A survey of topics in analysis and partial differential equations II

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Denise Huet
INSTITUT ELIE CARTAN NANCY
Universite Henri Poincare Nancy 1 B.P. 239
54506 Vandoeuvre les Nancy Cedex, FRANCE
Denise.Huet@univ-lorraine.fr

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Abstract

This paper is the result of investigations suggested by recent publications and completes D.Huet [33] and [34]. The topics, which are dealt with, concern some spaces of functions, and properties of solutions of linear and nonlinear, stationary and evoluton differential equations namely: existence, spectral properties, resonances, singular perturbations, boundary layers, inertal manifolds. The aim of this document, and of [33] and [34], is to be a useful reference for (youngs) researchers in mathematics and applied sciences.

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INTRODUCTION

The article is divided into several sections as follows

Contents

The development of each entry includes indications on history, definitions, an overview of main results, examples and applications, but is, of course, non exhaustive. Complements will be found in the references.

1 Birman-Schwinger operator

Definition 1. Consider the Schrödinger operator

$$H_\gamma = H_o + \gamma V(x), H_o = (-\Delta)^l \tag{1}$$

acting on $L^2(\mathbb{R}^d)$, $2l \geq d$, where $x \to V(x)$ is a real-valued continuous function defined on $\mathbb{R}^d$ which is non-negative, and tends to zero sufficiently fast as $|x| \to \infty$, γ is a small negative coupling constant. The operator H_o is self adjoint and its spectrum is $\sigma_o = [0, +\infty)$. The **Birman-Schwinger** operator associated to (1) is the operator

$$X_V(\lambda) = V^{\frac{1}{2}} R_\lambda(H_o) V^{\frac{1}{2}},$$

where $R_\lambda(H_o)$ is the resolvent of H_o in $(-\infty, 0)$. For each $\lambda \in (-\infty, 0)$, $X_V(\lambda)$ is self adjoint and compact (cf. J. Arazy, L. Zelenko [2]).

Application

In [2] , the authors consider the decomposition $X_V(\lambda) = \Phi(\lambda) + T(\lambda)$, where $\Phi(\lambda)$ is a finite rank operator and $T(\lambda)$ an Hilbert-Schmidt operator whose norm is uniformly bounded with respect to $\lambda \in (-\delta, 0)$ for some $\delta > 0$. An asymptotic expansion of the bottom virtual eigenvalue $\lambda_o(\gamma)$ of H_γ, as $\gamma < 0$ tends to zero, is deduced from this decomposition: if d is odd, it is of power type, while, when d is even , it involves the log function. Asymptotic estimates are obtained, as $\gamma \uparrow 0$, for the non-bottom virtual eigenvalues of H_γ, $\{\lambda_{\mathbf{k}}(\gamma), \mathbf{k} \in \mathbb{Z}^d_+; 0 < |\mathbf{k}| \leq m\}$, where $m = l - \frac{d+1}{2}$ if d is odd and $m = l - \frac{d}{2}$ if d is even. If d is odd, $\Phi(-t^{2l})$ is a meromorphic operator function, and the leading terms of the asymptotic estimates of $\lambda_{\mathbf{k}}(\gamma)$ are of power type. An algorithm, based on the Puiseux-Newton diagram (cf. H. Baumgärtel [4]), is proposed for an evaluation of the leading coefficients of these estimates. If d is even, two-sided estimates are obtained for eigenvalues with an exponential rate of decay; the rest of the eigenvalues have a power rate of decay. Estimates of Lieb-Thirring type are obtained for groups of eigenvalues which have the same rate of decay, when d is odd or even.

2 BMO spaces

BMO and related spaces

Definitions

Let $\sup_I$ be the supremum over all cubes $I \subset \mathbb{R}^n$ with edges parallel to the coordinate axes, $\ell(I)$ be the sidelength of I and f_I the mean value of f over I. The square form of John-Nirenberg's $BMO = BMO(\mathbb{R}^n)$ space (cf. F. John, L. Nirenberg [36]) is the space of localy integrable complex-valued functions f defined on $\mathbb{R}^n$, such that

$$||f||_{BMO} = \left(\sup_I (\ell(I))^{-n} \int_I |f(x) - f_I|^2 dx \right)^{\frac{1}{2}} < \infty.$$

The space of functions of bounded mean oscillation, modulo constants, equipped with the above norm, is a Banach space.

Real hardy space $H^1(\mathbb{R}^n)$

A function $f \in H^1(\mathbb{R}^n)$ if and only if

$$f = \sum c_j \alpha_j \tag{2}$$

where $c_j \in \mathbb{R}, \sum |c_j| < \infty$, each function α_j is supported on a ball B_j, has integral zero, and $\sup_{x \in B_j} |\alpha_j(x)| \leq \frac{1}{|B_j|}$. Functions α that satisfy the above properties are called 1-atom (cf. E.M. Stein, [53]).With the norm

$$||f||_{H^1} = \inf \sum |c_j| \tag{3}$$

for all decompositions of f of the form (2), $H^1(\mathbb{R}^n)$ is a Banach space. For definitions and properties of Hardy-spaces $H^p(\mathbb{R}^n)$, see C. Fefferman and E.M. Stein [27].
The definitions of the following spaces are recalled in J. Xiao [58].

$\mathcal{Q}_\alpha$ spaces

For $\alpha \in (-\infty, \infty)$, $\mathcal{Q}_\alpha(\mathbb{R}^n)$ is the Banach space of all measurable complex-valued functions on $\mathbb{R}^n$, modulo constants, such that

$$||f||_{\mathcal{Q}_\alpha} = \sup_I \left((\ell(I))^{2\alpha - n} \int_I \int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} dxdy \right)^{\frac{1}{2}} < \infty.$$

(cf. M. Essen, S. Janson, L. Peng, J. Xiao [23]).

Space $\dot{L}^2_\alpha, \alpha \in (0,1)$

The homogeneous Sobolev space $\dot{L}^2_\alpha(\mathbb{R}^n)$ is the space of complex-valued functions f such that

$$||f||_{\dot{L}^2_\alpha} = \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} dxdy \right)^{\frac{1}{2}} < \infty$$

Space $\mathcal{L}_{2,n-2\alpha}$, $\alpha \in (0,1)$.

It is the space of measurable complex-valued functions f on $\mathbb{R}^n$ such that

$$||f||_{\mathcal{L}_{2,n-2\alpha}} = \sup_I \left((\ell(I))^{2\alpha - n} \int_I |f(x) - f_I|^2 dx \right)^{\frac{1}{2}} < \infty$$

Space $\mathcal{Q}^{-1}_{\alpha;T}, \alpha \in (0,1), T \in (0, \infty)$

A temperated distribution f on $\mathbb{R}^n$ belongs to this space provided

$$||f||_{\mathcal{Q}^{-1}_{\alpha;T}} = \sup_{x \in \mathbb{R}^n, r \in (0,T)} \left(r^{2\alpha - n} \int_0^{r^2} \int_{|y-x|<r} |e^{t\Delta} f(y)|^2 t^{-\alpha} dydt \right)^{\frac{1}{2}} < \infty$$

where $e^{t\Delta}(x,y)$ is the heat kernel.

Relations between these spaces

1–In [27], it is proved that the dual of $H^1(\mathbb{R}^n)$ is BMO (theorem 2).
2–For $\alpha \in \mathbb{R}, f \in \mathcal{S}'(\mathbb{R}^n)$, modulo polynomials, $(-\Delta)^{-\frac{\alpha}{2}} f = \mathcal{F}^{-1}\left(|\xi|^{-\alpha}\hat{f}(\xi)\right) = I_{\alpha f}$ (cf. R. Strichartz [54]). In [58], when $\alpha \in (0,1)$, it is proved that
1) $\mathcal{Q}_\alpha = (-\Delta)^{-\frac{\alpha}{2}}\mathcal{L}_{2,n-2\alpha}$,
2)$\dot{L}^2_\alpha = (-\Delta)^{-\frac{\alpha}{2}} L^2$,
3) $\mathcal{Q}^{-1}_{\alpha;\infty} = \nabla\cdot(\mathcal{Q}_\alpha)^n$
and sharp estimates for the norms of the embeddings $\mathcal{Q}_\alpha \hookrightarrow BMO$, and $\dot{L}^2_\alpha \hookrightarrow L^{\frac{2n}{n-2\alpha}}$, are obtained .
We point out that 3) means that $f \in \mathcal{Q}^{-1}_{\alpha;\infty}$ if and only if there are $f_j \in \mathcal{Q}_\alpha$ such that $f = \sum_{j=1}^n \partial_j f_j$.

BMO nonlinearity

(cf. S-S. Byun and L. Wang [10]) Let $a(\xi, x)$ be a real valued function on $\mathbb{R}^n \times \mathbb{R}^n$. For $y \in \mathbb{R}^n$ and $\rho > 0, B_\rho(y)$ denotes the open ball of radius ρ centered at y. Set

$$\bar{a}_{B_{\rho(y)}}(\xi) = \frac{1}{|B_{\rho(y)}|}\int_{B_{\rho(y)}} a(\xi, x)dx \tag{4}$$

and define the function $\beta(a, B_{\rho(y)})$ by

$$\beta[a, B_{\rho(y)}](x) = \sup_{\xi\in\mathbb{R}^n} \frac{|a(\xi, x) - \bar{a}_{B_{\rho(y)}}(\xi)|}{|\xi| + 1} \tag{5}$$

Definition 2. Small BMO condition The vector field $a = a(\xi, x)$ satisfies the (δ, R)-**BMO condition** if

$$\sup_{0<\rho\le R}\sup_{y\in\mathbb{R}^n} \frac{1}{|B_{\rho(y)}|}\int_{B_{\rho(y)}} |\beta[a, B_{\rho(y)}]|^2(x)dx \le \delta^2 \tag{6}$$

Definition 3. Reifenberg domain A bounded open set Ω in $\mathbb{R}^n$ is (δ, R)-**Reifenberg flat** if, for every $x_o \in \partial\Omega$ and every $r \in (0, R]$, there exists a coordinate system $\{y_1, ...y_n\}$, (which can depend on r and x_o) so that x_o is the origin in this coordinate system and that

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\} \tag{7}$$

(cf.[9])

Applications

i)Let Ω be a bounded, open subset of $\mathbb{R}^n$, $f \in L^p(\Omega; \mathbb{R}^n), 2 \le p < \infty$, $a = a(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a vector field measurable in x for almost every ξ, and continuous in ξ for each x. In [10], the authors consider the non linear boundary value problem

$$\text{div } a(\nabla u, x) = \text{div} f, \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega. \tag{8}$$

The following conditions are imposed on $a(\xi, x)$

$$[a(\xi, x) - a(\eta, x)]\cdot(\xi - \eta) \ge c_o|\xi - \eta|^2 \tag{9}$$

for all $\xi, \eta \in \mathbb{R}^n$ and almost every $x \in \Omega$.

