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Basic microscopic plasma physics unified and simplified by N -body classical mechanics

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One considers an infinite plasma made up of the periodic replication of N electrons coupled by Coulomb forces in a volume L^3 with a neutralizing ionic background. Using Fourier and Laplace transforms, a rigorous fundamental equation is derived for the electrostatic potential. A first coarse graining of this equation reveals the potential to be the sum of the screened Coulomb potentials of the individual particles. A second coarse graining yields the classical Vlasovian expression including initial conditions in Landau contour calculations of Langmuir wave growth or damping. Using the screened potential, the collisional diffusion coefficient is computed by a convergent expression including the deflections for all impact parameters. Screening and collisional transport are found to be two related aspects of the repulsive deflections of electrons. The theory is extended to accommodate a correct description of trapping or chaos due to Langmuir waves. In the linear regime, the amplitude of such a wave is found to be ruled by Landau growth or damping and by spontaneous emission.

I. INTRODUCTION

From the outset, inspired by gas physics, plasma physicists derived kinetic equations to describe microscopic aspects of their physics, in particular the Vlasov equation. This trend has been the dominant one till nowadays. However, for plasmas where transport due to short range interactions is weak, it is possible to work directly with N -body classical mechanics. As will be recalled in section V, this approach led to a new description of wave-particle interaction making it more intuitive and unifying particle and wave evolutions, as well as collective and granular physics.

This paper develops this approach further by deriving a rigorous fundamental equation for the electrostatic potential (section II). It brings further unifications: Debye screening and Landau theory of linear waves (section III), Debye screening and collisional transport (section IV). Finally, the theory is extended to accommodate a correct description of trapping or chaos due to Langmuir waves, and to generalize to three dimensions a formula giving the time evolution of the amplitude of a Langmuir wave, unifying Landau growth or damping and spontaneous emission (section V). These results come with a considerable simplification of the mathematical framework with respect to textbooks, and with new insights into microscopic plasma physics. The derivations are elementary, and this paper is more oriented toward concepts and intuitive physics, than toward ready-to-use formulas.

II. FUNDAMENTAL EQUATION FOR THE POTENTIAL

We consider an infinite plasma with spatial periodicity L in three orthogonal directions with coordinates (x, y, z) , which is made up of N electrons in each elementary cube of volume L^3 . A uniform neutralizing ionic background is assumed to enable periodic boundary conditions. We are interested in $\varphi(\mathbf{r})$, the potential created by the N particles at any point where there is no particle. In the absence of neutralizing background and in infinite space, it would read

$$\varphi_C(\mathbf{r}) = \sum_{j \in S} \delta\varphi_C(\mathbf{r} - \mathbf{r}_j), \quad (1)$$

where S is the set of integers going from 1 to N ,

$$\delta\varphi_{C\infty}(\mathbf{r}) = -\frac{e}{4\pi\epsilon_0\|\mathbf{r}\|}. \quad (2)$$

Given the neutralizing background and the periodic boundary condition, the discrete Fourier transform of $\varphi(\mathbf{r})$ is actually readily obtained from the Poisson equation, and is given by $\tilde{\varphi}(\mathbf{0}) = 0$, and for $\mathbf{m} \neq \mathbf{0}$

$$\tilde{\varphi}(\mathbf{m}) = -\frac{e}{\epsilon_0 k_{\mathbf{m}}^2} \sum_{j \in S} \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_j), \quad (3)$$

where $-e$ is the electron charge, ϵ_0 is the vacuum permittivity, \mathbf{r}_j is the position of particle j , $\tilde{\varphi}(\mathbf{m}) = \int \varphi(\mathbf{r}) \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}) d^3\mathbf{r}$, with $\mathbf{k}_{\mathbf{m}} = (m_x \frac{2\pi}{L}, m_y \frac{2\pi}{L}, m_z \frac{2\pi}{L})$, $\mathbf{m} = (m_x, m_y, m_z)$ is a vector with three integer components, and $k_{\mathbf{m}} = \|\mathbf{k}_{\mathbf{m}}\|$.

Reciprocally,

$$\varphi(\mathbf{r}) = \frac{1}{L^3} \sum_{\mathbf{m}} \tilde{\varphi}(\mathbf{m}) \exp(i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}), \quad (4)$$

where $\sum_{\mathbf{m}}$ means the sum over all components of \mathbf{m} running from $-\infty$ to $+\infty$.

The dynamics of particle l is defined by Newton's equation

$$\ddot{\mathbf{r}}_l = \frac{e}{m_e} \nabla \varphi_l(\mathbf{r}_l), \quad (5)$$

where m_e is the electron mass, φ_l is the electrostatic potential acting on particle l , i.e. the one created by all other particles. It is given by Eq. (3) with the supplementary condition $j \neq l$. Let

$$\mathbf{r}_l^{(0)} = \mathbf{r}_{l0} + \mathbf{v}_l t \quad (6)$$

be a ballistic approximation of the dynamics of particle l . In the following, we consider two instances of this approximation: the one where \mathbf{r}_{l0} and \mathbf{v}_l are respectively the initial position and velocity of particle l , and the one where they are slightly shifted from these values by small amplitude Langmuir waves. Let $\delta\mathbf{r}_l = \mathbf{r}_l - \mathbf{r}_l^{(0)}$. With this notation and Eq. (4), Eq. (5) yields

$$\delta\ddot{\mathbf{r}}_l = \frac{ie}{L^3 m_e} \sum_{\mathbf{n}} \mathbf{k}_{\mathbf{n}} \tilde{\varphi}_l(\mathbf{n}) \exp[i\mathbf{k}_{\mathbf{n}} \cdot (\mathbf{r}_l^{(0)} + \delta\mathbf{r}_l)]. \quad (7)$$

We now introduce the time Laplace-transform which transforms a function $f(t)$ into $\hat{f}(\omega) = \int_0^\infty f(t) \exp(i\omega t) dt$. The Laplace-transform of Eq. (7) is

$$\omega^2 \delta\hat{\mathbf{r}}_l(\omega) = -\frac{ie}{L^3 m_e} \sum_{\mathbf{n}} \mathbf{k}_{\mathbf{n}} \exp(i\mathbf{k}_{\mathbf{n}} \cdot \mathbf{r}_{l0}) \Psi_l(\hat{\varphi}_l; \mathbf{n}, \omega + \omega_{\mathbf{n},l}) + i\omega \delta\mathbf{r}_l(0) - \delta\dot{\mathbf{r}}_l(0). \quad (8)$$

where carets indicate the Laplace transformed versions of the quantities in Eq. (7), $\omega_{\mathbf{n},l} = \mathbf{k}_{\mathbf{n}} \cdot \mathbf{v}_l$ comes from the time dependence of $\mathbf{r}_l^{(0)}$ in the exponent of Eq. (7), and the operator Ψ_l acting on a function $g(\mathbf{m}, \omega)$ is defined by

$$\Psi_l(g; \mathbf{n}, \cdot) = g(\mathbf{n}, \cdot) * T_l(\mathbf{n}, \cdot), \quad (9)$$

where $*$ is the convolution product, and $T_l(\mathbf{n}, \omega)$ is the Laplace transform of $\exp(i\mathbf{k}_{\mathbf{n}} \cdot \delta\mathbf{r}_l)$.

