_0(\delta(\sigma))^{1/3}h^{2/3} + \check{A}(\sigma,\tau) - h\frac{1}{6}\delta'\delta^{-1}(\tau D_\tau + D_\tau\tau)\right)\check{m}^{-1/2}.$$

- *Normal form* $\check{\mathcal{L}}(h)$. Therefore, the operator takes the form "à la Hörmander":

$$\check{\mathcal{L}}(h) = P_1(h)^2 + P_2(h)^2 - \frac{h^2 k(\sigma)^2}{4m(\sigma,\delta(\sigma)^{1/3}\tau)^2}, \tag{7.5}$$

where:

$$P_1(h) = \check{m}^{-1/2}\left(-hD_\sigma - \eta_0(\delta(\sigma))^{1/3}h^{2/3} + \check{A}(\sigma,\tau) - h\frac{1}{6}\delta'\delta^{-1}(\tau D_\tau + D_\tau\tau)\right)\check{m}^{-1/2},$$

$$P_2(h) = h\delta(\sigma)^{1/3}D_\tau.$$

Computing a commutator, we can rewrite $P_1(h)$:

$$P_1(h) = \check{m}^{-1}\left(-hD_\sigma - \eta_0(\delta(\sigma))^{1/3}h^{2/3} + \check{A}(\sigma,\tau) - h\frac{1}{6}\delta'\delta^{-1}(\tau D_\tau + D_\tau\tau)\right) + C_h, \tag{7.6}$$

where:

$$C_h = -h\check{m}^{-1/2}(D_\sigma\check{m}^{-1/2}) - \frac{h\delta'\delta^{-1}}{3}\tau\check{m}^{-1/2}(D_\tau\check{m}^{-1/2}).$$

Notation 7.2. The quadratic form corresponding to $\check{\mathcal{L}}(h)$ will be denoted by $\check{Q}$.

7.3. **Quasimodes.** We shall now construct quasimodes using the classical recipe.

7.3.1. *The homogenized operator $\hat{\mathcal{L}}$.* We perform the scaling:

$$\tau = h^{1/3}\hat{\tau}, \quad \sigma = h^{1/6}\hat{\sigma}. \tag{7.7}$$

Notation 7.3. The operator $h^{-4/3}\check{\mathcal{L}}$ will be denoted by $\hat{\mathcal{L}}$ in these new coordinates.

We expand the new operator in powers of $h^{1/6}$ in the sense of formal power series:

$$\delta_0^{-2/3}\hat{\mathcal{L}}(h) \sim \sum_{j\geq 0} \mathcal{L}_j h^{j/6},$$

with

$$\mathcal{L}_0 = D_{\hat{\tau}}^2 + \left(-\eta_0 + \frac{1}{2}\hat{\tau}^2\right)^2,$$

$$\mathcal{L}_1 = -2D_{\hat{\sigma}}\left(-\eta_0 + \frac{1}{2}\hat{\tau}^2\right),$$

$$\mathcal{L}_2 = D_{\hat{\sigma}}^2 + \frac{2}{3}\alpha\hat{\sigma}^2\mathcal{L}_0 + L,$$

where $\alpha = \frac{1}{2}\delta_0^{-1}\delta''(0) > 0$ and:

$$L = 2\kappa(0)\delta(0)^{-4/3}\left(\frac{\hat{\tau}^2}{2} - \eta_0\right)\hat{\tau}^3 + 2\hat{\tau}\delta(0)^{-1/3}k(0)\left(-\eta_0 + \frac{\hat{\tau}^2}{2}\right)^2.$$

We look for quasi eigenpairs in the form:

$$\lambda \sim h^{4/3}\sum_{j\geq 0}\theta_j h^{j/6},$$

$$\psi \sim \sum_{j\geq 0}\psi_j h^{j/6}$$

so that, in the sense of formal power series:

$$\hat{\mathcal{L}}(h)\psi \sim \lambda\psi. \tag{7.8}$$

7.3.2. *Solving the formal system.* Considering (7.8), we are led to solve an infinite formal system of PDE's which we will solve thanks a compatibility condition known as the Fredholm alternative.

- *Term in h^0.* We solve the equation:

$$\mathcal{L}_0\psi_0 = \theta_0\psi_0.$$

This provides:

$$\theta_0 = \nu_1(\eta_0)$$

and

$$\psi_0(\hat{\sigma},\hat{\tau}) = g_0(\hat{\sigma})u_{\eta_0}(\hat{\tau}).$$

• *Term in* $h^{1/6}$. We solve the equation:

$$(\mathcal{L}_0-\theta_0)\psi_1=(\theta_1-\mathcal{L}_1)\psi_0.$$

Using the Feynman-Hellmann formulas, we have:

$$(\mathcal{L}_0-\theta_0)(\psi_1+D_{\hat{\sigma}}g_0(\hat{\sigma})v_{\eta_0}(\hat{\tau}))=\theta_1\psi_0.$$

The Fredholm alternative (the r. h. s. is orthogonal to u_{η_0} for each $\hat{\sigma}$) implies:

$$\theta_1=0$$

and:

$$\psi_1+D_{\hat{\sigma}}g_0(\hat{\sigma})v_{\eta_0}(\hat{\tau})=g_1(\hat{\sigma})u_{\eta_0}(\hat{\tau}),$$

where g_1 shall be determined in a next step.

• *Term in* $h^{2/6}$. We solve the equation:

$$(\mathcal{L}_0-\theta_0)\psi_2=(\theta_2-\mathcal{L}_2)\psi_0-\mathcal{L}_1\psi_1. \tag{7.9}$$

Using the Feynman-Hellmann formulas, this equation rewrites:

$$\begin{aligned}&(\mathcal{L}_0-\theta_0)\left(\psi_2+D_{\hat{\sigma}}g_1v_{\eta_0}-D^2_{\hat{\sigma}}g_0\frac{w_{\eta_0}}{2}\right)\\&=\left(\theta_2g_0-\frac{\nu''(\eta_0)}{2}D^2_{\hat{\sigma}}g_0-\frac{2}{3}\alpha\nu_1(\eta_0)\hat{\sigma}^2g_0-g_0L(\hat{\tau},\partial_{\hat{\tau}})\right)u_{\eta_0}.\end{aligned}$$

The Fredholm condition implies that, for all $\hat{\sigma}$:

$$(\mathcal{H}+C_0)g_0=\theta_2g_0,$$

where C_0 is defined in (7.4) and where $\mathcal{H}$ denotes the effective harmonic oscillator (we recall (7.3) and that $\nu_1''(\eta_0)>0$ by (3.5)):

$$\mathcal{H}=\frac{\nu''(\eta_0)}{2}D^2_{\hat{\sigma}}+\frac{2}{3}\alpha\hat{\sigma}^2. \tag{7.10}$$

If we denote by $(\mu_n)_{n\geq 1}$ the increasing sequence of the eigenvalues of $\mathcal{H}$, we have by scaling:

$$\mu_n=(2n-1)\left(\frac{\alpha\nu_1''(\eta_0)}{3}\right)^{1/2}.$$

Anyway we choose

$$\theta_2=\mu_n+C_0$$

and for g_0, we take $g_{(n)}$ a corresponding L^2-normalized eigenfunction. With theses choices, we determine a unique function $\psi_2^\perp$ which is solution of (7.9) and satisfying $\langle\psi_2^\perp,u_{\eta_0}\rangle_{\hat{\tau}}=0$ so that ψ_2 can be written as:

$$\psi_2=\psi_2^\perp-D_{\hat{\sigma}}g_1v_{\eta_0}+D^2_{\hat{\sigma}}g_0\frac{w_{\eta_0}}{2}+g_2(\hat{\sigma})u_{\eta_0}(\hat{\tau}),$$

where g_2 has to be determined in a next step.

• *Further terms ("Grushin procedure")*. We leave the next step to the reader.

7.4. **A rough estimate.** Thanks to the "IMS" formula and a partition of unity, we may prove the following proposition (exercise: use Lemma 2.9).

Proposition 7.4. *For all $n \geq 1$, there exist $h_0 > 0$ and $C > 0$ such that, for $h \in (0, h_0)$:*

$$\lambda_n(h) \geq \delta_0^{2/3}\nu_1(\eta_0)h^{4/3} - Ch^{4/3+2/15}.$$

7.5. **Agmon Estimates.** Two kinds of Agmon's estimates can be proved using the stand partition of unity arguments. We leave their proofs to the reader.

Proposition 7.5. *Let (λ, ψ) be an eigenpair of $\mathcal{L}_{h,\mathbf{A}}$. There exist $h_0 > 0$, $C > 0$ and $\varepsilon_0 > 0$ such that, for $h \in (0, h_0)$:*

$$\int e^{2\varepsilon_0|t(x)|h^{-1/3}}|\psi|^2\, dx \leq C\|\psi\|^2 \tag{7.11}$$

and:

$$Q_{h,\mathbf{A}}(e^{\varepsilon_0|t(x)|h^{-1/3}}\psi) \leq Ch^{4/3}\|\psi\|^2. \tag{7.12}$$

Proposition 7.6. *Let (λ, ψ) be an eigenpair of $\mathcal{L}_{h,\mathbf{A}}$. There exist $h_0 > 0$, $C > 0$ and $\varepsilon_0 > 0$ such that, for $h \in (0, h_0)$:*

$$\int e^{2\chi(t(x))|s(x)|h^{-1/15}}|\psi|^2\, dx \leq C\|\psi\|^2 \tag{7.13}$$

and:

$$Q_{h,\mathbf{A}}(e^{\chi(t(x))|s(x)|h^{-1/15}}\psi) \leq Ch^{4/3}\|\psi\|^2, \tag{7.14}$$

where χ is a fixed smooth cutoff function being 1 near 0.

• *Introduction of cutoff functions.* From Propositions 7.5 and 7.6, we are led to introduce a cutoff function living near x_0. We take $\gamma > 0$ and we let:

$$\chi_{h,\gamma}(x) = \chi\left(h^{-1/3+\gamma}t(x)\right)\chi\left(h^{-1/15+\gamma}s(x)\right).$$

where χ is a fixed smooth cutoff function supported near 0.

Notation 7.7. We will denote by $\check{\psi}$ the function $\chi_{h,\gamma}(x)\psi(x)$ in the coordinates (σ, τ).

7.6. **Refined Estimates.** From the normal estimates of Agmon, we deduce the proposition:

Proposition 7.8. *For all $n \geq 1$, there exist $h_0 > 0$ and $C > 0$ s. t., for $h \in (0, h_0)$:*

$$\lambda_n(h) \geq \delta_0^{2/3}\nu_1(\eta_0)h^{4/3} - Ch^{5/3}.$$

We provide the proof of this proposition to understand the main idea of the lower bound.

Proof. We consider an eigenpair $(\lambda_n(h), \psi_{n,h})$ and we use the IMS formula:

$$\check{Q}(\check{\psi}_{n,h}) = \lambda_n(h)\|\check{\psi}_{n,h}\|^2 + O(h^\infty)\|\check{\psi}_{n,h}\|^2.$$

We have (cf. (8.2)):

$$\begin{aligned}&\check{Q}(\check{\psi}_{n,h}) \geq \\ &\int \check{m}^{-2}\left|\left(-hD_\sigma-\eta_0\delta^{1/3}h^{2/3}+\check{A}-\frac{h}{6}\delta'\delta^{-1}(\tau D_\tau+D_\tau\tau)+C_h\right)\check{\psi}_{n,h}\right|^2\,d\sigma d\tau\\ &+h^2\delta_0^{2/3}\|D_\tau\check{\psi}_{n,h}\|^2-Ch^2\|\check{\psi}_{n,h}\|^2.\end{aligned}$$

Let us deal with the terms involving C_h in the double product produced by the expansion of the square. We have to estimate:

$$h\left|\Re\langle\check{m}^{-2}\delta'\delta^{-1}(\tau D_\tau+D_\tau\tau)\check{\psi}_{n,h},C_h\check{\psi}_{n,h}\rangle\right|$$

We have :

$$\|C_h\check{\psi}_{n,h}\|=o(h)\|\check{\psi}_{n,h}\|$$

and, with the estimates of Agmon (and the fact that 0 is a critical point of δ):

$$\|\delta'\delta^{-1}(\tau D_\tau+D_\tau\tau)\check{\psi}_{n,h}\|=o(1)\|\check{\psi}_{n,h}\|.$$

Moreover, we have in the same way:

$$h\left|\Re\langle\check{A}\check{\psi}_{n,h},C_h\check{\psi}_{n,h}\rangle\right|=o(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

Then, we have the control:

$$h\left|\Re\langle hD_\sigma\check{\psi}_{n,h},C_h\check{\psi}_{n,h}\rangle\right|=o(h^{5/3})\|\check{\psi}_{n,h}\|^2,$$

where we have used the rough estimate:

$$\|hD_\sigma\check{\psi}_{n,h}\|\leq Ch^{2/3}\|\check{\psi}_{n,h}\|.$$

We have:

$$\begin{aligned}&\check{Q}(\check{\psi}_{n,h}) \geq \\ &\int \check{m}^{-2}\left|\left(-hD_\sigma-\eta_0\delta^{1/3}h^{2/3}+\check{A}-\frac{h}{6}\delta'\delta^{-1}(\tau D_\tau+D_\tau\tau)\right)\check{\psi}_{n,h}\right|^2\,d\sigma d\tau\\ &+h^2\delta_0^{2/3}\|D_\tau\check{\psi}_{n,h}\|^2+o(h^{5/3})\|\check{\psi}_{n,h}\|^2.\end{aligned} \tag{7.15}$$

We now deal with the term involving $\tau D_\tau+D_\tau\tau$. With the estimates of Agmon, we have:

$$h\left|\Re\langle\check{m}^{-2}\delta'\delta^{-1}(\tau D_\tau+D_\tau\tau)\check{\psi}_{n,h},(-hD_\sigma-\eta_0\delta^{1/3}h^{2/3}+\check{A})\check{\psi}_{n,h}\rangle\right|=o(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

This implies:

$$\begin{aligned}\check{Q}(\check{\psi}_{n,h}) \geq&\delta_0^{2/3}h^2\|D_\tau\check{\psi}_{n,h}\|^2+\int\check{m}^{-2}\left|\left(-hD_\sigma-\eta_0\delta^{1/3}h^{2/3}+\check{A}\right)\check{\psi}_{n,h}\right|^2\,d\sigma d\tau\\ &+o(h^{5/3})\|\check{\psi}_{n,h}\|^2.\end{aligned}$$

With the same kind of arguments, it follows:

$$\begin{aligned}\check{Q}(\check{\psi}_{n,h}) \geq&h^2\delta_0^{2/3}\|D_\tau\check{\psi}_{n,h}\|^2+\int\check{m}^{-2}\left|\left(-hD_\sigma-\eta_0\delta^{1/3}h^{2/3}+\delta^{1/3}\frac{\tau^2}{2}\right)\check{\psi}_{n,h}\right|^2\,d\sigma d\tau\\ &+O(h^{5/3})\|\check{\psi}_{n,h}\|^2\end{aligned} \tag{7.16}$$

and

$$\check{Q}(\check{\psi}_{n,h}) \geq h^2\delta_0^{2/3}\|D_\tau\check{\psi}_{n,h}\|^2 + \int \left|\left(-hD_\sigma - \eta_0\delta^{1/3}h^{2/3} + \delta^{1/3}\frac{\tau^2}{2}\right)\check{\psi}_{n,h}\right|^2 d\sigma d\tau + O(h^{5/3})\|\check{\psi}_{n,h}\|^2. \tag{7.17}$$

We get:

$$\check{Q}(\check{\psi}_{n,h}) \geq h^2\delta_0^{2/3}\|D_\tau\check{\psi}_{n,h}\|^2 + \int \delta_0^{2/3}\left|\left(-h\delta^{-1/3}D_\sigma - \eta_0 h^{2/3} + \frac{\tau^2}{2}\right)\check{\psi}_{n,h}\right|^2 d\sigma d\tau + O(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

Then, we write:

$$\delta^{-1/3}D_\sigma = \delta^{-1/6}D_\sigma\delta^{-1/6} + i\delta^{-1/6}(\delta^{-1/6})'$$

and deduce (by estimating the double product involved by $i\delta^{-1/6}(\delta^{-1/6})'$):

$$\check{Q}(\check{\psi}_{n,h}) \geq h^2\delta_0^{2/3}\|D_\tau\check{\psi}_{n,h}\|^2 + \int \delta_0^{2/3}\left|\left(-h\delta^{-1/6}D_\sigma\delta^{-1/6} - \eta_0 h^{2/3} + \frac{\tau^2}{2}\right)\check{\psi}_{n,h}\right|^2 d\sigma d\tau + o(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

We can apply the functional calculus to the self-adjoint operator $\delta^{-1/6}D_\sigma\delta^{-1/6}$ and the following lower bound follows:

$$\check{Q}(\check{\psi}_{n,h}) \geq h^{4/3}\delta_0^{2/3}\nu_1(\eta_0) + O(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

□

Exercise. Let δ be a smooth and bounded (so as its derivatives) and positive function on $\mathbb{R}$. Find a unitary transform which diagonalizes the self-adjoint realization of $\delta D_\sigma\delta$ on $L^2(\mathbb{R}, d\sigma)$. Notice that such a transform exists by the spectral theorem.

• *Introduction of the space generated by the truncated eigenfunctions.* For all $N \geq 1$, let us consider L^2-normalized eigenpairs $(\lambda_n(h), \psi_{n,h})_{1\leq n\leq N}$ such that $\langle\psi_{n,h}, \psi_{m,h}\rangle = 0$ if $n \neq m$. We consider the N dimensional space defined by:

$$\mathfrak{E}_N(h) = \underset{1\leq n\leq N}{\operatorname{span}}\, \check{\psi}_{n,h}.$$

The next two propositions provide control with respect to σ and D_σ. We leave the proof to the reader and refer to [31] and also to the spirit of the proof of Proposition 7.8.

Proposition 7.9. *There exist $h_0 > 0$, $C > 0$ such that, for $h \in (0, h_0)$ and for all $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\|\sigma\check{\psi}\| \leq Ch^{1/6}\|\check{\psi}\|.$$

Proposition 7.10. *There exist $h_0 > 0$, $C > 0$ such that, for $h \in (0, h_0)$ and for all $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\|D_\sigma\check{\psi}\| \leq Ch^{-1/6}\|\check{\psi}\|.$$

With Proposition 7.9, we have a better lower bound for the quadratic form.

Proposition 7.11. *There exists $h_0 > 0$ such that for $h \in (0, h_0)$ and $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\check{Q}(\check{\psi}) \geq \delta_0^{2/3} \int (1 + 2k_0 \tau \delta_0^{-1/3}) |(\delta^{-1/6} ih\partial_\sigma \delta^{-1/6} + \eta_0 h^{2/3} + \frac{\tau^2}{2} + \delta_0^{-4/3}\kappa(0)\tau^3)\check{\psi}|^2 \, d\sigma d\tau$$
$$+ \int \delta_0^{2/3} |hD_\tau \check{\psi}|^2 \, d\sigma d\tau + \frac{2}{3}\delta_0^{2/3}\alpha\nu_1(\eta_0)h^{4/3}\|\sigma\check{\psi}\|^2 + o(h^{5/3})\|\check{\psi}\|^2.$$

7.7. **Projection Method.** We can now prove an approximation result for the eigenfunctions. Let us recall the rescaled coordinates (see (8.3)):

$$\sigma = h^{1/6}\hat{\sigma}, \quad \tau = h^{1/3}\hat{\tau}. \tag{7.18}$$

Notation 7.12. $\hat{\mathcal{L}}(h)$ denotes $h^{-4/3}\check{\mathcal{L}}(h)$ in the coordinates $(\hat{\sigma}, \hat{\tau})$. The corresponding quadratic form will be denoted by $\hat{Q}$. We will use the notation $\hat{\mathfrak{E}}_N(h)$ to denote $\mathfrak{E}_N(h)$ after rescaling.

We introduce the Feshbach-Grushin projection:

$$\Pi_0\phi = \langle \phi, u_{\eta_0} \rangle_{\hat{\tau}} u_{\eta_0}(\hat{\tau}).$$

We will need to consider the quadratic form:

$$\hat{Q}_0(\phi) = \delta_0^{2/3} \int |D_{\hat{\tau}}\phi|^2 + \left| \left(-\eta_0 + \frac{\hat{\tau}^2}{2} \right) \phi \right|^2 \, d\hat{\sigma} d\hat{\tau}.$$

The fundamental approximation result is given in the following proposition.

Proposition 7.13. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and $\hat{\psi} \in \hat{\mathfrak{E}}_N(h)$:*

$$0 \leq \hat{Q}_0(\hat{\psi}) - \delta_0^{2/3}\nu_1(\eta_0)\|\hat{\psi}\|^2 \leq Ch^{1/6}\|\hat{\psi}\|^2 \tag{7.19}$$

and:

$$\begin{aligned} &\|\Pi_0\hat{\psi} - \hat{\psi}\| \leq Ch^{1/12}\|\hat{\psi}\| \\ &\|D_{\hat{\tau}}(\Pi_0\hat{\psi} - \hat{\psi})\| \leq Ch^{1/12}\|\hat{\psi}\|, \\ &\|\hat{\tau}^2(\Pi_0\hat{\psi} - \hat{\psi})\| \leq Ch^{1/12}\|\hat{\psi}\|. \end{aligned} \tag{7.20}$$

This permits to simplify the lower bound (see (7.4)).

Proposition 7.14. *There exist $h_0 > 0$, $C > 0$ such that, for $h \in (0, h_0)$ and $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\check{Q}(\check{\psi}) \geq \int \delta_0^{2/3} \left(|hD_\tau\check{\psi}|^2 + |(\delta^{-1/6} ih\partial_\sigma \delta^{-1/6} - \eta_0 h^{2/3} + \frac{\tau^2}{2})\check{\psi}|^2 \right) d\sigma d\tau$$
$$+ \frac{2}{3}\delta_0^{2/3}\alpha\nu_1(\eta_0)h^{4/3}\|\sigma\check{\psi}\|^2 + C_0 h^{5/3}\|\check{\psi}\|^2 + o(h^{5/3})\|\check{\psi}\|^2.$$

It remains to diagonalize $\delta^{-1/6} i\partial_\sigma \delta^{-1/6}$:

Corollary 7.15. *There exist* $h_0 > 0$, $C > 0$ *such that, for* $h \in (0, h_0)$ *and* $\check{\psi} \in \mathfrak{E}_N(h)$*:*

$$\check{Q}(\check{\psi}) \geq \int \delta_0^{2/3} \left(|hD_\tau \check{\phi}|^2 + |(-h\mu - \eta_0 h^{2/3} + \frac{\tau^2}{2})\check{\phi}|^2 \right) d\mu d\tau$$
$$+ \frac{2}{3}\delta_0^{2/3} \alpha \nu_1(\eta_0) h^{4/3} \|D_\mu \check{\phi}\|^2 + C_0 h^{5/3} \|\check{\phi}\|^2 + o(h^{5/3})\|\check{\phi}\|^2,$$

with $\check{\phi} = \mathcal{F}_\delta \check{\psi}$.