$$|a(\xi, x)| \le c_1(1 + |\xi|) \tag{10}$$

for all $\xi \in \mathbb{R}^n$ and almost $x \in \mathbb{R}^n$, and

$$|\nabla_\xi a(\xi, x)| \leq c_2 \quad \forall \xi \in \mathbb{R}^n, \text{ and almost every } x \in \mathbb{R}^n \tag{11}$$

for some positive constants c_o, c_1, c_2. Then it is proved that there exists $\delta > 0$ such that, if a satifies the (δ, R)- BMO condition and Ω is (δ, R)-Reifenberg flat, the weak solution $u \in H_o^1(\Omega)$ to (8) belongs to $W_o^{1,p}(\Omega)$ with the estimate

$$_|\nabla u||_{L^p(\Omega} \leq C(||f||_{L^p(\Omega)} + 1) \tag{12}$$

where C is independent of u and f.

ii)In [9], S-S Byun extends the previous results to Orlicz spaces.. He recalls the following definitions:

Definition 4. A positive function ϕ defined on $[0, \infty)$ is called a Young function if it is increasing, convex and satisfies

$$\phi(0) = 0, \quad \phi(\infty) = \lim_{t\to\infty} \phi(t) = \infty, \quad \lim_{t\to 0} \frac{\phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty \tag{13}$$

Definition 5. We say that the Young function $\phi \in \Delta_2 \cup \nabla_2$ if it satisfies the following conditions

$$\phi(2t) \leq \kappa\phi(t), \forall t \geq 0, \quad 2t_o\phi(t) \leq \phi(t_o t), \forall t > 0 \tag{14}$$

for some numbers $\kappa, t_o > 1$.

Definition 6. Let $\phi \in \Delta_2 \cup \nabla_2$ be a Young function. The Orlicz space $L^\phi(\Omega)$ is the linear space of all measurable functions $v : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \phi(|v|)dx < \infty \tag{15}$$

Equipped with the norm

$$||v|||_{L^\phi(\Omega)} = \inf\{\lambda > 0 : \int_\Omega \phi(\frac{|v|}{\lambda})dx \leq 1\} \tag{16}$$

$L^\phi(\Omega)$ is a Banach space.

The following result is proved in [9]: let $\phi \in \Delta_2 \cup \nabla_2$ be a Young function. There exists a small $\delta > 0$ and a positive constant C such that, if the nonlinearity a satisfies (9), (10), (11), (6), if Ω satisfies (7), and if $|f|^2 \in L^\phi(\Omega)$, then the unique weak solution $u \in H_o^1(\Omega)$ to (8) satifies $|\nabla u|^2 \in L^\phi(\Omega)$, with the estimate

$$\int_\Omega \phi(|\nabla u|^2)dx \leq C(1 + \int_\Omega \phi(|f|^2)dx)) \tag{17}$$

Remark 1. If $p > 2$ and $\phi(t) = t^{\frac{p}{2}}$ the inequality (17) is only (12)

3 Bounded variation (functions of)

The notion of functions of bounded variation is closely related to the notion of **measure** and the following usual definitions are useful.

Measures-Definitions

Let Ω be an open subset of $\mathbb{R}^n$, and $\mathcal{C}_c(\Omega,\mathbb{C})$ [resp. $\mathcal{C}_c(\Omega,\mathbb{R})$] be the space of continuous functions $\varphi:\Omega\to\mathbb{C}$ [resp. $\mathbb{R}$] with compact support in Ω.

i) A measure μ on Ω is a linear functional

$$\mu:\ \varphi\in\mathcal{C}_c(\Omega,\mathbb{C})\to\mu(\varphi)(\text{or}\langle\mu,\varphi\rangle)\in\mathbb{C} \tag{18}$$

that is continuous, in the following sense: for all K compact in Ω, there exists a constant C_K such that

$$|\langle\mu,\varphi\rangle|\leq C_K||\varphi||_\infty \tag{19}$$

for all $\varphi\in\mathcal{C}_c(\Omega,\mathbb{C})$ whose support is contained in K. We write $\mu\in\mathcal{M}(\Omega)$. In [51], L.Schwartz introduces a topology on $\mathcal{C}_c(\Omega,\mathbb{C})$ and μ is continuous on this topological space .

ii) μ is said **bounded** on Ω if, in(19), the constant is independent of K. The space of bounded measures on Ω is denoted by $\mathcal{M}^1(\Omega)$.

iii) The conjugate $\bar{\mu}$ of μ is given by

$$\langle\bar{\mu},\varphi\rangle=\overline{\langle\mu,\bar{\varphi}\rangle} \tag{20}$$

iv) μ is called **real**, if $\langle\mu,\varphi\rangle\in\mathbb{R},\forall\varphi\in\mathcal{C}_c(\Omega,\mathbb{R})$

v) μ is called positive if $\langle\mu,\varphi\rangle\geq 0,\forall\varphi\in\mathcal{C}_c(\Omega,\mathbb{R}),\varphi\geq 0$

Absolute value of a measure

If μ is a complex or real measure, its absolute value, denoted by $|\mu|$, is the map

$$\psi\in\mathcal{C}_c(\Omega,\mathbb{R}),\psi\geq 0\to\langle|\mu|,\psi\rangle=\sup_{\varphi\in\mathcal{C}_c(\Omega,\mathbb{C}),|\varphi|\leq\psi}\{|\langle\mu,\varphi\rangle|\} \tag{21}$$

Total variation of a positive bounded measure

Let μ be a **positive, bounded measure** on Ω. Its total variation, denoted by $|\mu|_\Omega$ or $\int_\Omega\mu$, is

$$|\mu|=\sup_{\varphi\in\mathcal{C}_c(\Omega,\mathbb{R}),0\leq\varphi\leq 1}\mu(\varphi) \tag{22}$$

Space $BV(I,\mathbb{R})$ or $BV(I)$

Let $I=(a,b)$ be an interval in $\mathbb{R}$. The following definitions are well known. A function $f:x\in I\to\mathbb{R}$ is of bounded variation

i) if it can be expressed in the form $\phi-\psi$, where ϕ,ψ are nondecreasing bounded functions

ii) if the interval (a,b) is divided up by points

$$a=x_o<x_1<...<x_{n-1}<x_n=b$$

then, there exists a constant $C>0$, independent of the mode of division, such that

$$\sum_{\nu=0}^{n-1}|f(x_{\nu+1})-f(x_\nu)|<C \tag{23}$$

and the upper bound of the sum (23) is called the total variation of f on I and is denoted by V(f,I)
For these definitions, see E.C. Titchmarsh [56] p. 355 and F.Riesz and B. Sz-Nagy [48] p.10

iii) If $f \in L^1_{loc}(\mathbb{R})$, f defines a distribution $T = T_f \in \mathcal{D}'(\mathbb{R})$. In order the derivative of T to be a measure, it is necessary and sufficient that f is of bounded variation on every finite interval, see L. Schwartz [51] p.53.

Space $BV(\Omega, \mathbb{R})$ or $BV(\Omega)$

Definitions

The following definitions are equivalent (cf. H. Brezis [8], p. 153):
A function $f \in W^1_1(\Omega)$, is of bounded variation

i) if all first derivatives of f, in the distributional sense i.e. in $\mathcal{D}'(\Omega)$, are bounded measures.

ii) there exists a constant C such that

$$|\int_\Omega f \frac{\partial \phi}{\partial x_i} dx| \leq C||\phi||L^\infty(\Omega) \quad \forall \phi \in C^\infty_c(\Omega), \quad i = 1, ..., n \tag{24}$$

iii) there exists a constant C such that

$$||\tau_h f - f||_{L^1(\omega)} \leq C|h| \tag{25}$$

for all open set $\omega \subset\subset \Omega$ and all $h \in \mathbb{R}^n$ with $|h| < \mathbf{dist}(\omega, \complement\Omega)$. Moreover, in (24) and(25) we can take $C = ||\nabla f||_{L^1(\Omega)}$. Here $\tau_h f(x) = f(x+h)$, and ∇f is the distributional gradient of f.

iv) a function $f \in L^1(\Omega)$ is of bounded variation if its first distributional derivatives are bounded measures. Then, the gradient ∇f is a bounded, vector-valued measure whose **absolute value** $|\nabla f|$ is the map

$$\psi \in \mathcal{C}_c(\Omega, \mathbb{R}), \psi \geq 0 \rightarrow \langle |\nabla f|, \psi \rangle = \sup_{\varphi \in \mathcal{C}_c(\Omega, \mathbb{C}^n), \sum_i^n |\varphi_i|^2 \leq \psi^2} \{|\langle \nabla f, \varphi \rangle|\} \tag{26}$$

where $\langle \nabla f, \varphi \rangle = \sum_i^n \frac{\partial f}{\partial x_i} \varphi_i$, $|\nabla f|$ is a bounded positive measure. The total variation of $|\nabla f|$ is

$$\int_\Omega |\nabla f| = ||\nabla f||_\Omega = \sup_{\psi \in \mathcal{C}_c(\Omega, \mathbb{R}), 0 \leq \psi \leq 1} \langle |\nabla f|, \psi \rangle \tag{27}$$

(cf. F. and G. Demengel [21], p.303)

The general case

Let Ω be an open subset of $\mathbb{R}^d$ with a smooth boundary. A function $f \in L^1_{loc}(\Omega, \mathbb{R}^n)$ has a bounded variation i.e.$f \in BV(\Omega, \mathbb{R}^n)$, if ∇f, in the distributional sense, is a vector-valued Radon measure of finite total mass. Let $|\mathbf{f}|_{\mathbf{BV}} = \int_\Omega |\nabla \mathbf{f}|$ **be a BV-semi-norm**. In J. Davila [20], the following property of $|f|_{BV}$ is proved: there exists a positive constant K, which depends on d, such that, for every family of non negative radial mollifiers $\rho_\epsilon \in L^1_{loc}((0, \infty), \mathbb{R}_+)$ satisfying

$$\int_0^\infty \rho_\epsilon(r) r^{d-1} dr = 1 \text{ and } \lim_{\epsilon \to 0} \int_\delta^\infty \rho_\epsilon(r) r^{d-1} dr = 0 \quad \forall \delta > 0,$$

we have

$$\lim_{\epsilon\to 0}\int_{\Omega^2}\frac{|f(x)-f(y)|}{|x-y|}\rho_\epsilon(|x-y|)dxdy = K|f|_{BV},\quad \forall f\in BV(\Omega,\mathbb{R}^q).$$

Application

In [40], B. Merlet shows, by means of the above property of $|f|_{BV}$, that, if $u\in BV(\Omega,S^1)$, there exists a lifting $\varphi\in BV(\Omega,\mathbb{R})$ of u (i.e. $u(x)=e^{i\varphi(x)},\forall x\in\Omega$) such that $|\varphi|_{BV}\leq 2|u|_{BV}$.