In the following, we are interested in cases where the $\delta\mathbf{r}_l$'s are small in some sense. To this end we split $\tilde{\varphi}_l(\mathbf{m})$ as

$$\tilde{\varphi}_l(\mathbf{m}) = \tilde{\phi}_l(\mathbf{m}) + \Delta\tilde{\varphi}_l(\mathbf{m}) \quad (10)$$

where

$$\tilde{\phi}_l(\mathbf{m}) = \sum_{j \in S; j \neq l} \delta\tilde{\phi}_j(\mathbf{m}), \quad (11)$$

with

$$\delta\tilde{\phi}_j(\mathbf{m}) = -\frac{e}{\varepsilon_0 k_{\mathbf{m}}^2} \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_j^{(0)}) (1 - i\mathbf{k}_{\mathbf{m}} \cdot \delta\mathbf{r}_j), \quad (12)$$

and

$$\Delta\tilde{\varphi}_l(\mathbf{m}) = -\frac{e}{\varepsilon_0 k_{\mathbf{m}}^2} \sum_{j \in S; j \neq l} \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_j^{(0)}) R_j(\mathbf{m}), \quad (13)$$

with

$$R_j(\mathbf{m}) = \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \delta\mathbf{r}_j) - 1 + i\mathbf{k}_{\mathbf{m}} \cdot \delta\mathbf{r}_j, \quad (14)$$

which is of order two in $\delta\mathbf{r}_j$. The Laplace transform of Eq. (10) yields

$$k_{\mathbf{m}}^2 \hat{\varphi}_l(\mathbf{m}, \omega) = k_{\mathbf{m}}^2 \hat{\phi}_l^{(00)}(\mathbf{m}, \omega) + \frac{ie}{\varepsilon_0} \sum_{j \in S; j \neq l} \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_{j0}) [\mathbf{k}_{\mathbf{m}} \cdot \delta\hat{\mathbf{r}}_j(\omega - \omega_{\mathbf{m},j}) + i\hat{R}_j(\mathbf{m}, \omega - \omega_{\mathbf{m},j})], \quad (15)$$

where $\hat{R}_j(\mathbf{m}, \omega)$ is the Laplace transform of R_j , $\hat{\phi}_l^{(00)}(\mathbf{m}, \omega)$ is $\hat{\phi}_l(\mathbf{m}, \omega)$ computed with another $\delta \mathbf{r}_j = 0$ for all j 's, and $\omega_{\mathbf{m},j} = \mathbf{k}_{\mathbf{m}} \cdot \mathbf{v}_j$ comes from the time dependence of $\mathbf{r}_l^{(0)}$ in the exponent of Eqs. (12,13). Substituting the $\delta \hat{\mathbf{r}}_j$'s with their expression provided by Eq. (8) yields

$$\begin{aligned} k_{\mathbf{m}}^2 \hat{\phi}_l(\mathbf{m}, \omega) &= \frac{e^2}{L^3 m_e \epsilon_0} \sum_{\mathbf{n}} \mathbf{k}_{\mathbf{m}} \cdot \mathbf{k}_{\mathbf{n}} \sum_{j \in S; j \neq l} \frac{\Psi_j(\hat{\phi}_j; \mathbf{n}, \omega + \omega_{\mathbf{n},j} - \omega_{\mathbf{m},j})}{(\omega - \omega_{\mathbf{m},j})^2} \exp[i(\mathbf{k}_{\mathbf{n}} - \mathbf{k}_{\mathbf{m}}) \cdot \mathbf{r}_{j0}] \\ &= k_{\mathbf{m}}^2 \hat{\phi}_l^{(0)}(\mathbf{m}, \omega) - \frac{e}{\epsilon_0} \sum_{j \in S; j \neq l} \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_{j0}) \hat{R}_j(\omega - \omega_{\mathbf{m},j}), \end{aligned} \quad (16)$$

where $\hat{\phi}_l^{(0)}(\mathbf{m}, \omega)$ is $\hat{\phi}_l(\mathbf{m}, \omega)$ computed with another $\delta \mathbf{r}_j = \delta \mathbf{r}_j(0) + \delta \dot{\mathbf{r}}_j(0)t$ for all j 's.

Summing Eq. (16) over $l = 1, \dots, N$ and dividing by $N - 1$, yields

$$\begin{aligned} k_{\mathbf{m}}^2 \hat{\phi}(\mathbf{m}, \omega) &= \frac{e^2}{L^3 m_e \epsilon_0} \sum_{\mathbf{n}} \mathbf{k}_{\mathbf{m}} \cdot \mathbf{k}_{\mathbf{n}} \sum_{j \in S} \frac{\Psi_j(\hat{\phi}; \mathbf{n}, \omega + \omega_{\mathbf{n},j} - \omega_{\mathbf{m},j})}{(\omega - \omega_{\mathbf{m},j})^2} \exp[i(\mathbf{k}_{\mathbf{n}} - \mathbf{k}_{\mathbf{m}}) \cdot \mathbf{r}_{j0}] \\ &= k_{\mathbf{m}}^2 \hat{\phi}^{(0)}(\mathbf{m}, \omega) - \frac{e}{\epsilon_0} \sum_{j \in S} \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_{j0}) \hat{R}_j(\omega - \omega_{\mathbf{m},j}) + k_{\mathbf{m}}^2 \hat{U}'(\mathbf{m}, \omega), \end{aligned} \quad (17)$$

with $\hat{U}'(\mathbf{m}, \omega) = 0$ as yet, and where $\hat{\phi}^{(0)}(\mathbf{m}, \omega)$ is $\hat{\phi}_l^{(0)}(\mathbf{m}, \omega)$ complemented by the missing l -th term. Equation (17) is of the type $\mathcal{E}\hat{\phi} = \text{source term}$, where \mathcal{E} is a linear operator, acting on the infinite dimensional array whose components are all the $\hat{\phi}(\mathbf{m}, \omega)$'s. It is the fundamental equation of this paper.

