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# Brownian motion in the quadrant with oblique repulsion from the sides

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## Abstract

We consider the problem of strong existence and uniqueness of a Brownian motion forced to stay in the quadrant by an electrostatic repulsion from the sides that works obliquely. When the repulsion works in a normal direction, the question has previously been solved by a convex analysis method. To construct the solution, we start from the normal case and then we use as main tool a comparison lemma. The results are reminiscent of the study of a Brownian motion with oblique reflection in a wedge. Actually, the same skew symmetry condition is involved when looking for a stationary distribution in product form. The terms of the product are now gamma distributions in place of exponential ones.

## 1 Introduction

In the late seventies the study of heavy traffic limits in open multi-station queueing networks has put the question of existence and properties of the Brownian motion obliquely reflected on the sides of a wedge and more generally on the faces of a polyhedron. In the following decade there was an extensive literary output on that topic, among which we mention the remarkable works of Harrison, Reiman, Williams and co-authors ([7],[8],[15],[17],[9],[5], to cite a few of them). But there is another way than normal or oblique reflection to prevent Brownian motion from overstepping a linear barrier. We may add as drift term the gradient of a concave function that explodes in the neighborhood of the faces of the polyhedron. To be more specific, let  $\mathbf{n}_1, \dots, \mathbf{n}_k$  be unit vectors in  $\mathbb{R}^d$  and  $b_1, \dots, b_k$  be real numbers. The state space  $S$  is defined by

$$S := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{n}_r \cdot \mathbf{x} \geq b_r, r = 1, \dots, k\}.$$

Let  $\phi_1, \dots, \phi_k$  be  $k$  convex  $C^1$  functions on  $(0, \infty)$  with  $\phi_r(0+) = +\infty$  for any  $r = 1, \dots, k$ . The potential function  $\Phi$  on  $S$  is defined by

$$\Phi(\mathbf{x}) := \sum_{r=1}^k \phi_r(\mathbf{n}_r \cdot \mathbf{x} - b_r).$$

From the results in [1],[2],[3],[12] we know there exists a unique strong solution living in  $S$  to the equation

$$\begin{aligned} d\mathbf{X}_t &= d\mathbf{B}_t - \nabla\Phi(\mathbf{X}_t)dt \\ \mathbf{X}_0 &\in S \end{aligned} \tag{1}$$

where  $\mathbf{B}$  is a Brownian motion in  $\mathbb{R}^d$ . Since

$$\nabla\Phi(\mathbf{x}) = \sum_{r=1}^k \phi_r'(\mathbf{n}_r \cdot \mathbf{x} - b_r) \mathbf{n}_r,$$

the repulsion from the faces  $F_r = \{\mathbf{x} \in S : \mathbf{n}_r \cdot \mathbf{x} = b_r\}$  points in a normal direction into the interior of  $S$ . If we introduce a set of vectors  $\mathbf{q}_1, \dots, \mathbf{q}_k$  with  $\mathbf{q}_r \cdot \mathbf{n}_r = 0$  for  $r = 1, \dots, k$ , we may consider a new drift function

$$-\sum_{r=1}^k \phi_r'(\mathbf{n}_r \cdot \mathbf{x} - b_r) (\mathbf{n}_r + \mathbf{q}_r). \quad (2)$$

This is a singular drift but convex analysis cannot be used as in [1] or [2] to get strong existence and uniqueness of the associated stochastic differential system.

In this paper we will only take the quadrant in  $\mathbb{R}^2$  as state space and an electrostatic drift term. We obtain strong existence and uniqueness for a large set of parameters. Our results are reminiscent of the thorough study in [15] about oblique reflection in a wedge. However a key tool in this work was an appropriate harmonic function that made a weak approach possible and fruitful. Thus full results were obtained, while our strong approach merely provides a partial answer to the crucial question of hitting the corner.

For theoretical as well as practical reasons, a great deal of interest was taken in the question of existence and computation of the invariant measure of the Brownian motion with a constant drift vector and oblique reflection ([8],[17],[9]). Under the assumption that the directions of reflection satisfy a skew symmetry condition, it was proved that the invariant measure has exponential product form density. Motivated by the so-called Atlas model of equity markets, some authors ([14],[10],[11]) have recently studied Brownian motions on the line with rank dependent local characteristics. As this model is strongly related to Brownian motion reflected in polyhedral domains, the above mentioned product form of the stationary distribution happens to be of paramount importance. Related interesting works are also [6] and [4].

A neighboring way has been recently explored in [13]. Here the process is a Brownian motion with a drift term that is continuous and depends obliquely, via a regular potential function, on the position of the process relative to a polyhedral domain. Under the same skew symmetry condition as in [9], the invariant density has an explicit product form again. In the last section of our paper, we consider a Brownian motion with a constant drift living in the quadrant with oblique electrostatic repulsion from the sides. Under the skew symmetry condition, there is still an invariant measure in product form. Now the terms of the product are two gamma distributions with explicit parameters.

## 2 The setting

The general state space is the quadrant  $S = \mathbb{R}_+ \times \mathbb{R}_+$ . The corner  $\mathbf{0} = (0, 0)$  will play a crucial role and in some cases it will be necessary to restrict the state space to  $S^0 = S \setminus \{\mathbf{0}\}$ .