4 Discrete energy

Let $\omega_N=(x_1,...,x_N), N\geq 2$ be a set of N points on the unit-sphere $S^2=\{x\in\mathbb{R}^3,|x|=1\}$.

Discrete energy of ω_N

Different discrete energies are associared with ω_N. Let $\alpha\in\mathbb{R}, -2<\alpha<2$.

Definition 7. The Coulomb [resp. logarithmic] energy associated with ω_N is

$$\mathcal{H}_C(\omega_N)=\sum_{i=1}^{N}\sum_{j=i+1}^{N}\frac{1}{|x_i-x_j|}$$

[resp.

$$\mathcal{H}_L(\omega_N)=-\sum_{i=1}^{N}\sum_{j=i+1}^{N}\log|x_i-x_j| \quad]$$

Definition 8. More generally, the α-energy associated with ω_N is

$$E(\alpha,\omega_N)=\begin{cases}\sum\limits_{i=1}^{N}\sum\limits_{j=i+1}^{N}|x_i-x_j|^\alpha & \text{if } \alpha\neq 0\\ \mathcal{H}_L(\omega_N) & \text{if } \alpha=0\end{cases}$$

Remark 2.

$$\mathcal{H}_C(\omega_N)=E(-1,\omega_N)$$

Remark 3. $E(\alpha,\omega_N)$ is the energy of the N points $x_1,...,x_N$, on the surface of the sphere, interacting through a potential $V=r^\alpha$ (cf E. A. Rakhmanov, E.B. Saff, Y.M. Zhou [47]).

The article [47] is devoted to extremal energy for ω_N:

$$\mathcal{E}(\alpha,N)=\begin{cases}\inf\limits_{\omega_N\in S^2}E(\alpha,\omega_N) & \text{if } \alpha\leq 0\\ \sup\limits_{\omega_N\in S^2}E(\alpha,\omega_N) & \text{if } \alpha<0\end{cases}$$

Bounds for $\mathcal{E}(\alpha,N), -2<\alpha<2$ and explicit formula for N points on S^2 that yields good estimates for $\mathcal{E}(\alpha,N)$ are obtained. The authors point out important applications of the determination of $\mathcal{E}(\alpha,N)$ to geometry, chemistry, physics, and cristallography, and give references for the history of related researchs.

A more general discrete energy-like function on the unit sphere

In [12], A.F. Cheviakov, M.J. Ward, R. Straube introduce the following definition

Definition 9. The discrete energy-like function associated with ω_N is

$$\mathcal{H}(\omega_N) = \mathcal{H}_C(\omega_N) + \frac{1}{2}\mathcal{H}_L(\omega_N) - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=i+1}^{N}\log(2+|x_i - x_j|)$$

Application

In [12], $\mathcal{H}(\omega_N)$ is related to the mean first passage time (MFPT) v(x) for a Brownian particle in the unit ball Ω in $\mathbb{R}^3$ that contains N small locally circular absorbing windows $\partial\Omega_{\epsilon_j}$ on its boundary $\partial\Omega = S^2$. Set $\partial\Omega_a = \bigcup_{j=1}^{N}\partial\Omega_{\epsilon_j}$. The function $v(x)$ is solution to the Dirichlet-Neumann problem :

$$\Delta v = -\frac{1}{D} \quad \text{in } \Omega$$
$$v = 0 \quad \text{on } \partial\Omega_a, \quad \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega \setminus \partial\Omega_a \tag{28}$$

where D is a diffusivity coefficient. The authors obtain three-term asymptotic expansions for $v(x)$, for the average MFPT $\bar{v} = \frac{1}{|\Omega|}\int_\Omega v(x)dx$, and for the principal eigenvalue λ of the Laplacian associated with the boundary conditions (28), when the area $|\partial\Omega_{\epsilon_j}| \sim \pi\epsilon^2 a_j^2$, $\partial\Omega_{\epsilon_j} \to x_j \in S^2$ as $\epsilon \to 0, j = 1, ..., N$ and $|x_i - x_j| = \mathcal{O}(1), i \neq j$. For instance, when the windows have common radius $\epsilon \ll 1$, they obtain:

$$\bar{v} = \frac{|\Omega|}{4\epsilon DN}[1 + \frac{\epsilon}{\pi}\log(\frac{2}{\epsilon}) + \frac{\epsilon}{\pi}(-\frac{9N}{5} + 2(N-2)\log 2 + \frac{3}{2}) + \frac{4}{N}\mathcal{H}(x_1, ..., x_N) + \mathcal{O}(\epsilon^2\log\epsilon)].$$

and

$$\lambda \sim \frac{1}{D\bar{v}}$$

Next, $\bar{v}$ is minimized and the corresponding λ is maximized at the configuration ω_N that minimizies $\mathcal{H}(\omega_N)$. The optimum arrangements $\{x_1, ..., x_N\}$ that minimize $\mathcal{H}(\omega_N)$ are numerically computed by different methods.

5 Equal- area condition

Equal-area type conditions appear, as sufficient or necessary conditions, in the formation of layers (internal or superficial) in stationary solutions to various singularly perturbed reaction-diffusion systems. In the recent works A.S. do Nascimento [43], J. Crema, A.S. do Nascimento [18] and R.J. de Moura, A.S. do Nascimento [41], the authors prove the necessity of suitable equal-area condition for the formation of internal or (and) superficial transition layers in this type of problems.

Example 1. A simple particular case of problems studied in [43] is the elliptic boundary value problem

$$\begin{cases} \epsilon\text{div}(h(x)\nabla u) + f(u) = 0 \quad x \in \Omega \\ \dfrac{\partial u}{\partial n} = 0 \ on \ \partial\Omega \end{cases} \tag{29}$$

where Ω is a smooth domain in $\mathbb{R}^N, N \geq 1, f : \mathbb{R} \to \mathbb{R}$ is such that there exist $\alpha, \beta, \alpha > \beta$,with $f(\alpha) = f(\beta) = 0$.

Let $\Gamma \subset \Omega$ be a smooth (N-1) dimensional compact manifold without boundary. It is proved that, if (29) has a family $\{u_\epsilon\}$ of solutions which develop an **internal** transition layer with interface Γ connecting the states α to β, then, necessarily, the simple **equal-area condition**

$$\int_\alpha^\beta f(s)ds = 0$$

is satisfied.

Example 2. In [18], the following stationary system is considered:

$$\begin{cases} \epsilon \text{div}(h(x)\nabla u) + f(x,u,\mathbf{v}) = 0 \quad x \in \Omega \\ \text{div}(\mathbf{k}(x)\nabla \mathbf{v}) + \mathbf{g}(x,u,\mathbf{v}) = 0 \quad x \in \Omega \\ \dfrac{\partial \mathbf{v}}{\partial n} = 0 \quad (\text{or } \mathbf{v} = \mathbf{0}) \quad \text{on } \partial\Omega. \end{cases} \tag{30}$$

where Ω is a smooth domain in $\mathbb{R}^N, N \geq 1$, **v,k,g** are sufficiently smooth $\mathbb{R}^n$-valued functions, and $\mathbf{k}\nabla\mathbf{v} = (k_1\nabla v_1, ...k_n\nabla v_n)$.

Let $\mathcal{U}$ be an open connected set in Ω, $\Gamma \subset \overline{\mathcal{U}}$ be an (N-1)-dimensional compact connected orientable manifold whose boundary $\partial\Gamma$ is such that $\partial\Gamma \cap \partial\Omega$ is an (N-2)-dimensional submanifold of $\partial\Omega$. A definition of a family of **internal** transition layer solutions $\{(u_\epsilon, \mathbf{v}_\epsilon)\}, 0 < \epsilon < \epsilon_o\}$ to (30) in $\mathcal{U}$ with interface Γ, depending on two functions $\alpha, \beta \in C^o(\mathcal{U})$, $\alpha(x) < \beta(x)$ on Γ is given. For such a family, there exists u_o [resp.$\mathbf{v}_o$] such that $u_\epsilon \to u_o$ [resp.$\mathbf{v}_\epsilon \to \mathbf{v}_o$] on compact sets of $\overline{\mathcal{U}} \setminus \Gamma$ [resp. in $\overline{\mathcal{U}}$]. It is proved that, if a family of internal transition layer solutions to (30) exists, then $f(x, u_o(x), \mathbf{v}_o(x)) = 0$ on $\mathcal{U} \setminus \Gamma$ and necessarily the **equal-area condition**

$$\int_\Gamma (\int_{\alpha(x)}^{\beta(x)} f(x,s,\mathbf{v}_o(x))\ ds)\ dS = 0$$

is satisfied. Several concrete applications of these results are presented in the paper.

Example 3. In [41], Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$ with C^2 boundary $\partial\Omega$, S is a C^2 (N-1)-dimensional surface with a boundary Σ which is assumed to be a C^2 (N-2)-dimensional compact surface without boundary with $\Sigma = \overline{S} \cap \partial\Omega$, and $\overline{S}$ intersects $\partial\Omega$ transversally The authors define a family of solutions $\{u_\epsilon\}, 0 < \epsilon < \epsilon_o$ to the elliptic boundary value problem:

$$\begin{cases} \epsilon \text{div}(a(x)\nabla v_\epsilon) + f(x, v_\epsilon) = 0, \quad x \in \Omega \\ \epsilon a(x)\dfrac{\partial v_\epsilon}{\partial \hat{n}} = g(x, v_\epsilon), \quad x \in \partial\Omega \end{cases}$$

which develops **internal and superficial** transition layers, depending on some smooth functions $\alpha, \beta, \alpha(x) < \beta(x), x \in \overline{\Omega}$, with interfaces S and Σ respectively.

Here $a \in C^1(\overline{\Omega}), a > 0, f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are of class C^1 and $\hat{n}$ is the exterior normal vector field on $\partial\Omega$. It is proved that the **equal-area conditions**:

$$\int_{\alpha(x)}^{\beta(x)} f(x,t)dt = 0\ \forall x \in S \text{ and } \int_{\alpha(y)}^{\beta(y)} g(y,t)dt = 0 \quad \forall y \in \Sigma$$

are necessary for the existence of such solutions.

6 Inertial manifolds

Let (E, d) be a metric space and $S(t)$ a continuous semigroup on E.

Definition 10. A set $M \subset E$ is an inertial manifold for $S(t)$ if
i) M is a finite-dimensional Lipschitz manifold in E
ii)M is of class C^1
iii)M is positively invariant under the flow i.e. $S(t)M \subset M, \forall t \geq 0$
iv)M is exponentially attracting i.e. there exists a constant c_o such that, for ecery $u_o \in E$, there exists a constant $c_1(u_o) > 0$ such that

$$dist_E(S(t)u_o, M) \leq c_1 e^{-c_o t}, \quad \forall t \geq 0$$

where dist_E is the Hausdorff semi-distance (see section 1.8 in D.Huet [34]) cf A. Bonfoh, M. Grasselli, A. Miranville [6] p.164.

