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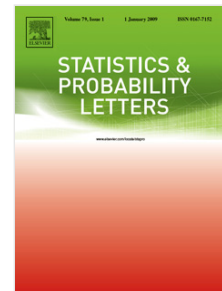
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## MOMENT EXPLOSION IN THE LIBOR MARKET MODEL

STEFAN GERHOLD

ABSTRACT. In the LIBOR market model, forward interest rates are log-normal under their respective forward measures. This note shows that their distributions under the other forward measures of the tenor structure have approximately log-normal tails.

### 1. INTRODUCTION

The LIBOR market model [3] is one of the most popular models for pricing and hedging interest rate derivatives. Its state variables are forward interest rates  $F_n(t) := F(t; T_{n-1}, T_n)$ , spanning time periods  $[T_{n-1}, T_n]$ , where

$$0 < T_0 < T_1 < \dots < T_M$$

is a fixed tenor structure. Under the  $T_M$ -forward measure  $\mathbb{Q}^M$ , which has as numeraire the zero coupon bond maturing at  $T_M$ , the dynamics of the forward rates are

$$\begin{aligned} dF_n(t) &= -\sigma_n(t)F_n(t) \sum_{j=n+1}^M \frac{\rho_{nj}\tau_j\sigma_j(t)F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_n(t)F_n(t)dW_n(t), \\ 1 &\leq n < M, \\ dF_M(t) &= \sigma_M(t)F_M(t)dW_M(t). \end{aligned}$$

Here,  $\sigma_n$  are some positive deterministic volatility functions, and  $W$  is a vector of standard Brownian motions with instantaneous correlations  $dW_i(t)dW_j(t) = \rho_{ij}dt$ . Moreover,  $\tau_n = \tau(T_{n-1}, T_n)$  denotes the year fraction between the tenor dates  $T_{n-1}$  and  $T_n$ .

Note that  $F_M$  is a geometric Brownian motion under  $\mathbb{Q}^M$ . More generally, each rate  $F_n$  is a geometric Brownian motion under its own forward measure, while it has a stochastic drift under the other forward measures. Therefore, pricing derivatives with the LIBOR market model often requires a Monte Carlo simulation. To avoid this, a popular approximation of the above dynamics is obtained by “freezing the drift” [2, 4]:

$$\begin{aligned} dF_n^{\text{fd}}(t) &= -\sigma_n(t)F_n^{\text{fd}}(t) \sum_{j=n+1}^M \frac{\rho_{nj}\tau_j\sigma_j(t)F_j(0)}{1 + \tau_j F_j(0)} dt + \sigma_n(t)F_n^{\text{fd}}(t)dW_n(t), \\ 1 &\leq n < M, \\ dF_M^{\text{fd}}(t) &= \sigma_M(t)F_M^{\text{fd}}(t)dW_M(t). \end{aligned}$$

Since the log-drifts are now deterministic, the new rates  $F_n^{\text{fd}}$  are just geometric Brownian motions, which allow for explicit pricing formulas for many interest-linked products. As a piece of evidence for the quality of this approximation, we

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show in the present note that, for fixed  $t > 0$ , the distribution of  $F_n^{\text{fd}}(t)$  has roughly the same tail heaviness as the distribution of  $F_n(t)$ .

## 2. MAIN RESULT

If  $X$  is any log-normal random variable, so that  $\log X \sim \mathcal{N}(\mu, \sigma^2)$  for some real  $\mu$  and positive  $\sigma$ , then

$$(1) \quad \sup\{v : \mathbf{E}[e^{v \log^2 X}] < \infty\} = \frac{1}{2\sigma^2}.$$

This follows from

$$\mathbf{E}[e^{v \log^2 X}] = \frac{1}{\sqrt{1 - 2\sigma^2 v}} \exp\left(\frac{\mu^2 v}{1 - 2\sigma^2 v}\right), \quad v < \frac{1}{2\sigma^2}.$$

Our main result shows that  $F_n(t)$  has approximately log-normal tails, in the sense that the left-hand side of (1) is finite and positive if  $X$  is replaced by  $F_n(t)$ . Furthermore, this ‘‘critical moment’’ is the same for  $F_n(t)$  and the frozen drift approximation  $F_n^{\text{fd}}(t)$ . We write  $\mathbf{E}^n$  for the expectation w.r.t. the  $T_n$ -forward measure  $\mathbb{Q}^n$ .