Let  $(B_t, C_t)$  be a Brownian motion in the plane starting from  $\mathbf{0}$ , adapted to a filtration  $\mathcal{F} = (\mathcal{F}_t)$  with usual conditions. Let  $\alpha, \beta, \gamma, \delta$  be four real constants with  $\alpha > 0, \delta > 0$ . We say that an  $\mathcal{F}$ -adapted continuous process  $(X, Y)$  with values in  $S$  is a Brownian motion with electrostatic oblique repulsion from the sides if for any  $t \geq 0$

$$\begin{aligned} X_t &= X_0 + B_t + \alpha \int_0^t \frac{ds}{X_s} + \beta \int_0^t \frac{ds}{Y_s} \geq 0 \\ Y_t &= Y_0 + C_t + \gamma \int_0^t \frac{ds}{X_s} + \delta \int_0^t \frac{ds}{Y_s} \geq 0 \end{aligned} \quad (3)$$

where  $X_0$  and  $Y_0$  are non-negative  $\mathcal{F}_0$ -measurable random variables. Each coordinate  $X_t$  or  $Y_t$  may vanish, so to make sense we must have a.s. for any  $t \geq 0$

$$\begin{aligned} \int_0^t \mathbf{1}_{\{X_s=0\}} ds &= 0 & \int_0^t \mathbf{1}_{\{Y_s=0\}} ds &= 0 \\ \int_0^t \mathbf{1}_{\{X_s>0\}} \frac{ds}{X_s} &< \infty & \int_0^t \mathbf{1}_{\{Y_s>0\}} \frac{ds}{Y_s} &< \infty. \end{aligned}$$

The drift in (3) is of type (2) with  $d = k = 2$  and

$$\begin{aligned} \phi_1(x) &= -\alpha \log(x) & \phi_2(y) &= -\delta \log(y) \\ b_1 &= 0 & b_2 &= 0 \\ \mathbf{n}_1 &= (1, 0) & \mathbf{n}_2 &= (0, 1) \\ \mathbf{q}_1 &= (0, \frac{\gamma}{\alpha}) & \mathbf{q}_2 &= (\frac{\beta}{\delta}, 0) \end{aligned}$$

The case where  $\beta = \gamma = 0$  is a particular case of (1). In the sequel, we will note  $(U, V)$  the solution of the system

$$\begin{aligned} U_t &= X_0 + B_t + \alpha \int_0^t \frac{ds}{U_s} \geq 0 \\ V_t &= Y_0 + C_t + \delta \int_0^t \frac{ds}{V_s} \geq 0. \end{aligned} \quad (4)$$

The processes  $U$  and  $V$  are independent Bessel processes (if  $X_0$  and  $Y_0$  are independent variables). Actually,  $U$  is a Bessel process with index  $\alpha - \frac{1}{2}$ , and the point 0 is instantaneously reflecting for  $U$  if  $\alpha < \frac{1}{2}$  and polar if  $\alpha \geq \frac{1}{2}$ . Moreover,  $U^2 + V^2$  is the square of a Bessel process with index  $\alpha + \delta$ , and so the corner  $\mathbf{0}$  is polar for  $(U, V)$  in any case.

Comparison between  $X$  and  $U$ ,  $Y$  and  $V$  will play a key role in the construction of the solution  $(X, Y)$  and the study of its behavior near the sides of the quadrant. The following simple lemma will be of constant use.

**Lemma 1** . For  $T > 0$ ,  $\alpha > 0$ , let  $x_1$  and  $x_2$  be nonnegative continuous solutions on  $[0, T]$  of the equations

$$\begin{aligned} x_1(t) &= v_1(t) + \alpha \int_0^t \frac{ds}{x_1(s)} \\ x_2(t) &= v_2(t) + \alpha \int_0^t \frac{ds}{x_2(s)} \end{aligned}$$

where  $v_1, v_2$  are continuous functions such that  $0 \leq v_1(0) \leq v_2(0)$ , and  $v_2 - v_1$  is nondecreasing. Then  $x_1(t) \leq x_2(t)$  on  $[0, T]$ .

**Proof.** Assume there exists  $t \in (0, T]$  such that  $x_2(t) < x_1(t)$ . Set

$$\tau := \max\{s \leq t : x_1(s) \leq x_2(s)\}.$$

Then,

$$\begin{aligned} x_2(t) - x_1(t) &= x_2(\tau) - x_1(\tau) + (v_2(t) - v_1(t)) - (v_2(\tau) - v_1(\tau)) + \int_\tau^t \left( \frac{1}{x_2(s)} - \frac{1}{x_1(s)} \right) ds \\ &\geq 0, \end{aligned}$$

a contradiction. ■

We will also need the following form of the results in [1],[2],[3],[12] on multivalued stochastic differential systems.