III. DEBYE SCREENING AND LANDAU DAMPING

We now specialize Eq. (17) by considering the lowest order contribution of the $\delta \mathbf{r}_j$'s, which makes the \hat{R}_j 's vanish, and reduces $\Psi_j(\hat{\phi}; \mathbf{n}, \omega)$ to $\hat{\phi}(\mathbf{n}, \omega)$. We introduce a smooth function $f(\mathbf{r}, \mathbf{v})$, the coarse-grained velocity distribution function at $t = 0$ such that the distribution

$$\sum_{l \in S} \bullet = \iint \bullet f(\mathbf{r}, \mathbf{v}) d^3 \mathbf{r} d^3 \mathbf{v} + W(\bullet), \quad (18)$$

where the distribution W yields a negligible contribution when applied to space dependent function which evolve slowly on the scale of the inter-particle distance; there the spatial integration is performed over the elementary cube of volume L^3 , and the velocity integration is over all velocities. This means we are considering a limit where L and N are large, while keeping constant the density $N/L^3 = L^{-3} \iint f(\mathbf{r}, \mathbf{v}) d^3 \mathbf{r} d^3 \mathbf{v}$.

We first apply this coarse-graining to \mathcal{E} only. Eq. (17) becomes

$$k_{\mathbf{m}}^2 \hat{\Phi}(\mathbf{m}, \omega) = k_{\mathbf{m}}^2 \hat{\phi}^{(0)}(\mathbf{m}, \omega) + \frac{e^2}{m_e \epsilon_0} \sum_{\mathbf{n}} \mathbf{k}_{\mathbf{m}} \cdot \mathbf{k}_{\mathbf{n}} \int \frac{\hat{\Phi}(\mathbf{n}, \omega + (\mathbf{k}_{\mathbf{n}} - \mathbf{k}_{\mathbf{m}}) \cdot \mathbf{v})}{(\omega - \mathbf{k}_{\mathbf{m}} \cdot \mathbf{v})^2} \tilde{f}(\mathbf{n} - \mathbf{m}, \mathbf{v}) d^3 \mathbf{v} + \hat{U}', \quad (19)$$

where $\hat{U}' = 0$ as yet, $\hat{\Phi}$ is this approximate and coarse-grained version of $\hat{\phi}$, and $\tilde{f}(\mathbf{m}, \mathbf{v})$ is the spatial Fourier transform of f . If the initial distribution f is a spatially uniform distribution function $f_0(\mathbf{v})$ plus a small perturbation of the order of Φ , the operator \mathcal{E} becomes diagonal with respect to both \mathbf{m} and ω , and linearizing Eq. (19) in $\hat{\Phi}$ yields

$$\epsilon(\mathbf{m}, \omega) \hat{\Phi}(\mathbf{m}, \omega) = \hat{\phi}^{(0)}(\mathbf{m}, \omega) + \hat{U}', \quad (20)$$

where $\hat{U}' = 0$ as yet and

$$\epsilon(\mathbf{m}, \omega) = 1 - \frac{e^2}{m_e \epsilon_0} \int \frac{f_0(\mathbf{v})}{(\omega - \mathbf{k}_{\mathbf{m}} \cdot \mathbf{v})^2} d^3 \mathbf{v}, \quad (21)$$

which is the classical plasma dielectric function. A first check of this can be obtained for a cold plasma: then $\epsilon(\mathbf{m}, \omega) = 1 - \frac{\omega_p^2}{\omega^2}$ where ω_p is the plasma frequency. The classical expression involving the gradient of f_0 in \mathbf{v} is obtained by a mere integration by parts.

The contribution of particle j to $\tilde{\phi}^{(0)}(\mathbf{m})$ is $\delta\tilde{\phi}_j^{(0)}(\mathbf{m}) = -\frac{e}{\varepsilon_0 k_{\mathbf{m}}^2} \exp[-i\mathbf{k}_{\mathbf{m}} \cdot (\mathbf{r}_{j0} + \mathbf{v}_j t)]$. Its Laplace transform is

$$\delta\hat{\phi}_j^{(0)}(\mathbf{m}, \omega) = -\frac{ie}{\varepsilon_0 k_{\mathbf{m}}^2} \frac{\exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_{j0})}{\omega - \mathbf{k}_{\mathbf{m}} \cdot \mathbf{v}_j}. \quad (22)$$

The corresponding part of $\hat{\Phi}(\mathbf{m}, \omega)$ is $\delta\hat{\Phi}_j(\mathbf{m}, \omega) = \delta\hat{\phi}_j^{(0)}(\mathbf{m}, \omega)/\epsilon(\mathbf{m}, \omega)$. This turns out to be the screened potential of particle j as first computed in section II.A of Ref. [1] and later on in [2]. By inverse Fourier-Laplace transform, after some transient described later in this section, the potential due to particle j becomes the screened Coulomb potential

$$\delta\Phi_j(\mathbf{r}) = \delta\Phi(\mathbf{r} - \mathbf{r}_{j0} - \mathbf{v}_j t, \mathbf{v}_j), \quad (23)$$

where

$$\delta\Phi(\mathbf{r}, \mathbf{v}) = -\frac{e}{L^3 \varepsilon_0} \sum_{\mathbf{m}} \frac{\exp(i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r})}{k_{\mathbf{m}}^2 \epsilon(\mathbf{m}, \mathbf{k}_{\mathbf{m}} \cdot \mathbf{v})}. \quad (24)$$

Therefore, after this transient, the full potential in the plasma turns out to be the sum of the screened Coulomb potentials of individual particles located at their ballistic positions. Since $\epsilon(\mathbf{m}, \mathbf{k}_{\mathbf{m}} \cdot \mathbf{v}) - 1 \simeq (k_{\mathbf{m}} \lambda_D)^{-2}$ where λ_D is the Debye length, there is no screening for $\|\mathbf{r}\| \ll \lambda_D$, since the main Fourier components in Eq. (24) are such that $k_{\mathbf{m}} \|\mathbf{r}\| \sim 1$. As was done for the bare potential of Eq. (3), the field acting on a given particle l is obtained by removing its own divergent contribution $\delta\Phi_l$ from Φ .