Let us introduce the operator on $L^2(\mathbb{R}^2, d\mu d\tau)$:

$$\frac{2}{3}\delta_0^{2/3} \alpha \nu_1(\eta_0) h^{4/3} D_\mu^2 + \delta_0^{2/3} \left(h^2 D_\tau^2 + \left(-h\mu - \eta_0 h^{2/3} + \frac{\tau^2}{2} \right)^2 \right) + C_0 h^{5/3}. \tag{7.21}$$

Exercise. Determine the asymptotic expansion of the lowest eigenvalues of this operator thanks to the Born-Oppenheimer theory and prove Theorem 7.1.

8. FROM THE MAGNETIC LAPLACIAN TO THE ELECTRIC LAPLACIAN: A NON REGULAR CASE IN 3D

8.1. **Motivation.** In this section we investigate the Neumann realization of the magnetic Laplacian $\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla + \mathbf{A})^2$ on Ω when Ω has the shape of a symmetric lens (with edge E, see Figures 5 and 6) and when the magnetic field is perpendicular to the symmetry plane of the sample. This model is a non smooth version of the paper of Helffer and Morame [61] where they apply their analysis to an ellipsoid. This is also somehow a generalization of the work of V. Bonnaillie-Noël in dimension 3.

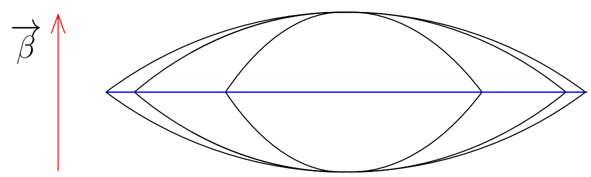


FIGURE 5. Lens with constant aperture in constant magnetic field.

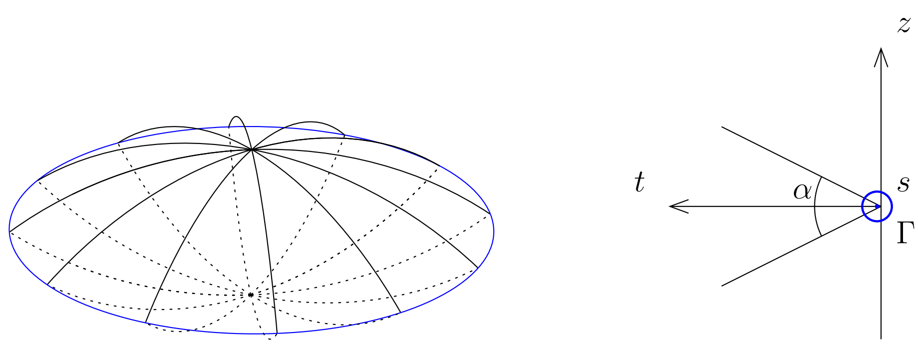


FIGURE 6. Lens with aperture α.

In this section we will assume that opening angle of the lens is variable.

As usual, one will be led to compare different models operators: in the interior of Ω, on the smooth boundary and on E. To catch the phenomenon due to the presence of the edge, we will make the following assumption. In particular, this will involve the reduction to a problem near the non smooth boundary.

Assumption 8.1.

$$\inf_{\mathbf{x}\in E} \nu(\alpha(\mathbf{x})) < \inf_{\mathbf{x}\in\partial\Omega\setminus E} \sigma(\theta(\mathbf{x})). \tag{8.1}$$

8.2. **Normal Form.** This is standard that the condition (8.1) leads to localization properties of the eigenfunctions near the edge E and more precisely near the points of the edge where $E \ni \mathbf{x} \mapsto \nu(\alpha(\mathbf{x}))$ is minimal (that is where α is maximal). We can introduce, near each $\mathbf{x}_0 \in E$, a local change of variables which transforms a neighborhood of $\mathbf{x}_0$ in Ω in a ε_0-neighborhood of $(0,0,0)$ of $\mathcal{E}_{\alpha(\mathbf{x}_0)}$, denoted by $\mathcal{E}_{\alpha(\mathbf{x}_0),\varepsilon_0}$. For the convenience of the reader, let us describe below the shape of the magnetic Laplacian in the new (local) coordinates $(\check{s},\check{t},\check{z})$. The magnetic Laplacian $\mathcal{L}_h$ is given by the Laplace-Beltrami expression (on $L^2(|\check{G}|^{1/2}d\check{s}d\check{t}d\check{z})$):

$$|\check{G}|^{-1/2}\check{\nabla}_h|\check{G}|^{1/2}\check{G}^{-1}\check{\nabla}_h \tag{8.2}$$

with boundary conditions:

$$|\check{G}|^{1/2}\check{G}^{-1}\check{\nabla}_h\check{\psi}\cdot\begin{pmatrix}-\tau'(\check{s})\check{t}\\ -\tau(\check{s})\\ \pm 1\end{pmatrix}=0 \quad \text{on } \partial_{\mathsf{Neu}}\mathcal{E}_{\alpha(\mathbf{x}_0),\varepsilon_0}$$

$$\check{\psi}=0 \quad \text{on } \partial_{\mathsf{Dir}}\mathcal{E}_{\alpha(\mathbf{x}_0),\varepsilon_0}$$

and where:

$$\check{\nabla}_h=\begin{pmatrix}hD_{\check{s}}\\ hD_{\check{t}}\\ hD_{\check{z}}\end{pmatrix}+\begin{pmatrix}-\check{t}-h\frac{\tau'}{2\tau}(\check{z}D_{\check{z}}+D_{\check{z}}\check{z})+\check{R}_1(\check{s},\check{t},\check{z})\\ 0\\ 0\end{pmatrix}.$$

We refer to [83] where the forms of the Taylor expansions of $\check{R}_1$ and $\check{G}^{-1}$ are analysed. Let us just mention that $\check{s}$ is the curvilinear coordinate along E and

$$\tau(\check{s})=\tan\left(\frac{\alpha(\check{s})}{2}\right),$$

where $\check{s}\mapsto\alpha(\check{s})$ is the variable opening angle along the edge.

Assumption 8.2. $\alpha : E \to (0,\pi)$ admits a unique and non-degenerate maximum at $\mathbf{x}_0$ denoted by α_0.

Notation 8.3. In order to shorten the notation we will denote by η_0 the number $\eta(\alpha_0)$ and by u_{η_0} the function $u_{\alpha_0,\eta(\alpha_0)}$. See Notation 3.5.

Theorem 8.4. *We assume Assumptions 3.8, 8.1 and 8.2. For all $n \geq 1$ there exist a sequence $(\mu_{j,n})_{j\geq 0}$ such that:*

$$\lambda_n(h) \underset{h\to 0}{\sim} h\sum_{j\geq 0}\mu_{j,n}h^{j/4}.$$

Moreover, we have:

$$\mu_{0,n}=\nu(\alpha_0,\eta_0),\quad \mu_{1,n}=0,\quad \mu_{2,n}=C_0+(2n-1)\sqrt{\kappa\tau_0^{-1}\|D_{\hat{z}}u_{\eta_0}\|^2\partial_\eta^2\nu(\alpha_0,\eta_0)},$$

where C_0 is a constant independent from n.

8.3. **Quasimodes.** Before starting the analysis, we use the following scaling:

$$\check{s} = h^{1/4}\hat{s}, \quad \check{t} = h^{1/2}\hat{t}, \quad \check{z} = h^{1/2}\hat{z} \tag{8.3}$$

so that we denote by $\hat{\mathcal{L}}(h)$ and $\hat{\mathcal{T}}(h)$ the operators $h^{-1}\mathcal{L}^{\mathsf{Normal}}(h)$ and $h^{-1/2}\mathcal{T}^{\mathsf{Normal}}(h)$ in the coordinates $(\hat{s}, \hat{t}, \hat{z})$. We can write in the sense of formal power series:

$$\hat{\mathcal{L}}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \mathcal{L}_j h^{j/4}$$

and

$$\hat{\mathcal{T}}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \mathcal{T}_j h^{j/4},$$

where the first $\mathcal{L}_j$ and $\mathcal{T}_j$ are given by:

$$\mathcal{L}_0 = D_{\hat{t}}^2 + D_{\hat{z}}^2 + (\hat{t} - \eta_0)^2, \tag{8.4}$$

$$\mathcal{L}_1 = -2(\hat{t} - \eta_0)D_{\hat{s}}, \tag{8.5}$$

$$\mathcal{L}_2 = D_{\hat{s}}^2 + 2\kappa\tau_0^{-1}\hat{s}^2 D_{\hat{z}}^2 + L_2, \tag{8.6}$$

where

$$\hat{P} = \begin{pmatrix} \eta_0 - \hat{t} \\ D_{\hat{t}} \\ D_{\hat{z}} \end{pmatrix}, \quad L_2 = 2(\eta_0 - \hat{t})\hat{r}_1 - \frac{\hat{l}}{2}\hat{P}\hat{P} + \hat{P}\frac{\hat{l}}{2}\hat{P} + \hat{P}\hat{L}\hat{P}. \tag{8.7}$$

and:

$$\mathcal{T}_0 = (-\hat{t} + \eta_0, D_{\hat{t}}, D_{\hat{z}}),$$
$$\mathcal{T}_1 = (D_{\hat{s}}, 0, 0),$$
$$\mathcal{T}_2 = (0, 0, \kappa\tau_0^{-1}\hat{s}^2 D_{\hat{z}}) + \frac{\hat{l}}{2}\hat{P} + \hat{L}\hat{P},$$

where $\kappa = -\frac{\tau''(0)}{2} > 0$. We have used the notation

$$\hat{r}_1(\hat{t}, \hat{z}) = h^{-1}\check{r}_1(h^{1/2}\hat{t}, h^{1/2}\hat{z}), \tag{8.8}$$

$$\hat{l}(\hat{t}, \hat{z}) = h^{-1/2}\check{l}(h^{-1/2}\hat{t}, h^{-1/2}\hat{z}), \tag{8.9}$$

$$\hat{L}(\hat{t}, \hat{z}) = h^{-1/2}\check{L}(h^{-1/2}\hat{t}, h^{-1/2}\hat{z}). \tag{8.10}$$

We will also use an asymptotic expansion of the normal $\hat{\boldsymbol{n}}$:

$$\hat{\boldsymbol{n}} \underset{h\to 0}{\sim} \sum_{j\geq 0} \boldsymbol{n}_j h^{j/4},$$

with:

$$\boldsymbol{n}_0 = (0, -\tau_0, \pm 1), \quad \boldsymbol{n}_1 = (0, 0, 0), \quad \boldsymbol{n}_2 = (0, \kappa\hat{s}^2, 0). \tag{8.11}$$

We look for $(\hat{\lambda}(h), \hat{\psi}(h))$ in the form:

$$\hat{\lambda}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_j h^{j/4},$$

$$\hat{\psi}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \psi_j h^{j/4},$$

which satisfies, in the sense of formal series:

$$\hat{\mathcal{L}}(h)\hat{\psi}(h) \underset{h\to 0}{\sim} \hat{\lambda}(h)\hat{\psi}(h).$$

This provides an infinite system of PDE's.

- *Terms in* h^0. We solve the equation:

$$\mathcal{L}_0\psi_0 = \lambda_0\psi_0, \quad \boldsymbol{n}_0 \cdot \mathcal{T}_0\psi_0 = 0, \text{ on } \partial_{\mathsf{Neu}}\mathcal{E}_{\alpha_0}.$$

We notice that the boundary condition is exactly the Neumann condition. We are led to choose $\lambda_0 = \nu(\alpha_0, \eta_0)$ and $\psi_0(\hat{s}, \hat{t}, \hat{z}) = u_{\eta_0}(\hat{t}, \hat{z}) f_0(\hat{s})$.

- *Terms in* $h^{1/4}$. Collecting the terms of size $h^{1/4}$, we find the equation:

$$(\mathcal{L}_0 - \lambda_0)\psi_1 = (\lambda_1 - \mathcal{L}_1)\psi_0, \quad \boldsymbol{n}_0 \cdot \mathcal{T}_0\psi_0 = 0, \text{ on } \partial_{\mathsf{Neu}}\mathcal{E}_{\alpha_0}.$$

As in the previous step, the boundary condition is just the Neumann condition. We deduce with the Feynman-Hellmann formulas:

$$(\mathcal{L}_0 - \lambda_0)(\psi_1 + v_{\alpha,\eta_0}(\hat{t}, \hat{z})D_{\hat{s}}f_0(\hat{s})) = \lambda_1\psi_0, \quad \boldsymbol{n}_0 \cdot \mathcal{T}_0\psi_1 = 0, \text{ on } \partial_{\mathsf{Neu}}\mathcal{E}_{\alpha_0}.$$

Taking the scalar product of the r.h.s. of the first equation with u_{α,η_0} with respect to $(\hat{t}, \hat{z})$ we find: $\lambda_1 = 0$. This leads to choose:

$$\psi_1(\hat{s}, \hat{t}, \hat{z}) = -v_{\alpha,\eta_0}(\hat{t}, \hat{z})D_{\hat{s}}f_0(\hat{s}) + f_1(\hat{s})u_{\alpha,\eta_0}(\hat{t}, \hat{z}),$$

where f_1 will be determined in a next step.

- *Terms in* $h^{1/2}$. Let us now deal with the terms of order $h^{1/2}$:

$$(\mathcal{L}_0 - \lambda_0)\psi_2 = (\lambda_2 - \mathcal{L}_2)\psi_0 - \mathcal{L}_1\psi_1, \quad \boldsymbol{n}_0 \cdot \mathcal{T}_0\psi_2 = -\boldsymbol{n}_0 \cdot \mathcal{T}_2\psi_0 - \boldsymbol{n}_2 \cdot \mathcal{T}_0\psi_0, \text{ on } \partial_{\mathsf{Neu}}\mathcal{E}_{\alpha_0}.$$

We analyze the boundary condition:

$$\begin{aligned}
\boldsymbol{n}_0 \cdot \mathcal{T}_2\psi_0 + \boldsymbol{n}_2 \cdot \mathcal{T}_0\psi_0 &= \pm\kappa\tau_0^{-1}\hat{s}^2 D_{\hat{z}}\psi_0 + \kappa\hat{s}^2 D_{\hat{t}}\psi_0 + \boldsymbol{n}_0 \cdot \frac{\hat{l}}{2}\hat{P}\psi_0 + \boldsymbol{n}_0 \cdot \hat{L}\hat{P}\psi_0 \\
&= \kappa\hat{s}^2\tau_0^{-1}(\pm D_{\hat{z}} + \tau_0 D_{\hat{t}})\psi_0 + \boldsymbol{n}_0 \cdot \frac{\hat{l}}{2}\hat{P}\psi_0 + \boldsymbol{n}_0 \cdot \hat{L}\hat{P}\psi_0 \\
&= \pm 2\kappa\hat{s}^2\tau_0^{-1}D_{\hat{z}}\psi_0 + \boldsymbol{n}_0 \cdot \frac{\hat{l}}{2}\hat{P}\psi_0 + \boldsymbol{n}_0 \cdot \hat{L}\hat{P}\psi_0.
\end{aligned}$$

Then, we use the Feynman-Hellmann formulas to get:
(8.12)

$$(\mathcal{L}_0 - \lambda_0)(\psi_2 + v_{\alpha,\eta_0}D_{\hat{s}}f_1 - w_{\alpha,\eta_0}D_{\hat{s}}^2 f_0) = \lambda_2\psi_0 - \frac{\partial_\eta^2\nu(\alpha_0, \eta_0)}{2}D_{\hat{s}}^2\psi_0 - 2\kappa\tau_0^{-1}\hat{s}^2 D_{\hat{z}}^2\psi_0 - L_2\psi_0,$$

with boundary condition:

$$\boldsymbol{n}_0 \cdot \mathcal{T}_0\psi_2 = \mp 2\kappa\hat{s}^2\tau_0^{-1}D_{\hat{z}}\psi_0 - \boldsymbol{n}_0 \cdot \frac{\hat{l}}{2}\hat{P}\psi_0 - \boldsymbol{n}_0 \cdot \hat{L}\hat{P}\psi_0, \text{ on } \partial S_0.$$

We use the Fredholm condition by taking the scalar product of the r.h.s. of (8.12) with u_{α_0,η_0} with respect to $(\hat{t},\hat{z})$. Integrating by parts (the boundary terms cancel), this provides the equation:

$$\mathcal{H}f_0 = (\lambda_2 - C_0)f_0,$$

with:

$$\mathcal{H} = \frac{\partial_\eta^2\nu(\alpha_0,\eta_0)}{2}D_{\hat{s}}^2 + 2\kappa\tau_0^{-1}\|D_{\hat{z}}u_{\alpha_0,\eta_0}\|^2\hat{s}^2$$

and:

$$C_0 = \langle 2(\eta_0-\hat{t})\hat{r}_1 u_{\eta_0}, u_{\eta_0}\rangle - \nu(\eta_0)\int\frac{\hat{l}}{2}u_{\eta_0}^2 + \int\frac{\hat{l}}{2}\hat{P}u_{\eta_0}\hat{P}u_{\eta_0} + \int\hat{L}\hat{P}u_{\eta_0}\hat{P}u_{\eta_0}. \tag{8.13}$$

Therefore for λ_2 we take:

$$\lambda_2 = C_0 + (2n-1)\sqrt{\kappa\tau_0^{-1}c_0\partial_\eta^2\nu(\alpha_0,\eta_0)}$$

and for f_0 the corresponding eigenfunction. With this choice we deduce the existence of $\psi_2^\perp$ such that:

$$(\mathcal{L}_0-\lambda_0)\psi_2^\perp = \lambda_2\psi_0 - \frac{\partial_\eta^2\nu(\alpha_0,\eta_0)}{2}D_{\hat{s}}^2\psi_0 - 2\kappa\tau_0^{-1}\hat{s}^2D_{\hat{z}}^2\psi_0, \text{ and } \langle\psi_2^\perp, u_{\alpha_0,\eta_0}\rangle_{\hat{t},\hat{z}} = 0. \tag{8.14}$$

We can write ψ_2 in the form:

$$\psi_2 = \psi_2^\perp - v_{\alpha,\eta_0}D_{\hat{s}}f_1 + w_{\alpha,\eta_0}D_{\hat{s}}^2f_0 + f_2(\hat{s})u_{\alpha_0,\eta_0},$$

where f_2 has to be determined in a next step.

- *Further terms.* The construction can be continued (exercise).

8.4. **Agmon Estimates.** Thanks to a standard partition of unity, we can establish the following estimate for the eigenvalues.

Proposition 8.5. *There exist C and $h_0>0$ such that, for $h\in(0,h_0)$:*

$$\lambda_n(h) \geq \nu(\alpha_0)h - Ch^{5/4}.$$

From Proposition 8.5, we infer a localization near E.

Proposition 8.6. *There exist $\varepsilon_0>0, h_0>0$ and $C>0$ such that for all $h\in(0,h_0)$:*

$$\int e^{2\varepsilon_0 h^{-1/2}d(\mathbf{x},E)}|\psi|^2\,d\mathbf{x} \leq C\|\psi\|^2,$$

$$Q_h(e^{\varepsilon_0 h^{-1/2}d(\mathbf{x},E)}\psi) \leq Ch\|\psi\|^2.$$

As a consequence, we get:

Proposition 8.7. *For all $n\geq 1$, there exists $h_0>0$ such that for $h\in(0,h_0)$, we have:*

$$\lambda_n(h) = \nu(\alpha_0,\eta_0)h + O(h^{3/2}).$$

Proof. We have:

$$\check{Q}_h(\check{\psi}) = \langle \check{G}^{-1}\check{\nabla}_h\check{\psi}, \check{\nabla}_h\check{\psi}\rangle_{L^2(d\check{s}d\check{t}d\check{z})}.$$

With the Taylor expansion of $\check{G}^{-1}$ and $|\check{G}|$ and the estimates of Agmon with respect to $\check{t}$ and $\check{z}$, we infer:

$$\check{Q}_h(\check{\psi}) \geq Q^{\mathsf{flat},h}(\check{\psi}) - Ch^{3/2}\|\check{\psi}\|^2.$$

where:

$$\check{Q}_h^{\mathsf{flat}}(\check{\psi}) = \|hD_{\check{t}}\check{\psi}\|^2 + \|h\tau_0\tau(\check{s})^{-1}D_{\check{z}}\check{\psi}\|^2 + \|(hD_{\check{s}} + \eta_0 h^{1/2} - \check{t})\check{\psi}\|^2.$$

Moreover, we have:

$$\check{Q}_h^{\mathsf{flat}}(\check{\psi}) \geq \|hD_{\check{t}}\check{\psi}\|^2 + \|hD_{\check{z}}\check{\psi}\|^2 + \|(hD_{\check{s}} + \eta_0 h^{1/2} - \check{t})\check{\psi}\|^2 \geq \nu(\alpha_0, \eta_0)h.$$

□

A rough localization estimate is given by the following proposition.

Proposition 8.8. *There exist* $\varepsilon_0 > 0, h_0 > 0$ *and* $C > 0$ *such that for all* $h \in (0, h_0)$*:*

$$\int e^{\chi(\mathbf{x})h^{-1/8}|s(\mathbf{x})|}|\psi|^2\, d\mathbf{x} \leq C\|\psi\|^2,$$
$$Q_h(e^{\chi(\mathbf{x})h^{-1/8}|s(\mathbf{x})|}\psi) \leq Ch\|\psi\|^2,$$

where χ *is a smooth cutoff function supported in a fixed neighborhood of* E.