**Proposition 2** . Let  $\alpha > 0$ ,  $\delta \geq 0$ ,  $\sigma = (\sigma_j^i; i, j = 1, 2)$  a  $2 \times 2$ -matrix,  $(B_1, B_2)$  a Brownian motion in the plane,  $b_1$  and  $b_2$  two Lipschitz functions on  $\mathbb{R}^2$ ,  $Z_0^1$  and  $Z_0^2$  two  $\mathcal{F}_0$ -measurable nonnegative random variables. There exists a unique solution  $(Z^1, Z^2)$  to the system

$$\begin{aligned} Z_t^1 &= Z_0^1 + \sigma_1^1 B_t^1 + \sigma_2^1 B_t^2 + \alpha \int_0^t \frac{ds}{Z_s^1} + \int_0^t b_1(Z_s^1, Z_s^2) ds \\ Z_t^2 &= Z_0^2 + \sigma_1^2 B_t^1 + \sigma_2^2 B_t^2 + \delta \int_0^t \frac{ds}{Z_s^2} + \int_0^t b_2(Z_s^1, Z_s^2) ds \end{aligned} \quad (5)$$

with the conditions  $Z_t^1 \geq 0$  if  $\delta = 0$  and  $Z_t^1 \geq 0, Z_t^2 \geq 0$  if  $\delta > 0$ .

### 3 Avoiding the corner

We shall see in the next section that existence and uniqueness to the solution to (3) are easily obtained as soon as the solution process keeps away from the corner. Thus the question of hitting the corner in finite time is of great interest.

**Theorem 3** . *Let  $(X, Y)$  be a solution to (3) in the interval  $[0, \tau] \cap [0, \infty)$  where  $\tau$  is a  $\mathcal{F}$ -stopping time. We set*

$$\tau^{\mathbf{0}} := \inf\{t \in (0, \tau] \cap (0, \infty) : (X_t, Y_t) = \mathbf{0}\}.$$

Then  $\mathbb{P}(\tau^{\mathbf{0}} < \infty) = 0$  if one of the following conditions is satisfied:

1.  $C_1$  :  $\beta \geq 0$  and  $\gamma \geq 0$
2.  $C_{2a}$  :  $\alpha \geq \frac{1}{2}$  and  $\beta \geq 0$
3.  $C_{2b}$  :  $\delta \geq \frac{1}{2}$  and  $\gamma \geq 0$
4.  $C_3$  : There exist  $\lambda > 0$  and  $\mu > 0$  such that
  - $\lambda\alpha + \mu\gamma \geq 0$
  - $\lambda\beta + \mu\delta \geq 0$
  - $\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta) - \frac{1}{2}(\lambda^2 + \mu^2) \geq -2\sqrt{\lambda\mu(\lambda\beta + \mu\delta)(\lambda\alpha + \mu\gamma)}$ .

**Proof.**

*Condition  $C_1$ .* From Lemma 1 we get  $X_t \geq U_t$ ,  $Y_t \geq V_t$ . where  $(U, V)$  is the solution to (4), and we know that  $\mathbf{0}$  is polar for  $(U, V)$ .

*Condition  $C_{2a}$  (resp.  $C_{2b}$ ).* From Lemma 1 we get  $X_t \geq U_t$  (resp.  $Y_t \geq V_t$ ) and in this case  $\mathbf{0}$  is polar for  $U$  (resp.  $V$ ), so  $U_t > 0$  (resp.  $V_t > 0$ ) for  $t \in (0, \tau] \cap (0, \infty)$ .

*Condition  $C_3$ .* For  $\epsilon > 0$  let

$$\begin{aligned} \sigma^\epsilon &= \mathbf{1}_{\{(X_0, Y_0) = \mathbf{0}\}} \inf\{t \in (0, \tau] \cap (0, \infty) : X_t + Y_t \geq \epsilon\} \\ \tau^{0, \epsilon} &= \inf\{t \in (\sigma^\epsilon, \tau] \cap (0, \infty) : (X_t, Y_t) = \mathbf{0}\}. \end{aligned}$$

As  $\epsilon \downarrow 0$ ,  $\sigma^\epsilon \downarrow 0$  on  $\{\tau > 0\}$  and  $\tau^{0, \epsilon} \downarrow \tau^{\mathbf{0}}$ . We set  $S_t = \lambda X_t + \mu Y_t$  for  $t \in [0, \tau] \cap [0, \infty)$ . From Ito formula we get for  $t \in [\sigma^\epsilon, \tau^{0, \epsilon}] \cap [0, \tau]$

$$\begin{aligned} &\log S_t \\ &= \log S_{\sigma^\epsilon} + \int_{\sigma^\epsilon}^t \frac{\lambda dB_s + \mu dC_s}{S_s} + (\lambda\alpha + \mu\gamma) \int_{\sigma^\epsilon}^t \frac{ds}{X_s S_s} + (\lambda\beta + \mu\delta) \int_{\sigma^\epsilon}^t \frac{ds}{Y_s S_s} - \frac{1}{2}(\lambda^2 + \mu^2) \int_{\sigma^\epsilon}^t \frac{ds}{S_s^2} \\ &= \log S_{\sigma^\epsilon} + M_t + \int_{\sigma^\epsilon}^t \frac{P(X_s, Y_s)}{X_s Y_s S_s^2} ds \end{aligned}$$

where  $M$  is a continuous local martingale and  $P(x, y)$  is the second degree homogeneous polynomial

$$P(x, y) = \lambda(\lambda\beta + \mu\delta)x^2 + \mu(\lambda\alpha + \mu\gamma)y^2 + (\lambda(\lambda\alpha + \mu\gamma) + \mu(\lambda\beta + \mu\delta) - \frac{1}{2}(\lambda^2 + \mu^2))xy.$$