Definition 11. (see R. Temam [55]) Let $B \subset E$, and U an open subset of E such that $B \subset U$; B is said **absorbing** for $S(t)$ in U if, for any bounded set B_o in U, there exists $t_1(B_o)$ such that $S(t)B_o \subset B$, for all $t \geq t_1$. In particular, if B is bounded, $S(t)B \subset B$, for $t \geq t_1(B)$

Remark 4. Similar definitions were given in [55] and M.Luskin, G.R. Sell [38], in the case of Hilbert spaces. In [55] the smoothness of M and the continuity of the semi-group $S(t)$ are not parts of the definition of an inertial manifold.

In a Hilbert space H, an inertial manifold M for a semi-group $S(t)$ is usually viewed as the graph of a suitable smooth function $\Phi : PH \to QH$, where P is an orthogonal projection on H with finite-dimensional range, and $Q = I - P$.
Usually, the semigroup $S(t)$ is associated with an evolution equation of the form

$$u'_t + Au + Ru = 0 \tag{31}$$

with the initial condition $u(0) = u_o$, where A is a linear operator and R a nonlinear one. When an inertial manifold M exists for problem (31), the restriction of (31) to M reduces to a finite dimensional ordinary differential equation, which is an exact copy of the initial system.

Construction of an inertial manifold

An Hilbert space case

(cf.[55]). In a Hilbert space H whose norm is denoted by $|.|$, consider problem (31) where A is an unbounded linear operator in H, strictly positive (i.e. $\exists c > 0$ such that $(Au, u) \geq c||u||_H^2$ for all u in the domain $D(A)$ of A), and self-adjoint. It is assumed that A^{-1} is compact. Thanks to assumptions on A , it is possible to define its powers A^s defined on $D(A^s)$, for all $s \in \mathbb{R}$. When $0 < s < 1$ see also D. Huet [32],T. Kato [37], M. Schechter [49]. For $r > 0$, we denote by $B_{s,r}$ the ball, in $D(A^s)$, with center 0 of radius r, i.e.

$$B_{s,r} = \{v \in D(A^s); ||v||_s = |A^s v| \leq r\} \tag{32}$$

It is assumed that R satisfies the following properties:

H_1: $\exists \alpha \in \mathbb{R}$ such that R is Lipschitz on the bounded sets of $D(A^\alpha)$ with values in $D(A^{\alpha-\frac{1}{2}})$ and

$$\begin{aligned} |A^{\alpha-\frac{1}{2}}R(u) - A^{\alpha-\frac{1}{2}}R(v)| \leq C_K |A^\alpha(u-v)|, \\ |A^\alpha u| \leq K \quad |A^\alpha v| \leq K \quad \forall u, v \in D(A^\alpha) \end{aligned} \tag{33}$$

H_2: If $u_o \in D(A^\alpha)$ problem (31) has a unique solution $u \in C(\mathbb{R}^+; D(A^\alpha)) \cap L^2((O,T); D(A^{\alpha+\frac{1}{2}}))$, $\forall T \geq 0$ and the map $S(t) : u_o \to u(t)$ is continuous from $D(A^\alpha)$ into itself, $\forall t \geq 0$.

H_3: The semi-group $S(t)$ possesses an absorbing set $B_o \subset D(A^\alpha)$ such that
1-$S(t)B_o \subset B_o$, $\forall t \geq O$
2-The ω-limit set of B_o

$$\omega(B_o) = \bigcap_{\beta \geq 0} \overline{\cup_{t \geq \beta} S(t)B_o} \tag{34}$$

is the maximal attractor for $S(t)$ in $D(A^\alpha)$
3-$\rho > 0$ is chosen such that B_o is included in the ball $B_{\alpha,\frac{\rho}{2}}$ of $D(A^\alpha)$ (cf (32)).

The prepared equation

The prepared equation is equivalent to the original one for t large. Let $\theta : \mathbb{R}^+ \to [0,1]$ be a C^∞ function such that

$$\theta(s) = \begin{cases} 1 & \text{for } 0 \leq s \leq 1 \\ 0 & \text{for } s \geq 2 \end{cases}$$

and

$$\sup_{s \geq 0} |\theta'(s)| \leq 2 \tag{35}$$

The aim of the prepared equation is to avoid the difficulties related to the behavior of the nonlinear term $R(u)$ for large values of $|A^\alpha(u)|$. Let $\theta_r(s) = \theta(\frac{s}{r}), r > 0$ and

$$R_r(u) = \theta_r(|A^\alpha(u)|)R(u) \quad \forall u \in D(A^\alpha) \tag{36}$$

The prepared equation associated to (31) is of the form

$$u'_t + Au + R_\rho u = 0 \tag{37}$$

Let $S_\rho(t)$ be the semi-group associated with (37). The following is assumed

H_4: The ball $B_{\alpha,\rho}$ is absorbing for $S_\rho(t)$

Thanks to the assumptions on R, R_ρ is a bounded operator, and a Lipschitz mapping from $D(A^\alpha)$ into $D(A^{\alpha-\frac{1}{2}})$. In particular

$$|R_\rho(u_1) - R_\rho(u_2)| \leq C_\rho |A^\alpha(u_1 - u_2)| \quad \forall u_1, u_2 \in D(A^\alpha) \tag{38}$$

where C_ρ is a constant which depend of ρ.

Space $\mathcal{F}^\alpha_{Nbl}$

Let N $\in \mathbb{N}$, and $b, l > 0$. Under the assumptions on A, there exists an orthonormal basis w_j in H, where w_j is the eigenvector of A corresponding to the eigenvalue λ_J, with $0 < \lambda_1 \leq \lambda_2 ... \leq \lambda_J \to \infty$, as $j \to \infty$. Set

$$P = P_N \quad Q = I - P = I - P_N$$

where P_N is the orthogonal projector, in H, onto the space spanned by $w_1, ...w_N$. The projections P and Q commute with $A^\beta, \forall \beta \in \mathbb{R}$.

Definition 12.

$$\mathcal{F} = \mathcal{F}^{\alpha}_{Nbl} = \{\varphi : PD(A^{\alpha}) \to QD(A^{\alpha})\} \tag{39}$$

where φ is a Lipschitz function with supp $(\varphi) \subset \{p \in PD(A^{\alpha}), |A^{\alpha}p| \leq 2\rho\}$, that satisfies

$$|A^{\alpha}\varphi(p)| \leq b \quad |A^{\alpha}\varphi(p_1) - A^{\alpha}\varphi(p_2)| \leq l|A^{\alpha}(p_1 - p_2)| \tag{40}$$

$\forall p \in PD(A^{\alpha}), p_1, p_2 \in D(A^{\alpha})$

Remark 5. 1-$PD(A^{\alpha})$ is finite dimensional.
2- $\mathcal{F}$ is a complete metric space for the distance

$$d(\varphi_1, \varphi_2) = \sup_{p \in PD(A^{\alpha})} |A^{\alpha}\left(\varphi_1(p) - \varphi_2(p)\right)| \tag{41}$$

Construction of a map $\mathcal{T} : \mathcal{F} \to \mathcal{F}$

Let $\varphi \in \mathcal{F}, p_o \in PD(A^{\alpha})$.Thanks to assumptions on φ and R_{ρ}, the problem:

$$\frac{dp}{dt} + Ap + PR_{\rho}(p + \varphi(p)), \quad p(0) = p_o \tag{42}$$

has a unique solution $p_{p_o,\varphi}(t) \in PD(A^{\alpha})$. Then, problem

$$\frac{dq}{dt} + Aq = -QR_{\rho}(p_{p_o,\varphi}(t) + \varphi(p_{p_o,\varphi}(t)) \tag{43}$$

has a unique solution $q_{p_o,\varphi}(t) \in C_b(\mathbb{R}; QD(A^{\alpha}))$. Therefore $q_{p_o,\varphi}(0) \in QD(A^{\alpha})$ (cf lemma 2.3., p.420 in [55]). The mapping $\mathcal{T}$ is defined by $\psi = \mathcal{T}(\varphi)$ with

$$\psi : p_o \in PD(A^{\alpha}) \to q_{p_o,\varphi}(0) \in QD(A^{\alpha}) \tag{44}$$

Let $\sigma(t)$ denote the right member of (43). Then

$$\psi(p_o) = -\int_{-\infty}^{0} e^{tA}\sigma(\tau)d\tau \tag{45}$$

Existence of an inertial manifold for (31) or (37)

Under asumptions $H_1 - H_4$, if it is possible to find N,b,l, such that $\mathcal{T}$ is a strict contraction of $\mathcal{F}$ into itself i.e. $\exists L < 1$ with $d(\mathcal{T}\varphi_1, \mathcal{T}\varphi_2) \leq Ld(\varphi_1, \varphi_2), \forall \varphi_1, \varphi_2 \in \mathcal{F}$, then, $\mathcal{T}$ has a fixed point $\Phi \in \mathcal{F}$, and the graph of Φ is an inertial manifold for (31) or (37)

Example 4. (cf. A. Bonfoh [7]) Set $\Omega = \prod_{i=1}^{n}(0, L_i), L_i > 0, n \leq 2$ and denote by N the operator $-\Delta : H^2(\Omega) \cap H^1_o(\Omega) \to L^2(\Omega)$. The following singularly perturbed problem (P_{ϵ}) is studied, in Ω :

$$\begin{cases} \delta\phi_t + N\phi + g(\phi) = u \\ \epsilon u_t + \phi_t + Nu = 0 \end{cases} \tag{46}$$

with the boundary conditions

$$\phi, u|_{\partial\Omega} = 0 \tag{47}$$

and the initial conditions

$$\phi|_{t=0} = \phi_o, \quad u|_{t=0} = u_o \tag{48}$$

where $\delta > 0, \epsilon \in (0, \epsilon_o,], g \in C^2(\mathbb{R})$ satisfies the conditions

$$|g'(s)| \leq C_1(|s|^p + 1), \quad g'(s) \geq -C_2, \quad C_1 > 0, C_2 \geq 0, p > 0, \forall s \in \mathbb{R} \tag{49}$$

The limit problem (P_o) is given by (46) where $\epsilon = 0$:

$$\begin{cases} \delta\phi_t + N\phi + g(\phi) = u \\ \phi_t + Nu = 0 \end{cases} \tag{50}$$

(47), and the initial condition $\phi|_{t=0} = \phi_o$. Set

$$H_1 = H_o^1(\Omega), \quad H_2 = H^2(\Omega) \cap H_o^1(\Omega), \quad H_3 = \{q \in H_3(\Omega) \cap H_o^1(\Omega); \Delta q \in H_o^1(\Omega)\} \tag{51}$$