We use a cutoff function $\chi_h(\mathbf{x})$ near $\mathbf{x}_0$ such that:

$$\chi_h(\mathbf{x}) = \chi_0(h^{1/8-\gamma}\check{s}(\mathbf{x}))\chi_0(h^{1/2-\gamma}\check{t}(\mathbf{x}))\chi_0(h^{1/2-\gamma}\check{z}(\mathbf{x})).$$

• *Space of the eigenfunctions.* For all $N \geq 1$, let us consider L^2-normalized eigenpairs $(\lambda_n(h), \psi_{n,h})_{1\leq n\leq N}$ such that $\langle\psi_{n,h}, \psi_{m,h}\rangle = 0$ when $n \neq m$. We consider the N dimensional space defined by:

$$\mathfrak{E}_N(h) = \underset{1\leq n\leq N}{\operatorname{span}}\, \tilde{\psi}_{n,h}, \quad \text{where} \quad \tilde{\psi}_{n,h} = \chi_h\psi_{n,h}.$$

Notation 8.9. We will denote by $\tilde{\psi}(=\chi_h\psi)$ the elements of $\mathfrak{E}_N(h)$.

8.5. **Refined Estimates.** Let us state a proposition providing the localization of the eigenfunctions with respect to $D_{\check{s}}$ (the proof is left to the reader as an exercise).

Proposition 8.10. *There exist* $h_0 > 0$ *and* $C > 0$ *such that, for* $h \in (0, h_0)$ *and* $\check{\psi} \in \check{\mathfrak{E}}_N(h)$*, we have:*

$$\|D_{\check{s}}\check{\psi}\| \leq Ch^{-1/4}\|\check{\psi}\|.$$

8.6. **Projection Method.** The result of Proposition 8.10 implies an approximation result for the eigenfunctions. Let us recall the scaling defined in (8.3):

$$\check{s} = h^{1/4}\hat{s}, \quad \check{t} = h^{1/2}\hat{t}, \quad \check{z} = h^{1/2}\hat{z}. \tag{8.15}$$

Notation 8.11. We will denote by $\hat{\mathfrak{E}}_N(h)$ the set of the rescaled elements of $\check{\mathfrak{E}}_N(h)$. The elements of $\hat{\mathfrak{E}}_N(h)$ will be denoted by $\hat{\psi}$. Moreover we will denote by $\hat{\mathcal{L}}_h$ the operator $h^{-1}\check{\mathcal{L}}_h$ in the rescaled coordinates. The corresponding quadratic form will be denoted by $\hat{Q}_h$.

Lemma 8.12. *There exist $h_0 > 0$ and $C > 0$ such that, for $h \in (0, h_0)$ and $\hat{\psi} \in \hat{\mathfrak{E}}_N(h)$, we have:*

$$\|\hat{\psi} - \Pi_0\hat{\psi}\| + \|D_{\hat{t}}(\hat{\psi} - \Pi_0\hat{\psi})\| + \|D_{\hat{z}}(\hat{\psi} - \Pi_0\hat{\psi})\| \leq Ch^{1/8}\|\hat{\psi}\| \tag{8.16}$$

$$\|\hat{s}(\hat{\psi} - \Pi_0\hat{\psi})\| + \|\hat{s}D_{\hat{t}}(\hat{\psi} - \Pi_0\hat{\psi})\| + \|\hat{s}D_{\hat{z}}(\hat{\psi} - \Pi_0\hat{\psi})\| \leq Ch^{1/8-\gamma}(\|\hat{\psi}\| + (\|\hat{s}\hat{\psi}\|), \tag{8.17}$$

where Π_0 is the projection on u_{η_0}:

$$\Pi_0\hat{\psi} = \langle \hat{\psi}, u_{\eta_0}\rangle_{\hat{t},\hat{z}} u_{\eta_0}.$$

This approximation result allows us to catch the behavior of the eigenfunction with respect to $\check{s}$. In fact, this is the core of the dimension reduction process of the next proposition. Indeed $\hat{s}^2 D_{\hat{z}}^2$ is not an elliptic operator, but, once projected on u_{η_0}, it becomes elliptic.

Proposition 8.13. *There exist $h_0 > 0$ and $C > 0$ such that, for $h \in (0, h_0)$ and $\check{\psi} \in \check{\mathfrak{E}}_N(h)$, we have:*

$$\|\check{s}\check{\psi}\| \leq Ch^{1/4}\|\check{\psi}\|.$$

Proof. It is equivalent to prove that:

$$\|\hat{s}\hat{\psi}\| \leq C\|\hat{\psi}\|.$$

The proof of Proposition 8.7 provides the inequality:

$$\|D_{\hat{t}}\hat{\psi}\|^2 + \|\tau_0\tau(h^{1/4}\hat{s})^{-1}D_{\hat{z}}\hat{\psi}\|^2 + \|(h^{1/4}D_{\hat{s}} + \eta_0 - \hat{t})\hat{\psi}\|^2 \leq (\nu(\eta_0) + Ch^{1/2})\|\hat{\psi}\|^2.$$

From the non-degeneracy of the maximum of α, we deduce the existence of $c > 0$ such that:

$$\|\tau_0\tau(h^{1/4}\hat{s})^{-1}D_{\hat{z}}\hat{\psi}\|^2 \geq \|D_{\hat{z}}\hat{\psi}\|^2 + ch^{1/2}\|\hat{s}D_{\hat{z}}\hat{\psi}\|^2$$

so that we have:

$$ch^{1/2}\|\hat{s}D_{\hat{z}}\hat{\psi}\|^2 \leq Ch^{1/2}\|\hat{\psi}\|^2$$

and:

$$\|\hat{s}D_{\hat{z}}\hat{\psi}\| \leq \tilde{C}\|\hat{\psi}\|.$$

It remains to use Lemma 8.12 and especially (8.17). In particular, we have:

$$\|\hat{s}D_{\hat{z}}(\hat{\psi} - \Pi_0\hat{\psi})\| \leq Ch^{1/8-\gamma}(\|\hat{\psi}\| + (\|\hat{s}\hat{\psi}\|).$$

We infer:

$$\|\hat{s}D_{\hat{z}}\Pi_0\hat{\psi}\| \leq \tilde{C}\|\hat{\psi}\| + Ch^{1/8-\gamma}(\|\hat{\psi}\| + (\|\hat{s}\hat{\psi}\|).$$

Let us write

$$\Pi_0\hat{\psi} = f_h(\hat{s})u_{\eta_0}(\hat{t}, \hat{z}).$$

We have:

$$\|\hat{s}D_{\hat{z}}\Pi_0\hat{\psi}\| = \|D_{\hat{z}}u_{\eta_0}\|\|\hat{s}f_h\|_{L^2(d\hat{s})} = \|D_{\hat{z}}u_{\eta_0}\|\|\hat{s}f_h u_{\eta_0}\| = \|D_{\hat{z}}u_{\eta_0}\|\|\hat{s}\Pi_0\hat{\psi}\|.$$

We use again Lemma 8.12 to get:

$$\|\hat{s}D_{\hat{z}}\Pi_0\hat{\psi}\| = \|D_{\hat{z}}u_{\eta_0}\|\|\hat{s}\hat{\psi}\| + O(h^{1/8-\gamma})(\|\hat{\psi}\| + \|\hat{s}\hat{\psi}\|).$$

We deduce:

$$\|D_{\hat{z}}u_{\eta_0}\|\|\hat{s}\hat{\psi}\| \leq \tilde{C}\|\hat{\psi}\| + 2Ch^{1/8-\gamma}(\|\hat{\psi}\| + (\|\hat{s}\hat{\psi}\|)$$

and the conclusion follows. □

Proposition 8.14. *There exists $h_0 > 0$ such that for $h \in (0, h_0)$ and $\hat{\psi} \in \hat{\mathfrak{E}}_N(h)$, we have:*

$$\hat{Q}_h(\hat{\psi}) \geq \|D_{\hat{t}}\hat{\psi}\|^2 + \|D_{\hat{z}}\hat{\psi}\|^2 + \|(h^{1/4}D_{\hat{s}} - \hat{t} + \eta_0)\hat{\psi}\|^2 + h^{1/2}\tau_0^{-1}\kappa\|D_{\hat{z}}u_{\eta_0}\|^2\hat{s}^2 + \tilde{C}_0 h^{1/2}\|\hat{\psi}\|^2 + o(h^{1/2})\|\hat{\psi}\|^2,$$

with:

$$\tilde{C}_0 = \langle(2(\eta_0 - \hat{t})\hat{r}_1 u_{\eta_0}, u_{\eta_0}\rangle_{L^2(d\hat{t}d\hat{z})} + \int \frac{\hat{l}}{2}\hat{P}u_{\eta_0}\hat{P}u_{\eta_0}\, d\hat{t}d\hat{z} + \int \hat{L}\hat{P}u_{\eta_0}\hat{P}u_{\eta_0}\, d\hat{t}d\hat{z}, \tag{8.18}$$

where $\hat{P}, \hat{l}, \hat{L}$ and $\hat{r}_j$ are homogeneous polynomials defined in (8.7) *and* (8.8).

Let us introduce the operator:

$$D_{\hat{t}}^2 + D_{\hat{z}}^2 + (h^{1/4}D_{\hat{s}} - \hat{t} + \eta_0)^2 + h^{1/2}\tau_0^{-1}\kappa\|D_{\hat{z}}u_{\eta_0}\|^2\hat{s}^2 + C_0h^{1/2}. \tag{8.19}$$

After Fourier transform with respect to $\hat{s}$, the operator (8.19) becomes:

$$D_{\hat{t}}^2 + D_{\hat{z}}^2 + (h^{1/4}\xi - \hat{t} + \eta_0)^2 + h^{1/2}\tau_0^{-1}\kappa\|D_{\hat{z}}u_{\eta_0}\|^2 D_{\xi}^2 + C_0h^{1/2}. \tag{8.20}$$

Exercise. Use the Born-Oppenheimer approximation to estimate the lowest eigenvalues of this last operator and deduce Theorem 8.4.

9. Another Approach: The Semiclassical Birkhoff Normal Form

The aim of this section is to enlighten in a geometrical way the phenomenon of Sections 7 and 8: In each case we have reduced the analysis to the "Born-Oppenheimer" framework.

For the background of symplectic geometry that we need, we refer to the classical references [74] and [4]. For the elements of pseudo-differential calculus that we will need, we refer to [91, 30, 73].

We study the magnetic Laplacian $\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla - \mathbf{A})^2$ on $\mathbb{R}^2$. Its symbol is given by

$$H(q,p) = \|p - \mathbf{A}(q_1,q_2)\|^2 = (p_1 - A_1(q_1,q_2))^2 + (p_2 - A_2(q_1,q_2))^2.$$

The operator $\mathcal{L}_{h,\mathbf{A}}$ is gauge invariant so that its spectrum only depends on $\beta = \nabla \times \mathbf{A}$ and so that we can assume that $A_1 = A_2(0,0) = 0$. We let :

$$\omega_0 = dp \wedge dq = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$

We consider the zero set of the Hamiltonian function H:

$$\Sigma = \{(q,p) \in \mathbb{R}^4 : H(q,p) = 0\} = \{(q,p) \in \mathbb{R}^4 : p_1 = 0, \quad p_2 = A_2(q_1,q_2)\}.$$

With our choice of gauge, we have: $(0,0,0,0) \in \Sigma$. For simplicity, we assume that the magnetic field is at least 1.

9.1. **Symplectic Magnetic Geometry.**

Lemma 9.1. *Σ is a symplectic submanifold of $\mathbb{R}^4$. In other words, the 2-form $\omega_{0|\Sigma}$ is non degenerate. In fact, we have:*

$$\omega_{0|\Sigma} = \beta(q_1,q_2) dq_1 \wedge dq_2 \neq 0.$$

Proof. We have:

$$dp_i = \partial_{q_1} A_i \, dq_1 + \partial_{q_2} A_i \, dq_2$$

and we infer:

$$\omega_{0|\Sigma} = (\partial_{q_1} A_2 - \partial_{q_2} A_1) dq_1 \wedge dq_2.$$

□

Lemma 9.2. *There exists a change of coordinates $\hat{\Phi}^{-1}$ defined in a neighborhood $\mathcal{V}$ of $(0,0,0,0)$ which sends $\mathcal{V} \cap \Sigma$ on $\hat{x}_1 = \hat{\xi}_1 = 0$ and so that $(\hat{x}_2, \hat{\xi}_2)$ is a parametrization of Σ and:*

$$\hat{\Phi}^* \omega_0 = \omega_0 + O(\hat{x}_1^2).$$

Proof. The application

$$\varphi : (q_1,q_2) \mapsto (q_1, q_2, 0, A_2(q_1,q_2))$$

is a parametrization of Σ. We have (see Lemma 9.1):

$$\omega_0(\partial_{q_1}\varphi, \partial_{q_2}\varphi) = \beta(q_1,q_2).$$

Let us change the parametrization of Σ. We let:

$$\tilde{x}_2 = -\int_0^{q_1} \beta(q, q_2)\, dq = -A_2(q_1,q_2), \quad \tilde{\xi}_2 = q_2.$$

The inverse change of variables is given by:

$$q_1 = f(\tilde{x}_2, \tilde{\xi}_2), \quad q_2 = \tilde{\xi}_2.$$

We deduce the new parametrization of Σ:

$$\tilde{\varphi} : (\tilde{x}_2, \tilde{\xi}_2) \mapsto (f(\tilde{x}_2, \tilde{\xi}_2), \tilde{\xi}_2, 0, A_2(f(\tilde{x}_2, \tilde{\xi}_2), \tilde{\xi}_2)).$$

Computations provide:

$$u_2(\tilde{x}_2, \tilde{\xi}_2) = \partial_{\tilde{x}_2}\tilde{\varphi} = (-\beta^{-1}, 0, 0, -1)$$
$$v_2(\tilde{x}_2, \tilde{\xi}_2) = \partial_{\tilde{\xi}_2}\tilde{\varphi} = (-\beta^{-1}\partial_2 A_2, 1, 0, 0).$$

We get:

$$\omega_0(u_2, v_2) = -1.$$

Let us complete (u_2, v_2) in a symplectic basis.

The form ω_0 being non degenerate this is clear that the symplectic orthogonal of $T_{\tilde{\varphi}(\tilde{x}_2,\tilde{\xi}_2)}\Sigma$ is 2-dimensional. We can write the equations of this orthogonal:

$$\omega_0(v, u_2) = \omega_0(v, v_2) = 0.$$

We let:

$$u_1 = (0, \beta^{-1}, 1, \beta^{-1}\partial_2 A_2) \tag{9.1}$$
$$v_1 = (-1, 0, 0, 0). \tag{9.2}$$

The vectors u_1 and v_1 form a basis of the symplectic orthogonal and satisfy:

$$\omega_0(u_1, v_1) = -1.$$

This leads to introduce the following application:

$$\tilde{\Phi} : (\tilde{x}_1, \tilde{x}_2, \tilde{\xi}_1, \tilde{\xi}_2) \mapsto \tilde{\varphi}(\tilde{x}_2, \tilde{\xi}_2) + \tilde{x}_1 u_1 + \tilde{\xi}_1 v_1. \tag{9.3}$$

The Jacobian admits the form:

$$[u_1, u_2 + \tilde{x}_1\partial_{\tilde{x}_2}u_1, v_1, v_2 + \tilde{x}_1\partial_{\tilde{\xi}_2}u_1].$$

This matrix is invertible at $(0,0,0,0)$ so that $\tilde{\Phi}$ defines a local diffeomorphism. The surface Σ locally becomes $\tilde{x}_1 = \tilde{\xi}_1 = 0$. Moreover, on $\tilde{x}_1 = 0$, the Jacobian is a symplectic matrix.

In fact we can describe how the symplectic form transforms itself:

$$\tilde{\Phi}^*\omega_0 = \omega_0$$
$$+ \tilde{x}_1\omega_0(u_1, \partial_{\tilde{x}_2}u_1)d\tilde{x}_1 \wedge d\tilde{x}_2 + \tilde{x}_1\omega_0(u_1, \partial_{\tilde{\xi}_2}u_1)d\tilde{x}_1 \wedge d\tilde{\xi}_2 + \tilde{x}_1^2\omega_0(\partial_{\tilde{x}_2}u_1, \partial_{\tilde{\xi}_2}u_1)d\tilde{x}_2 \wedge d\tilde{\xi}_2.$$

We infer:

$$\tilde{\Phi}^*\omega_0 = d\tilde{\xi}_1 \wedge d\tilde{x}_1 + d\tilde{\xi}_2 \wedge d\tilde{x}_2 + a(\tilde{x}_2, \tilde{\xi}_2)\tilde{x}_1 d\tilde{x}_1 \wedge d\tilde{x}_2 + b(\tilde{x}_2, \tilde{\xi}_2)\tilde{x}_1 d\tilde{x}_1 \wedge d\tilde{\xi}_2 + O(\tilde{x}_1^2).$$

We get:

$$\tilde{\Phi}^*\omega_0 = d\tilde{\xi}_1 \wedge d\tilde{x}_1 + (d\tilde{\xi}_2 + a(\tilde{x}_2, \tilde{\xi}_2)\tilde{x}_1 d\tilde{x}_1) \wedge (d\tilde{x}_2 - b(\tilde{x}_2, \tilde{\xi}_2)\tilde{x}_1 d\tilde{x}_1) + O(\tilde{x}_1^2).$$

We introduce the change of variables $\hat{\psi}^{-1}$:

$$\hat{x}_1 = \tilde{x}_1, \quad \hat{\xi}_1 = \tilde{\xi}_1, \quad \hat{x}_2 = \tilde{x}_2 - b(\tilde{x}_2, \tilde{\xi}_2)\frac{\tilde{x}_1^2}{2}, \quad \hat{\xi}_2 = \tilde{\xi}_2 + a(\tilde{x}_2, \tilde{\xi}_2)\frac{\tilde{x}_1^2}{2} \tag{9.4}$$

We have:

$$\hat{\psi}^*\tilde{\Phi}^*\omega_0 = d\hat{\xi}_1 \wedge d\hat{x}_1 + d\hat{\xi}_2 \wedge d\hat{x}_2 + O(\hat{x}_1^2).$$

□

Lemma 9.3. *Let us consider ω_0 and ω_1 two 2-forms on $\mathbb{R}^4$ which are closed and non degenerate. Let us assume that $\omega_{1|\hat{x}_1=0} = \omega_{0|\hat{x}_1=0}$. There exist a neighborhood of $(0,0,0,0)$ and a change of coordinates ψ_1 such that:*

$$\psi_1^*\omega_1 = \omega_0 \quad \textit{and} \quad \psi_{1|\hat{x}_1=0} = \mathsf{Id}_{|\hat{x}_1=0}.$$

Proof. The proof is rather standard but we recall it for completeness (see [74, p. 92]).

• *Poincaré's Lemma.* Let us begin to prove that we can find a 1-form σ defined in a neighborhood of $(0,0,0,0)$ such that:

$$\tau := \omega_1 - \omega_0 = d\sigma \quad \text{and} \quad \sigma_{|\hat{x}_1=0} = 0.$$

We introduce the family of diffeomorphisms $(\phi_t)_{0<t\leq 1}$ defined by:

$$\phi_t(\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2) = (t\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2)$$

and we let:

$$\phi_0(\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2) = (0, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2).$$

We have:

$$\phi_0^*\tau = 0 \quad \text{and} \quad \phi_1^*\tau = \tau.$$

Let us denote by X_t the vector field associated with ψ_t:

$$X_t = \frac{d\phi_t}{dt}(\phi_t^{-1}) = (t^{-1}x_1, 0, 0, 0).$$

Let us compute the Lie derivative of τ along X_t:

$$\frac{d}{dt}\phi_t^*\tau = \phi_t^*\mathcal{L}_{X_t}\tau.$$

From the Cartan formula, we have:

$$\mathcal{L}_{X_t} = \iota(X_t)d\tau + d(\iota(X_t)\tau).$$

Since τ is closed on $\mathbb{R}^4$, we have $d\tau = 0$. Therefore it follows:

$$\frac{d}{dt}\phi_t^*\tau = d(\phi_t^*\iota(X_t)\tau).$$

We consider the 1-form $\sigma_t := \phi_t^*\iota(X_t)\tau$ which vanishes on $\tilde{x}_1 = 0$. We denote $\sigma = \int_0^1 \sigma_t\, dt$ and we have:

$$\frac{d}{dt}\phi_t^*\tau = d\sigma_t \quad \text{and} \quad \tau = d\sigma.$$

• *Conclusion.* We use Moser's argument. We let: $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. The 2-form ω_t is closed and non degenerate (up to choose a neighborhood of $(0,0,0,0)$ small enough). We look for ψ_t such that:

$$\psi_t^* \omega_t = \omega_0.$$

For that purpose, let us determine a vector field X_t such that:

$$\frac{d}{dt}\psi_t = X_t(\psi_t).$$

By using again the Cartan formula, we get:

$$0 = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \iota(X_t)d\omega_t + d(\iota(X_t)\omega_t)\right).$$

This becomes:

$$\omega_0 - \omega_1 = d(\iota(X_t)\omega_t).$$

We are led to solve:

$$\iota(X_t)\omega_t = -\sigma.$$

By non degeneracy of ω_t, this determines X_t. Choosing a neighborhood of $(0,0,0,0)$ small enough, we infer that ψ_t exists until the time $t = 1$ and that it satisfies $\psi_t^*\omega_t = \omega_0$ (so that this is a diffeomorphism). Since $\sigma_{|\hat{x}_1=0} = 0$, we get $\psi_t = \mathsf{Id}_{|\hat{x}_1=0}$. More precisely we get: $\psi_1 = \mathsf{Id} + O(x_1^2)$.

□

Proposition 9.4. *There exists a symplectic change of coordinates Φ^{-1} defined in a neighborhood $\mathcal{V}$ of $(0,0,0,0)$ which sends $\mathcal{V}\cap\Sigma$ on $x_1 = \xi_1 = 0$ and so that (x_2, ξ_2) is a parametrization of Σ.*

Proof. We have just to apply Lemma 9.3 to the 2-form defined in Lemma 9.2 by $\omega_1 = \hat{\Phi}^*\omega_0$. We have $\Phi = \hat{\Phi} \circ \psi_1$. □

9.2. **A Reduction of the Magnetic Symbol.** Let us now analyze the form taken by the Hamiltonian in the normal symplectic coordinates.