Condition  $C_3$  is exactly the condition for  $P$  being nonnegative in  $S$ . Therefore

$$0 \leq \int_{\sigma^\epsilon}^t \frac{P(X_s, Y_s)}{X_s Y_s S_s^2} ds < \infty$$

and so

$$0 \leq \int_{\sigma^\epsilon}^{\tau^{0,\epsilon} \wedge \tau} \frac{P(X_s, Y_s)}{X_s Y_s S_s^2} ds \leq \infty.$$

As  $t \rightarrow \tau^{0,\epsilon} \wedge \tau$ , the local martingale  $M$  either converges to a finite limit or oscillates between  $+\infty$  and  $-\infty$ . It cannot converge to  $-\infty$  and thus  $S_{\tau^{0,\epsilon} \wedge \tau} > 0$  on  $\{\tau^{0,\epsilon} \wedge \tau < \infty\}$ , proving that  $\mathbb{P}(\tau^{0,\epsilon} < \infty) = 0$  and finally  $\mathbb{P}(\tau^0 < \infty) = 0$ .  $\blacksquare$

**Example.** When  $\alpha = \delta$  and  $|\beta| = |\gamma|$ , Condition  $C_3$  is satisfied (with  $\lambda = \mu$ ) if

$$\begin{aligned} \bullet \quad & \beta^2 \leq \alpha - \frac{1}{4} \quad \text{when } \beta = -\gamma \\ \bullet \quad & -\beta \leq \alpha - \frac{1}{4} \quad \text{when } \beta = \gamma < 0. \end{aligned} \quad (6)$$

We may also be interested in hitting a single side. Then we set

$$\begin{aligned} \tau_X^0 &:= \inf\{t \in (0, \tau] \cap (0, \infty) : X_t = 0\} \\ \tau_Y^0 &:= \inf\{t \in (0, \tau] \cap (0, \infty) : Y_t = 0\}. \end{aligned} \quad (7)$$

We already know that  $\mathbb{P}(\tau_X^0 < \infty) = 0$  if  $\alpha \geq \frac{1}{2}$  and  $\beta \geq 0$ . In a symmetric way we can prove that  $\mathbb{P}(\tau_X^0 < \infty) > 0$  if  $\alpha < \frac{1}{2}$  and  $\beta \leq 0$  ( $= 1$  if moreover  $\tau = \infty$  a.s.). If we know that the corner is not hit and  $\alpha \geq \frac{1}{2}$ , we can get rid of the nonnegativity assumption on  $\beta$ .

**Proposition 4 .** Assume  $\mathbb{P}(\tau^0 < \infty) = 0$ . If  $\alpha \geq \frac{1}{2}$ , then  $\mathbb{P}(\tau_X^0 < \infty) = 0$ .

**Proof.** For  $\eta > 0$  let

$$\begin{aligned} \theta_X^\eta &= \mathbf{1}_{\{X_0=0\}} \inf\{t \in (0, \tau] \cap (0, \infty) : X_t \geq \eta\} \\ \tau_X^{0,\eta} &= \inf\{t \in (\theta_X^\eta, \tau] \cap (0, \infty) : X_t = 0\}. \end{aligned}$$

As  $\eta \downarrow 0$ ,  $\theta_X^\eta \downarrow 0$  on  $\{\tau > 0\}$  and  $\tau_X^{0,\eta} \downarrow \tau_X^0$ . For  $t \in [\theta_X^\eta, \tau_X^{0,\eta}) \cap [0, \tau]$ ,

$$\log X_t = \log X_{\theta_X^\eta} + \int_{\theta_X^\eta}^t \frac{dB_s}{X_s} + \left(\alpha - \frac{1}{2}\right) \int_{\theta_X^\eta}^t \frac{ds}{X_s^2} + \beta \int_{\theta_X^\eta}^t \frac{ds}{X_s Y_s}. \quad (8)$$

Since  $\mathbb{P}(\tau^0 < \infty) = 0$ , we have  $Y_{\tau_X^{0,\eta}} > 0$  on  $\{\tau_X^{0,\eta} < \infty\}$ . On this set,

$$\int_{\theta_X^\eta}^{\tau_X^{0,\eta}} \frac{ds}{X_s} < \infty, \quad \int_{\theta_X^\eta}^{\tau_X^{0,\eta}} \frac{ds}{Y_s} < \infty$$

and  $X_s > 0$  on  $[\theta_X^\eta, \tau_X^{0,\eta})$ , which proves that

$$\beta \int_{\theta_X^\eta}^{\tau_X^{0,\eta}} \frac{ds}{X_s Y_s} > -\infty.$$

As  $t \rightarrow \tau_X^{0,\eta}$ , the local martingale in the R.H.S. of (8) cannot converge to  $-\infty$ . This entails that  $\mathbb{P}(\tau_X^{0,\eta} < \infty) = 0$  and therefore  $\mathbb{P}(\tau_X^0 < \infty) = 0$ .  $\blacksquare$

## 4 Existence and uniqueness

We now proceed to the question of existence and uniqueness of a global solution to (3). We consider separately the three cases:  $\beta \geq 0$  and  $\gamma \geq 0$ ,  $\beta \geq 0$  and  $\gamma < 0$ ,  $\beta < 0$  and  $\gamma < 0$ .

#### 4.1 Case $\beta \geq 0$ and $\gamma \geq 0$

This is exactly Condition  $C_1$ .