Let $||\cdot||$ denote the usual norm in $L^2(\Omega)$ and set $||q||_r = ||N^{\frac{r}{2}} q||, r = 1, 2, 3$. The norms $||\cdot||_r$ are equivalent to the usual Sobolev $H^r(\Omega)$ norms on H_r. Several Banach spaces whose norms depend on ϵ are introduced,in particular

$$\mathcal{H}_\epsilon^o = H_1 \times L^2(\Omega), \text{ with the norm } ||(p,q)||_{\mathcal{H}_\epsilon^o} = (||p||_1^2 + \epsilon||q||^2)^{\frac{1}{2}} \tag{52}$$

Under additional assumptions on the regularity of the data, there exists semigroups $S_\epsilon(t)$, global attractors $\mathcal{A}_\epsilon$ and exponential attractors $\mathcal{M}_\epsilon$ for problems $(P_\epsilon), 0 \leq \epsilon \leq \epsilon_o$ (cf C.V. Chepyzhov, M.I. Vishik [11]). It is proved that $\mathcal{A}_\epsilon$ and $\mathcal{M}_\epsilon$ converge as $\epsilon \to 0$, to some lifting of $\mathcal{A}_o$ and $\mathcal{M}_o$, in $\mathcal{H}_1^o$ (cf (52)). Moreover, there exists $r > 0$, independent of ϵ, such that

$$B = \{u \in H_3, ||u||_3 \leq r\} \quad B_\epsilon = \{(\phi, u) \in \mathcal{H}_\epsilon^2, ||(\phi, u)||_{\mathcal{H}_\epsilon^2} \leq r\} \tag{53}$$

are bounded absorbing sets for $S(t)$ in H_3 and $S_\epsilon(t)$ in $\mathcal{H}_\epsilon^2$, respectively, where

$$\mathcal{H}_\epsilon^2 = H_3 \times H_2 \text{ with the norm } ||(p,q)||_{\mathcal{H}_\epsilon^2}^2 = ||p||_3^2 + \epsilon||q||_2^2 \tag{54}$$

and there exists $t_1 > 0$ such that $S(t)B \subset B, S_\epsilon(t)B_\epsilon \subset B_\epsilon$, for $t \geq t_1$, in particular:

$$\tilde{B} = S(t_1)B \subset B \quad \tilde{B}_\epsilon = S_\epsilon(t_1)B_\epsilon \subset B_\epsilon \tag{55}$$

Following the above construction, the authors associate an inertial manifold M^r to problem (P_o), and a family of inertial manifolds M_ϵ^r to problems (P_ϵ) such that

$$M^r \subset \tilde{B}, \quad M_\epsilon^r \subset \tilde{B}_\epsilon \tag{56}$$

Let $(M^r)_o$ be the following lifting of M^r:

$$(M^r)_o = \{(\phi, (I + \delta N)^{-1}(N\phi + g(\phi)) \in \mathcal{H}_\epsilon^1, \phi \in M^r\} \tag{57}$$

Then, it is proved that
1) For all $0 < \epsilon_2 < \epsilon_1$ small enouph, $M_{\epsilon_2}^r$ is lower and upper semi-continuous at ϵ_2, with respect to the metric induced by the $\mathcal{H}_{\epsilon_1}^0$ norm
2)M_ϵ^r converge, in a suitable sense, to $(M^r)_o$ with respect to the metric induced by the $\mathcal{H}_1^0$norm, as ϵ goes to 0.

7 Mathieu-Hill type equations

Mathieu and Hill Equations

The real Mathieu [resp. Hill] equation has the form

$$y" + (a + b\cos 2\pi x)y = 0$$

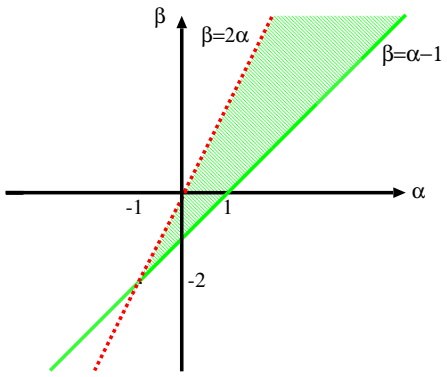

Figure 1: The $\alpha - \beta$ plane

where a and b are constants, [resp.

$$y" + (a + bq(x))y = 0]$$

where q is any smooth periodic function of period 1 with mean 0 (see E.A. Coddington, N. Levinson [14] and Z.X. Wang, D.R. Guo [57]). Physical problems leading to Mathieu or Hill equations often require solutions with periodicity, called oscillatory solutions . Therefore, to find conditions on the data for which the above equations have a fundamental system of periodic solutions is a central problem.

Its Equation

In [35], A.R. Its considers the Schrödinger equation on the positive semi-axis

$$y" - [x^{\beta}p(x^{1+\alpha}) + cx^{-2}]y = 0 \tag{58}$$

where p is a smooth periodic function, with period 1 and mean 0, c is a real number and the parameters α, β satisfy the relations $\beta - \alpha \geq -1$ and $2\alpha - \beta > 0$ (cf.Figure 1).

He proves that equation (58) has oscillatory solutions when $\beta > \alpha - 1$. If $\beta = \alpha - 1$, the solutions are oscillatory or not. In all cases, asymptotic formulas for the solutions are stated, as $x \to \infty$. His method is based on a transformation which leads to a Hill-type equation and Floquet functions.

Method of diagonalisation

In [5], S. Bodine and D.A. Lutz investigate equation (58) by a general method of diagonalization with a L^1-regularity for p. Set

$$t = x^{1+\alpha},\ \rho(t) = \int_0^t \frac{p(s)}{1+\alpha}ds,\ \mu = \int_0^1 \rho(t)dt,\ \nu = \int_0^1 (\rho(t))^2 dt, (\mu^2 < \nu).$$

Let

$$\lambda_{1,2} = (\frac{1}{2} \pm \sqrt{c + \mu^2 - \nu + \frac{1}{4}})/(1+\alpha)$$

be the eigenvalues of a suitable matrix which depends on α, c, μ, ν, and Λ be the crucial diagonal matrix $diag\{\lambda_1, \lambda_2\}$. When $\beta = \alpha - 1$, three cases are studied separately according to $\lambda_1 - \lambda_2$ belongs to $\mathbb{N}$, does not belong to $\mathbb{N}_0$ or is equal to 0. In each case, very sharp asymptotic formulas are obtained for a fundamental system of solutions to (58). In particular, when $\lambda_1 - \lambda_2 \in \mathbb{N}_o$, a logarithmic term appears in the formulas. For instance, when $\lambda_1 - \lambda_2 \in \mathbb{N}$ the following formulas are obtained:

$$y_1(x) = x^{\lambda_1(1+\alpha)}[1 + \sum_{r=1}^{N-1} \frac{p_r^1(x^{1+\alpha})}{x^{r(1+\alpha)}} + \mathcal{O}(\frac{1}{x^{N(1+\alpha)}})] + \eta(\ln x) y_2(x)$$

$$y_2(x) = x^{\lambda_2(1+\alpha)}[1 + \sum_{r=1}^{N-1} \frac{p_r^2(x^{1+\alpha})}{x^{r(1+\alpha)}} + \mathcal{O}(\frac{1}{x^{N(1+\alpha)}})]$$

where $p_r^{1,2}$ are scalar valued , bounded, continuous, periodic functions with period 1 which are recursively calculated. This logarithmic term is missing in Its formulas.

8 Memory (equations with)

Definition

Definition 13. (cf. B. D. Coleman and W. Noll [15]) A function h is called an **influence function** if
i) $h(s)$ is defined for $0 \leq s < \infty$, $h(s) > 0$
ii)There exists $r > 0$ such that $\lim_{s \to \infty} s^r h(s) = 0$, monotonically for large s.
The function $h(s)$ is said to be of order r.

Example 5. 1) $h(s) = (s+1)^{-p}$ is an influence function for $r < p$
2) $h(s) = e^{-\beta s}, \beta > 0$ is an influence function of any order

Dafermos results

Let H be a Hilbert space with inner product $\langle . \rangle$ and norm $||.||$.

An abstract integrodifferential equation

In [19], C.M. Dafermos investigates solutions $t \in (-\infty, +\infty) \to u(t) \in H$, on $[0, \infty)$, to the integrodifferential equation

$$\frac{d}{dt}(\rho \dot{u}(t)) + Cu(t) + \int_{-\infty}^{t} G(t-\tau)u(\tau)d\tau = 0 \tag{59}$$

with specified "history "

$$u(\tau) = v(\tau), \quad \tau \in (-\infty, 0] \tag{60}$$

Here $\rho, C, G(t)$ are self adjoint operators in H, ρ is bounded, C and $G(t)$ are unbounded with dense domains $\mathcal{D}(C)$ and $\mathcal{D}[G(t)]$ respectively. Moreover, they satisfy the following conditions

$$\begin{array}{l} \langle \rho w, w \rangle \geq \rho_o ||w||^2, \quad \forall w \in H \\ \langle Cw, w \rangle \geq c||w||^2, \quad \forall w \in \mathcal{D}(C) \\ \langle G(t)w, w \rangle \leq 0, \quad \forall w \in \mathcal{D}[G(t)], \quad t \in [0, \infty). \end{array} \tag{61}$$

with $\rho_o > , c > 0$. Let H_2 [resp.H_1] be the Hilbert space defined as $\mathcal{D}(C)$ [resp.$\mathcal{D}(C^{\frac{1}{2}})$],equipped with the natural norms, and H_{-1} the dual of H_1. The injection $H_2 \to H$ is assumed to be

compact. More regularity assumptions on $G(t)$ and $\dot{G}(t)$ imply the existence of a decreasing influence function $h \in C^0[0,\infty)$ such that

$$\int_0^\infty [||G(t)||_{\mathcal{L}(H_1,H_{-1})} + ||\dot{G}(t)||_{\mathcal{L}(H_1,H_{-1})}]h^{-2}(t)dt < \infty \tag{62}$$

In the following, h is a fixed influence function which satisfies (62)

Spaces of fading memory type; stability results

Set $I = (-\infty, 0]$. For $k = 0, 1, ...,$, in [19], the author introduces the Banach spaces of **fading memory type** (cf.[15]) $\mathcal{C}_k$ [resp. $\mathcal{B}_k$] of sufficiently smooth functions $w : I \to H_1$ [resp.$w : I \to H_2$] such that:

$$||w||_{\mathcal{C}_k} = \sum_{j=0}^{1} \sum_{i=0}^{k+1-j} \sup_I [h(-t)||w^{(i)}(t)||_j] < \infty \tag{63}$$

[resp.

$$||w||_{\mathcal{B}_k} = \sum_{j=0}^{2} \sum_{i=0}^{k+2-j} \sup_I ||w^{(i)}(t)||_j < \infty] \tag{64}$$