Proposition 9.5. *We let: $\mathcal{H} = H \circ \Phi$. We have:*

$$\mathcal{H}(z_1, z_2) = \mathcal{H}_{z_2}^{\mathsf{quad}}(z_1) + O(|z_1|^3),$$

where:

$$\mathcal{H}_{z_2}^{\mathsf{quad}}(z_1) = \tilde{\beta}(x_2, \xi_2)^2 x_1^2 + \xi_1^2.$$

Proof. We can notice that the differential of H vanishes on Σ so that the differential of $\mathcal{H}$ vanishes on $z_1 = 0$. From the definition of H and Σ, we have: $D_{\Phi(0,x_2,0,\xi_2)}H = 0$. We infer that:

$$D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{x_2}\Phi, \partial_{x_2}\Phi) = 0, \quad D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{x_2}\Phi, \partial_{\xi_2}\Phi) = 0,$$
$$D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{\xi_2}\Phi, \partial_{\xi_2}\Phi) = 0, \quad D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{x_1}\Phi, \partial_{x_2}\Phi) = 0,$$
$$D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{x_1}\Phi, \partial_{\xi_2}\Phi) = 0, \quad D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{\xi_1}\Phi, \partial_{\xi_2}\Phi) = 0.$$

Using the explicit expression of $D^2_{\Phi(0,x_2,0,\xi_2)}H$:

$$\begin{bmatrix} 2\beta^2 & 2\beta\partial_2 A_2 & 0 & -2\beta \\ 2\beta\partial_2 A_2 & 2(\partial_2 A_2)^2 & 0 & -2\partial_2 A_2 \\ 0 & 0 & 2 & 0 \\ -2\beta & -2\partial_2 A_2 & 0 & 2 \end{bmatrix} \tag{9.5}$$

and the fact that, on Σ, we have $\partial_{x_1}\Phi = \partial_{\tilde{x}_1}\tilde{\Phi} = u_1$ and $\partial_{\xi_1}\Phi = \partial_{\tilde{\xi}_1}\tilde{\Phi} = v_1$, we deduce:

$$\begin{bmatrix} D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{x_1}\Phi, \partial_{x_1}\Phi) & D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{x_1}\Phi, \partial_{\xi_1}\Phi) \\ D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{x_1}\Phi, \partial_{\xi_1}\Phi) & D^2_{\Phi(0,x_2,0,\xi_2)}H(\partial_{\xi_1}\Phi, \partial_{\xi_1}\Phi) \end{bmatrix} = \begin{bmatrix} 2\beta^2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{9.6}$$

□

Let us now analyze the quadratic form $\mathcal{H}^{\mathsf{quad}}_{z_2}$.

Lemma 9.6. *There exist local symplectic coordinates near* $(0,0,0,0)$ *denoted by* $(\hat{x}_1, \hat{x}_2, \hat{\xi}_1, \hat{\xi}_2)$ *such that:*

$$\mathcal{H}^{\mathsf{quad}}_{z_2}(z_1) = \hat{\mathcal{H}}^{\mathsf{quad}}_{\hat{z}_2}(\hat{z}_1) + O(|\hat{z}_1|^3),$$

where:

$$\hat{\mathcal{H}}^{\mathsf{quad}}_{\hat{z}_2}(\hat{z}_1) = \tilde{\beta}(\hat{x}_2, \hat{\xi}_2)\left(\hat{x}_1^2 + \hat{\xi}_1^2\right).$$

Proof. The proof is divided into two main steps.

• *An almost symplectic transform.* We let:

$$f = \tilde{\beta}^{1/2} \text{ and } g = \ln f.$$

We introduce the change of coordinates $(\check{x}_1, \check{x}_2, \check{\xi}_1, \check{\xi}_2) = C_1(x_1, x_2, \xi_1, \xi_2)$ define by:

$$\begin{cases} \check{x}_1 &= f x_1, \\ \check{\xi}_1 &= f^{-1}\xi_1, \\ \check{x}_2 &= x_2 + \frac{\partial g}{\partial \xi_2} x_1\xi_1, \\ \check{\xi}_2 &= \xi_2 - \frac{\partial g}{\partial x_2} x_1\xi_1, \end{cases}$$

We want to know at which point this transformation is symplectic. Therefore we shall compute $d\check{\xi}_1 \wedge d\check{x}_1 + d\check{\xi}_2 \wedge d\check{x}_2$. We have:

$$\begin{aligned} d\check{\xi}_1 \wedge d\check{x}_1 &= (\xi_1 d(f^{-1}) + f^{-1}d\xi_1) \wedge (x_1 df + ddx_1) \\ &= d\xi_1 \wedge dx_1 + (\xi_1 dx_1 + x_1 d\xi_1) \wedge \frac{df}{f} \\ &= d\xi_1 \wedge dx_1 + dP \wedge dg, \end{aligned}$$

where $P = x_1\xi_1$. Moreover, we get:

$$\begin{aligned} d\check{\xi}_2 \wedge d\check{x}_2 &= d\left(\xi_2 - \frac{\partial g}{\partial x_2}P\right) \wedge d\left(x_2 + \frac{\partial g}{\partial \xi_2}P\right) \\ &= d\xi_2 \wedge dx_2 + dg \wedge dP + \mathcal{R}, \end{aligned}$$

where

$$\begin{aligned}\mathcal{R} &= -Pd\left(\frac{\partial g}{\partial x_2}\right)\wedge dx_2 - Pd\left(\frac{\partial g}{\partial \xi_2}\right)\wedge d\xi_2 - d\left(\frac{\partial g}{\partial x_2}P\right)\wedge d\left(\frac{\partial g}{\partial \xi_2}P\right)\\ &= -Pd(dg) - d\left(\frac{\partial g}{\partial x_2}P\right)\wedge d\left(\frac{\partial g}{\partial \xi_2}P\right)\\ &= -d\left(\frac{\partial g}{\partial x_2}P\right)\wedge d\left(\frac{\partial g}{\partial \xi_2}P\right).\end{aligned}$$

Then, this is clear that $\mathcal{R} = O(|x_1\xi_1|)$. We infer:

$$d\check{\xi}_1 \wedge d\check{x}_1 + d\check{\xi}_2 \wedge d\check{x}_2 = d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2 + O(|x_1\xi_1|).$$

In other words, we have:

$$C_1^*\omega_0 = \omega_0 + O(|x_1\xi_1|).$$

Let us write $\mathcal{H}^{\mathsf{quad}}_{z_2}$ in these new coordinates. We notice that:

$$\begin{cases} x_1 &= f^{-1}\left(\check{x}_2 - \frac{\partial g}{\partial \xi_2}\check{x}_1\check{\xi}_1, \check{\xi}_2 + \frac{\partial g}{\partial x_2}\check{x}_1\check{\xi}_1\right)\check{x}_1,\\ \check{\xi}_1 &= f^{-1}\left(\check{x}_2 - \frac{\partial g}{\partial \xi_2}\check{x}_1\check{\xi}_1, \check{\xi}_2 + \frac{\partial g}{\partial x_2}\check{x}_1\check{\xi}_1\right)\xi_1,\\ x_2 &= \check{x}_2 - \frac{\partial g}{\partial \xi_2}\check{x}_1\check{\xi}_1,\\ \xi_2 &= \check{\xi}_2 + \frac{\partial g}{\partial x_2}\check{x}_1\check{\xi}_1,\end{cases}$$

Using a Taylor formula with respect to z_1, we find:

$$\mathcal{H}^{\mathsf{quad}}_{z_2}(z_1) = \check{\mathcal{H}}^{\mathsf{quad}}_{\check{z}_2}(\check{z}_1) + O(|\check{z}_1|^4),$$

where:

$$\check{\mathcal{H}}^{\mathsf{quad}}_{\check{z}_2}(\check{z}_1) = \tilde{\beta}(\check{x}_2,\check{\xi}_2)\left(\check{x}_1^2 + \check{\xi}_1^2\right).$$

• *How to make C_1 become symplectic.* We let $\omega_1 = C_1^*\omega_0$. The 2-forms ω_0 and ω_1 coincide on $\check{x}_1 = 0$, they are closed and non degenerate. We let $\tau = \omega_1 - \omega_0$. We can use exactly the same argument as in the proof of Lemma 9.3 and we find σ such that $\tau = d\sigma$ with a σ vanishing on $\check{x}_1 = 0$ and satisfying even $\sigma = O(|\check{x}_1\check{\xi}_1|)$. It remains to use Moser's argument as in Lemma 9.3 and we deduce the existence of a local diffeomorphism C_2 such that:

$$C_2^*\omega_1 = \omega_0$$

and $(\hat{x}_1,\hat{x}_2,\hat{\xi}_1,\hat{\xi}_2) = C_2(\check{x}_1,\check{x}_2,\check{\xi}_1,\check{\xi}_2) = (\check{x}_1,\check{x}_2,\check{\xi}_1,\check{\xi}_2) + O(|\check{x}_1\check{\xi}_1|)$. The change of coordinates C_1C_2 satisfies:

$$(C_1C_2)^*\omega_0 = \omega_0.$$

In these coordinates, we can write:

$$\check{\mathcal{H}}^{\mathsf{quad}}_{\check{z}_2}(\check{z}_1) = \hat{\mathcal{H}}^{\mathsf{quad}}_{\hat{z}_2}(\hat{z}_1) + O(|\hat{z}_1|^3).$$

□

9.3. **The Normal Form.** The procedure of last subsection, known as the Birkhoff normal form, can be continued at any order with respect to $|\hat{z}_1|^2$.

Let us consider the space of the formal power series in $\hat{x}_1, \hat{\xi}_1, h$ with coefficients smoothly depending on $(\hat{x}_2, \hat{\xi}_2)$: $\mathcal{E} = \mathbb{R}_{\hat{x}_2,\hat{\xi}_2}[\hat{x}_1, \hat{\xi}_1, h]$. We refer to [21, Section 2] for details.

Notation 9.7. The degree of $\hat{x}_1^\alpha \hat{\xi}_1^\beta h^l$ is $\alpha + \beta + 2l$. $\mathcal{D}_N$ denotes the space of the monomials of degree N. $\mathcal{O}_N$ is the space of formal series with valuation at least N.

Notation 9.8. We denote by $\sigma(L)$ the Taylor series of the symbol L with respect to $(\hat{x}_1, \hat{\xi}_1, h)$ at $(0, 0, 0)$.

Notation 9.9. We let:

$$\mathsf{ad}_A = [A, \cdot],$$

where the bracket between two formal series is the formal power series obtained through the composition of pseudo-differential operators in the Weyl quantization.

Let us fix a symbol L such that $\sigma(L) \in \mathcal{O}_3$.

Proposition 9.10. *There exist formal power series $\tau, \kappa \in \mathcal{O}_3$ such that:*

$$e^{ih^{-1}\mathsf{ad}_\tau}(H_2 + \sigma(L)) = H_2 + \kappa,$$

with: $[\kappa, H_2] = 0$.

Proof. Let $N \geq 1$. Assume that we have, for $N \geq 1$ and $\tau_N \in \mathcal{O}_3$:

$$e^{ih^{-1}\mathsf{ad}_{\tau_N}}(H_2 + \sigma(L)) = H_2 + K_3 + \cdots + K_{N+1} + R_{N+2} + \mathcal{O}_{N+3},$$

where $K_i \in \mathcal{D}_i$ commutes with $|\hat{z}_1|^2$ and where $R_{N+2} \in \mathcal{D}_{N+2}$.

Let $\tau' \in \mathcal{D}_{N+2}$. A computation provides:

$$e^{ih^{-1}\mathsf{ad}_{\tau_N+\tau'}}(H_2 + \sigma(L)) = H_2 + K_3 + \cdots + K_{N+1} + K_{N+2} + \mathcal{O}_{N+3},$$

with:

$$K_{N+2} = R_{N+2} + \tilde{\beta}(\hat{z}_2) ih^{-1}\mathsf{ad}_{\tau'}|\hat{z}_1|^2 = R_{N+2} - \tilde{\beta}(\hat{z}_2) ih^{-1}\mathsf{ad}_{|\hat{z}_1|^2}\tau'.$$

We can write:

$$R_{N+2} = K_{N+2} + \tilde{\beta}(\hat{z}_2) ih^{-1}\mathsf{ad}_{|\hat{z}_1|^2}\tau'.$$

Since $\tilde{\beta}(\hat{z}_2) \neq 0$, we deduce the existence of τ' and K_{N+2} such that K_{N+2} commutes with $|\hat{z}_1|^2$. Note that $ih^{-1}\mathsf{ad}_{|\hat{z}_1|^2} = \{|\hat{z}_1|^2, \cdot\}$. □

9.4. **Localization and Micro-Localization Estimates.** We must now justify that the eigenfunctions are micro-localized near Σ to make the formal construction of the previous subsection less formal.

9.4.1. *Space localization.* We begin by proving a space localization.

Proposition 9.11. *Let us assume that:*

$$\beta(x) \geq C_1 > 0 \text{ for } |x| \geq \varepsilon_0. \tag{9.7}$$

Let us fix $0 < C_0 < C_1$ *and* $\alpha \in (0, 1/2)$*. There exist* $C, h_0 > 0$ *such that for all eigenpair* (λ, ψ) *such that* $\lambda \leq C_0 h$*, we have:*

$$\int |e^{\chi(x)h^{-\alpha}|x|}\psi|^2 \, dx \leq C\|\psi\|^2,$$

where χ *is zero for* $|x| \leq \varepsilon_0$ *and* 1 *for* $|x| \geq 2\varepsilon_0$*. Moreover, we also have the* H^1 *estimate:*

$$\int |e^{\chi(x)h^{-\alpha}|x|}h\nabla\psi|^2 \, dx \leq Ch\|\psi\|^2.$$

9.4.2. *Microlocalization of the eigenfunctions near* Σ*.* In the following, we will use the Weyl quantization of a symbol $a(x, \xi) \in \mathcal{S}(\mathbb{R}^n, \mathbb{R}^n)$ defined by the expression:

$$\mathsf{Op}_h^w(a)u(x) = (2\pi)^{-n} \int_{\xi\in\mathbb{R}^n} \int_{y\in\mathbb{R}^n} a\left(\frac{x+y}{2}, \xi\right) e^{ih^{-1}(x-y)\xi} u(y) \, dy d\xi.$$

We refer to [30] and [73] where the basic properties of this quantization are discussed.

We now investigate the microlocalization properties of the eigenfunctions.

Proposition 9.12. *Let* (λ, ψ) *be an eigenpair with* $\lambda \leq C_0 h$*. Let us consider* $\delta \in \left(0, \frac{1}{2}\right)$*. Then, we have:*

$$\psi = \mathsf{Op}_h^w\left(\chi_0(x_1, x_2)\chi_1\left(\frac{\xi_1^2 + (\xi_2 - A_2)^2}{h^{2\delta}}\right)\right)\psi + O(h^\infty),$$

where χ_0 *is a smooth cutoff function supported in a neighborhood of* $(0, 0)$ *of size* $4\varepsilon_0$ *and* χ_1 *a smooth cutoff function being* 1 *near* 0*.*

Proof. We start by proving that:

$$\chi\left(\frac{\mathcal{L}}{h^{2\delta}}\right)(\chi_0(x)\psi) = O(h^\infty), \tag{9.8}$$

where χ is zero near 0. By the space localization, we have:

$$\mathcal{L}(\chi_0(x)\psi) = \lambda\chi_0(x)\psi + O(h^\infty)\|\psi\|.$$

Then, we get:

$$\chi\left(\frac{\mathcal{L}}{h^{2\delta}}\right)\mathcal{L}(\chi_0(x)\psi) = \lambda\chi\left(\frac{\mathcal{L}}{h^{2\delta}}\right)(\chi_0(x)\psi) + O(h^\infty)\|\psi\|.$$

We have:

$$h^{2\delta}\|\mathcal{L}(\chi_0(x)\psi)\|^2 \leq \mathcal{Q}\left(\mathcal{L}(\chi_0(x)\psi)\right) \leq C_0 h\|\mathcal{L}(\chi_0(x)\psi)\|^2 + O(h^\infty)\|\psi\|^2.$$

Since $\delta \in \left(0, \frac{1}{2}\right)$, we deduce (9.8). We can also notice that:

$$\chi_0(x)\chi\left(\frac{\mathcal{L}}{h^{2\delta}}\right)\psi = 0.$$

Using the h-pseudo-differential calculus, with parameter $\varepsilon = h^\delta$ (with $\delta \in \left(0, \frac{1}{2}\right)$), it can be shown that:

$$\psi = \mathsf{Op}_h^w \left(\chi_0(x_1, x_2)\chi_1 \left(\frac{\xi_1^2 + (\xi_2 - A_2)^2}{\varepsilon^2} \right) \right) \psi + O(h^\infty).$$

□

9.5. **Application to the Spectral Theory.** Without going into the details (see [38]), let us describe the philosophy. The main ingredient is the theorem of Egorov (see [73, Theorems 5.5.5 and 5.5.9]). Associated with the change of coordinates $\hat{\Phi}$ defined in Lemma 9.6, there exists a Fourier integral operator U_h (depending on h) such that the pseudo-differential operator $U_h^{-1} H U_h$ admits as principal symbol $H \circ \hat{\Phi}$ which has Taylor expansion $H_2 + \mathcal{O}(|\hat{z}_1|^3)$. In terms of power series at 0, we can write the symbol of $U_h^{-1} H U_h$ as:

$$H_2 + \sigma(L),$$

where $\sigma(L) \in \mathcal{O}_3$. We now use Proposition 9.10. Thanks to a Borel argument, we can find a bounded operator A whose symbol is τ. By Egorov's theorem (see [73, Theorem 5.5.5]), $e^{ih^{-1}A} U_h^{-1} H U_h e^{-ih^{-1}A}$ is a pseudo-differential operator and its formal power series at 0 is $H_2 + \kappa$. We have used the formula (see for instance [21, p. 482]):

$$e^{ih^{-1}A} Q e^{-ih^{-1}A} = e^{\mathsf{ad}_{ih^{-1}A}} Q.$$

In conclusion, the bottom of the spectrum of $\mathcal{L}_{h,\mathbf{A}}$ can be described by the one of:

$$\tilde{\beta}(\hat{z}_2)|\hat{z}_1|^2 + (hc_0(\hat{z}_2) + \gamma(\hat{z}_2)|z_1|^4 + ch^2) + \cdots$$

The next sections introduce to problems related to the Dirichlet Laplacian on triangles and waveguides.

10. A Related Topics: Semiclassical Triangles

Let us explain how we can be led to study the so-called "semiclassical triangles". As we will see, this topics is closely related to "broken waveguides" or "waveguides with corners" (see Section 11). In fact, from a heuristic point of view, we are led to investigate such waveguides when analyzing the spectral behavior of $\mathcal{L}_\theta$ defined in Subsection 3.4 when $\theta \to 0$ (see [10] and [28]). Indeed the potential V_θ creates an effective broken waveguide whose corner can be described by a triangle with Dirichlet conditions (see Section 11).

10.1. A Brief State of the Art.

This subject is already dealt with in [42, Theorem 1] where four-term asymptotics is proved for the lowest eigenvalue, whereas a three-term asymptotics for the second eigenvalue is provided in [42, Section 2]. We can mention the papers [43, 44] whose results provide two-term asymptotics for the thin rhombi and also [14] which deals with a regular case (thin ellipse for instance), see also [15]. We also invite the reader to take a look to [63].

Let us finally mention the case of the cones studied in [78].

10.2. Main result.

Let us define the isosceles triangle in which we are interested:

$$\mathsf{Tri}_\theta = \left\{(x_1, x_2) \in \mathbb{R}_- \times \mathbb{R} : x_1 \tan\theta < |x_2| < \left(x_1 + \frac{\pi}{\sin\theta}\right)\tan\theta\right\}. \tag{10.1}$$

We will use the coordinates

$$x = x_1\sqrt{2}\sin\theta, \quad y = x_2\sqrt{2}\cos\theta, \tag{10.2}$$

which transform Tri_θ into $\mathsf{Tri}_{\pi/4}$. The operator becomes:

$$\mathcal{D}_{\mathsf{Tri}}(h) = 2\sin^2\theta\, \partial_x^2 - 2\cos^2\theta\, \partial_y^2,$$

with Dirichlet condition on the boundary of Tri. We let $h = \tan\theta$; after a division by $2\cos^2\theta$, we get the new operator:

$$\mathcal{L}_{\mathsf{Tri}}(h) = -h^2\partial_x^2 - \partial_y^2. \tag{10.3}$$

We state the result for the scaled operator $\mathcal{L}_{\mathsf{Tri}}(h)$.

Theorem 10.1. *The eigenvalues of $\mathcal{L}_{\mathsf{Tri}}(h)$, denoted by $\lambda_{\mathsf{Tri},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{Tri},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \beta_{j,n} h^{j/3} \quad \text{with } \beta_{0,n} = \frac{1}{8},\ \beta_{1,n} = 0,\ \text{and } \beta_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathbf{A}}(n),$$

the terms of odd rank being zero for $j \leq 8$. The corresponding eigenvectors have expansions in powers of $h^{1/3}$ with both scales $x/h^{2/3}$ and x/h.

10.3. **Born-Oppenheimer Approximation.** Let us introduce the Dirichlet realization on the space $L^2((-\pi\sqrt{2},0))$ of:

$$\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h) = -h^2\partial_x^2 + \frac{\pi^2}{4(x+\pi\sqrt{2})^2}\,. \tag{10.4}$$

This operator is the Born-Oppenheimer approximation of the operator $\mathcal{L}_{\mathsf{Tri}}(h)$ on the triangle Tri.