**Theorem 1.** *In the log-normal LIBOR market model, we have for all  $t > 0$  and all  $1 \leq n \leq M$*

$$\begin{aligned} \sup\{v : \mathbf{E}^M[e^{v \log^2 F_n(t)}] < \infty\} &= \sup\{v : \mathbf{E}^M[e^{v \log^2 F_n^{\text{fd}}(t)}] < \infty\} \\ &= \frac{1}{2 \int_0^t \sigma_n(s)^2 ds}. \end{aligned}$$

*Proof.* Note that the latter equality is obvious from (1), since  $F_n^{\text{fd}}(t)$  is log-normal with log-variance parameter  $\sigma^2 = \int_0^t \sigma_n(s)^2 ds$ . We now show the first equality. Recall that the measure change from the  $T_n$ -forward measure to the  $T_{n-1}$ -forward measure is effected by the likelihood process [1]

$$\frac{d\mathbb{Q}^n}{d\mathbb{Q}^{n-1}} \Big|_{\mathcal{F}_t} = \frac{1 + \tau_n F_n(0)}{1 + \tau_n F_n(t)}.$$

Therefore, putting  $\phi(x) = \exp(\log^2 x)$ , we obtain

$$\begin{aligned} \mathbf{E}^M[\phi(F_n(t))^v] &= \mathbf{E}^{M-1} \left[ \phi(F_n(t))^v \times \frac{1 + \tau_M F_M(0)}{1 + \tau_M F_M(t)} \right] \\ &= \dots = \\ &= \mathbf{E}^n \left[ \phi(F_n(t))^v \prod_{i=n+1}^M \frac{1 + \tau_i F_i(0)}{1 + \tau_i F_i(t)} \right] \\ &\leq \mathbf{E}^n[\phi(F_n(t))^v] \prod_{i=n+1}^M (1 + \tau_i F_i(0)), \end{aligned}$$

hence

$$\sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} \leq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}.$$

On the other hand, for  $1 < k \leq M$  and  $v \in \mathbb{R}$  we have

$$\begin{aligned} \mathbf{E}^{k-1}[\phi(F_n(t))^v] &= \mathbf{E}^k \left[ \phi(F_n(t))^v \times \frac{1 + \tau_k F_k(t)}{1 + \tau_k F_k(0)} \right] \\ &= \frac{1}{1 + \tau_k F_k(0)} \left( \mathbf{E}^k[\phi(F_n(t))^v] + \tau_k \mathbf{E}^k[F_k(t) \phi(F_n(t))^v] \right). \end{aligned}$$

Now let  $\varepsilon > 0$  be arbitrary, and define  $q$  by  $\frac{1}{q} + \frac{1}{1+\varepsilon} = 1$ . Then Hölder’s inequality yields

$$\mathbf{E}^k[F_k(t) \phi(F_n(t))^v] \leq \mathbf{E}^k[F_k(t)^q]^{1/q} \times \mathbf{E}^k[\phi(F_n(t))^{v(1+\varepsilon)}]^{1/(1+\varepsilon)}.$$

By the finite moment assumption, we obtain the implication

$$\mathbf{E}^k[\phi(F_n(t))^{v(1+\varepsilon)}] < \infty \implies \mathbf{E}^{k-1}[\phi(F_n(t))^v] < \infty, \quad v \in \mathbb{R}.$$

(Note that the left-hand side implies  $\mathbf{E}^k[\phi(F_n(t))^v] < \infty$ .)

Inductively, this leads to the implication

$$\mathbf{E}^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty \implies \mathbf{E}^n[\phi(F_n(t))^v] < \infty, \quad v \in \mathbb{R}.$$

Therefore, we find

$$\begin{aligned} \sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} &\geq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty\} \\ &= \frac{1}{(1+\varepsilon)^{M-n}} \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}. \end{aligned}$$

Since  $\varepsilon$  was arbitrary,

$$\sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} \geq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}$$

follows, which finishes the proof.  $\square$

The following corollary explicitly shows the implication of Theorem 1 on the distribution tails of  $F_n(t)$ : At least one of the two tails is log-normal, and neither tail is heavier than log-normal. We are talking here about the dominating factor of the tail asymptotics; a lower order factor (a power of  $x$ , e.g.) could make the tails (slightly) heavier or lighter than log-normal. These refined asymptotics are left for future research. Note also that our result does so far not exclude the possibility that *one* of the two tails is (significantly) lighter than log-normal.

**Corollary 2.** *Under the assumptions of Theorem 1, the supremum of all real  $v$  that satisfy the condition*

$$\mathbf{P}[F_n(t) > x] + \mathbf{P}[F_n(t) < \frac{1}{x}] = O(e^{-v \log^2 x}), \quad x \rightarrow \infty,$$

is given by

$$v^* = \frac{1}{2 \int_0^t \sigma_n(s)^2 ds}.$$

The same holds for  $F_n^{\text{fd}}(t)$  instead of  $F_n(t)$ .

*Proof.* From Theorem 1 we obtain

$$v^* = \sup\{v : \mathbf{P}[e^{\log^2 F_n(t)} > x] = O(x^{-v})\}.$$

Now for  $x > 1$  we have

$$\mathbf{P}[e^{\log^2 F_n(t)} > x] = \mathbf{P}[F_n(t) > e^{\sqrt{\log x}}] + \mathbf{P}[F_n(t) < e^{-\sqrt{\log x}}],$$

and the claim follows after replacing  $x$  by  $\exp(\log^2 x)$ . The same reasoning works for the frozen drift rate  $F_n^{\text{fd}}(t)$ .  $\square$

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