We now apply the coarse-graining to $\tilde{\phi}^{(0)}(\mathbf{m}, \omega)$ too. As a result of Eq. (11-12), this yields

$$\tilde{\Phi}^{(0)}(\mathbf{m}) = -\frac{e}{\varepsilon_0 k^2} \iint \exp[-i\mathbf{k}_{\mathbf{m}} \cdot (\mathbf{r} + \mathbf{v}t)] f(\mathbf{r}, \mathbf{v}) d^3\mathbf{r} d^3\mathbf{v} \quad (25)$$

whose Laplace transform is

$$\hat{\Phi}^{(0)}(\mathbf{m}, \omega) = -\frac{ie}{\varepsilon_0 k^2} \int \frac{\tilde{f}(\mathbf{m}, \mathbf{v})}{\omega - \mathbf{k}_{\mathbf{m}} \cdot \mathbf{v}} d^3\mathbf{v}, \quad (26)$$

which shows this second coarse-graining makes Eq. (20) with $\hat{U}' = 0$ to become the expression including initial conditions in Landau contour calculations of Langmuir wave growth or damping, obtained by linearizing Vlasov equation and using Fourier-Laplace transform, as described in many textbooks. Therefore in these calculations, $\hat{\Phi}(\mathbf{m}, \omega)$ turns out to be the coarse-grained version of the actual screened potential in the plasma.

It is worth noting the physical meaning of the linearization performed in this section. It corresponds to the approximation of the true dynamics with an approximate one ruled by

$$\delta\ddot{\mathbf{r}}_l = \frac{e}{m_e} \nabla \phi_l(\mathbf{r}_l^{(0)} + \delta\mathbf{r}_l), \quad (27)$$

where $\phi_l(\mathbf{r}) = \sum_{j \in S; j \neq l} \delta\phi_j(\mathbf{r})$ is the inverse Fourier transform of $\tilde{\phi}_l$ with

$$\delta\phi_j(\mathbf{r}) = -\frac{e}{4\pi\varepsilon_0 \|\mathbf{r} - \mathbf{r}_j^{(0)}\|} - \frac{e\delta\mathbf{r}_j \cdot (\mathbf{r} - \mathbf{r}_j^{(0)})}{4\pi\varepsilon_0 \|\mathbf{r} - \mathbf{r}_j^{(0)}\|^3}. \quad (28)$$

The j th component of the approximate electric field acting over particle l turns out to be due to a particle located at $\mathbf{r}_j^{(0)}$ instead of \mathbf{r}_j , and is made up of a Coulombian part and of a dipolar part with a dipole moment $-e\delta\mathbf{r}_j$. The cross-over between the two contributions occurs for $\|\mathbf{r}_l - \mathbf{r}_j^{(0)}\|$ on the order of $\|\delta\mathbf{r}_j\|$, i.e. when the distance between particle l and the ballistic particle j is equal to the distance between the latter and the true particle j . For larger values of $\|\mathbf{r}_l - \mathbf{r}_j^{(0)}\|$, the dipolar component is subdominant. For smaller ones, it is dominant, but with a direction which is a priori random with respect to the Coulombian one ($\mathbf{r}_l - \mathbf{r}_j^{(0)}$ is almost independent from $\delta\mathbf{r}_j$). Since the $\|\delta\mathbf{r}_j\|$'s are assumed small, the latter case should be rare since it corresponds to a very close encounter between particle l and the ballistic particle j . As a result the approximate electric field stays dominantly of Coulombian nature, but with a small mismatch of the charge positions with respect to the actual ones.

In order to provide a complementary view to Debye screening, we now turn back to the mechanical description of microscopic dynamics with the full Coulomb potential of Eq. (3). In order to compute the dynamics, we use Picard iteration technique. From Eq. (5), $\mathbf{r}_l^{(n)}$, the n th iterate of \mathbf{r}_l , is computed by

$$\ddot{\mathbf{r}}_l^{(n)} = \frac{e}{m_e} \nabla \phi_l^{(n-1)}(\mathbf{r}_l^{(n-1)}), \quad (29)$$

where $\phi_l^{(n-1)}$ is computed by the inverse Fourier transform of Eq. (3) with the \mathbf{r}_j 's substituted with the $\mathbf{r}_j^{(n-1)}$'s. The iteration starts with the ballistic approximation of the dynamics defined by Eq. (6). It is convenient to write Eq. (29) as

$$\ddot{\mathbf{r}}_l^{(n)} = \sum_{j \in S; j \neq l} \ddot{\mathbf{r}}_{lj}^{(n)}, \quad (30)$$

with

$$\ddot{\mathbf{r}}_{lj}^{(n)} = \mathbf{a}_C(\Delta \mathbf{r}_{lj}^{(n-1)}), \quad (31)$$

where $\Delta \mathbf{r}_{lj}^{(n-1)} = \mathbf{r}_l^{(n-1)} - \mathbf{r}_j^{(n-1)}$ and

$$\mathbf{a}_C(\mathbf{r}) = -\frac{e}{m_e} \nabla \delta \varphi_C(\mathbf{r}), \quad (32)$$

with $\delta \varphi_C$ given by

$$\delta \varphi_C(\mathbf{r}) = -\frac{e}{\varepsilon_0 k_{\mathbf{m}}^2 L^3} \sum_{\mathbf{m}} \exp(i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}), \quad (33)$$

such that $\varphi(\mathbf{r}) = \sum_{j \in S} \delta \varphi_C(\mathbf{r} - \mathbf{r}_j)$. Then $\ddot{\mathbf{r}}_l^{(1)} = \sum_{j \in S; j \neq l} \mathbf{a}_C(\Delta \mathbf{r}_{lj}^{(0)})$ and for $n \geq 2$

$$\ddot{\mathbf{r}}_l^{(n)} = \ddot{\mathbf{r}}_l^{(1)} + \sum_{j \in S; j \neq l} \nabla \mathbf{a}_C(\Delta \mathbf{r}_{lj}^{(0)}) \cdot \Delta^2 \mathbf{r}_{lj}^{(n-1)} + O(a^3), \quad (34)$$

where $\Delta^2 \mathbf{r}_{lj}^{(n-1)} = \Delta \mathbf{r}_{lj}^{(n-1)} - \Delta \mathbf{r}_{lj}^{(0)}$ and a is the order of magnitude of the total Coulombian acceleration. It is also convenient to introduce $\delta \mathbf{r}_l^{(n)} = \mathbf{r}_l^{(n)} - \mathbf{r}_l^{(0)} = \sum_{l \neq j} \delta \mathbf{r}_{lj}^{(n)}$ with $\delta \mathbf{r}_{lj}^{(n)}(t) = \int_0^t \int_0^{t'} \ddot{\mathbf{r}}_{lj}^{(n)}(t'') dt'' dt'$; note that $\delta \mathbf{r}_{lj}^{(n)} = -\delta \mathbf{r}_{jl}^{(n)}$ is the contribution of particle j to the perturbation of the motion of particle l , while $\Delta^2 \mathbf{r}_{lj}^{(n)}$ is the difference between the perturbation of particle l and the perturbation of particle j .