Then, $\mathcal{B}_k \subset \mathcal{C}_{k+1} \subset \mathcal{C}_k$, and the injection $\mathcal{B}_k \to \mathcal{C}_k$ is compact. Let $T > 0$ and set $I_T = (-\infty, T]$. If $v \in \mathcal{C}_k$ [resp. $\mathcal{B}_k$], he proves the existence of a unique solution u to (59) on $[0, T]$ which satisfies (60) on $(-\infty, 0]$. Moreover, si $v \in \mathcal{C}_k$ [resp $\mathcal{B}_k$], u satisfies

$$\sum_{j=0}^{1} \sum_{i=0}^{k+1-j} \sup_{[0,T]} ||u^{(i)}(t)||_j \leq c||v||_{\mathcal{C}_k} \tag{65}$$

[resp.

$$\sum_{j=0}^{2} \sum_{i=0}^{k+2-j} \sup_{[0,T]} ||u^{(i)}(t)||_j \leq c||v||_{\mathcal{B}_k}] \tag{66}$$

The new function

$$\omega : (v, \xi) \in \mathcal{C}_k \times [0,\infty) \to \omega[(v, \xi)](\tau) = u(\xi + \tau) \in \mathcal{C}_k, \quad \tau \in I$$

where u is the solution to (59) on $[0,\infty)$ with condition (60) on $(-\infty, 0]$, satisfies

$$\begin{aligned} &\omega(v, 0) = v \quad \forall v \in \mathcal{C}_k \\ &\omega(v, \xi + \xi') = \omega[\omega(v, \xi), \xi'] \quad , \forall \xi, \xi' \in [0,\infty), v \in \mathcal{C}_k \end{aligned}$$

i.e. w looks like a dynamical system on $\mathcal{C}_k$ (or $\mathcal{B}_k$), but it is not necessarily continuous on $\mathcal{C}_k \times [0,\infty)$. Therefore, w is not a classical dynamical system (cf. J.K. Hale [30]) Nevertheless, it is possible to define a Lyapunov function $V_k(v)$, for w, on $\mathcal{C}_k$ and prove, with suitable spectral conditions on G, via an adaptation of J.K. Hale's results [30], the following stability results for the solution u to (59) on $[0,\infty)$ with condition (60) on $(-\infty, 0]$, when $v \in \mathcal{C}_m, m = 0, 1, ...,$:

$$\begin{aligned} &u^{(i)}(t) \to 0 \text{ in } H_1 \text{ as } t \to \infty \quad i = 0, 1, ...m \\ &u^{m+1}(t) \to 0 \text{ in } H \text{ as } t \to \infty \end{aligned}$$

Equations with memory and singular perturbations

Example 6. In [28], S. Gatti, A. Miranville, V. Pata, S. Zelik consider a family of Banach spaces H_ϵ and a family of semi-groups $S_\epsilon(t) : H_\epsilon \to H_\epsilon$, $\epsilon \in [0,1]$. They state sufficient conditions on H_ϵ and S_ϵ which imply the existence of exponential attractors $\mathcal{E}_\epsilon$ for $S_\epsilon(t)$, $\epsilon \in [0,1]$ such that

$$\text{dist-sym}_{H_\epsilon}(\mathcal{E}_\epsilon, \mathcal{E}_o) \leq C\epsilon^\delta$$

and

$$\text{dist-sym}_{H_{\epsilon_2}}(\mathcal{E}_{\epsilon_1}, \mathcal{E}_{\epsilon_2}) \leq C(\epsilon_1 - \epsilon_2)^\delta \quad \forall \epsilon_1 \geq \epsilon_2 > 0$$

with $C \geq 0$ and $\delta \in (0,1)$.The results are applied to evolution equations with **memory** of the form

$$\partial_t x(t) + \int_0^\infty k_\epsilon(s) B_o(x(t-s))ds + B_1(x(t)) = O, \quad t \in [0,\infty) \tag{67}$$

with specified history

$$x(t) = v(t), \quad t \in (-\infty, 0] \tag{68}$$

where v is a given function, $k : [0,\infty) \to [0,\infty)$ is a convex summable function with $\int_0^\infty k(s)ds = 1$ and $k_\epsilon(s) = \frac{1}{\epsilon}k(\frac{s}{\epsilon})$, B_0, B_1 are densely defined, possibly nonlinear, operators in a suitable Banach space. As $\epsilon \to 0$, the formal limit to problem (67) and (68) is

$$\partial_t x(t) + B_o(x(t)) + B_1(x(t)) = 0, \quad x(0) = v(0) \tag{69}$$

Example 7. In [16], M. Conti, V. Pata and M. Squassina are interested in the convergence, in an appropriate sense, as $\epsilon \to 0$, of the solution $u_\epsilon(x,t)$, $x \in \Omega$, $t \in \mathbb{R}$ to problem (p_ϵ):

$$u_t - \omega\Delta u - (1-\omega)\int_0^\infty k_\epsilon(s)\Delta u(t-s)ds + \varphi(u) = f, \ t > 0,$$

with Dirichlet boundary conditions on the boundary of the smooth bounded domain Ω in $\mathbb{R}^3, \omega > 0, \varphi$ is a suitable nonlinearity, f is a time independent source term , and the **memory** kernel k_ϵ converges to the Dirac distribution at the origin, $u(x,t)$ is supposed to be a given datum for $t \leq 0$. Let $A = -\Delta$ on $L^2(\Omega)$ with domain $\mathcal{D}(A) = H_o^1(\Omega) \cap H^2(\Omega)$, $H^r = \mathcal{D}(A^{\frac{r}{2}}), r \in \mathbb{R}$. Following C. Dafermos [19] and M. Grasselli and V. Pata [29], they introduce, under additional assumptions on k, φ, f, ($\varphi(x) = x^3 - x$ is allowed), the auxiliary variable $\eta^t(x,s) = \int_o^s u(x,t-y)dy$, the functions $\mu(s) = -(1-\omega)k'(s), \mu_\epsilon(s) = \frac{1}{\epsilon^2}\mu(\frac{s}{\epsilon})$, the Hilbert spaces $\mathcal{M}_\epsilon^r = L^2_{\mu_\epsilon}(\mathbb{R}^+, H^{r+1})$ and $\mathcal{H}_\epsilon^r = H^r \times \mathcal{M}_\epsilon^r$ for $\epsilon > 0$, $\mathcal{H}_o^r = H^r$. The correct reformulation of p_ϵ, in the frame work of dynamical systems, is(P_ϵ) : find $(u_\epsilon, \eta_\epsilon) \in C([0,\infty), \mathcal{H}_\epsilon^o)$ solution to

$$u_t + \omega Au + \int_o^\infty \mu_\epsilon(s)A\eta(s)ds + \varphi(u) = f, \quad \partial_t\eta = -\partial_s\eta + u$$

for $t > 0$, associated with the initial condition $(u_o, \eta_o) \in \mathcal{H}_\epsilon^o$. The existence, for $\epsilon > 0$, of a strongly continuous semi-group $S_\epsilon(t)$ on $\mathcal{H}_\epsilon^o$ corresponding to P_ϵ and of an exponential attractor $\mathcal{E}_\epsilon$ for $S_\epsilon(t)$ are proved. The convergence of $\mathcal{E}_\epsilon$ is also studied.

9 Nodes, Nodal

Nodal points, curves, surfaces, or nodes

Definition 14. In R. Courant, D. Hilbert [17], they are defined as points [resp. curves, surfaces], on which some eigenfunctions of a differential problem, in a domain $G \in \mathbb{R}^N, N = 1$ [resp. $N = 2, 3$], vanishes.

Example 8. In the 1970 edition of [17], p.452, a second order Sturm-Liouville problem is considered. It is proved that the nodes of the n^{th} eigenfunction u_n divide the domain into no more than n subdomains.

Nodal sets, Nodal domains, Nodal solutions

Example 9. In [31], M. and T. Hoffman-Ostenhof and N. Nadirashvili study the Dirichlet eigenvalue problem

$$-\Delta u_i = \lambda_i u_i, \quad i = 1, 2, ... \quad u \in H_o^1(D),$$

where D is a bounded domain in $\mathbb{R}^n$, . The **nodal set** u_i is defined as

$$\mathcal{N}(u_i) = \overline{\{x \in D : u_i(x) = 0\}}$$

The **nodal domains** of u_i are the connected components of $D \backslash \mathcal{N}(u_i)$.The authors construct a domain $D \subset \mathbb{R}^2$ on which the second eigenvalue has a nodal set disjoint from the boundary, whereas the nodal line conjecture, first mentioned by L.E. Payne [45], stated that

$$\mathcal{N}(u_2) \cap \partial D \neq \emptyset.$$

Example 10. In [3], T. Bartsch and T.Weth consider the nonlinear elliptic Dirichlet problem:

$$-\epsilon^2 \Delta u + au = f(u) \text{ in } \Omega, \ u \in H_o^1(\Omega) \tag{70}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, $a > 0$, ϵ is a small positive parameter, and f grows superlinearly and subcritically. They study the number of **nodal solutions** i.e. sign-changing solutions of (70) and their nodal domains. They show that the number of nodal solutions can be expressed as a Lyusternik-Schnirelman category (cf. L. A. Lyusternik, L. Schnirelman [39]), of a suitable inclusion between two spaces which involve the shape of $\partial\Omega$.

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Example 11. The Ehrenberg-Siday-Aharonov-Bohm effect
This effect is a quantum mechanical phenomenom in which an electrically charged particle is affected by an electromagnetic field (E, B) despite being confined to a region in which both the magnetic field B and the electric field E are zero (cf. Y. Aranov, D. Bohm [1], W. Ehrenberg, R.E. Siday [22]) and M. Peshkin, A. Tonomura [46]).
In [44], B. Noris and S. Terracini consider the following problem of Ehrenberg-Siday - Aharonov-Bohm type:

$$H_{a,V} u \equiv (i\nabla + A_a)^2 u + Vu = 0 \text{ in } D_a, \quad u = \Gamma \text{ on } \partial D. \tag{71}$$

acting in $L^2(D; \mathbb{C})$ where D is the open unit disk in $\mathbb{R}^2, a = (a_1, a_2) \in D, D_a = D \backslash \{a\}$, $V \in W^{1,\infty}(D)$ is the scalar potential, A_a is the following vectorial potential

$$A_a(x_1, x_2) = \frac{2n+1}{2}\left(-\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right) + \nabla\phi, \tag{72}$$

$\Gamma \in W^{1,\infty}(D;\mathbb{C}), n \in \mathbb{Z}, \phi \in C^2(D)$. The magnetic fied associate to A_a is a Dirac distribution at a, directed orthogonally to the plane. The magnetic potential A_a satisfies $\nabla \times A_a = 0$ in D_a, and the half integer circulation

$$\frac{1}{2\pi} \oint_\sigma A_a \cdot dx = \frac{2n+1}{2}, \quad n \in \mathbb{Z}, \tag{73}$$

for every closed path σ which winds one around the pole. The authors investigate the nodal set $\mathcal{N}(u)$ for a class of solutions to (71) whose nodal set consists of regular arcs connecting the singular points with the boundary and , in particular, when $\mathcal{N}(u)$ consists of exactly three arcs, intersecting in a.