**Theorem 5** . Assume  $\beta \geq 0$  and  $\gamma \geq 0$ .

1. There is a unique solution to (3) in  $S^0$ .
2. There is a solution to (3) in  $S$  starting from  $\mathbf{0}$ .
3. If  $\alpha\delta \geq \beta\gamma$ , there is a unique solution to (3) in  $S$ .

**Proof.** 1. Let  $a > 0$ ,  $\epsilon > 0$  and define for  $(x, z) \in \mathbb{R}_+ \times \mathbb{R}$

$$\psi_\epsilon(x, z) := \max\left(\frac{1}{\gamma x + z}, \frac{1}{\alpha\epsilon}\right).$$

This is a Lipschitz function. From Proposition 2 we know that the system

$$\begin{aligned} X_t^\epsilon &= X_0 + B_t + \alpha \int_0^t \frac{ds}{X_s^\epsilon} + \alpha\beta \int_0^t \psi_\epsilon(X_s^\epsilon, Z_s^\epsilon) ds \geq 0 \\ Z_t^\epsilon &= -\gamma X_0 + \alpha(Y_0 + \mathbf{1}_{\{Y_0=0\}}a) - \gamma B_t + \alpha C_t + \alpha(\alpha\delta - \beta\gamma) \int_0^t \psi_\epsilon(X_s^\epsilon, Z_s^\epsilon) ds \end{aligned} \quad (9)$$

has a unique solution. Let

$$\tau_Y^\epsilon := \inf\{t > 0 : \gamma X_t^\epsilon + Z_t^\epsilon < \alpha\epsilon\}.$$

If  $0 < \eta < \epsilon < a$  we deduce from the uniqueness that  $(X^\epsilon, Z^\epsilon)$  and  $(X^\eta, Z^\eta)$  are identical on  $[0, \tau_Y^\epsilon]$ . Patching together we can set

$$\begin{aligned} X_t &:= \lim_{\epsilon \rightarrow 0} X_t^\epsilon \\ Y_t &:= \lim_{\epsilon \rightarrow 0} \frac{1}{\alpha}(\gamma X_t^\epsilon + Z_t^\epsilon) \end{aligned}$$

on  $\{Y_0 > 0\} \times [0, \tau_Y^0)$ , where

$$\tau_Y^0 := \lim_{\epsilon \rightarrow 0} \tau_Y^\epsilon.$$

On this set,  $(X, Y)$  is the unique solution to (3). As we noted in the proof of Condition  $C_1$ , we have  $X_t \geq U_t$  and  $Y_t \geq V_t$ . Therefore, on  $\{Y_0 > 0\} \cap \{\tau_Y^0 < \infty\}$ ,

$$\int_0^{\tau_Y^0} \frac{ds}{X_s} \leq \int_0^{\tau_Y^0} \frac{ds}{U_s} < \infty \quad \text{and} \quad \int_0^{\tau_Y^0} \frac{ds}{Y_s} \leq \int_0^{\tau_Y^0} \frac{ds}{V_s} < \infty$$

and we can define

$$\begin{aligned} X_{\tau_Y^0} &:= \lim_{t \rightarrow \tau_Y^0} X_t = X_0 + B_{\tau_Y^0} + \alpha \int_0^{\tau_Y^0} \frac{ds}{X_s} + \beta \int_0^{\tau_Y^0} \frac{ds}{Y_s} \\ Y_{\tau_Y^0} &:= \lim_{t \rightarrow \tau_Y^0} Y_t = Y_0 + C_{\tau_Y^0} + \gamma \int_0^{\tau_Y^0} \frac{ds}{X_s} + \delta \int_0^{\tau_Y^0} \frac{ds}{Y_s}. \end{aligned} \quad (10)$$

We have  $Y_{\tau_Y^0} = 0$  and as  $\mathbf{0}$  is polar for  $(U, V)$ , then  $X_{\tau_Y^0} > 0$ . In exactly the same way we can construct a solution on  $\{Y_0 > 0\}$  in the interval  $[T_1, T_2]$ , where  $T_1 = \tau_Y^0$ ,  $T_2 = \inf\{t > T_1 : X_t = 0\}$ . Iterating, we get a solution on  $\{Y_0 > 0\} \cap [0, \lim_{n \rightarrow \infty} T_n)$  where

$$\begin{aligned} T_{2p} &:= \inf\{t > T_{2p-1} : X_t = 0\} \\ T_{2p+1} &:= \inf\{t > T_{2p} : Y_t = 0\}. \end{aligned}$$

On  $\{Y_0 > 0\} \cap \{\lim_{n \rightarrow \infty} T_n < \infty\}$  we set  $X_{\lim_{n \rightarrow \infty} T_n} := \lim_{p \rightarrow \infty} X_{T_{2p}} = 0$  and  $Y_{\lim_{n \rightarrow \infty} T_n} := \lim_{p \rightarrow \infty} Y_{T_{2p+1}} = 0$ . The polarity of  $\mathbf{0}$  entails this is not possible in finite time and thus

$\lim_{n \rightarrow \infty} T_n = \infty$ . So we have obtained a unique global solution on  $\{Y_0 > 0\}$ . In the same way we obtain a unique global solution on  $\{X_0 > 0\}$  and as  $\mathbb{P}((X_0, Y_0) = \mathbf{0}) = 0$  the proof is complete.