Equation (34) may be written

$$\ddot{\mathbf{r}}_l^{(n)} = \sum_{j \in S; j \neq l} [(\ddot{\mathbf{r}}_{lj}^{(1)} + M_{lj}^{(n-1)}) + 2\nabla \mathbf{a}_C(\Delta \mathbf{r}_{lj}^{(0)}) \cdot \delta \mathbf{r}_{lj}^{(n-1)}] + O(a^3), \quad (35)$$

where $M_{lj}^{(n-1)} = \nabla \mathbf{a}_C(\Delta \mathbf{r}_{lj}^{(0)}) \cdot \sum_{i \in S; i \neq l, j} (\delta \mathbf{r}_{li}^{(n-1)} - \delta \mathbf{r}_{ji}^{(n-1)})$ is the effect of particle j on particle l mediated by all other particles. Since the above screened potential was found by first order perturbation theory, it is felt in the acceleration of particles computed to second order. This acceleration is provided by Eq. (35) for $n = 2$. Therefore its term in brackets is the screened acceleration of particle l due to particle j . As a result, though the summation runs over all particles, its effective part is only due to particles j typically inside the Debye sphere about particle l . Starting from the third iterate of Picard scheme, the summation of $\nabla \mathbf{a}_C(\Delta \mathbf{r}_{lj}^{(0)}) \cdot \Delta^2 \mathbf{r}_{lj}^{(n-1)}$ ranges inside this Debye sphere, since the $\Delta^2 \mathbf{r}_{lj}^{(n-1)}$'s are screened ones. This justifies the use of the screened potential to compute collisional transport in the following section. It is worth noting that in Eq. (23) it would be better to compute the screened potential at the actual positions of the particles, since the true screened potential corresponding to $n \rightarrow \infty$ diverges exactly like the original Coulomb one at the actual position of the particles.

The preceding calculation yields the following interpretation of screening. At $t = 0$ consider a set of randomly distributed particles. Consider a particle l . At a later time t , it has deflected all particles which made a closest approach to it with an impact parameter $b \lesssim v_{\text{th}} t$ where v_{th} is the thermal velocity. This part of their global deflection due to particle l reduces the number of particles inside the sphere $S(t)$ of radius $v_{\text{th}} t$ about it. Therefore the effective charge of particle l as seen out of $S(t)$ is reduced: the charge of particle l is screened due to these

deflections. This screening effect increases with t , and thus with the distance to particle l . As a result, the typical time-scale for screening to set in when starting from random particle positions is the time for a thermal particle to cross a Debye sphere, i.e. ω_p^{-1} , which sets the duration of the above transient. Furthermore, screening is a cooperative dynamical process: it results from the accumulation of independent repulsive deflections with the same qualitative impact on the effective electric field of particle l (if ions were added, the attractive deflection of charges with opposite signs would have the same effect). It is a cooperative effect, but not a collective one. As a result, screening and collisional transport are two aspects of the same two-body repulsive process.

IV. DEBYE SCREENING AND COLLISIONAL TRANSPORT

We now focus on the case where the particles have random initial positions, i.e. where the plasma has a uniform density, and for simplicity we consider the plasma to be in thermal equilibrium. Then the dynamics of particles has no collective aspect, but is ruled by the cumulative effect of two-body deflections. More specifically, we choose random \mathbf{r}_{l0} 's, and vanishing $\delta\mathbf{r}_l(0)$'s and $\delta\dot{\mathbf{r}}_l(0)$'s, and we assume that at $t = 0$ all particles are in the same cube of volume L^3 , with $L \gg \lambda_D$. To the contrary each particle has a well defined velocity, in such a way that the overall initial velocity distribution is close to f . We focus on particle l which is assumed to be close to the center of the cube. In this section we approximate the true dynamics by that due to the screened Coulombian interactions, i.e. we write

$$\delta\ddot{\mathbf{r}}_l = \sum_{j \in S; j \neq l} \mathbf{a}(\mathbf{r}_l - \mathbf{r}_j, \mathbf{v}_j), \quad (36)$$

where

$$\mathbf{a}(\mathbf{r}, \mathbf{v}) = \frac{e}{m_e} \nabla \delta\Phi(\mathbf{r}, \mathbf{v}), \quad (37)$$

where $\delta\Phi(\mathbf{r}, \mathbf{v})$ is given by Eq. (24). Here we make the slight generalization of Eq. (37) suggested after Eq. (35) where $\mathbf{r}_j^{(0)}(t)$ is substituted with $\mathbf{r}_j(t)$: the screened potential now has its divergences at the actual positions of particles, as it should. We compute particle l deflection in a series of steps. First, we use first order perturbation theory in $\delta\Phi$, which shows the total deflection to be the sum of the individual deflections due to all other particles. For an impact parameter much smaller than λ_D , the deflection due to a particle turns out to be the perturbative value of the Rutherford deflection due to this particle if it were alone. Second, for a close encounter with particle n , we show that the deflection of particle l is exactly the one it would undergo if the other $N - 2$ particles were absent. Third, the deflection for an impact parameter of order λ_D is given by the Rutherford expression multiplied by some function of the impact parameter reflecting screening. This yields an analytical expression for deflection whatever be the impact parameter. This procedure is reminiscent of that in [4], but avoids invoking the cancellation of three infinite integrals.

We first compute $\delta\mathbf{r}_l$ by first order perturbation theory in $\delta\Phi$, taking the ballistic motion defined by Eq. (6) as zeroth order approximation. This yields

$$\delta\dot{\mathbf{r}}_{l1}(t) = \sum_{j \in S; j \neq l} \delta\dot{\mathbf{r}}_{lj1}(0, t), \quad (38)$$

where

$$\delta\dot{\mathbf{r}}_{lj1}(t_1, t_2) = \int_{t_1}^{t_2} \mathbf{a}[\mathbf{r}_l^{(0)}(t') - \mathbf{r}_j^{(0)}(t'), \mathbf{v}_j] dt'. \quad (39)$$

It is convenient to write $\mathbf{r}_l^{(0)}(t') - \mathbf{r}_j^{(0)}(t') = \mathbf{b}_{lj} + \Delta\mathbf{v}_{lj}(t' - t_{lj})$, where t_{lj} is the time of closest approach of the two ballistic orbits. Then $b_{lj} = \|\mathbf{b}_{lj}\|$ is the impact parameter of these two orbits when singled out. The initial random positions of the particles translate into random values of \mathbf{b}_{lj} and of t_{lj} . For a given b_{lj} , the deflection of particle l given by Eq. (39) is maximum if t_{lj} is in the interval $[0, t]$, or more precisely for $t \gg \Delta t_{lj} \equiv b_{lj}/\Delta v_{lj}$, in the interval $[\alpha\Delta t_{lj}, t - \alpha\Delta t_{lj}]$, where $\Delta v_{lj} = \|\Delta\mathbf{v}_{lj}\|$ and α is about a few units. Indeed Δt_{lj} is the order of magnitude of the duration of the main contribution of the integral in Eq. (39) to the deflection. This duration is about the inverse of the plasma frequency for $b_{lj} \sim \lambda_D$ and Δv_{lj} on the order of the thermal velocity.