Theorem 10.2. *The eigenvalues of $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$, denoted by $\lambda_{\mathsf{BO},\mathsf{Tri},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{BO},\mathsf{Tri},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \hat{\beta}_{j,n} h^{2j/3}, \quad \textit{with} \quad \hat{\beta}_{0,n} = \frac{1}{8} \ \textit{and} \ \hat{\beta}_{1,n} = (4\pi\sqrt{2})^{-2/3} z_{\boldsymbol{A}}(n).$$

10.4. **When the triangle becomes a rectangle...** We first perform a change of variables to transform the triangle into a rectangle:

$$u = x \in (-\pi\sqrt{2},0), \quad t = \frac{y}{x+\pi\sqrt{2}} \in (-1,1). \tag{10.5}$$

so that Tri is transformed into

$$\mathsf{Rec} = (-\pi\sqrt{2},0) \times (-1,1). \tag{10.6}$$

The operator $\mathcal{L}_{\mathsf{Tri}}(h)$ becomes:

$$\mathcal{L}_{\mathsf{Rec}}(h)(u,t;\partial_u,\partial_t) = -h^2\Big(\partial_u - \frac{t}{u+\pi\sqrt{2}}\,\partial_t\Big)^2 - \frac{1}{(u+\pi\sqrt{2})^2}\,\partial_t^2, \tag{10.7}$$

with Dirichlet boundary conditions on ∂Rec. The equation $\mathcal{L}_{\mathsf{Tri}}(h)\psi_h = \beta_h\psi_h$ is transformed into the equation

$$\mathcal{L}_{\mathsf{Rec}}(h)\hat{\psi}_h = \beta_h\hat{\psi}_h \quad \text{with} \quad \hat{\psi}_h(u,t) = \psi_h(x,y).$$

10.5. **Quasimodes and boundary layer.** We want to construct quasimodes (β_h,ψ_h) for the operator $\mathcal{L}_{\mathsf{Tri}}(h)(\partial_x,\partial_y)$. It will be more convenient to work on the rectangle Rec with the operator $\mathcal{L}_{\mathsf{Rec}}(h)(u,t;\partial_u,\partial_t)$. We introduce the new scales

$$s = h^{-2/3}u \quad \text{and} \quad \sigma = h^{-1}u, \tag{10.8}$$

and we look quasimodes $(\beta_h,\hat{\psi}_h)$ in the form of series

$$\beta_h \sim \sum_{j\geq 0} \beta_j h^{j/3} \quad \text{and} \quad \hat{\psi}_h(u,t) \sim \sum_{j\geq 0} \big(\Psi_j(s,t) + \Phi_j(\sigma,t)\big)h^{j/3} \tag{10.9}$$

in order to solve $\mathcal{L}_{\mathsf{Rec}}(h)\hat{\psi}_h = \beta_h\hat{\psi}_h$ in the sense of formal series. As will be seen hereafter, an Ansatz containing the scale $h^{-2/3}u$ alone (like for the Born-Oppenheimer operator $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$) is not sufficient to construct quasimodes for $\mathcal{L}_{\mathsf{Rec}}(h)$. Expanding the operator in powers of $h^{2/3}$, we obtain the formal series:

$$\mathcal{L}_{\mathsf{Rec}}(h)(h^{2/3}s,t;h^{-2/3}\partial_s,\partial_t) \sim \sum_{j\geq 0} \mathcal{L}_{2j} h^{2j/3} \quad \text{with leading term} \quad \mathcal{L}_0 = -\frac{1}{2\pi^2}\partial_t^2 \tag{10.10}$$

and in powers of h:

$$\mathcal{L}_{\mathsf{Rec}}(h)(h\sigma,t;h^{-1}\partial_\sigma,\partial_t)\sim\sum_{j\geq 0}\mathcal{N}_{3j}h^j \quad\text{with leading term}\quad \mathcal{N}_0=-\partial_\sigma^2-\frac{1}{2\pi^2}\partial_t^2. \tag{10.11}$$

In what follows, in order to finally ensure the Dirichlet conditions on the triangle Tri, we will require for our Ansatz the boundary conditions, for any $j\in\mathbb{N}$:

$$\Psi_j(0,t)+\Phi_j(0,t)=0,\quad -1\leq t\leq 1 \tag{10.12}$$

$$\Psi_j(s,\pm 1)=0,\ \ s<0\quad\text{and}\quad \Phi_j(\sigma,\pm 1)=0,\ \ \sigma\leq 0. \tag{10.13}$$

More specifically, we are interested in the ground energy $\lambda=\frac{1}{8}$ of the Dirichlet problem for $\mathcal{L}_0$ on the interval $(-1,1)$. Thus we have to solve Dirichlet problems for the operators $\mathcal{N}_0-\frac{1}{8}$ and $\mathcal{L}_0-\frac{1}{8}$ on the half-strip

$$\mathsf{Hst}=\mathbb{R}_-\times(-1,1), \tag{10.14}$$

and look for *exponentially decreasing solutions*. The situation is similar to that encountered in thin structure asymptotics with Neumann boundary conditions. The following lemma shares common features with the Saint-Venant principle, see for example [25, §2].

Lemma 10.3. *We denote the first normalized eigenvector of $\mathcal{L}_0$ on $H_0^1((-1,1))$ by c_0:*

$$c_0(t)=\cos\left(\frac{\pi t}{2}\right).$$

Let $F=F(\sigma,t)$ be a function in $L^2(\mathsf{Hst})$ with exponential decay with respect to σ and let $G\in H^{3/2}((-1,1))$ be a function of t with $G(\pm 1)=0$. Then there exists a unique $\gamma\in\mathbb{R}$ such that the problem

$$\left(\mathcal{N}_0-\frac{1}{8}\right)\Phi=F\ \ \textit{in}\ \ \mathsf{Hst},\quad \Phi(\sigma,\pm 1)=0,\quad \Phi(0,t)=G(t)+\gamma c_0(t),$$

admits a (unique) solution in $H^2(\mathsf{Hst})$ with exponential decay. There holds

$$\gamma=-\int_{-\infty}^{0}\int_{-1}^{1}F(\sigma,t)\,\sigma c_0(t)\,d\sigma dt-\int_{-1}^{1}G(t)\,c_0(t)\,dt.$$

The following two lemmas are consequences of the Fredholm alternative.

Lemma 10.4. *Let $F=F(s,t)$ be a function in $L^2(\mathsf{Hst})$ with exponential decay with respect to s. Then, there exist solution(s) Ψ such that:*

$$\left(\mathcal{L}_0-\frac{1}{8}\right)\Psi=F\ \ \textit{in}\ \ \mathsf{Hst},\quad \Psi(s,\pm 1)=0$$

if and only if $\left\langle F(s,\cdot),c_0\right\rangle_t=0$ for all $s<0$. In this case, $\Psi(s,t)=\Psi^\perp(s,t)+g(s)c_0(t)$ where $\Psi^\perp$ satisfies $\left\langle\Psi(s,\cdot),c_0\right\rangle_t\equiv 0$ and has also an exponential decay.

Lemma 10.5. *Let $n\geq 1$. We recall that $z_{\boldsymbol{A}}(n)$ is the n-th zero of the reverse Airy function, and we denote by*

$$g_{(n)}=\boldsymbol{A}\big((4\pi\sqrt{2})^{-1/3}s+z_{\boldsymbol{A}}(n)\big)$$

the eigenvector of the operator $-\partial_s^2 - (4\pi\sqrt{2})^{-1}s$ *with Dirichlet condition on* $\mathbb{R}_-$ *associated with the eigenvalue* $(4\pi\sqrt{2})^{-2/3}z_{\mathbf{A}}(n)$. *Let* $f = f(s)$ *be a function in* $L^2(\mathbb{R}_-)$ *with exponential decay and let* $c \in \mathbb{R}$. *Then there exists a unique* $\beta \in \mathbb{R}$ *such that the problem:*

$$\left(-\partial_s^2 - \frac{s}{4\pi\sqrt{2}} - (4\pi\sqrt{2})^{-2/3}z_{\mathbf{A}}(n)\right) g = f + \beta g_{(n)} \ \textit{ in } \ \mathbb{R}_-, \ \textit{with } g(0) = c,$$

has a solution in $H^2(\mathbb{R}_-)$ *with exponential decay.*

Now we can start the construction of the terms of our Ansatz (10.9).

- *Terms in* h^0. The equations provided by the constant terms are:

$$\mathcal{L}_0\Psi_0 = \beta_0\Psi_0(s,t), \quad \mathcal{N}_0\Phi_0 = \beta_0\Phi_0(s,t)$$

with boundary conditions (10.12)-(10.13) for $j = 0$, so that we choose $\beta_0 = \frac{1}{8}$ and $\Psi_0(s,t) = g_0(s)c_0(t)$. The boundary condition (10.12) provides: $\Phi_0(0,t) = -g_0(0)c_0(t)$ so that, with Lemma 10.3, we get $g_0(0) = 0$ and $\Phi_0 = 0$. The function $g_0(s)$ will be determined later.

- *Terms in* $h^{1/3}$. Collecting the terms of order $h^{1/3}$, we are led to:

$$(\mathcal{L}_0 - \beta_0)\Psi_1 = \beta_1\Psi_0 - \mathcal{L}_1\Psi_1 = \beta_1\Psi_0, \quad (\mathcal{N}_0 - \beta_0)\Phi_1 = \beta_1\Phi_0 - \mathcal{N}_1\Phi_1 = 0$$

with boundary conditions (10.12)-(10.13) for $j = 1$. Using Lemma 10.4, we find $\beta_1 = 0$, $\Psi_1(s,t) = g_1(s)c_0(t)$, $g_1(0) = 0$ and $\Phi_1 = 0$.

- *Terms in* $h^{2/3}$. We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_2 = \beta_2\Psi_0 - \mathcal{L}_2\Psi_0, \quad (\mathcal{N}_0 - \beta_0)\Phi_2 = 0,$$

where $\mathcal{L}_2 = -\partial_s^2 + \frac{s}{\pi^3\sqrt{2}}\partial_t^2$ and with boundary conditions (10.12)-(10.13) for $j = 2$. Lemma 10.4 provides the equation in s variable

$$\big\langle(\beta_2\Psi_0 - \mathcal{L}_2\Psi_0(s,\cdot)), c_0\big\rangle_t = 0, \quad s < 0.$$

Taking the formula $\Psi_0 = g_0(s)c_0(t)$ into account this becomes

$$\beta_2 g_0(s) = \left(-\partial_s^2 - \frac{s}{4\pi\sqrt{2}}\right) g_0(s).$$

This equation leads to take $\beta_2 = (4\pi\sqrt{2})^{-2/3}z_{\mathbf{A}}(n)$ and for g_0 the corresponding eigenfunction $g_{(n)}$. We deduce $(\mathcal{L}_0 - \beta_0)\Psi_2 = 0$, then get $\Psi_2(s,t) = g_2(s)c_0(t)$ with $g_2(0) = 0$ and $\Phi_2 = 0$.

- *Terms in* $h^{3/3}$. We get:

$$(\mathcal{L}_0 - \beta_0)\Psi_3 = \beta_3\Psi_0 + \beta_2\Psi_1 - \mathcal{L}_2\Psi_1, \quad (\mathcal{N}_0 - \beta_0)\Phi_3 = 0,$$

with boundary conditions (10.12)-(10.13) for $j = 3$. The scalar product with c_0 (Lemma 10.4) and then the scalar product with g_0 (Lemma 10.5) provide $\beta_3 = 0$ and $g_1 = 0$. We deduce: $\Psi_3(s,t) = g_3(s)c_0(t)$, and $g_3(0) = 0$, $\Phi_3 = 0$.

- *Terms in $h^{4/3}$.* We get:

$$(\mathcal{L}_0-\beta_0)\Psi_4=\beta_4\Psi_0+\beta_2\Psi_2-\mathcal{L}_4\Psi_0-\mathcal{L}_2\Psi_2,\quad (\mathcal{N}_0-\beta_0)\Phi_4=0,$$

where

$$\mathcal{L}_4=\frac{\sqrt{2}}{\pi}\,t\partial_t\partial_s-\frac{3}{4\pi^4}s^2\partial_t^2,$$

and with boundary conditions (10.12)-(10.13) for $j=4$. The scalar product with c_0 provides an equation for g_2 and the scalar product with g_0 determines β_4. By Lemma 10.4 this step determines $\Psi_4=\Psi_4^\perp+c_0(t)g_4(s)$ with a non-zero $\Psi_4^\perp$ and $g_4(0)=0$. Since by construction $\left\langle\Psi_4^\perp(0,\cdot),\,c_0\right\rangle_t=0$, Lemma 10.3 yields a solution Φ_4 with exponential decay. Note that it also satisfies $\left\langle\Phi_4(\sigma,\cdot),\,c_0\right\rangle_t=0$ for all $\sigma<0$.

- *Further terms.* We leave the obtention of the other terms as an exercise.

10.6. **Agmon Estimates.** Let us provide the estimates of Agmon which can be proved.

Proposition 10.6. *Let $\Gamma_0>0$. There exist $h_0>0$, $C_0>0$ and $\eta_0>0$ such that for $h\in(0,h_0)$ and all eigenpair (λ,ψ) of $\mathcal{L}_{\mathsf{Tri}}(h)$ satisfying $|\lambda-\frac{1}{8}|\leq\Gamma_0h^{2/3}$, we have:*

$$\int_{\mathsf{Tri}}e^{\eta_0h^{-1}|x|^{3/2}}\left(|\psi|^2+|h^{2/3}\partial_x\psi|^2\right)dxdy\leq C_0\|\psi\|^2.$$

Proposition 10.7. *Let $\Gamma_0>0$. There exist $h_0>0$, $C_0>0$ and $\rho_0>0$ such that for $h\in(0,h_0)$ and all eigenpair (λ,ψ) of $\mathcal{L}_{\mathsf{Tri}}(h)$ satisfying $|\lambda-\frac{1}{8}|\leq\Gamma_0h^{2/3}$, we have:*

$$\int_{\mathsf{Tri}}(x+\pi\sqrt{2})^{-\rho_0/h}\left(|\psi|^2+|h\,\partial_x\psi|^2\right)dxdy\leq C_0\|\psi\|^2.$$

10.7. **Projection Method.** Let us consider the first N_0 eigenvalues of $Ł_{\mathsf{Rec}}(h)$ (shortly denoted by λ_n). In each corresponding eigenspace, we choose a normalized eigenfunction $\hat{\psi}_n$ so that $\langle\hat{\psi}_n,\hat{\psi}_m\rangle=0$ if $n\neq m$. We introduce:

$$\mathfrak{E}_{N_0}(h)=\operatorname{span}(\hat{\psi}_1,\ldots,\hat{\psi}_{N_0}).$$

Let us define Q^0_{Rec} the following quadratic form:

$$Q^0_{\mathsf{Rec}}(\hat{\psi})=\int_{\mathsf{Rec}}\left(\frac{1}{2\pi^2}|\partial_t\hat{\psi}|^2-\frac{1}{8}|\hat{\psi}|^2\right)(u+\pi\sqrt{2})\,dudt,$$

associated with the operator $\mathcal{L}^0_{\mathsf{Rec}}=\mathsf{Id}_u\otimes\left(-\frac{1}{2\pi^2}\partial_t^2-\frac{1}{8}\right)$ on $L^2(\mathsf{Rec},(u+\pi\sqrt{2})dudt)$. We consider the projection on the eigenspace associated with the eigenvalue 0 of $-\frac{1}{2\pi^2}\partial_t^2-\frac{1}{8}$:

$$\Pi_0\hat{\psi}(u,t)=\left\langle\hat{\psi}(u,\cdot),c_0\right\rangle_t c_0(t),\tag{10.15}$$

where we recall that $c_0(t)=\cos\left(\frac{\pi}{2}t\right)$. We can now state a first approximation result:

Proposition 10.8. *There exist $h_0>0$ and $C>0$ such that for $h\in(0,h_0)$ and all $\hat{\psi}\in\mathfrak{E}_{N_0}(h)$:*

$$0\leq Q^0_{\mathsf{Rec}}(\hat{\psi})\leq Ch^{2/3}\|\hat{\psi}\|^2$$

and

$$\|(\mathsf{Id}-\Pi_0)\hat{\psi}\|+\|\partial_t(\mathsf{Id}-\Pi_0)\hat{\psi}\|\leq Ch^{1/3}\|\hat{\psi}\|.$$

Moreover, $\Pi_0 : \mathfrak{E}_{N_0}(h) \to \Pi_0(\mathfrak{E}_{N_0}(h))$ *is an isomorphism.*

Let us consider an eigenpair (λ, ψ) of $\mathcal{L}_{\mathsf{Tri}}(h)$. We let $\hat{\psi}(u,t) = \psi(x,y)$. Then, $(\lambda, \hat{\psi})$ satisfies:

$$-h^2 \left(\partial_u^2 - \frac{2t\partial_u\partial_t}{u+\pi\sqrt{2}} + \frac{2t\partial_t}{(u+\pi\sqrt{2})^2} + \frac{t^2\partial_t^2}{(u+\pi\sqrt{2})^2}\right)\hat{\psi} - \frac{1}{(u+\pi\sqrt{2})^2}\partial_t^2\hat{\psi} = \lambda\hat{\psi}.$$

The main idea is to determine the (differential) equation satisfied by $\Pi_0\hat{\psi}$. In other words we will compute and control the commutator between the operator and the projection Π_0.

Proposition 10.9. *Let* $\Gamma_0 > 0$. *There exist* $h_0 > 0$ *and* $C > 0$ *such that for* $h \in (0, h_0)$ *and all eigenpair* (λ, ψ) *of* $\mathcal{L}_{\mathsf{Tri}}(h)$ *satisfying* $|\lambda - \frac{1}{8}| \leq \Gamma_0 h^{2/3}$, *we have:*

$$\left\| \left(-h^2\partial_u^2 + \frac{\pi^2}{4(u+\pi\sqrt{2})^2} - \lambda\right)\Pi_0\hat{\psi}\right\| \leq Ch\|\Pi_0\hat{\psi}\|.$$

This is then enough to deduce Theorem 10.1.

11. Another Related Topics: Semiclassical Waveguides

We refer to our review [88].

11.1. Discrete Spectrum of Waveguides: The Result of Duclos-Exner. Quantum waveguides refer to meso- or nanoscale wires (or thin sheets) inside electronic devices. They can be modelled by one-electron Schrödinger operators with potentials having high contrast in their values. In many situations, such Schrödinger operators can be approximated by a simple Laplace operator with Dirichlet conditions on the boundary of the wires [33]. The presence of bound states is an undesirable effect which is nevertheless frequent and useful to predict. The same Laplace-Dirichlet problems arise for TE modes in electromagnetic waveguides [18].

This is a well-known fact, from the papers [35, 33, 19, 22], that curvature makes discrete spectrum to appear in waveguides. Moreover the analysis of this spectrum can be accurately performed in the thin tube limit (in dimension 2 and 3, see [33, Section 5]).

Curvature inducing discrete spectrum, this is then a natural question to ask what happens in dimension 2 when there is corner (infinite curvature): does discrete spectrum always exist? This question is investigated with the L-shape waveguide in [36] where the existence of discrete spectrum is proved. For an arbitrary angle too, this existence is proved in [5] and an asymptotic study of the ground energy is done when θ goes to $\frac{\pi}{2}$ (where θ is the semi-opening of the waveguide). Another question which arises is the estimate of the lowest eigenvalues in the regime $\theta \to 0$. This problem is analyzed in [18] through matched asymptotic expansions and electromagnetic experiments. We also refer to our work [27, 28].

For the case of dimension 3, we can cite the paper [37] which deals with the Dirichlet Laplacian in a conical layer (see also [77]).

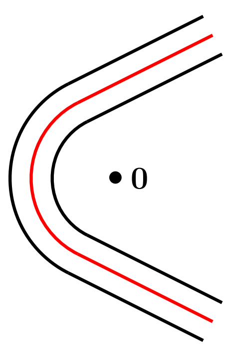


FIGURE 7. Curved guide

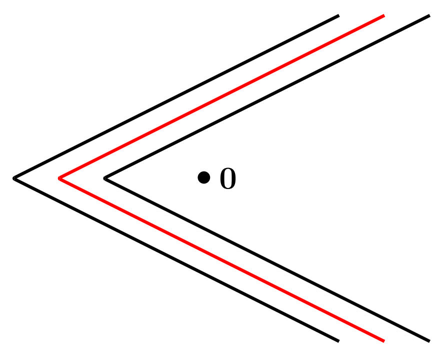


FIGURE 8. Broken guide

11.2. Broken Waveguides. Let us denote by (x_1, x_2) the Cartesian coordinates of the plane and by $\mathbf{0} = (0,0)$ the origin. The positive Laplace operator is given by $-\partial_1^2 - \partial_2^2$. The domains of interest are the "broken waveguides" which are infinite V-shaped open sets: For any angle $\theta \in \left(0, \frac{\pi}{2}\right)$ we introduce

$$\Omega_\theta = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \tan\theta < |x_2| < \left(x_1 + \frac{\pi}{\sin\theta}\right)\tan\theta\right\}. \tag{11.1}$$

Note that its width is independent from θ, normalized to π, and θ represents the (half) opening of the V, see Fig. 9. The limit case where $\theta = \frac{\pi}{2}$ corresponds to the straight strip $(-\pi, 0) \times \mathbb{R}$. The aim of this paper is the investigation of the lowest eigenvalues of the *positive* Dirichlet Laplacian $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ in the small angle limit $\theta \to 0$.

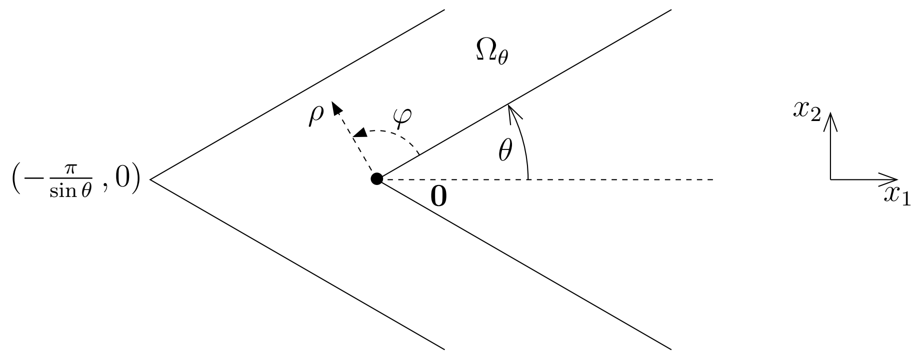


FIGURE 9. The broken guide Ω_θ (here $\theta = \frac{\pi}{6}$). Cartesian and polar coordinates.