2. Assume now  $X_0 = Y_0 = 0$ . Let  $(y_n)_{n \geq 1}$  be a sequence of real numbers (strictly) decreasing to 0. Consider for any  $n \geq 1$  the unique solution  $(X^n, Y^n)$  to the system

$$\begin{aligned} X_t^n &= B_t + \alpha \int_0^t \frac{ds}{X_s^n} + \beta \int_0^t \frac{ds}{Y_s^n} \\ Y_t^n &= y_n + C_t + \gamma \int_0^t \frac{ds}{X_s^n} + \delta \int_0^t \frac{ds}{Y_s^n}. \end{aligned}$$

From the continuity of  $Y_t$  there exists a positive stopping time  $\rho$  such that a.s.  $Y_t^{n+1} < Y_t^n$  on the interval  $[0, \rho]$ . From Lemma 1 we deduce that  $X_t^{n+1} \geq X_t^n$  on  $[0, \rho]$ . Let

$$\tau := \inf\{t > 0 : X_t^{n+1} < X_t^n\}.$$

Using again Lemma 1 we obtain  $Y_t^{n+1} \leq Y_t^n$  on  $[0, \tau]$ . On  $\{Y_\tau^{n+1} = Y_\tau^n\} \cap \{\tau < \infty\}$ , since  $X_\tau^{n+1} = X_\tau^n$ , we note that  $(X_\tau^n, Y_\tau^n) \in S^0$ . Therefore the uniqueness of the solution starting at time  $\tau$  implies that  $X_t^{n+1} = X_t^n$  and  $Y_t^{n+1} = Y_t^n$  on  $[\tau, \infty)$ . On  $\{Y_\tau^{n+1} < Y_\tau^n\} \cap \{\tau < \infty\}$ , the same proof as above entails there exists  $\rho' > 0$  such that  $X_t^{n+1} \geq X_t^n$  on  $[\tau, \tau + \rho']$ , a contradiction to the definition of  $\tau$ . Therefore  $X_t^{n+1} \geq X_t^n$  and  $Y_t^{n+1} \leq Y_t^n$  for any  $t \in [0, \infty)$ . We define

$$X_t := \lim_{n \rightarrow \infty} \uparrow X_t^n \quad Y_t := \lim_{n \rightarrow \infty} \downarrow Y_t^n.$$

As  $Y_t^n \geq V_t$  where  $(U, V)$  is the solution to (4) with  $X_0 = Y_0 = 0$ , we have

$$\begin{aligned} X_t &= B_t + \alpha \lim_{n \rightarrow \infty} \int_0^t \frac{ds}{X_s^n} + \beta \lim_{n \rightarrow \infty} \int_0^t \frac{ds}{Y_s^n} \\ &= B_t + \alpha \int_0^t \frac{ds}{X_s} + \beta \int_0^t \frac{ds}{Y_s} \\ &< \infty \end{aligned}$$

and also

$$\begin{aligned} Y_t &= \lim_{n \rightarrow \infty} y_n + C_t + \gamma \lim_{n \rightarrow \infty} \int_0^t \frac{ds}{X_s^n} + \delta \lim_{n \rightarrow \infty} \int_0^t \frac{ds}{Y_s^n} \\ &= C_t + \gamma \int_0^t \frac{ds}{X_s} + \delta \int_0^t \frac{ds}{Y_s} \\ &< \infty. \end{aligned}$$

3. Assume finally  $\alpha\delta - \beta\gamma \geq 0$ . As the conclusion holds true if  $\beta = \gamma = 0$ , we may also assume that  $\beta > 0$ . Let  $(X, Y)$  be the solution to (3) with  $X_0 = Y_0 = 0$  obtained in the previous paragraph and let  $(X', Y')$  be another solution. Considering  $(X^n, Y^n)$  again and replacing  $(X^{n+1}, Y^{n+1})$  with  $(X', Y')$ , the previous proof works and we finally obtain  $X'_t \geq X_t$  and  $Y'_t \leq Y_t$ . Then,

$$\begin{aligned} &(\delta(X_t - X'_t) - \beta(Y_t - Y'_t))^2 \\ &= 2 \int_0^t (\delta(X_s - X'_s) - \beta(Y_s - Y'_s))(\alpha\delta - \beta\gamma) \left(\frac{1}{X_s} - \frac{1}{X'_s}\right) ds \\ &\leq 0 \end{aligned} \tag{11}$$

and thus  $X'_t = X_t$  and  $Y'_t = Y_t$ , proving uniqueness. Same conclusion with a small change in equation (11) if  $\gamma > 0$ .  $\blacksquare$

## 4.2 Case $\beta \geq 0$ and $\gamma < 0$

**Theorem 6** . Assume  $\beta \geq 0$ ,  $\gamma < 0$  and one of the Conditions  $C_{2a}$  or  $C_3$  is satisfied. Then, there exists a unique solution to (3) in  $S^0$ .



**Proof.** The proof is similar to the proof of 1 in Theorem 5. The only change is that now  $Y_t \leq V_t$ . Therefore, on  $\{Y_0 > 0\} \cap \{\tau_Y^0 < \infty\}$ ,

$$\delta \int_0^{\tau_Y^0} \frac{ds}{Y_s} \leq V_{\tau_Y^0} - Y_0 - C_{\tau_Y^0} - \gamma \int_0^{\tau_Y^0} \frac{ds}{U_s} < \infty$$

and we can define  $X_{\tau_Y^0}$  and  $Y_{\tau_Y^0}$  as previously. ■

### 4.3 Case $\beta < 0$ and $\gamma < 0$

In this case we can give a full answer to the question of existence and uniqueness. Our condition of existence is exactly the condition found in [16] for the reflected Brownian in a wedge being a semimartingale, i.e. there is a convex combination of the directions of reflection that points into the wedge.