For the sake of brevity, we compute here just the trace of the diffusion tensor for the particle velocities. To this end, we perform an average over all the \mathbf{r}_{l0} 's to get

$$\langle \delta\dot{\mathbf{r}}_{l1}^2(t) \rangle = \sum_{j \in S; j \neq l} \langle \delta\dot{\mathbf{r}}_{lj1}^2(t) \rangle, \quad (40)$$

taking into account Eq. (24), and the fact that the initial positions are independently random, as well as the $\mathbf{r}_i - \mathbf{r}_j$'s for $i \neq j$. Therefore, though being due to the simultaneous scattering of particle l with the many particles inside its Debye sphere, $\langle \delta \dot{\mathbf{r}}_{l1}^2(t) \rangle$ turns out to be the sum of individual two-body deflections. Hence the contribution to $\langle \delta \dot{\mathbf{r}}_{l1}^2(t) \rangle$ of particles with given b_{lj} and Δv_{lj} can be computed as if it would result from successive two-body collisions, as was done in Ref. [3] and in many textbooks.

For an impact parameter much smaller than λ_D , the main contribution of $\mathbf{a}[\mathbf{r}_l^{(0)}(t') - \mathbf{r}_j^{(0)}(t'), \mathbf{v}_j]$ to the deflection of particle l comes from times t' where $\|\mathbf{r}_l^{(0)}(t') - \mathbf{r}_j^{(0)}(t')\| \ll \lambda_D$. Therefore $\mathbf{a}(\mathbf{r}, \mathbf{v})$ takes on its bare Coulombian value, and $\langle \delta \dot{\mathbf{r}}_{l1}^2(t) \rangle$ is a first order approximation of the effect on particle l of a Rutherford collision with particle j . The comparison of this approximate value with the exact one shows the perturbative calculation to be correct for $b_{lj} \gg \lambda_{\text{ma}} = \frac{e^2}{\pi m_e \varepsilon_0 \Delta v_{lj}^2}$, the distance of minimum approach of two electrons in a Rutherford collision.

Second, we consider the case of the close approach of particle n to particle l , i.e. $b_{ln} \sim \lambda_{\text{ma}}$. We write the acceleration of particle l as

$$\ddot{\mathbf{r}}_l = \mathbf{a}(\mathbf{r}_l - \mathbf{r}_n) + \sum_{j \in S; j \neq l, n} \mathbf{a}(\mathbf{r}_l - \mathbf{r}_j). \quad (41)$$

For particle n we write the same equation by exchanging indices l and n . Since the two particles are at distances much smaller than the inter-particle distance $d = n^{-1/3} = L/N^{1/3}$, the accelerations they get from all other particles are almost the same. Therefore, when subtracting the two rigorous equations of motion, the two summations over j almost cancel, yielding

$$\frac{d^2(\mathbf{r}_l - \mathbf{r}_j)}{dt^2} = 2\mathbf{a}(\mathbf{r}_l - \mathbf{r}_n), \quad (42)$$

which is the equation describing the Rutherford collision of these two particles in their center of mass frame, in the absence of all other particles (at such distances the screened potential is the bare Coulomb one). Since $b_{ln} \ll d$, Δt_{ln} is much smaller than the Δt_{lj} s of the other particles. Therefore they produce a negligible deflection of the center of mass during the Rutherford. As a result, the deflection of particle l during this collision is exactly that of a Rutherford two-body collision. The contribution of such collisions to $\langle \delta \dot{\mathbf{r}}_l^2(t) \rangle$ was calculated in Ref. [3].

Now, since the deflection of particle l due to particle j as computed by the above perturbation theory is an approximation of the Rutherford deflection for the same impact parameter, we may approximate the perturbative deflection one by the full Rutherford one, which provides a trivial matching of the theories for $b_{lj} \sim \lambda_{\text{ma}}$ and for $\lambda_D \gg b_{lj} \gg \lambda_{\text{ma}}$: we may use the estimate of [3] in the whole domain $b_{lj} \ll \lambda_D$.

Third, we deal with impact parameters of the order of λ_D . Then the deflection due to particle j must be computed with Eq. (39). For the sake of simplicity, we make the calculation for the case where \mathbf{v}_j is small, which makes $\delta\Phi(\mathbf{r}, \mathbf{v}) \simeq \delta\Phi(\mathbf{r}, \mathbf{0})$ which is the Yukawa potential $\delta\Phi_Y(\mathbf{r}) = -\frac{e}{4\pi\varepsilon_0\|\mathbf{r}\|} \exp(-\frac{\|\mathbf{r}\|}{\lambda_D})$ (Eq. (18) of Ref. [1]). The first order correction in $\mathbf{k}_m \cdot \mathbf{v}_j$ to this approximation is a dipolar potential with an electric dipole moment proportional to \mathbf{v}_j . Since a Maxwellian distribution is symmetrical in \mathbf{v} , these individual dipolar contributions cancel globally. As a result, the first relevant correction to the Yukawa potential is of second order in $\mathbf{k}_m \cdot \mathbf{v}_j$. This should make the Yukawa approximation a relevant one for a large part of the bulk of the Maxwellian distribution.

In the small deflection limit, the full deflection of particle l due to particle j is provided by

$$\delta \dot{\mathbf{r}}_{lj1}(-\infty, +\infty) = \frac{e^2}{4\pi m_e \varepsilon_0} \mathbf{b}_{lj} \int_{-\infty}^{+\infty} \left[\frac{1}{r^3(t)} + \frac{1}{\lambda_D r^2(t)} \right] \exp\left[-\frac{r(t)}{\lambda_D}\right] dt, \quad (43)$$

where $r(t) = (b_{lj}^2 + \Delta v_{lj}^2 t^2)^{1/2}$. Defining $\theta = \arcsin[\Delta v_{lj} t / r(t)]$, this equation becomes

$$\delta \dot{\mathbf{r}}_{lj1}(-\infty, +\infty) = -\frac{2e^2}{4\pi m_e \varepsilon_0 \Delta v_{lj}} \frac{h(b_{lj})}{b_{lj}^2} \mathbf{b}_{lj}, \quad (44)$$

where

$$h(b) = \frac{1}{2} \int_{-\pi/2}^{+\pi/2} \left[\cos(\theta) + \frac{b}{\lambda_D} \right] \exp\left[-\frac{b}{\lambda_D \cos(\theta)}\right] d\theta, \quad (45)$$

During time $t \gg \Delta t_{lj}$, a volume $2\pi \Delta v_{lj} t b_{lj} \delta b_{lj}$ of particles with velocity \mathbf{v}_j and impact parameters between b_{lj} and $b_{lj} + \delta b_{lj}$ produce the deflection of particle l given by Eq. (44), and a contribution scaling like $\frac{h^2(b_{lj})}{b_{lj}} \delta b_{lj}$ to $\langle \delta \dot{\mathbf{r}}_{l1}^2(t) \rangle$.