The operator $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ is a positive unbounded self-adjoint operator with domain

$$\mathsf{Dom}(\Delta^{\mathsf{Dir}}_{\Omega_\theta}) = \{\psi \in H^1_0(\Omega_\theta) : \quad \Delta\psi \in L^2(\Omega_\theta)\}.$$

When $\theta \in \big(0, \frac{\pi}{2}\big)$, the boundary of Ω_θ is not smooth, it is polygonal. The presence of the non-convex corner with vertex $\mathbf{0}$ is the reason for the space $\mathsf{Dom}(\Delta^{\mathsf{Dir}}_{\Omega_\theta})$ to be distinct from $H^2 \cap H^1_0(\Omega_\theta)$. Nevertheless this domain can be precisely characterized as follows. Let us introduce polar coordinates (ρ, φ) centered at the origin, with $\varphi = 0$ coinciding with the upper part $x_2 = x_1 \tan\theta$ of the boundary of Ω_θ. Let χ be a smooth radial cutoff function with support in the region $x_1 \tan\theta < |x_2|$ and $\chi \equiv 1$ in a neighborhood of the origin. We introduce the explicit *singular function*

$$\psi^\theta_{\mathsf{sing}}(x_1, x_2) = \chi(\rho)\, \rho^{\pi/\omega} \sin\frac{\pi\varphi}{\omega}, \quad \text{with} \quad \omega = 2(\pi - \theta). \tag{11.2}$$

Then there holds, see the classical references [68, 46]:

$$\mathsf{Dom}(\Delta^{\mathsf{Dir}}_{\Omega_\theta}) = \big(H^2 \cap H^1_0(\Omega_\theta)\big) \oplus [\psi^\theta_{\mathsf{sing}}] \tag{11.3}$$

where $[\psi^\theta_{\mathsf{sing}}]$ denotes the space generated by $\psi^\theta_{\mathsf{sing}}$.

When $\theta = \frac{\pi}{2}$, we simply have $\mathsf{Dom}(\Delta^{\mathsf{Dir}}_{\Omega_\theta}) = H^2 \cap H^1_0(\Omega_\theta)$.

We gather in the following statement several important preliminary properties for the spectrum of $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$. All these results are proved in the literature.

Proposition 11.1. (i) *If* $\theta = \frac{\pi}{2}$, $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ *has no discrete spectrum. Its essential spectrum is the closed interval* $[1, +\infty)$.

(ii) *For any* θ *in the open interval* $(0, \frac{\pi}{2})$ *the essential spectrum of* $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ *coincides with* $[1, +\infty)$.

(iii) *For any* $\theta \in (0, \frac{\pi}{2})$*, the discrete spectrum of* $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ *is* nonempty *and* finite. *In other words,* $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ *has at least one eigenvalue below* 1*, but a finite number of them.*

(iv) *For any* $\theta \in (0, \frac{\pi}{2})$ *and any eigenvalue in the discrete spectrum of* $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$*, the associated eigenvectors* ψ *are* even *with respect to the horizontal axis:* $\psi(x_1, -x_2) = \psi(x_1, x_2)$.

(v) *For any* $\theta \in (0, \frac{\pi}{2})$*, let* $\mu_{\mathsf{Gui},n}(\theta)$*,* $n = 1, \ldots$*, be the* n*-th Rayleigh quotient of* $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$*. Then, for any* $n \geq 1$*, the function* $\theta \mapsto \mu_{\mathsf{Gui},n}(\theta)$ *is continuous and increasing.*

11.2.1. *The half-guide.* As a consequence of the parity properties of the eigenvectors of $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$, cf. point (iv) of Proposition 11.1, we can reduce the spectral problem to the half-guide

$$\Omega^+_\theta = \{(x_1, x_2) \in \Omega_\theta :\ x_2 > 0\}\,. \tag{11.4}$$

We define the Dirichlet part of the boundary by $\partial_{\mathsf{Dir}}\Omega^+_\theta = \partial\Omega_\theta \cap \partial\Omega^+_\theta$, and the corresponding variational space (the form domain)

$$H^1_{\mathsf{Mix}}(\Omega^+_\theta) = \big\{\psi \in H^1(\Omega^+_\theta) : \quad \psi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\Omega^+_\theta\big\}.$$

Then the new operator of interest, denoted by $\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}$, is the Laplacian with mixed Dirichlet-Neumann conditions on Ω^+_θ. Its domain is:

$$\mathsf{Dom}(\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}) = \big\{\psi \in H^1_{\mathsf{Mix}}(\Omega^+_\theta) : \ \ \Delta\psi \in L^2(\Omega^+_\theta) \ \text{ and } \ \partial_2\psi = 0 \ \text{ on } \ x_2 = 0\big\}.$$

Then the operators $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ and $\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}$ have the same eigenvalues below 1 and the eigenvectors of the latter are the restriction to Ω^+_θ of the former.

11.2.2. *Rescaling of the half-guide.* In order to analyze the asymptotics $\theta \to 0$, it is useful to rescale the integration domain and transfer the dependence on θ into the coefficients of the operator. For this reason, let us perform the following linear change of coordinates:

$$x = x_1\sqrt{2}\sin\theta, \quad y = x_2\sqrt{2}\cos\theta, \tag{11.5}$$

which maps Ω^+_θ onto $\Omega^+_{\pi/4}$ which will serve as reference domain, see Fig. 10. That is why we set for simplicity

$$\Omega := \Omega^+_{\pi/4}\,, \ \ \partial_{\mathsf{Dir}}\Omega = \partial_{\mathsf{Dir}}\Omega^+_{\pi/4}\,, \ \text{ and } \ H^1_{\mathsf{Mix}}(\Omega) = \big\{\psi \in H^1(\Omega) : \psi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\Omega\big\}. \tag{11.6}$$

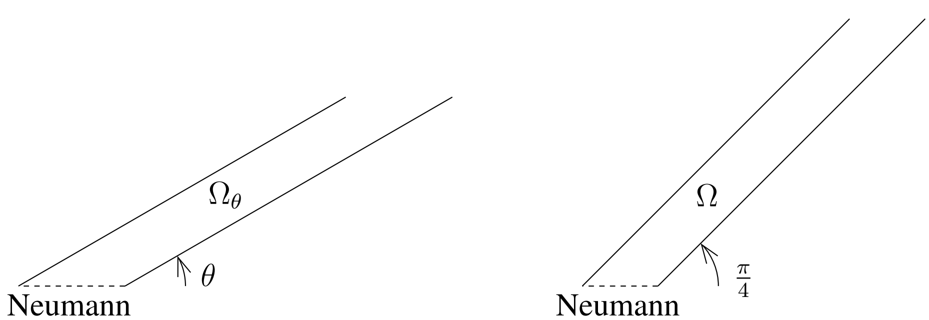


FIGURE 10. The half-guide Ω^+_θ for $\theta = \frac{\pi}{6}$ and the reference domain Ω.

Then, $\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}$ is unitarily equivalent to the operator defined on Ω by:

$$\mathcal{D}_{\mathsf{Gui}}(\theta) := -2\sin^2\theta\,\partial_x^2 - 2\cos^2\theta\,\partial_y^2, \tag{11.7}$$

with Neumann condition on $y = 0$ and Dirichlet everywhere else on the boundary of Ω. We let $h = \tan\theta$; after a division by $2\cos^2\theta$, we get the new operator:

$$\mathcal{L}_{\mathsf{Gui}}(h) = -h^2\partial_x^2 - \partial_y^2, \tag{11.8}$$

with domain:

$$\mathsf{Dom}(\mathcal{L}_{\mathsf{Gui}}(h)) = \left\{\psi \in H^1_{\mathsf{Mix}}(\Omega) : \ \mathcal{L}_{\mathsf{Gui}}(h)\psi \in L^2(\Omega) \ \text{ and } \ \partial_y\psi = 0 \ \text{ on } \ y = 0\right\}.$$

11.3. **A finite number of eigenvalues.** In this subsection, we provide the proof of the following proposition (see [27]).

Proposition 11.2. *For any $\theta \in (0, \frac{\pi}{2})$, the number of eigenvalues of $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ below 1, denoted by $\mathcal{N}(\Delta^{\mathsf{Dir}}_{\Omega_\theta}, 1)$, is finite.*

Thus in any case $\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ has a nonzero finite number of eigenvalues under its essential spectrum.

Proof. For the proof of Proposition 11.2 we use a similar method as [76, Theorem 2.1].

Instead we introduce the open set $\widetilde{\Omega}_\theta$ isometric to Ω^+_θ, see Figure 11,

$$\widetilde{\Omega}_\theta = \left\{(\tilde{x}, \tilde{y}) \in \left(-\frac{\pi}{\tan\theta}, +\infty\right) \times (0, \pi) : \quad \tilde{y} < \tilde{x}\tan\theta + \pi \ \text{ if } \ \tilde{x} \in \left(-\frac{\pi}{\tan\theta}, 0\right)\right\}.$$

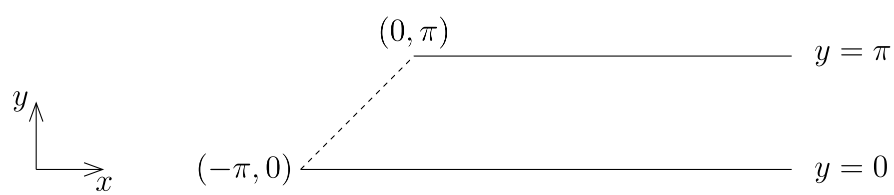


FIGURE 11. The reference half-guide $\widetilde{\Omega} := \widetilde{\Omega}_{\pi/4}$.

The part $\partial_{\mathsf{Dir}}\widetilde{\Omega}_\theta$ of the boundary carrying the Dirichlet condition is the union of its horizontal parts. Let us now perform the change of variable:

$$x = \tilde{x}\tan\theta, \quad y = \tilde{y},$$

so that the new integration domain $\widetilde{\Omega} := \widetilde{\Omega}_{\pi/4}$ is independent of θ. The bilinear gradient form b on $\widetilde{\Omega}_\theta$ is transformed into the anisotropic form b_θ on the fixed set $\widetilde{\Omega}$:

$$b_\theta(\psi, \psi') = \int_{\widetilde{\Omega}} \tan^2\theta\,(\partial_x\psi\,\partial_x\psi') + (\partial_y\psi\,\partial_y\psi')\,dxdy, \tag{11.9}$$

with associated form domain

$$V := \{\psi \in \mathrm{H}^1(\widetilde{\Omega}) : \psi = 0 \text{ on } \partial_{\mathsf{Dir}}\widetilde{\Omega}\} \tag{11.10}$$

independent of θ.

The opening θ being fixed, we drop the index θ in the notation of quadratic forms and write simply as Q the quadratic form associated with b_θ:

$$Q(\psi) = b_\theta(\psi,\psi) = \int_{\widetilde{\Omega}} \tan^2\theta\,|\partial_x\phi|^2 + |\partial_y\phi|^2\,dxdy.$$

We recall that the form domain V is the subspace of $\psi \in H^1(\widetilde{\Omega})$ which satisfy the Dirichlet condition on $\partial_{\mathsf{Dir}}\widetilde{\Omega}$. We want to prove that

$$\mathcal{N}(Q,1) \quad \text{is finite.}$$

We consider a partition of unity (χ_0,χ_1) such that

$$\chi_0(x)^2 + \chi_1(x)^2 = 1$$

with $\chi_0(x) = 1$ for $x < 1$ and $\chi_0(x) = 0$ for $x > 2$. For $R > 0$ and $\ell \in \{0,1\}$, we introduce:

$$\chi_{\ell,R}(x) = \chi_\ell(R^{-1}x).$$

Thanks to the IMS formula, we can split the quadratic form as:

$$Q(\psi) = Q(\chi_{0,R}\psi) + Q(\chi_{1,R}\psi) - \|\chi'_{0,R}\psi\|^2_{\widetilde{\Omega}} - \|\chi'_{1,R}\psi\|^2_{\widetilde{\Omega}}\,. \tag{11.11}$$

We can write

$$|\chi'_{0,R}(x)|^2 + |\chi'_{1,R}(x)|^2 = R^{-2}W_R(x) \quad \text{with} \quad W_R(x) = |\chi'_0(R^{-1}x)|^2 + |\chi'_1(R^{-1}x)|^2\,.$$

Then

$$\begin{aligned}\|\chi'_{0,R}\psi\|^2_{\widetilde{\Omega}} + \|\chi'_{1,R}\psi\|^2_{\widetilde{\Omega}} &= \int_{\widetilde{\Omega}} R^{-2}W_R(x)|\psi|^2\,dxdy\\ &= \int_{\widetilde{\Omega}} R^{-2}W_R(x)\big(|\chi_{0,R}\psi|^2 + |\chi_{1,R}\psi|^2\big)\,dxdy.\end{aligned} \tag{11.12}$$

Let us introduce the subsets of $\widetilde{\Omega}$:

$$\mathcal{O}_{0,R} = \{(x,y)\in\widetilde{\Omega} : x < 2R\} \quad \text{and} \quad \mathcal{O}_{1,R} = \{(x,y)\in\widetilde{\Omega} : x > R\}$$

and the associated form domains

$$V_0 = \Big\{\phi \in H^1(\mathcal{O}_{0,R}) : \quad \phi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\widetilde{\Omega}\cap\partial\mathcal{O}_{0,R} \ \text{ and on } \ \{2R\}\times(0,\pi)\Big\}$$
$$V_1 = \mathrm{H}^1_0(\mathcal{O}_{1,R}).$$

We define the two quadratic forms $Q_{0,R}$ and $Q_{1,R}$ by

$$Q_{\ell,R}(\phi) = \int_{\mathcal{O}_{\ell,R}} \tan^2\theta|\partial_x\phi|^2 + |\partial_y\phi|^2 - R^{-2}W_R(x)|\phi|^2\,dxdy \quad \text{for} \quad \psi \in V_\ell,\ \ \ell = 0,1. \tag{11.13}$$

As a consequence of (11.11) and (11.12) we find

$$Q(\psi) = Q_{0,R}(\chi_{0,R}\psi) + Q_{1,R}(\chi_{1,R}\psi) \quad \forall\psi\in V. \tag{11.14}$$

Let us prove

Lemma 11.3. *We have:*

$$\mathcal{N}(Q,1) \le \mathcal{N}(Q_{0,R},1) + \mathcal{N}(Q_{1,R},1).$$

Proof. We recall the formula for the j-th Rayleigh quotient of Q:

$$\lambda_j = \inf_{\substack{E\subset V\\ \dim E=j}} \sup_{\psi\in E} \frac{Q(\psi)}{\|\psi\|^2_{\widetilde{\Omega}}}\,.$$

The idea is now to give a lower bound for λ_j. Let us introduce:

$$\mathcal{J} : \begin{cases} V & \to \quad V_0\times V_1\\ \psi & \mapsto \quad (\chi_{0,R}\psi\,,\,\chi_{1,R}\psi)\,.\end{cases}$$

As $(\chi_{0,R},\chi_{1,R})$ is a partition of the unity, $\mathcal{J}$ is injective. In particular, we notice that $\mathcal{J}: V \to \mathcal{J}(V)$ is bijective so that we have:

$$\begin{aligned}\lambda_j &= \inf_{\substack{F\subset\mathcal{J}(V)\\ \dim F=j}} \sup_{\psi\in\mathcal{J}^{-1}(F)} \frac{Q(\psi)}{\|\psi\|^2_{\widetilde{\Omega}}}\\ &= \inf_{\substack{F\subset\mathcal{J}(V)\\ \dim F=j}} \sup_{\psi\in\mathcal{J}^{-1}(F)} \frac{Q_{0,R}(\chi_{0,R}\psi)+Q_{1,R}(\chi_{1,R}\psi)}{\|\chi_{0,R}\psi\|^2_{\widetilde{\Omega}}+\|\chi_{1,R}\psi\|^2_{\widetilde{\Omega}}}\\ &= \inf_{\substack{F\subset\mathcal{J}(V)\\ \dim F=j}} \sup_{(\psi_0,\psi_1)\in F} \frac{Q_{0,R}(\psi_0)+Q_{1,R}(\psi_1)}{\|\psi_0\|^2_{\mathcal{O}_{0,R}}+\|\psi_1\|^2_{\mathcal{O}_{1,R}}}.\end{aligned}$$

As $\mathcal{J}(V)\subset V_0\times V_1$, we deduce:

$$\lambda_j \geq \inf_{\substack{F\subset V_0\times V_1\\ \dim F=j}} \sup_{(\psi_0,\psi_1)\in F} \frac{Q_{0,R}(\psi_0)+Q_{1,R}(\psi_1)}{\|\psi_0\|^2_{\mathcal{O}_{0,R}}+\|\psi_1\|^2_{\mathcal{O}_{1,R}}} =: \nu_j, \tag{11.15}$$

Let $A_{\ell,R}$ be the self-adjoint operator with domain $\mathsf{Dom}(A_{\ell,R})$ associated with the coercive bilinear form corresponding to the quadratic form $Q_{\ell,R}$ on V_ℓ. We see that ν_j in (11.15) is the j-th Rayleigh quotient of the diagonal self-adjoint operator A_R

$$\begin{pmatrix} A_{0,R} & 0\\ 0 & A_{1,R}\end{pmatrix} \quad \text{with domain} \quad \mathsf{Dom}(A_{0,R})\times\mathsf{Dom}(A_{1,R})\,.$$

The Rayleigh quotients of $A_{\ell,R}$ are associated with the quadratic form $Q_{\ell,R}$ for $\ell=0,1$. Thus ν_j is the j-th element of the ordered set

$$\{\lambda_k(Q_{0,R}),\ k\geq 1\}\cup\{\lambda_k(Q_{1,R}),\ k\geq 1\}.$$

Lemma 11.3 follows. □

The operator $A_{0,R}$ is elliptic on a bounded open set, hence has a compact resolvent. Therefore we get:

Lemma 11.4. *For all $R>0$, $\mathcal{N}(Q_{0,R},1)$ is finite.*

To achieve the proof of Proposition 11.2, it remains to establish the following lemma:

Lemma 11.5. *There exists $R_0>0$ such that, for $R\geq R_0$, $\mathcal{N}(Q_{1,R},1)$ is finite.*

Proof. For all $\phi \in V_1$, we write:

$$\phi = \Pi_0\phi + \Pi_1\phi,$$

where

$$\Pi_0\phi(x,y) = \Phi(x)\sin y \quad \text{with} \quad \Phi(x) = \int_0^\pi \phi(x,y)\sin y\, dy \tag{11.16}$$

is the projection on the first eigenvector of $-\partial_y^2$ on $\mathrm{H}^1_0(0,\pi)$, and $\Pi_1 = \mathsf{Id} - \Pi_0$. We have, for all $\varepsilon > 0$:

$$\begin{aligned} Q_{1,R}(\phi) &= Q_{1,R}(\Pi_0\phi) + Q_{1,R}(\Pi_1\phi) - 2\int_{\mathcal{O}_{1,R}} R^{-2}W_R(x)\Pi_0\phi\,\Pi_1\phi\, dxdy \\ &\geq Q_{1,R}(\Pi_0\phi) + Q_{1,R}(\Pi_1\phi) - \varepsilon^{-1}\int_{\mathcal{O}_{1,R}} R^{-2}W_R(x)|\Pi_0\phi|^2\, dxdy \\ &\quad - \varepsilon\int_{\mathcal{O}_{1,R}} R^{-2}W_R(x)|\Pi_1\phi|^2\, dxdy. \end{aligned} \tag{11.17}$$

Since the second eigenvalue of $-\partial_y^2$ on $\mathrm{H}^1_0(0,\pi)$ is 4, we have:

$$\int_{\mathcal{O}_{1,R}} |\partial_y\Pi_1\phi|^2\, dxdy \geq 4\|\Pi_1\phi\|^2_{\mathcal{O}_{1,R}}.$$

Denoting by M the maximum of W_R (which is independent of R), and using (11.13) we deduce

$$Q_{1,R}(\Pi_1\phi) \geq (4 - MR^{-2})\|\Pi_1\phi\|^2_{\mathcal{O}_{1,R}}.$$

Combining this with (11.17) where we take $\varepsilon = 1$, and with the definition (11.16) of Π_0, we find

$$Q_{1,R}(\phi) \geq q_R(\Phi) + (4 - 2MR^{-2})\|\Pi_1\phi\|^2_{\mathcal{O}_{1,R}},$$

where

$$\begin{aligned} q_R(\Phi) &= \int_R^\infty \tan^2\theta|\partial_x\Phi|^2 + |\Phi|^2 - R^{-2}W_R(x)|\Phi|^2\, dx \\ &\geq \int_R^\infty \tan^2\theta|\partial_x\Phi|^2 + |\Phi|^2 - R^{-2}M\mathbb{1}_{[R,2R]}|\Phi|^2\, dx. \end{aligned}$$

We choose $R = \sqrt{M}$ so that $(4 - 2MR^{-2}) = 2$, and then

$$Q_{1,R}(\phi) \geq \tilde{q}_R(\Phi) + 2\|\Pi_1\phi\|^2_{\mathcal{O}_{1,R}}, \tag{11.18}$$

where now

$$\tilde{q}_R(\Phi) = \int_R^\infty \tan^2\theta|\partial_x\Phi|^2 + (1 - \mathbb{1}_{[R,2R]})|\Phi|^2\, dx. \tag{11.19}$$

Let $\tilde{a}_R$ denote the 1D operator associated with the quadratic form $\tilde{q}_R$. From (11.18)-(11.19), we deduce that the j-th Rayleigh quotient of $A_{1,R}$ admits as lower bound the j-th Rayleigh quotient of the diagonal operator:

$$\begin{pmatrix} \tilde{a}_R & 0 \\ 0 & 2\,\mathsf{Id} \end{pmatrix}$$

so that we find:

$$\mathcal{N}(Q_{1,R},1) \leq \mathcal{N}(\tilde{q}_R,1).$$

Finally, the eigenvalues < 1 of $\tilde{a}_R$ can be computed explicitly and this is an elementary exercise to deduce that $\mathcal{N}(\tilde{q}_R,1)$ is finite. □

This concludes the proof of Proposition 11.2. □

11.4. **Main result.** Let us now state the main results concerning the asymptotic expansion of the eigenvalues of the broken waveguide.

Theorem 11.6. *For all N_0, there exists $h_0 > 0$, such that for $h \in (0, h_0)$ the N_0 first eigenvalues of $\mathcal{L}_{\mathsf{Gui}}(h)$ exist. These eigenvalues, denoted by $\lambda_{\mathsf{Gui},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{Gui},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \gamma_{j,n} h^{j/3} \quad \textit{with} \;\; \gamma_{0,n} = \frac{1}{8}, \;\; \gamma_{1,n} = 0, \;\; \textit{and} \;\; \gamma_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathbf{A}}(n)$$

and the term of order h is not zero. The corresponding eigenvectors have expansions in powers of $h^{1/3}$ with the scale x/h when $x > 0$, and both scales $x/h^{2/3}$ and x/h when $x < 0$.