N-mode solution

In [42], K. Nakashima, M. Urano and Y. Yamada, consider the following ordinary differential problem :

$$\epsilon^2 u"(x) + f(x,u(x)) = 0 \quad x \in]0,1[; \; u'(0) = u'(1) = 0 \tag{74}$$

where $f(x,u) = u(1-u)(u-a(x))$; a is a $C^2[0,1]$ function such that $0 < a(x) < 1$, $a'(0) = a'(1) = 0$, and the subsets Σ and Λ of $(0,1)$ where $a(x) = \frac{1}{2}$ and $a'(x) = 0$, respectively, are finite sets and $a'(x) \neq 0$ for any $x \in \Sigma$. The authors study the asymptotic behavior, as $\epsilon \to 0$, of n-mode solutions u_ϵ of (74) i.e. solutions such that $v_\epsilon = u_\epsilon - a$ has exactly n zero-points in $(0,1)$, and show that any transition layer [resp. spike] is located in a neighborhood of a point of Σ [resp. Λ].

10 Resonances

Only a few aspects of this large topic are presented here. More aspects are in preparation.

The case of a vibrating string

Free motion

(cf. L. Schwartz [50]) Consider a homogeneous vibrating string , with linear density ρ, subjected to a constant tension μ.The position of the string, fixed at the end points $x = 0, x = L$ is given by solutions of the wave equation

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad x \text{ in } [0,L] \tag{75}$$

with the boundary conditions

$$u(t,0) = u(t,L) = 0 \tag{76}$$

where $v = \sqrt{\frac{\mu}{\rho}}$ has the dimension of a velocity. To solve problem (75)(76) the initial values of the position and of the velocity of the string are needed, i.e.

$$u(0,x) = u_o(x) \quad \frac{\partial u}{\partial t}(0,x) = u_1(x) \tag{77}$$

Solutions $u(x,t)$ of the above initial value problem of the form $u(x,t) = U(x)V(t)$ are

$$u_k(x,t) = \sin\frac{k\pi x}{L}(A_k \cos\frac{k\pi v t}{L} + B_k \sin\frac{k\pi v t}{L})$$

where $k > 0$ is an integer and A_k, B_k are constants.The numbers $\lambda = \dfrac{k^2\pi^2}{L^2}$, such that

$$U''(x) + \lambda U(x) = 0, \quad U(0) = U(L) = 0 \tag{78}$$

are called the **eigenvalues** and $\sin \dfrac{k\pi x}{L}$ the **eigenfunctions** of the problem. The constants A_k, B_k are the Fourier coefficients of suitable extensions of u_o and u_1 to periodic functions .

The case of a forced motion: resonance

(cf. R. Courant and D. Hilbert [17]) We suppose that the vibrating string is under the influence of an external force which has the Fourier expansion:

$$\mathcal{Q}(x,t) = \sum_{k=1}^{\infty} Q_k(t) \sin \frac{k\pi x}{L}$$

Then, the deflection of the string is solution to

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \mathcal{Q}(x,t), \quad u(0,t) = u(L,t) = 0$$

with the initial conditions (77). We look for a solution of the form

$$u(x,t) = \sum_{k=1}^{\infty} N_k(t) \sin \frac{k\pi x}{L}. \tag{79}$$

The coefficients $N_k(t)$ must be solutions of the equations

$$\frac{1}{v^2} N_k''(t) + \frac{k^2\pi^2}{L^2} N_k(t) = Q_k(t) \tag{80}$$

The general solution of the homogeneous equation associated to (80) is

$$a_k \cos \frac{k\pi v t}{L} + b_k \sin \frac{k\pi v t}{L}$$

A particular solution of (80)is obtained by the method of variation of constants (cf E.A. Coddington [13]). Finaly,

$$N_k(t) = \frac{1}{\frac{k\pi}{L}} \int_0^t Q_k(u) \sin \frac{k\pi}{L}(t-u)du + a_k \cos \frac{k\pi v t}{L} + b_k \sin \frac{k\pi v t}{L}$$

where the constants a_k, b_k are determined by the initial conditions (77)

Now, suppose $\mathcal{Q}_k(t) = \alpha \cos \omega t + \beta \sin \omega t$. Then, if $\omega \neq \dfrac{k\pi}{L}$, $u(x,t)$, in (79), is a linear combination of a sinusoidal function of frequancy ω, and one of frequency $\dfrac{k\pi}{L}$. But, if $\omega = \dfrac{k\pi}{L}$, $N_k(t)$ contains terms of the forms $t \sin t$ and $t \cos t$ which are unbounded, we say that **resonance** occurs.

Resonances as poles of complex functions

(cf. M.Zworski [59]). Set $X = [0, L]$ with $L = \pi$, for simplicity, $P_V = -\frac{\partial^2}{\partial x^2} + V(x), V \in C^\infty(X), V(x) \geq 0$ and

$$U(x,t) = \begin{pmatrix} u(t,x) \\ -i\partial_t u(t,x) \end{pmatrix} \qquad \mathcal{P}_V = \begin{pmatrix} 0 & 1 \\ P_V & 0 \end{pmatrix} \tag{81}$$

Consider the system

$$\frac{1}{i}\partial_t U = \mathcal{P}_V U \qquad U(t,0) = U(t,\pi) = 0 \qquad U(0,x) = \begin{pmatrix} u_o(x) \\ -iu_1(x) \end{pmatrix} \tag{82}$$

The solution to system (82) involves **eigenvalues** and **eigenfunctions** of

$$\mathcal{P}_V W = \lambda W, \quad W|_{\partial X} = 0$$

Consider the Hilbert space $\mathcal{H}$, closure of $C_c^\infty(X) \times C_c^\infty(X)$ for the inner product

$$\langle \Phi, \Psi \rangle = \langle P_V \phi_1, \psi_1 \rangle + \langle \phi_2, \psi_2 \rangle, \quad \Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

Then $\mathcal{P}_V$ is an unbounded self-adjoint operator on $\mathcal{H}$ and its eigenvalues are real. Let $\sigma(\mathcal{P}_V)$ be the spectrum of $\mathcal{P}_V$. It is pointed out, in [59] that

$$\lambda \in \sigma(\mathcal{P}_V) \Longleftrightarrow \lambda \text{ is a pole of } (\mathcal{P}_V - z)^{-1} : \mathcal{H} \to \mathcal{H}$$

Remark 6. When $V(x) = 0$, system (82) is equivalent to problem (75)– (77), and the eigenvalues of

$$\mathcal{P}_0 W = \lambda W, \quad W|_{\partial X} = 0$$

are $\lambda = k \in \mathbb{Z} - \{0\}$ which correspond to the eigenvalues k^2 of problem (78)

More generally, consider the problem with a dissipative term $a\partial_t, a > 0$:

$$(-\frac{\partial^2}{\partial t^2} + a\frac{\partial}{\partial t} - P_V)u(t,x) = 0, \quad t > 0, \quad x \in X, \quad u|_{\partial X} = 0$$

It can be rewritten as

$$\frac{1}{i}\partial_t U(t,x) = \mathcal{P}_a U(t,x), \quad u|_{\partial X} = 0 \tag{83}$$

where U is defined in (81) and

$$\mathcal{P}_a = \begin{pmatrix} 0 & 1 \\ P_V & ia \end{pmatrix}$$

The operator $\mathcal{P}_a$ is not self-adjoint on $\mathcal{H}$. The **eigenvalues** and the **eigenfunctions** of $\mathcal{P}_a$ are still defined by

$$\mathcal{P}_a W = \lambda W \quad W|_{\partial X} = 0 \tag{84}$$

and the solutions of (83) are given by superpositions of solutions of the form

$$U(t,x) = exp(-it\lambda)W(x) \tag{85}$$

but the eigenvalues $\lambda = \alpha + i\beta$ are not real any more. An elementary calculus shows that $\lambda(\lambda - ia) = \mu^2$, where $\mu \in \mathbb{R}$, is an eigenvalue of P_V, i.e. $\beta = \frac{a}{2}, \alpha^2 = \mu^2 - \frac{a^2}{4}$. Here, again,

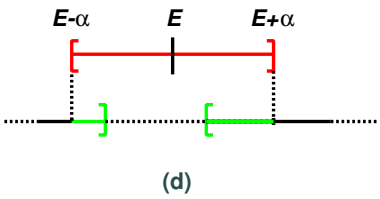

Figure 2: Spectral band

the eigenvalues of $\mathcal{P}_a$ are poles of $(\mathcal{P}_a - z)^{-1} : \mathcal{H} \to \mathcal{H}$. Now, since $\alpha \notin \sigma(\mathcal{P}_a)$, for a given f the equation

$$(\mathcal{P}_a - \alpha)u = f \tag{86}$$

has a solution. But, if a is small enough, $\lambda = \alpha + \frac{ia}{2}$ is a pole of $(\mathcal{P}_a - z)^{-1}$ very close to α, and, for a suitable choice of f , the solution of equation (86) can be "enormous "(cf. [59]). We say that a **resonance** occurs

Quasi-periodic Schrödinger operator, the resonant case

In their papers [24], [25] and [26] A. Fedotov and F. Klop study the spectrum of quasi periodic schrödinger operators of the form

$$H_{z,\epsilon} = -\frac{d^2}{dx^2} + V(x - z) + \alpha \cos(\epsilon x) \tag{87}$$

where, $\alpha > 0, V \in L^2_{loc}(\mathbb{R})$ is periodic with period $1, z$ is a real parameter indexing the equations of the family; $\epsilon > 0$ is such that $\frac{2\pi}{\epsilon}$ is irrational. Let

$$H_o = -\frac{d^2}{dx^2} + V(x)$$

be the periodic Schrödinger unperturbed operator. Its spectrum, on $L^2(\mathbb{R})$, consists of the union of intervals $[E_{2n-1}, E_{2n}], n \geq 1$, such that $E_n \to \infty$, as $n \to \infty$. It is assumed that all the spectral gaps (E_{2n}, E_{2n+1}) are open. The spectrum of (87), at an energy E, depends on the relative position of the spectral window $\mathcal{F}(E) = [E - \alpha, E + \alpha]$ with respect to the spectrum of H_o. In [24] four cases are distinguished according as $\mathcal{F}(E)$ lies inside or completely covers a spectral band of H_o or contains exactly one edge or two edges (Figure 2 (d)) of a spectral band.