Let b_{\min} be such that $\lambda_D \gg b_{\min} \gg \lambda_{\text{ma}}$. The contribution of all impact parameters between b_{\min} and some b_{\max} is thus scaling like the integral $\int_{b_{\min}}^{b_{\max}} h^2(b)/b db$. Since for b small $h(0) \simeq 1$, if $b_{\max} \ll \lambda_D$, this is the non-screened contribution of orbits relevant to the above perturbative calculation. Since, by approximating it by the Rutherford-like result of Ref. [3], this contribution matches that for impact parameters on the order of λ_{ma} , the contribution of all impact parameters between λ_{ma} and some b_{\max} small with respect to λ_D is thus scaling like the integral $\int_{\lambda_{\text{ma}}}^{b_{\max}} 1/b db$ as was computed in Ref. [3]. The matching of this result for $b \sim \lambda_D$ is simply accomplished by setting a factor $h^2(b)$ in the integrand which makes the integral converge for $b \rightarrow \infty$. Taking this limit, one finds that the Coulomb logarithm $\ln(\lambda_D/\lambda_{\text{ma}})$ of the second Eq. (14) of Ref. [3] becomes $\ln(\lambda_D/\lambda_{\text{ma}}) + C$ where C is of order unity. If the full dependence of the screening on \mathbf{v}_j were taken into account, the modification of the Coulomb logarithm would be velocity dependent.

For the sake of brevity, we do not develop here the calculation of the dynamical friction. It requires using second order perturbation theory, but follows the same lines as those for the diffusion coefficient.

V. WAVE-PARTICLE DYNAMICS

In section III, the existence of Langmuir waves is asserted by connection with Landau theory. For an inhomogeneous plasma, the acceleration of particle l may be split into a homogeneous and a wave part. To this end, we still consider that the \mathbf{r}_{l0} 's are random, but we allow for Langmuir waves by allowing for non zero $\delta\mathbf{r}_j(0)$'s and $\delta\dot{\mathbf{r}}_j(0)$'s for the $\delta\mathbf{r}_j$'s in Eq. (12). Therefore, in the formulas of section III, the \mathbf{r}_{j0} and \mathbf{v}_j 's are slightly shifted with respect to the initial $\mathbf{r}_j(0)$'s and $\dot{\mathbf{r}}_j(0)$'s due to Langmuir waves. When using the linearized versions of Eq. (17) and subsequent ones in section III, we can split all ϕ 's and Φ 's into a homogeneous part and an independent inhomogeneous one. Therefore the diffusion coefficient and the dynamical friction estimated by perturbative calculation of the dynamics up to second order are the sum of the previous collisional contribution and one due to waves, as calculated for instance in Refs. [5, 7]. For the sake of brevity, we do not develop this point here. We rather introduce another path for a better description of wave-particle dynamics.

Indeed, resonant particles may experience trapping or chaotic dynamics, which imply $\mathbf{k}_m \cdot \delta\mathbf{r}_l$'s of the order of 2π or larger for wave \mathbf{k}_m 's. To describe such a dynamics, it is not appropriate to expand ϕ as was done in Eqs. (11-12) for such particles. However, this expansion may still be justified for non resonant particles over times where trapping and chaos show up for resonant ones. In order to keep the capability to describe the latter effects, we now split the set of N particles into bulk and tail. The bulk is defined as the set of particles which are not resonant with Langmuir waves. We then perform the analysis of section II for the N_{bulk} particles, while keeping the exact contribution of the N_{tail} particles to the electrostatic potential. To this end, we number the tail particles from 1 to N_{tail} , the bulk ones from $N_{\text{tail}} + 1$ to N , and we call these respective sets of integer S_{tail} and S_{bulk} . For $l \in S_{\text{bulk}}$, we now substitute Eq. (11) with

$$\tilde{\phi}_l(\mathbf{m}) = U(\mathbf{m}) + \sum_{j \in S_{\text{bulk}}; j \neq l} \delta\tilde{\phi}_j(\mathbf{m}), \quad (46)$$

with

$$U(\mathbf{m}) = -\frac{e}{\varepsilon_0 k_{\mathbf{m}}^2} \sum_{j \in S_{\text{tail}}} \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_j), \quad (47)$$

and we perform the calculation of section II by substituting the previous summations with index running from 1 to N by ones where the index runs over S_{bulk} , while keeping the exclusion of $j = l$ where indicated. The previous division by $N - 1$ preceding Eq. (17) is now a division by $N_{\text{bulk}} - 1$. This yields Eq. (17) where $\hat{U}'(\mathbf{m}, \omega)$ now is $N_{\text{bulk}}/(N_{\text{bulk}} - 1)$ times $\hat{U}(\mathbf{m}, \omega)$, the Laplace transform of $U(\mathbf{m})$.

Let $\hat{\Phi}_{\text{bulk}}(\mathbf{m}, \omega)$ be the solution of Eq. (20) for $\hat{U}'(\mathbf{m}, \omega) = 0$, and $\tilde{\Phi}_{\text{bulk}}(\mathbf{m}, t)$ be its inverse Laplace transform. We now derive an amplitude equation for $\tilde{\Phi}_{\text{bulk}}(\mathbf{m}, t)$ in a way similar to Refs. [6, 7]. Let $\omega_{\mathbf{m}}$ be such that $\epsilon(\mathbf{m}, \omega_{\mathbf{m}}) = 0$. Because of the definition of the bulk, this frequency is real. Then $\tilde{\Phi}_{\text{bulk}}(\mathbf{m}, t) = A \exp(-i\omega_{\mathbf{m}}t)$, where A is a constant, and

$$\hat{\phi}^{(0)}(\mathbf{m}, \omega) = \frac{iA}{\omega - \omega_{\mathbf{m}}}, \quad (48)$$

according to Eq. (20) with $\hat{U}'(\mathbf{m}, \omega) = 0$.