**Theorem 7 .** *Assume  $\beta < 0$  and  $\gamma < 0$ .*

1. *If  $\alpha\delta > \beta\gamma$ , there exists a unique solution to (3) in  $S$ .*
2. *If  $\alpha\delta \leq \beta\gamma$ , there does not exist any solution.*

**Proof.** 1. *Existence.* Let  $(h_n, n \geq 1)$  be a (strictly) increasing sequence of bounded positive nonincreasing Lipschitz functions converging to  $1/x$  on  $(0, \infty)$  and to  $+\infty$  on  $(-\infty, 0]$ . For instance we can take

$$\begin{aligned} h_n(x) &= (1 - \frac{1}{n})\frac{1}{x} && \text{on } [\frac{1}{n}, \infty) \\ &= n - 1 && \text{on } (-\infty, \frac{1}{n}]. \end{aligned}$$

We consider for each  $n \geq 1$  the system

$$\begin{aligned} X_t^n &= X_0 + B_t + \alpha \int_0^t \frac{ds}{X_s^n} + \beta \int_0^t h_n(Y_s^n) ds \\ Y_t^n &= Y_0 + C_t + \gamma \int_0^t h_n(X_s^n) ds + \delta \int_0^t \frac{ds}{Y_s^n}. \end{aligned} \tag{12}$$

From Proposition 2 we deduce there exists a unique solution to this system. As  $h_{n+1}(X_0) > h_n(X_0)$  and  $h_{n+1}(Y_0) > h_n(Y_0)$ , by continuity there exists an a.s. positive stopping time  $\rho$  such that  $h_{n+1}(X_s^{n+1}) \geq h_n(X_s^n)$  and  $h_{n+1}(Y_s^{n+1}) \geq h_n(Y_s^n)$  on  $[0, \rho]$ . A first double application of Lemma 1 shows that  $X_t^{n+1} \leq X_t^n$  and  $Y_t^{n+1} \leq Y_t^n$  on the interval  $[0, \rho]$ . Now we set

$$\tau := \inf\{s > 0 : X_s^{n+1} > X_s^n\}.$$

We have  $\tau \geq \rho > 0$  and a second application of Lemma 1 shows that  $Y_t^{n+1} \leq Y_t^n$  on  $[0, \tau]$ . As in the proof of 2 in Theorem 5 and since  $h_{n+1}(Y_\tau^{n+1}) > h_n(Y_\tau^n)$  on  $\{\tau < \infty\}$ , a third application of Lemma 1 shows that there exists  $\rho' > 0$  such that  $X_t^{n+1} \leq X_t^n$  on  $[\tau, \tau + \rho']$ , a contradiction to the definition of  $\tau$ . Thus  $\tau = \infty$  proving that on the whole  $[0, \infty)$  we have  $X_t^{n+1} \leq X_t^n$  and  $Y_t^{n+1} \leq Y_t^n$ . We can set for any  $t \in [0, \infty)$

$$X_t := \lim_{n \rightarrow \infty} X_t^n \quad \text{and} \quad Y_t := \lim_{n \rightarrow \infty} Y_t^n.$$

If  $\alpha\delta > \beta\gamma$ , there is a convex combination of the directions of repulsion that points into the quadrant, i.e. there exist  $\lambda > 0$  and  $\mu > 0$  such that  $\lambda\alpha + \mu\gamma > 0$  and  $\mu\delta + \lambda\beta > 0$ . For  $n \geq 1$  and  $t \geq 0$ ,

$$\begin{aligned} \lambda U_t + \mu V_t &\geq \lambda X_t^n + \mu Y_t^n \\ &\geq \lambda X_0 + \mu Y_0 + \lambda B_t + \mu C_t + (\lambda\alpha + \mu\gamma) \int_0^t \frac{ds}{X_s^n} + (\mu\delta + \lambda\beta) \int_0^t \frac{ds}{Y_s^n}. \end{aligned} \tag{13}$$

Letting  $n \rightarrow \infty$  in (13) we obtain

$$\int_0^t \frac{ds}{X_s} < \infty \quad \text{and} \quad \int_0^t \frac{ds}{Y_s} < \infty.$$

Then we may let  $n$  go to  $\infty$  in (12) proving that  $(X, Y)$  is a solution to (3).

*Uniqueness.* Let  $(X', Y')$  be another solution to (3). Replacing  $(X^{n+1}, Y^{n+1})$  with  $(X', Y')$  we follow the above proof to obtain for  $t \in [0, \infty)$  and  $n \geq 1$

$$X'_t \leq X_t^n \quad \text{and} \quad Y'_t \leq Y_t^n$$

Letting  $n \rightarrow \infty$  we conclude

$$X'_t \leq X_t \quad \text{and} \quad Y'_t \leq Y_t.$$

With the same  $\lambda > 0$  and  $\mu > 0$  as above,

$$\begin{aligned} & (\lambda(X_t - X'_t) + \mu(Y_t - Y'_t))^2 \\ &= 2 \int_0^t (\lambda(X_s - X'_s) + \mu(Y_s - Y'_s)) [(\lambda\alpha + \mu\gamma)(\frac{1}{X_s} - \frac{1}{X'_s}) + (\mu\delta + \lambda\beta)(\frac{1}{Y_s} - \frac{1}{Y'_s})] ds \\ &\leq 0 \end{aligned}$$

and therefore  $X'_t = X_t, Y'_t = Y_t$ .