In each case, under specific additional assumptions, the authors describe the nature of the spectrum of (87), and state asymptotic formulas, as $\epsilon \to 0$.
In [25], they are interested in the spectrum of (87), in intervals J such that for all $E \in J$, $\mathcal{F}(E)$ covers the edges of two neighboring spectral bands of H_o and the spectral gap located between them (Figure 2 (d))). Let Γ_R be the real iso-energy curve associated to (87) and γ_0, γ_π, be the connected components of Γ_R in a periodicity cell. To each of these loops, one associates a sequence of energies in J, E^n_a, $a = \{0, \pi\}$, and, near each E^n_a, an exponentally small interval I^n_a such that the spectrum of (87) , in J, is contained in the union of these intervals. The location and the nature of the spectrum of (87) are investigated in the union $I^n_0 \cup I^m_\pi$, in the **resonant case**, i.e. when I^n_0 and I^m_π intersect each other.

References

[1] Y. Aharonov, D. Bohm, Significance of electromagnetic potentials in quantum theory, *Physical Review*, **115** no.3 (1959), 485–491 19

[2] J. Arazy and L. Zelenko, Virtual eigenvalues of high order Schrödinger operator, *Integral Equations and Operator Theory*, **55** (2006), 189–231 and 305-345. 2

[3] T. Bartsch, T. Weth, The effect of the domain's configuration of nodal solutions of singularly perturbed elliptic equations, *Topological Methods in Nonlinear Analysis*, **26** no.1 (2005), 109–133. 19

[4] H. Baumgärtel, *Analytic perturbation theory for matrices and operators*, Operator Theory, Advances and Applications, vol.15, Birkhäuser, (1985). 2

[5] S. Bodine and D.A. Lutz, Asymptotic Analysis of solutions of a radial Schrödinger equation with oscillating potential, *Mathematische Nachrichten*, **279** no. 15 (2006), 1641–1663. 15

[6] A. Bonfoh, M. Grasselli, A. Miranville, Inertial manifolds for a singular perturbation of the viscous Cahn-Hilliard-Gurtin equation, *Topological methods in Nonlinear Analysis*, **35** (2010), 155–185. 11

[7] A. Bonfoh, The singular limit dynamics of the phase-field equations, *Annali di Matematica*, **190** (2011) 105–144. 13

[8] H. Brezis, *Analyse Functionnelle, Théorie et Applications*, Collection Mathématiques Appliquées pour la maîtrise, Masson (1983). 7

[9] S-S. Byun, Gradient estimates in orlicz spaces for nonlinear elliptic equations with BMO Nonlinearity in nonsmooth domains, *Forum Mathematicum*, **23** (2011), 693–711. 4, 5

[10] S-S. Byun, L. Wang, Elliptic equations with BMO nonlinearity in Reifenberg domains, *Advances in Mathematics*, **219** (2008), 1937–1971 4

[11] V.V. Chepyzhov , M.I. Vishik, *Attractors for Equations of Mathematical Physics*, Colloquium Publications, American Mathematical Society **49** (2002). 14

[12] A.F. Cheviakov, M.J. Ward, R. Straube, *An asymptotic analysis of the mean first passage for narrow escape problems: Part II the sphere*, Multiscale Modeling and Simulation **8** no.3 (2010), 836–870. 9

[13] E.A. Coddington, *An introduction to ordinary differential equations*, Prentice-Hall, Inc. (1961). 21

[14] E.A. Coddington and N. Levinson *Theory of Ordinary Differential Equations*, MacGrall-Hill Book Company (1955). 15

[15] B.D. Coleman, W. Noll, Foundations of linear viscoelasticity, *Reviews of Modern Physics*, **33** no.2, (1961), 239–249 16, 17

[16] M. Conti, V. Pata, M. Squassina, Singular limit of differential systems with memory, *Indiana University Mathematics Journal*, **55** no. 1 (2006). 169–215. 18

[17] R. Courant and D. Hilbert *Method of mathematical Physics vol. 1*, Interscience (1953). 19, 21

[18] : J. Crema, A.S. do Nascimento, On the role of the equal-area condition in internal layer stationary solutions to a class of reaction-diffusion systems, *Electronic Journal of differential Equations*, **99** (2004), 1–13 . 9, 10

[19] C. Dafermos, Asymptotic stability in viscoelasticity, *Archive for rational mechanics and analysis*, **37** (1970), 297–308. 16, 17, 18

[20] J. Davila, On an open question about functions of bounded variation, *Calculus of Variations and Partial Differential Equations*, **15** no. 4 (2002), 519–527. 7

[21] F. and G. Demengel, *Functional spaces for the theory of elliptic partial differential equations*, Universitex. Springer, **xviii**, 465 p. EUR 64.15 (2012) 7

[22] W. Ehrenberg, R.E. Siday, The refractive index in electron optics and the principles of dynamics. *Proceedings of the Physical Society B*, **62**, 1949, 8–21 19

[23] M. Essen, S. Janson, L. Peng, J. Xiao, $\mathcal{Q}$ spaces of several real variables, *Indiana University Mathematics Journal*, **49** no.2 (2000), 575–615. 3

[24] A. Fedotov, F. Klop, The spectral theory of adiabatic quasi-periodic operators on the real line, *Markov Processes and Related Fields*, **9** (2003), 579-614. 23

[25] A. Fedotov, F. Klop, Level repulsion and spectral type for one-dimensional adiabatic quasi-periodic Schrödinger operators, In *Lecture notes in Physics*, Springer, Berlin **690** (2006), 383–402. 23

[26] A. Fedotov, F. Klopp, Weakly resonant tunneling interactions for adiabatic quasi-periodic Schrödinger operators, *memoires de la Socit Mathmatique de France*, **104** (2006), 105 pp. 23

[27] C. Fefferman, E.M. Stein, H^p spaces of several variables, *Acta mathematica*, **129** (1972), 137–193 3, 4

[28] S. Gatti, A. Miranville, V. Pata, S. Zelik, Continuous families of exponential attractors for singularly perturbed equations with memory, *Proceedings of the Royal Society of Edinfurgh*, **140A** (2010), 329–366. 18

[29] M. Grasselli, V. Pata, Uniform attractors of nonautonomous dynamical systems with memory, In *Progress in Nonlinear Differential Equations and their applications*, **50**, Birkhäuser, Basel (2002), 155–178. 18

[30] J.K. Hale, Dynamical systems and stability,*Journal of Mathematical Analysis and Applications*, **26** (1969), 39–59 17

[31] M. and T. Hoffman-Ostenhof, N. Nadirashvili, On the nodal line conjecture, *Advances in Differential Equations and Mathematical Physics, Comtemporary Mathematics*, **217** (1998), 33-48. 19

[32] D. Huet, *Décomposition spectrale et opérateurs,* Presses Universitaires de France (1976). 11

[33] D. Huet, A survey of topics in analysis and differential equations. *Prepublications de l'Institut Elie Cartan* (2009) and Hal-00394244 (2009) 1

[34] D. Huet, A survey of topics in analysis and differential equations. *International Journal of Pure and Applied Mathematics*, **62** no. 1 (2010), 79–128 1, 11

[35] A.R. Its, Asymptotic behavior of the solutions of the radial Schrödinger equation with oscillating potential for the zero value of the energy, *Problems in Mathematical Physics, Leningrad*, **9** (1979), 30–41 and 183–184, translated in *Selecta Mathematica Sovietica*, **3** (1984), 291–300. 15

[36] F. John, L. Nirenberg, On functions of bounded mean oscillation, *Communications on pure and applied Mathematics* , **14** (1961), 415–426. 2

[37] T. Kato, *Perturbation theory for linear operators*, Springer (1966). 11

[38] M. Luskin, G.R. Sell, Approximation theories for inertial manifolds, *RAIRO Modlisation Mathmatique et Analyse Numrique* ,**23**, no.3 (1989), 445–461 11

[39] L.A. Lyusternik, L. Schnirelman, *Methodes toptologiques dans les problemes variationnels*, Gauthier-Villars (1934). 19

[40] B. Merlet, Two remarks on liftings of maps with values into S^1, *Comptes rendus des Séances de l'Académie des Sciences, Paris, Série I*, **343** (2006), 467–472. 8

[41] R.J. de Moura, A.S. do Nascimento, The role of the equal-area condition in internal and superficial layered solutions to some nonlinear boundary value elliptic problems, In *Progress in Nonlinear Differential Equations and their Applications*, Birkhäuser **66** (2006), 415–427. 9, 10

[42] K. Nakashima, M. Urano, Y. Yamada, Transitions layers and spikes for a bistable reaction-diffusion equation, *Advances in Mathematical Sciences and Applications*, **15** no.2 (2005), 683–707. 20

[43] A.S. do Nascimento, Inner transition layers in an elliptic boundary value problem: a necessary condition, *Nonlinear Analysis*, **44** (2001), 487–497. 9

[44] B. Noris, S. Terracini, Nodal sets of magnetic Schödinger operators of Aharonov-Bohm type and energy minimizing partition. *Indiana University Mathematics Journal,* **59** no. 4 (2010), 1361–1403 19

[45] L.E. Payne, Isoperimetric inequalities and their applications, *Siam Review*, **9** (1967), 453–488. 19

[46] M. Peshkin, A. Tonomura,*The Aharonov-Bohm effect*, Lecture Notes in Physics, 340, Springer, Berlin (1989). 19

[47] E.A. Rakhmanov, E.B. Saff, Y.M. Zhou, Minimal discrete energy on the sphere, *Mathematical Research Letters* , **1** (1994), 647–662 8

[48] F. Riesz, B.Sz-Nagy, *Leçons d'analyse fonctionnelle*, Gauthier-Villars, Paris 1968 7

[49] M. Schechter, *Principles of functional analysis*, Academic Press, New-York (1971). 11

[50] L. Schwartz, *Méthodes mathématiques pour les sciences physiques*, Hermann, Paris, (1961) 20

[51] L. Schwartz, *théorie des distributions*, Hermann, Paris (1966). 6, 7

[52] L. Schwartz, *Analyse, Topologie Générale and analyse fonctionnelle*, Hermann (1970).

[53] A.M. Stein, Harmonic Analysis, Princeton University Press (1993) 3

[54] R. Strichartz, Bounded mean oscillation and Sobolev space, *Indiana University Mathematics Journal*, **29** no.4 (1980), 539–558. 4

[55] R.Temam *Infinite-dimensional dynamical systems in mechanics and physics.* Applied Mathematical Sciences, 68, Springer-Verlag, New York (1988). 11, 13

[56] E.C. Titchmarsh, *The Theory of Functions*, second edition, Oxford University Press (1988). 7

[57] Z.X. Wang, D.R. Guo , *Special functions*, World Scientific (1989). 15

[58] J. Xiao, Homothetic variant of fractional Sobolev space with application to Navier-Stokes system, *Dynamics of Partial Differential Equations*, **4** no.3 (2007), 227–245. 3, 4

[59] M. Zworski, Resonances in physics and geometry, *Notices of the American Mathematical Society*, **46** no.3 (1999), 319–328 22, 23