Let $\tilde{\Phi}(\mathbf{m}, t)$ be the inverse Laplace transform of $\hat{\Phi}(\mathbf{m}, \omega)$ and $g(\mathbf{m}, t) = \tilde{\Phi}(\mathbf{m}, t)/\tilde{\Phi}_{\text{bulk}}(\mathbf{m}, t)$. Therefore $\hat{\Phi}(\mathbf{m}, \omega) = A\hat{g}(\omega - \omega_{\mathbf{m}})$, which together with Eq. (20) and (48) yields

$$A\epsilon(\mathbf{m}, \omega_{\mathbf{m}} + \omega')[\hat{g}(\mathbf{m}, \omega') - \frac{i}{\omega'}] = \hat{U}'(\mathbf{m}, \omega_{\mathbf{m}} + \omega'), \quad (49)$$

where $\omega' = \omega - \omega_{\mathbf{m}}$. If $N_{\text{tail}} \ll N_{\text{bulk}}$, $g(\mathbf{m}, t)$ is a slowly evolving amplitude, and the support of $\hat{g}(\mathbf{m}, \omega)$ is narrow about zero. This justifies Taylor-expanding $\epsilon(\mathbf{m}, \omega_{\mathbf{m}} + \omega')$ about $\omega' = 0$ in Eq. (49), which yields $\frac{\partial \epsilon(\mathbf{m}, \omega_{\mathbf{m}})}{\partial \omega} \omega'$ to lowest order. Setting this in Eq. (49) and performing the inverse Laplace transform finally yields an amplitude equation for $\tilde{\Phi}(\mathbf{m}, t)$

$$\frac{\partial \tilde{\Phi}(\mathbf{m}, t)}{\partial t} + i\omega_{\mathbf{m}}\tilde{\Phi}(\mathbf{m}, t) = \frac{ieN_{\text{bulk}}}{\epsilon_0 k_{\mathbf{m}}^2 (N_{\text{bulk}} - 1) \frac{\partial \epsilon(\mathbf{m}, \omega_{\mathbf{m}})}{\partial \omega}} \sum_{j \in S_{\text{tail}}} \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_j). \quad (50)$$

The self-consistent dynamics of the potential and of the tail particles is ruled by this equation and by the equation of motion of these particles

$$\ddot{\mathbf{r}}_j = \frac{ie}{L^3 m_e} \sum_{\mathbf{n}} \mathbf{k}_{\mathbf{n}} \tilde{\Phi}_j(\mathbf{n}) \exp(i\mathbf{k}_{\mathbf{n}} \cdot \mathbf{r}_j). \quad (51)$$

These two sets of equations generalize to 3 dimensions the self-consistent dynamics defined in Refs. [6, 7]. For the sake of brevity, we do not develop here the full generalization of the analysis in these papers; it is lengthy, but rather trivial. However, since it unifies spontaneous emission and Landau growth and damping, we give the result ruling the evolution of the amplitude of a Langmuir wave provided by perturbation calculation where the r. h. s. of Eqs. (50,51) are considered as small of order one. This is natural for Eq. (50) since $N_{\text{tail}} \ll N_{\text{bulk}}$, and for Eq. (51) if the Langmuir waves are small. Let $J(\mathbf{m}, t) = \langle \tilde{\Phi}(\mathbf{m}, t)\tilde{\Phi}(-\mathbf{m}, t) \rangle$, where the average is over the initial positions of the tail particles (their distribution is assumed spatially uniform). Then a second order calculation yields

$$\frac{dJ(\mathbf{m}, t)}{dt} = 2\gamma_{\mathbf{m}L}J(\mathbf{m}, t) + S_{\mathbf{m} \text{spont}}, \quad (52)$$

where γ_{jL} is the Landau growth or damping rate given by

$$\gamma_{\mathbf{m}L} = \alpha_{\mathbf{m}} \frac{df_{\text{red}}}{dv} \left(\frac{\omega_{\mathbf{m}}}{k_{\mathbf{m}}}; \mathbf{m} \right) \quad (53)$$

with

$$\alpha_{\mathbf{m}} = \frac{\pi e^2}{m_e \epsilon_0 k_{\mathbf{m}}^2 \frac{\partial \epsilon(\mathbf{m}, \omega_{\mathbf{m}})}{\partial \omega}}, \quad (54)$$

and $f_{\text{red}}(v; \mathbf{m})$ is the reduced coarse-grained distribution function $f_{\text{red}}(v; \mathbf{m}) = \iint f(v\hat{\mathbf{k}}_{\mathbf{m}} + \mathbf{v}_{\perp}) d^2\mathbf{v}_{\perp}$ where $\hat{\mathbf{k}}_{\mathbf{m}}$ is the unit vector along $\mathbf{k}_{\mathbf{m}}$ and \mathbf{v}_{\perp} is the component of the velocity perpendicular to $\mathbf{k}_{\mathbf{m}}$; $S_{\mathbf{m} \text{spont}}$ is given by

$$S_{\mathbf{m} \text{spont}} = \frac{2\alpha_{\mathbf{m}}^2}{\pi e^2 k_{\mathbf{m}} n} f_{\text{red}} \left(\frac{\omega_{\mathbf{m}}}{k_{\mathbf{m}}} \right), \quad (55)$$

where $n = N/L^3$ is the plasma density. It corresponds to the spontaneous emission of waves by particles and induces an exponential relaxation of the waves to the thermal level in the case of Landau damping. The second order calculation for the particles yields the diffusion and friction coefficients of the Fokker-Planck equation ruling the tail dynamics. This equation corresponds to the classical quasilinear result, plus a dynamical friction term mirroring the spontaneous emission of waves by particles.

VI. CONCLUSION

This theory brings unification and simplification in basic microscopic plasma physics, and may be useful for pedagogical purposes. One might think about trying to apply the above approach to plasmas with more species, or with a magnetic field, or where particles experience trapping and chaotic dynamics. The first generalization sounds rather

trivial, and the second one is under way, at least in one dimension (see a pedestrian introduction in [8] and more specific results in [9, 10]).

Furthermore, we used only a very specific part of the fundamental equation (17): the one involving linearization and coarse-graining. It would be interesting to study the effect of the coupling of Fourier components with both coherent and incoherent effects. In particular to perform the analysis of the previous section by substituting $k_{\mathbf{m}}^2 \hat{U}'(\mathbf{m}, \omega)$ with $-\frac{e}{\epsilon_0} \sum_{j \in S} \exp(-i\mathbf{k}_{\mathbf{m}} \cdot \mathbf{r}_{j0}) \hat{R}_j(\omega - \omega_{\mathbf{m},j})$.

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