11.5. **Born-Oppenheimer Approximation.** The Born-Oppenheimer approximation is:

$$\mathcal{H}_{\mathsf{BO,Gui}}(h) = -h^2\partial_x^2 + V(x), \tag{11.20}$$

where

$$V(x) = \begin{cases} \dfrac{\pi^2}{4(x+\pi\sqrt{2})^2} & \text{when } x \in (-\pi\sqrt{2}, 0), \\ \dfrac{1}{2} & \text{when } x \geq 0. \end{cases}$$

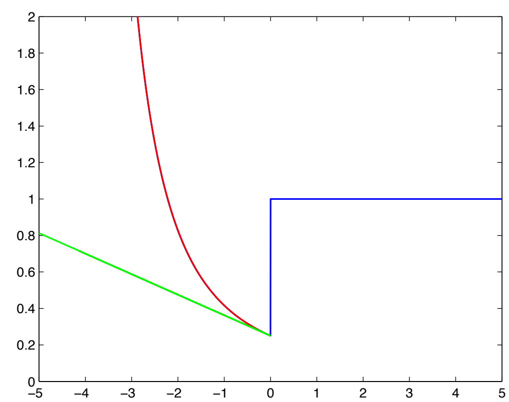


FIGURE 12. The Born-Oppenheimer potential V and its left tangent at $x = 0$.

11.6. **Quasimodes.** As usual we shall introduce appropriate quasimodes. As we will see, we will have to introduce the notion of Dirichlet-to-Neumann operators to analyze the transmission between the corner and the "guiding part" of the waveguide.

11.6.1. *Preliminaries.*

• *Ansatz, boundary and transmission conditions.* In order to construct quasimodes for $\mathcal{L}_{\mathsf{Gui}}(h)$ of the form (γ_h, ψ_h), we use the coordinates (u,t) on the left and (u,τ) on the right and look for quasimodes $\hat{\psi}_h(u,t,\tau) = \psi_h(x,y)$. Such quasimodes will have the form on the left:

$$\psi_{\mathsf{lef}}(u,t) \sim \sum_{j\geq 0} h^{j/3}\left(\Psi_{\mathsf{lef},j}(h^{-2/3}u,t) + \Phi_{\mathsf{lef},j}(h^{-1}u,t)\right), \tag{11.21}$$

and on the right:

$$\psi_{\mathsf{rig}}(u,\tau) \sim \sum_{j\geq 0} h^{j/3}\Phi_{\mathsf{rig},j}(h^{-1}u,\tau) \tag{11.22}$$

associated with quasi-eigenvalues:

$$\gamma_h \sim \sum_{j\geq 0}\gamma_j h^{j/3}.$$

We will denote $s = h^{-2/3}u$ and $\sigma = h^{-1}u$. Since ψ_h has no jump across the line $x = 0$, we find that ψ_{lef} and ψ_{rig} should satisfy two transmission conditions on the line $u = 0$:

$$\psi_{\mathsf{lef}}(0,t) = \psi_{\mathsf{rig}}(0,t) \quad \text{and} \quad \left(\partial_u - \frac{t}{\pi\sqrt{2}}\partial_t\right)\psi_{\mathsf{lef}}(0,t) = \left(\partial_u - \frac{\partial_\tau}{\pi\sqrt{2}}\right)\psi_{\mathsf{rig}}(0,t),$$

for all $t \in (0,1)$. For the Ansätze (11.21)-(11.22) these conditions write for all $j \geq 0$

$$\Psi_{\mathsf{lef},j}(0,t) + \Phi_{\mathsf{lef},j}(0,t) = \Phi_{\mathsf{rig},j}(0,t) \tag{11.23}$$

$$\begin{aligned} \partial_\sigma\Phi_{\mathsf{lef},j}(0,t) + \partial_s\Psi_{\mathsf{lef},j-1}(0,t) - \frac{t\partial_t}{\pi\sqrt{2}}\Phi_{\mathsf{lef},j-3}(0,t) - \frac{t\partial_t}{\pi\sqrt{2}}\Psi_{\mathsf{lef},j-3}(0,t) \\ = \partial_\sigma\Phi_{\mathsf{rig},j}(0,t) - \frac{\partial_\tau}{\pi\sqrt{2}}\Phi_{\mathsf{rig},j-3}(0,t), \end{aligned} \tag{11.24}$$

where we understand that the terms associated with a negative index are 0.

Notation 11.7. We still set $s = h^{-2/3}u$ and $\sigma = h^{-1}u$. Like in the case of the triangle Tri, the operators $\mathcal{L}^{\mathsf{lef}}_{\mathsf{Gui}}$ and $\mathcal{L}^{\mathsf{rig}}_{\mathsf{Gui}}$, written in variables (s,t) and (σ,t) expand in powers of $h^{2/3}$ and h, respectively. Now we have three operator series:

- $\mathcal{L}^{\mathsf{lef}}_{\mathsf{Gui}}(h)(h^{2/3}s,t;h^{-2/3}\partial_s,\partial_t) \sim \sum_{j\geq 0}\mathcal{L}_{2j}h^{2j/3}$. The operators are the same as for Tri, but they are defined now on the half-strip Hlef $:= (-\infty,0)\times(0,1)$.
- $\mathcal{L}^{\mathsf{lef}}_{\mathsf{Gui}}(h)(h\sigma,t;h^{-1}\partial_\sigma,\partial_t) \sim \sum_{j\geq 0}\mathcal{N}^{\mathsf{lef}}_{3j}h^j$ defined on Hlef.
- $\mathcal{L}^{\mathsf{rig}}_{\mathsf{Gui}}(h)(h\sigma,\tau;h^{-1}\partial_\sigma,\partial_\tau) \sim \sum_{j\geq 0}\mathcal{N}^{\mathsf{rig}}_{3j}h^j$ defined on Hrig $:= (0,\infty)\times(0,1)$.

We agree to incorporate the boundary conditions on the horizontal sides of Hlef in the definition of the operators $\mathcal{L}_j$, $\mathcal{N}^{\mathsf{lef}}_j$, and $\mathcal{N}^{\mathsf{rig}}_j$:

- Neumann-Dirichlet $\partial_t\Psi(s,0) = 0$ and $\Psi(s,1) = 0$ $(s < 0)$ for $\mathcal{L}_j$,
- Neumann-Dirichlet $\partial_t\Phi(\sigma,0) = 0$ and $\Psi(\sigma,1) = 0$ $(\sigma < 0)$ for $\mathcal{N}^{\mathsf{lef}}_j$,
- Pure Dirichlet $\Phi(\sigma,0) = 0$ and $\Psi(\sigma,1) = 0$ $(\sigma > 0)$ for $\mathcal{N}^{\mathsf{rig}}_j$.

Note that

$$\mathcal{N}_0^{\mathsf{lef}} = -\partial_\sigma^2 - \frac{1}{2\pi^2}\partial_t^2 \quad \text{and} \quad \mathcal{N}_0^{\mathsf{rig}} = -\partial_\sigma^2 - \frac{1}{2\pi^2}\partial_\tau^2 \,. \tag{11.25}$$

• *Dirichlet-to-Neumann operators.* Here we introduce the Dirichlet-to-Neumann operators T^{rig} and T^{lef} which we use to solve the problems in the variables (σ, t). We denote by I the interface $\{0\} \times (0,1)$ between Hrig and Hlef.

On the right, and with Notation 11.7, we consider the problem:

$$\left(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\right)\Phi_{\mathsf{rig}} = 0 \ \text{ in } \ \mathsf{Hrig} \quad \text{and} \quad \Phi_{\mathsf{rig}}(0,t) = G(t)$$

where $G \in H^{1/2}_{00}(I)$. Since the first eigenvalue of the transverse part of $\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}$ is positive, this problem has a unique exponentially decreasing solution Φ_{rig}. Its exterior normal derivative $-\partial_\sigma \Phi_{\mathsf{rig}}$ on the line I is well defined in $H^{-1/2}(I)$. We define:

$$T^{\mathsf{rig}} G = \partial_n \Phi_{\mathsf{rig}} = -\partial_\sigma \Phi_{\mathsf{rig}}.$$

We have:

$$\langle T^{\mathsf{rig}} G, G\rangle = Q_{\mathsf{rig}}(\Phi_{\mathsf{rig}}) \geq C\|G\|^2_{H^{1/2}_{00}(I)}.$$

On the left, we consider the problem:

$$\left(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\right)\Phi_{\mathsf{lef}} = 0 \ \text{ in } \ \mathsf{Hlef} \quad \text{and} \quad \Phi_{\mathsf{lef}}(0,t) = G(t)$$

where $G \in H^{1/2}_{00}(I)$.

For all $G \in H^{1/2}_{00}(I)$ such that $\Pi_0 G = 0$ (where Π_0 is defined in (10.15)), this problem has a unique exponentially decreasing solution Φ_{lef}. Its exterior normal derivative $\partial_\sigma \Phi_{\mathsf{lef}}$ on the line I is well defined in $H^{-1/2}(I)$. We define:

$$T^{\mathsf{lef}} G = \partial_n \Phi_{\mathsf{lef}} = \partial_\sigma \Phi_{\mathsf{lef}}.$$

We have:

$$\langle T^{\mathsf{lef}} G, G\rangle = Q_{\mathsf{lef}}(\Phi_{\mathsf{lef}}) \geq 0.$$

Proposition 11.8. *The operator $T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1$ is coercive on $H^{1/2}_{00}(I)$ with $\Pi_1 = \mathsf{Id} - \Pi_0$. In particular, it is invertible from $H^{1/2}_{00}(I)$ onto $H^{-1/2}(I)$.*

This proposition allows to prove the following lemma which is in the same spirit as Lemma 10.3, but now for transmission problems on $\mathsf{Hlef} \cup \mathsf{Hrig}$ (we recall that $c_0(t) = \cos(\frac{\pi}{2}t)$):

Lemma 11.9. *Let $F_{\mathsf{lef}} = F_{\mathsf{lef}}(\sigma, t)$ and $F_{\mathsf{rig}} = F_{\mathsf{rig}}(\sigma, \tau)$ be real functions defined on Hlef and Hrig, respectively, with exponential decay with respect to σ. Let $G^0 \in H^{1/2}_{00}(I)$ and $H \in H^{-1/2}(I)$ be data on the interface $I = \partial\mathsf{Hlef} \cap \partial\mathsf{Hrig}$. Then there exists a unique coefficient $\zeta \in \mathbb{R}$ and a unique trace $G \in H^{1/2}_{00}(I)$ such that the transmission problem*

$$\begin{cases} (\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8})\Phi_{\mathsf{lef}} = F_{\mathsf{lef}} \quad \textit{in } \mathsf{Hlef}, & \Phi_{\mathsf{lef}}(0,t) = G(t) + G^0(t) + \zeta c_0(t), \\ (\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8})\Phi_{\mathsf{rig}} = F_{\mathsf{rig}} \quad \textit{in } \mathsf{Hrig}, & \Phi_{\mathsf{rig}}(0,t) = G(t), \\ \partial_\sigma \Phi_{\mathsf{lef}}(0,t) - \partial_\sigma \Phi_{\mathsf{rig}}(0,t) = H(t) \quad \textit{on } \ I, & \end{cases}$$

admits a (unique) solution $(\Phi_{\mathsf{lef}}, \Phi_{\mathsf{rig}})$ *with exponential decay.*

Proof. Let $(\Phi^0_{\mathsf{lef}}, \zeta_0)$ be the solution provided by Lemma 10.3 for the data $F = F_{\mathsf{lef}}$ and $G = 0$. Let Φ^0_{rig} be the unique exponentially decreasing solution of the problem

$$\left(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\right)\Phi^0_{\mathsf{rig}} = F_{\mathsf{rig}} \quad \text{in} \quad \mathsf{Hrig}, \quad \Phi^0_{\mathsf{rig}}(0,t) = 0.$$

Let H^0 be the jump $\partial_\sigma \Phi^0_{\mathsf{rig}}(0,t) - \partial_\sigma \Phi^0_{\mathsf{lef}}(0,t)$. If we define the new unknowns $\Phi^1_{\mathsf{rig}} = \Phi_{\mathsf{rig}} - \Phi^0_{\mathsf{rig}}$ and $\Phi^1_{\mathsf{lef}} = \Phi_{\mathsf{lef}} - \Phi^0_{\mathsf{lef}}$, the problem we want to solve becomes

$$\begin{aligned}
&\left(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\right)\Phi^1_{\mathsf{lef}} = 0 \quad \text{in} \quad \mathsf{Hlef}, \quad \Phi^1_{\mathsf{lef}}(0,t) = G(t) + (\zeta - \zeta_0)c_0(t),\\
&\left(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\right)\Phi^1_{\mathsf{rig}} = 0 \quad \text{in} \quad \mathsf{Hrig}, \quad \Phi^1_{\mathsf{rig}}(0,t) = G(t),\\
&\partial_\sigma \Phi^1_{\mathsf{rig}}(0,t) - \partial_\sigma \Phi^1_{\mathsf{lef}}(0,t) = H(t) - H^0(t) \quad \text{on} \quad I.
\end{aligned}$$

Using Proposition 11.8 we can set $G = (T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1)^{-1}(H - H_0)$, which ensures the solvability of the above problem. ☐

11.6.2. *Construction of quasimodes.*

- *Terms of order* h^0. Let us write the "interior" equations:

$$\begin{aligned}
\mathsf{lef}_s : \quad & \mathcal{L}_0 \Psi_{\mathsf{lef},0} = \gamma_0 \Psi_{\mathsf{lef},0}\\
\mathsf{lef}_\sigma : \quad & \mathcal{N}_0^{\mathsf{lef}} \Phi_{\mathsf{lef},0} = \gamma_0 \Phi_{\mathsf{lef},0}\\
\mathsf{rig} : \quad & \mathcal{N}_0^{\mathsf{rig}} \Phi_{\mathsf{rig},0} = \gamma_0 \Phi_{\mathsf{rig},0}\,.
\end{aligned}$$

The boundary conditions are:

$$\Psi_{\mathsf{lef},0}(0,t) + \Phi_{\mathsf{lef},0}(0,t) = \Phi_{\mathsf{rig},0}(0,t),$$
$$\partial_\sigma \Phi_{\mathsf{lef},0}(0,t) = \partial_\sigma \Phi_{\mathsf{rig},0}(0,t).$$

We get:

$$\gamma_0 = \frac{1}{8}, \quad \Psi_{\mathsf{lef},0} = g_0(s)c_0(t).$$

We now apply Lemma 11.9 with $F_{\mathsf{lef}} = 0$, $F_{\mathsf{rig}} = 0$, $G_0 = 0$, $H = 0$ to get

$$G = 0 \quad \text{and} \quad \zeta = 0.$$

We deduce: $\Phi_{\mathsf{lef},0} = 0$, $\Phi_{\mathsf{rig},0} = 0$ and, since $\zeta = -g_0(0)$, $g_0(0) = 0$. At this step, we do not have determined g_0 yet.

- *Terms of order* $h^{1/3}$. The interior equations read:

$$\begin{aligned}
\mathsf{lef}_s : \quad & \mathcal{L}_0 \Psi_{\mathsf{lef},1} = \gamma_0 \Psi_{\mathsf{lef},1} + \gamma_1 \Psi_{\mathsf{lef},0}\\
\mathsf{lef}_\sigma : \quad & \mathcal{N}_0^{\mathsf{lef}} \Phi_{\mathsf{lef},1} = \gamma_0 \Phi_{\mathsf{lef},1} + \gamma_1 \Phi_{\mathsf{lef},0}\\
\mathsf{rig} : \quad & \mathcal{N}_0^{\mathsf{rig}} \Phi_{\mathsf{rig},1} = \gamma_0 \Phi_{\mathsf{rig},1} + \gamma_1 \Phi_{\mathsf{rig},0}.
\end{aligned}$$

Using Lemma 10.4, the first equation implies:

$$\gamma_1 = 0, \quad \Psi_{\mathsf{lef},1}(s,t) = g_1(s)c_0(t).$$

The boundary conditions are:

$$g_1(0)c_0(t) + \Phi_{\mathsf{lef},1}(0,t) = \Phi_{\mathsf{rig},1}(0,t),$$
$$g_0'(0)c_0(t) + \partial_\sigma\Phi_{\mathsf{lef},1}(0,t) = \partial_\sigma\Phi_{\mathsf{rig},1}(0,t).$$

The system becomes:

$$\mathsf{lef}_\sigma : \quad \Big(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\Big)\Phi_{\mathsf{lef},1} = 0,$$
$$\mathsf{rig} : \quad \Big(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\Big)\Phi_{\mathsf{rig},1} = 0.$$

We apply Lemma 11.9 with $F_{\mathsf{lef}} = 0$, $F_{\mathsf{rig}} = 0$, $G_0 = 0$, $H = -g_0'(0)c_0(t)$ to get:

$$G = -g_0'(0)(T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1)^{-1}c_0.$$

Since $G = \Phi_{\mathsf{rig},1}$ and $\zeta = -g_1(0)$, this determines $\Phi_{\mathsf{lef},1}$, $\Phi_{\mathsf{rig},1}$ and $g_1(0)$.

- *Terms of order $h^{2/3}$.* The interior equations write:

$$\mathsf{lef}_s : \quad \mathcal{L}_2\Psi_{\mathsf{lef},0} + \mathcal{L}_0\Psi_{\mathsf{lef},2} = \sum_{l+k=2}\gamma_l\Psi_{\mathsf{lef},k}$$
$$\mathsf{lef}_\sigma : \quad \mathcal{N}_0^{\mathsf{lef}}\Phi_{\mathsf{lef},2} = \sum_{l+k=2}\gamma_l\Phi_{\mathsf{lef},k}$$
$$\mathsf{rig} : \quad \mathcal{N}_0^{\mathsf{rig}}\Phi_{\mathsf{rig},2} = \frac{1}{8}\Phi_{\mathsf{rig},2},$$

with

$$\mathcal{L}_2\Psi_{\mathsf{lef},0} = -g_0''(s)c_0(t) + \frac{1}{\pi^3\sqrt{2}}sg_0(s)\partial_t^2(c_0).$$

Lemma 10.4 and then Lemma 10.5 imply:

$$-g_0'' - \frac{1}{4\pi\sqrt{2}}sg_0 = \gamma_2 g_0. \tag{11.26}$$

Thus, γ_2 is one of the eigenvalues of the Airy operator and g_0 an associated eigenfunction. In particular, this determines the unknown functions of the previous steps. We are led to take:

$$\Psi_{\mathsf{lef},2}(s,t) = \Psi_{\mathsf{lef},2}^{\perp} + g_2(s)c_0(t), \text{ with } \Psi_{\mathsf{lef},2}^{\perp} = 0$$

and to the system:

$$\mathsf{lef}_\sigma : \quad \Big(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\Big)\Phi_{\mathsf{lef},2} = 0$$
$$\mathsf{rig} : \quad \Big(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\Big)\Phi_{\mathsf{rig},2} = 0.$$

Using Lemma 11.9, we find

$$G = -g_1'(0)(T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1)^{-1}c_0.$$

This determines $\Phi_{\mathsf{rig},2}$, $\Phi_{\mathsf{lef},2}$ and $g_2(0)$. The function g_1 is still unknown at this step.

- *Further terms.* The next steps are left to the reader.

11.7. **Reduction to Triangles.** In this last subsection, we prove Theorem 11.6. For that purpose, we first state Agmon estimates to show that the first eigenfunctions are essentially living in the triangle Tri so that we can compare the problem in the whole guide with the triangle.

Proposition 11.10. *Let (λ, ψ) be an eigenpair of $\mathcal{L}_{\mathsf{Gui}}(h)$ such that $|\lambda - \frac{1}{8}| \leq Ch^{2/3}$. There exist $\alpha > 0$, $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$, we have:*

$$\int_{x \geq 0} e^{\alpha h^{-1} x} \Big(|\psi|^2 + |h\partial_x \psi|^2 \Big)\, dxdy \leq C\|\psi\|^2.$$

Proof. The proof is left to the reader, the main ingredients being the IMS formula and the fact that $\mathcal{H}_{\mathsf{BO,Gui}}$ is a lower bound of $\mathcal{L}_{\mathsf{Gui}}(h)$ in the sense of quadratic forms. See also [27, Proposition 6.1] for a more direct method. □

• *Proof of Theorem* 11.6. Let ψ_n^h be an eigenfunction associated with $\lambda_{\mathsf{Gui},n}(h)$ and assume that the ψ_n^h are orthogonal in $L^2(\Omega)$, and thus for the bilinear form $B_{\mathsf{Gui},h}$ associated with the operator $\mathcal{L}_{\mathsf{Gui}}(h)$.