2. If  $\alpha\delta \leq \beta\gamma$  there exist  $\lambda > 0$  and  $\mu > 0$  such that  $\lambda\alpha + \mu\gamma \leq 0$  and  $\mu\delta + \lambda\beta \leq 0$ . Thus, if  $(X, Y)$  is a solution to (3),

$$0 \leq \lambda X_t + \mu Y_t \leq \lambda X_0 + \mu Y_0 + \lambda B_t + \mu C_t.$$

This is not possible since the paths of a Brownian motion are not bounded below. So there is no global solution.  $\blacksquare$

## 5 Product-form stationary distribution

We introduce an additional constant drift  $(-\mu, -\nu)$  in the quadrant and consider the system

$$\begin{aligned} X_t &= X_0 + \alpha \int_0^t \frac{ds}{X_s} + \beta \int_0^t \frac{ds}{Y_s} - \mu t \\ Y_t &= Y_0 + \gamma \int_0^t \frac{ds}{X_s} + \delta \int_0^t \frac{ds}{Y_s} - \nu t \end{aligned} \quad (14)$$

with the conditions  $X_t \geq 0, Y_t \geq 0$ . If now

$$\begin{aligned} U_t &= X_0 + \alpha \int_0^t \frac{ds}{U_s} - \mu t \\ V_t &= Y_0 + \delta \int_0^t \frac{ds}{V_s} - \nu t \end{aligned} \quad (15)$$

with  $U_t \geq 0, V_t \geq 0$ , we can check that  $\mathbf{0}$  is still polar for  $(U, V)$  (as well,  $0$  is polar for  $U$  if  $\alpha \geq \frac{1}{2}$  and for  $V$  if  $\delta \geq \frac{1}{2}$ ). Therefore the results of the previous sections are still valid for the solution to (14). We are now looking for conditions on the set of parameters in order to obtain a stationary distribution for the Markov process  $(X, Y)$  in the form of a product of two gamma distributions.

**Theorem 8** . *Assume there exists a unique solution to (14) in  $S^0$  or in  $S$ . This process has an invariant distribution in the form  $\Gamma(a, c) \otimes \Gamma(b, d)$  if and only if*

- $\alpha\beta + \gamma\delta = 0$  (skew symmetry)
- $a = 2\alpha + 1, \quad b = 2\delta + 1$
- $c = 2\delta \frac{\mu\alpha + \nu\gamma}{\alpha\delta - \beta\gamma}, \quad d = 2\alpha \frac{\mu\beta + \nu\delta}{\alpha\delta - \beta\gamma}$
- $\mu\alpha + \nu\gamma > 0, \quad \mu\beta + \nu\delta > 0$ .

**Proof.** Let

$$\rho(x, y) = x^{a-1} e^{-cx} y^{b-1} e^{-dy} \quad \text{for } x \geq 0, y \geq 0.$$

The infinitesimal generator of the diffusion (14) is given by

$$L = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \left( \frac{\alpha}{x} + \frac{\beta}{y} - \mu \right) \frac{\partial}{\partial x} + \left( \frac{\gamma}{x} + \frac{\delta}{y} - \nu \right) \frac{\partial}{\partial y}.$$

By a density argument it is enough to check that

$$\int_0^\infty \int_0^\infty Lf(x, y) \rho(x, y) dx dy = 0$$

for any  $f(x, y) = g(x)h(y)$  with  $g, h \in C_c^2((0, \infty))$ . Integrating by parts, we get

$$\int_0^\infty \int_0^\infty L(gh)(x, y) \rho(x, y) dx dy = \int_0^\infty \int_0^\infty g(x)h(y) J(x, y) dx dy$$

where

$$J(x, y) = \rho(x, y) [A + Bx^{-1} + Cx^{-2} + Dy^{-1} + Ey^{-2} + Fx^{-1}y^{-1}]$$

with

$$\begin{aligned} A &= \frac{1}{2}c^2 + \frac{1}{2}d^2 - \mu c - \nu d \\ B &= -(a-1)c + \mu(a-1) + \alpha c + \gamma d \\ C &= \frac{1}{2}(a-1)(a-2) - \alpha(a-2) \\ D &= -(b-1)d + \nu(b-1) + \beta c + \delta d \\ E &= \frac{1}{2}(b-1)(b-2) - \delta(b-2) \\ F &= \beta(a-1) + \gamma(b-1). \end{aligned}$$

Letting  $A = B = C = D = E = F = 0$  we obtain the skew symmetry condition  $\alpha\beta + \gamma\delta = 0$  and the specified values for  $a, b, c, d$ . The last condition in the statement of the theorem is written out so that the invariant density  $\rho$  is integrable on  $S$ . Note that when  $\mu = \nu = 0$ , the invariant density  $\rho(x, y) = x^{2\alpha}y^{2\delta}$ , which is not integrable, does not depend on the obliqueness parameter  $\beta$ . ■

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