We choose $\varepsilon \in (0, \frac{1}{3})$ and introduce a smooth cutoff χ^hat the scale $h^{1-\varepsilon}$ for positive x

$$\chi^h(x) = \chi(xh^{\varepsilon-1}) \quad \text{with} \quad \chi \equiv 1 \text{ if } x \leq \tfrac{1}{2}, \quad \chi \equiv 0 \text{ if } x \geq 1$$

and we consider the functions $\chi^h \psi_n^h$. We denote:

$$\mathfrak{E}_{N_0}(h) = \operatorname{span}(\chi^h\psi_1^h, \ldots, \chi^h\psi_{N_0}^h).$$

We have:

$$Q_{\mathsf{Gui},h}(\psi_n^h) = \lambda_{\mathsf{Gui},n}(h)\|\psi_n^h\|^2$$

and deduce by the Agmon estimates of Proposition 11.10:

$$Q_{\mathsf{Gui},h}(\chi^h\psi_n^h) = \big(\lambda_{\mathsf{Gui},n}(h) + O(h^\infty)\big)\|\chi^h\psi_n^h\|^2.$$

In the same way, we get the "almost"-orthogonality, for $n \neq m$:

$$B_{\mathsf{Gui},h}(\chi^h\psi_n^h, \chi^h\psi_m^h) = O(h^\infty).$$

We deduce, for all $v \in \mathfrak{E}_{N_0}(h)$:

$$Q_{\mathsf{Gui},h}(v) \leq \big(\lambda_{\mathsf{Gui},N_0}(h) + O(h^\infty)\big)\|v\|^2.$$

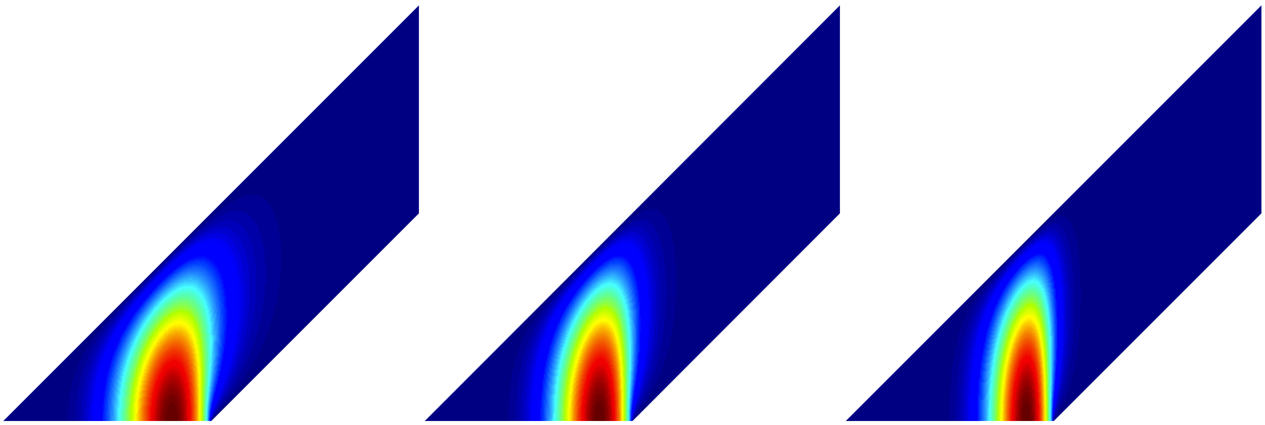
We can extend the elements of $\mathfrak{E}_{N_0}(h)$ by zero so that $Q_{\mathsf{Gui},h}(v) = Q_{\mathsf{Tri}_{\varepsilon,h}}(v)$ for $v \in \mathfrak{E}_{N_0}(h)$ where $\mathsf{Tri}_{\varepsilon,h}$ is the triangle with vertices $(-\pi\sqrt{2}, 0)$, $(h^{1-\varepsilon}, 0)$ and $(h^{1-\varepsilon}, h^{1-\varepsilon} + \pi\sqrt{2})$. A dilation reduces us to:

$$\left(1 + \frac{h^{1-\varepsilon}}{\pi\sqrt{2}}\right)^{-2} (-h^2\partial_{\tilde{x}}^2 - \partial_{\tilde{y}}^2)$$

on the triangle Tri. The lowest eigenvalues of this new operator admits the lower bounds $\frac{1}{8} + z_{\mathbf{A}}(n)h^{2/3} - Ch^{1-\varepsilon}$; in particular, we deduce:

$$\lambda_{\mathsf{Gui},N_0}(h) \geq \frac{1}{8} + z_{\mathbf{A}}(N_0)h^{2/3} - Ch^{1-\varepsilon}.$$

11.8. **Numerical simulations.** Below we provide numerical simulations of the first eigenfunction (see [27] for more numerical simulations). In particular, we can observe the jump of the potential which creates a wall for the eigenfunctions in the semiclassical regime.

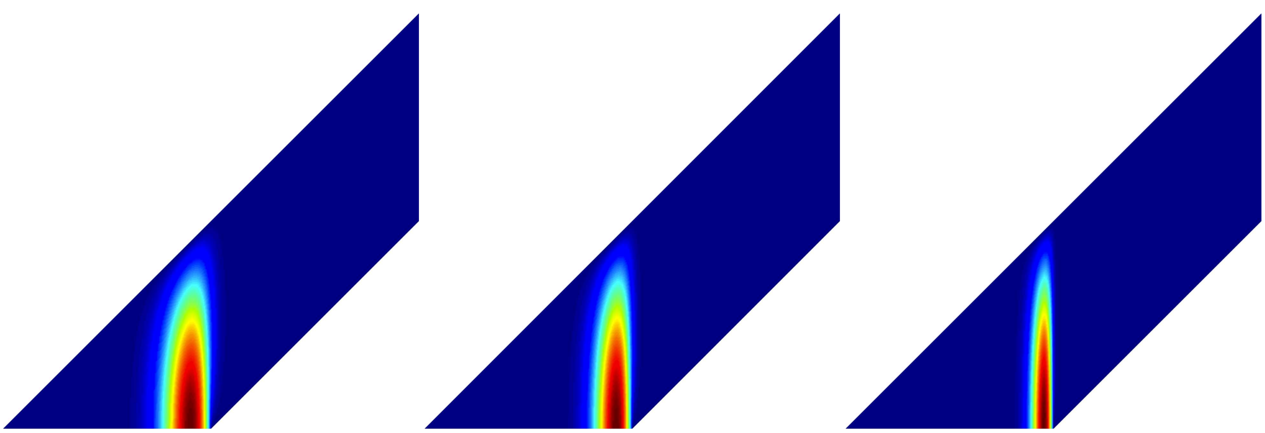


$\theta = 0.1482 * \pi/2$
$\lambda_1^{\mathsf{cpt}}(\theta) = 0.56209$

$\theta = 0.1032 * \pi/2$
$\lambda_1^{\mathsf{cpt}}(\theta) = 0.48754$

$\theta = 0.0701 * \pi/2$
$\lambda_1^{\mathsf{cpt}}(\theta) = 0.42763$

FIGURE 13. Computations for small angles. Plots in the computational domain Ω.



$\theta = 0.0416 * \pi/2$
$\lambda_1^{\mathsf{cpt}}(\theta) = 0.37085$

$\theta = 0.0270 * \pi/2$
$\lambda_1^{\mathsf{cpt}}(\theta) = 0.33845$

$\theta = 0.0112 * \pi/2$
$\lambda_1^{\mathsf{cpt}}(\theta) = 0.29766$

FIGURE 14. Computations for very small angles. Plots in the computational domain Ω.

REFERENCES

[1] Shmuel Agmon. *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, volume 29 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1982.

[2] Shmuel Agmon. Bounds on exponential decay of eigenfunctions of Schrödinger operators. In *Schrödinger operators (Como, 1984)*, volume 1159 of *Lecture Notes in Math.*, pages 1–38. Springer, Berlin, 1985.

[3] Stan Alama, Lia Bronsard, and Bernardo Galvão-Sousa. Thin film limits for Ginzburg-Landau with strong applied magnetic fields. *SIAM J. Math. Anal.*, 42(1):97–124, 2010.

[4] V. I. Arnol′d. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 199? Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition.

[5] Y. Avishai, D. Bessis, B. G. Giraud, and G. Mantica. Quantum bound states in open geometries. *Phys. Rev. B*, 44(15):8028–8034, Oct 1991.

[6] P. Bauman, D. Phillips, and Q. Tang. Stable nucleation for the Ginzburg-Landau system with an applied magnetic field. *Arch. Rational Mech. Anal.*, 142(1):1–43, 1998.

[7] Andrew Bernoff and Peter Sternberg. Onset of superconductivity in decreasing fields for general domains. *J. Math. Phys.*, 39(3):1272–1284, 1998.

[8] Catherine Bolley and Bernard Helffer. The Ginzburg-Landau equations in a semi-infinite superconducting film in the large κ limit. *European J. Appl. Math.*, 8(4):347–367, 1997.

[9] Virginie Bonnaillie. On the fundamental state energy for a Schrödinger operator with magnetic field in domains with corners. *Asymptot. Anal.*, 41(3-4):215–258, 2005.

[10] Virginie Bonnaillie, Monique Dauge, Nicolas Popoff, and Nicolas Raymond. Discrete spectrum of a model schrödinger operator on the half-plane with neumann conditions. *Z. Ange. Math. Phys.*, 2011.

[11] V. Bonnaillie-Noël, M. Dauge, D. Martin, and G. Vial. Computations of the first eigenpairs for the Schrödinger operator with magnetic field. *Comput. Methods Appl. Mech. Engrg.*, 196(37-40):3841–3858, 2007.

[12] V. Bonnaillie-Noël and S. Fournais. Superconductivity in domains with corners. *Rev. Math. Phys.*, 19(6):607–637, 2007.

[13] Virginie Bonnaillie-Noël and Monique Dauge. Asymptotics for the low-lying eigenstates of the Schrödinger operator with magnetic field near corners. *Ann. Henri Poincaré*, 7(5):899–931, 2006.

[14] Denis Borisov and Pedro Freitas. Singular asymptotic expansions for Dirichlet eigenvalues and eigenfunctions of the Laplacian on thin planar domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(2):547–560, 2009.

[15] Denis Borisov and Pedro Freitas. Asymptotics of Dirichlet eigenvalues and eigenfunctions of the Laplacian on thin domains in $\mathbb{R}^d$. *J. Funct. Anal.*, 258:893–912, 2010.

[16] M. Born and R. Oppenheimer. Zur Quantentheorie der Molekeln. *Ann. Phys.*, 84:457–484, 1927.

[17] Haïm Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[18] John P. Carini, J. T. Londergan, Kieran Mullen, and D. P. Murdock. Multiple bound states in sharply bent waveguides. *Phys. Rev. B*, 48(7):4503–4515, Aug 1993.

[19] G. Carron, P. Exner, and D. Krejčiřík. Topologically nontrivial quantum layers. *J. Math. Phys.*, 45(2):774–784, 2004.

[20] S. Jonathan Chapman, Qiang Du, and Max D. Gunzburger. On the Lawrence-Doniach and anisotropic Ginzburg-Landau models for layered superconductors. *SIAM J. Appl. Math.*, 55(1):156–174, 1995.

[21] Laurent Charles and San Vũ Ngọc. Spectral asymptotics via the semiclassical Birkhoff normal form. *Duke Math. J.*, 143(3):463–511, 2008.

[22] B. Chenaud, P. Duclos, P. Freitas, and D. Krejčiřík. Geometrically induced discrete spectrum in curved tubes. *Differential Geom. Appl.*, 23(2):95–105, 2005.

[23] Jean-Michel Combes, Pierre Duclos, and Ruedi Seiler. The Born-Oppenheimer approximation. *Rigorous atomic and molecular physics (eds G. Velo, A. Wightman).*, pages 185–212, 1981.

[24] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer-Verlag, Berlin, study edition, 1987.
[25] Monique Dauge and Isabelle Gruais. Asymptotics of arbitrary order for a thin elastic clamped plate. II. Analysis of the boundary layer terms. *Asymptot. Anal.*, 16(2):99–124, 1998.
[26] Monique Dauge and Bernard Helffer. Eigenvalues variation. I. Neumann problem for Sturm-Liouville operators. *J. Differential Equations*, 104(2):243–262, 1993.
[27] Monique Dauge, Yvon Lafranche, and Nicolas Raymond. Quantum waveguides with corners. In *Actes du Congrès SMAI 2011)*, ESAIM Proc. EDP Sciences, Les Ulis, 2012.
[28] Monique Dauge and Nicolas Raymond. Plane waveguides with corners : small angle limit. *Submitted*, 2012.
[29] Manuel del Pino, Patricio L. Felmer, and Peter Sternberg. Boundary concentration for eigenvalue problems related to the onset of superconductivity. *Comm. Math. Phys.*, 210(2):413–446, 2000.
[30] Mouez Dimassi and Johannes Sjöstrand. *Spectral asymptotics in the semi-classical limit*, volume 268 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1999.
[31] Nicolas Dombrowski and Nicolas Raymond. Semiclassical analysis with vanishing magnetic fields. *To appear in JST*, 2012.
[32] Mauro M. Doria and Sarah C. B. de Andrade. Virial theorem for the anisotropic Ginzburg-Landau theory. *Phys. Rev. B*, 53:3440–3454, Feb 1996.
[33] P. Duclos and P. Exner. Curvature-induced bound states in quantum waveguides in two and three dimensions. *Rev. Math. Phys.*, 7(1):73–102, 1995.
[34] Ju. V. Egorov. Canonical transformations and pseudodifferential operators. *Trudy Moskov. Mat. Obšč.*, 24:3–28, 1971.
[35] P. Exner and P. Šeba. Bound states in curved quantum waveguides. *J. Math. Phys.*, 30(11):2574–2580, 1989.
[36] P. Exner, P. Šeba, and P. Šťovíček. On existence of a bound state in an L-shaped waveguide . *Czech. J. Phys.*, 39(11):1181–1191, 1989.
[37] P. Exner and M. Tater. Spectrum of Dirichlet Laplacian in a conical layer. *J. Phys.*, A43, 2010.
[38] Frédéric Faure, Nicolas Raymond, and San Vũ Ngọc. Normal form for the 2D magnetic Laplacian. *In preparation*, 2012.
[39] Soeren Fournais and Bernard Helffer. Accurate eigenvalue asymptotics for the magnetic Neumann Laplacian. *Ann. Inst. Fourier (Grenoble)*, 56(1):1–67, 2006.
[40] Søren Fournais and Bernard Helffer. *Spectral methods in surface superconductivity*. Progress in Nonlinear Differential Equations and their Applications, 77. Birkhäuser Boston Inc., Boston, MA, 2010.
[41] Søren Fournais and Mikael Persson. Strong diamagnetism for the ball in three dimensions. *Asymptot. Anal.*, 72(1-2):77–123, 2011.
[42] Pedro Freitas. Precise bounds and asymptotics for the first Dirichlet eigenvalue of triangles and rhombi . *J. Funct. Anal.*, 251:376–398, 2007.
[43] L. Friedlander and M. Solomyak. On the spectrum of narrow periodic waveguides. *Russ. J. Math. Phys.*, 15(2):238–242, 2008.
[44] Leonid Friedlander and Michael Solomyak. On the spectrum of the Dirichlet Laplacian in a narrow strip. *Israel J. Math.*, 170:337–354, 2009.
[45] T. Giorgi and D. Phillips. The breakdown of superconductivity due to strong fields for the Ginzburg-Landau model. *SIAM J. Math. Anal.*, 30(2):341–359 (electronic), 1999.
[46] P. Grisvard. *Boundary Value Problems in Non-Smooth Domains*. Pitman, London, 1985.
[47] V. V. Grušin. Hypoelliptic differential equations and pseudodifferential operators with operator-valued symbols. *Mat. Sb. (N.S.)*, 88(130):504–521, 1972.
[48] B. Helffer. The Montgomery model revisited. *Colloq. Math.*, 118(2):391–400, 2010.
[49] B. Helffer and Y. A. Kordyukov. Spectral gaps for periodic Schrödinger operators with hypersurface magnetic wells: analysis near the bottom. *J. Funct. Anal.*, 257(10):3043–3081, 2009.
[50] B. Helffer and Y. A. Kordyukov. Eigenvalue estimates for a three-dimensional magnetic schrödinger operator. *Preprint*, 2012.

[51] B. Helffer and Y. A. Kordyukov. Semiclassical spectral asymptotics for a two-dimensional magnetic schroedinger operator. ii the case of degenerate wells. *Comm. PDE*, 37:1057–1095, 2012.
[52] B. Helffer and J. Sjöstrand. Multiple wells in the semiclassical limit. I. *Comm. Partial Differential Equations*, 9(4):337–408, 1984.
[53] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique. II. Interaction moléculaire. Symétries. Perturbation. *Ann. Inst. H. Poincaré Phys. Théor.*, 42(2):127–212, 1985.
[54] B. Helffer and J. Sjöstrand. Effet tunnel pour l'équation de Schrödinger avec champ magnétique. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 14(4):625–657 (1988), 1987.
[55] Bernard Helffer. *Semi-classical analysis for the Schrödinger operator and applications*, volume 1336 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.
[56] Bernard Helffer. Introduction to semi-classical methods for the Schrödinger operator with magnetic field. In *Aspects théoriques et appliqués de quelques EDP issues de la géométrie ou de la physique*, volume 17 of *Sémin. Congr.*, pages 49–117. Soc. Math. France, Paris, 2009.
[57] Bernard Helffer and Yuri A. Kordyukov. Semiclassical spectral asymptotics for a two-dimensional magnetic Schrödinger operator: the case of discrete wells. In *Spectral theory and geometric analysis*, volume 535 of *Contemp. Math.*, pages 55–78. Amer. Math. Soc., Providence, RI, 2011.
[58] Bernard Helffer and Abderemane Mohamed. Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells. *J. Funct. Anal.*, 138(1):40–81, 1996.
[59] Bernard Helffer and Abderemane Morame. Magnetic bottles in connection with superconductivity. *J. Funct. Anal.*, 185(2):604–680, 2001.
[60] Bernard Helffer and Abderemane Morame. Magnetic bottles for the Neumann problem: the case of dimension 3. *Proc. Indian Acad. Sci. Math. Sci.*, 112(1):71–84, 2002. Spectral and inverse spectral theory (Goa, 2000).
[61] Bernard Helffer and Abderemane Morame. Magnetic bottles for the Neumann problem: curvature effects in the case of dimension 3 (general case). *Ann. Sci. École Norm. Sup. (4)*, 37(1):105–170, 2004.
[62] Bernard Helffer and Mikael Persson. Spectral properties of higher order Anharmonic Oscillators. *J. Funct. Anal.*, 165(1), 2010.
[63] Luc Hillairet and Chris Judge. Spectral simplicity and asymptotic separation of variables. *Comm. Math. Phys.*, 302(2):291–344, 2011.
[64] Lars Hörmander. *The analysis of linear partial differential operators. III*. Classics in Mathematics. Springer, Berlin, 2007. Pseudo-differential operators, Reprint of the 1994 edition.
[65] Hala T. Jadallah. The onset of superconductivity in a domain with a corner. *J. Math. Phys.*, 42(9):4101–4121, 2001.
[66] Tosio Kato. *Perturbation theory for linear operators*. Die Grundlehren der mathematischen Wissenschaften, Band 132. Springer-Verlag New York, Inc., New York, 1966.
[67] M. Klein, A. Martinez, R. Seiler, and X. P. Wang. On the Born-Oppenheimer expansion for polyatomic molecules. *Comm. Math. Phys.*, 143(3):607–639, 1992.
[68] V. A. Kondrat'ev. Boundary-value problems for elliptic equations in domains with conical or angular points. *Trans. Moscow Math. Soc.*, 16:227–313, 1967.
[69] Pierre Lévy-Bruhl. *Introduction à la théorie spectrale*. Sciences Sup. Dunod, Paris, 2003.
[70] Kening Lu and Xing-Bin Pan. Eigenvalue problems of Ginzburg-Landau operator in bounded domains. *J. Math. Phys.*, 40(6):2647–2670, 1999.
[71] Kening Lu and Xing-Bin Pan. Surface nucleation of superconductivity in 3-dimensions. *J. Differential Equations*, 168(2):386–452, 2000. Special issue in celebration of Jack K. Hale's 70th birthday, Part 2 (Atlanta, GA/Lisbon, 1998).
[72] A. Martinez. Développements asymptotiques et effet tunnel dans l'approximation de Born-Oppenheimer. *Ann. Inst. H. Poincaré Phys. Théor.*, 50(3):239–257, 1989.
[73] André Martinez. *An introduction to semiclassical and microlocal analysis*. Universitext. Springer-Verlag, New York, 2002.
[74] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[75] Richard Montgomery. Hearing the zero locus of a magnetic field. *Comm. Math. Phys.*, 168(3):651–675, 1995.

[76] Abderemane Morame and Françoise Truc. Remarks on the spectrum of the Neumann problem with magnetic field in the half-space. *J. Math. Phys.*, 46(1):012105, 13, 2005.

[77] Thomas Ourmières. Guides d'ondes quantiques (thèse de doctorat). *En cours.*, 2011.

[78] Thomas Ourmières. Dirichlet eigenvalues of cones in the small aperture limit. *In preparation.*, 2012.

[79] Xing-Bin Pan. Upper critical field for superconductors with edges and corners. *Calc. Var. Partial Differential Equations*, 14(4):447–482, 2002.

[80] Xing-Bin Pan and Keng-Huat Kwek. Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains. *Trans. Amer. Math. Soc.*, 354(10):4201–4227 (electronic), 2002.

[81] Arne Persson. Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator. *Math. Scand.*, 8:143–153, 1960.

[82] Nicolas Popoff. Sur l'opérateur de Schrödinger magnétique dans un domaine diédral. (thèse de doctorat). *Université de Rennes 1*, 2012.

[83] Nicolas Popoff and Nicolas Raymond. When the 3D magnetic Laplacian meets a curved edge in the semiclassical limit. *In preparation.*, 2012.

[84] Nicolas Raymond. Sharp asymptotics for the Neumann Laplacian with variable magnetic field: case of dimension 2. *Ann. Henri Poincaré*, 10(1):95–122, 2009.

[85] Nicolas Raymond. On the semiclassical 3D Neumann Laplacian with variable magnetic field. *Asymptot. Anal.*, 68(1-2):1–40, 2010.

[86] Nicolas Raymond. From the Laplacian with variable magnetic field to the electric Laplacian in the semiclassical limit. *To appear in APDE*, 2012.

[87] Nicolas Raymond. Semiclassical 3D Neumann Laplacian with variable magnetic field : a toy model. *Comm. in PDE*, 37(9):1528–1552, 2012.

[88] Nicolas Raymond. Une excursion semi-classique dans l'univers des guides d'ondes. *Gazette des Mathématiciens*, 2012.

[89] Michael Reed and Barry Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.

[90] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[91] Didier Robert. *Autour de l'approximation semi-classique*, volume 68 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1987.

[92] D. Saint-James, G. Sarma, and E.J Thomas. *Type II Superconductivity*. Pergamon, Oxford, 1969.

[93] San Vũ Ngọc. *Systèmes intégrables semi-classiques: du local au global*, volume 22 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2006.

[94] San Vũ Ngọc. Quantum Birkhoff normal forms and semiclassical analysis. In *Noncommutativity and singularities*, volume 55 of *Adv. Stud. Pure Math.*, pages 99–116. Math. Soc. Japan, Tokyo, 2009.

[95] Joachim Weidmann. The virial theorem and its application to the spectral theory of Schrödinger operators. *Bull. Amer. Math. Soc.*, 73:452–456, 1967.

IRMAR (UMR 6625), UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX (FRANCE)