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MOMENT EXPLOSION IN THE LIBOR MARKET MODEL

STEFAN GERHOLD

ABSTRACT. In the LIBOR market model, forward interest rates are log-normal under their respective forward measures. This note shows that their distributions under the other forward measures of the tenor structure have approximately log-normal tails.

1. INTRODUCTION

The LIBOR market model [3] is one of the most popular models for pricing and hedging interest rate derivatives. Its state variables are forward interest rates $F_n(t) := F(t; T_{n-1}, T_n)$, spanning time periods $[T_{n-1}, T_n]$, where

$$0 < T_0 < T_1 < \dots < T_M$$

is a fixed tenor structure. Under the T_M -forward measure \mathbb{Q}^M , which has as numeraire the zero coupon bond maturing at T_M , the dynamics of the forward rates are

$$\begin{aligned} dF_n(t) &= -\sigma_n(t)F_n(t) \sum_{j=n+1}^M \frac{\rho_{nj}\tau_j\sigma_j(t)F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_n(t)F_n(t)dW_n(t), \\ 1 \leq n &< M, \\ dF_M(t) &= \sigma_M(t)F_M(t)dW_M(t). \end{aligned}$$

Here, σ_n are some positive deterministic volatility functions, and W is a vector of standard Brownian motions with instantaneous correlations $dW_i(t)dW_j(t) = \rho_{ij}dt$. Moreover, $\tau_n = \tau(T_{n-1}, T_n)$ denotes the year fraction between the tenor dates T_{n-1} and T_n .

Note that F_M is a geometric Brownian motion under \mathbb{Q}^M . More generally, each rate F_n is a geometric Brownian motion under its own forward measure, while it has a stochastic drift under the other forward measures. Therefore, pricing derivatives with the LIBOR market model often requires a Monte Carlo simulation. To avoid this, a popular approximation of the above dynamics is obtained by “freezing the drift” [2, 4]:

$$\begin{aligned} dF_n^{\text{fd}}(t) &= -\sigma_n(t)F_n^{\text{fd}}(t) \sum_{j=n+1}^M \frac{\rho_{nj}\tau_j\sigma_j(t)F_j(0)}{1 + \tau_j F_j(0)} dt + \sigma_n(t)F_n^{\text{fd}}(t)dW_n(t), \\ 1 \leq n &< M, \\ dF_M^{\text{fd}}(t) &= \sigma_M(t)F_M^{\text{fd}}(t)dW_M(t). \end{aligned}$$

Since the log-drifts are now deterministic, the new rates F_n^{fd} are just geometric Brownian motions, which allow for explicit pricing formulas for many interest-linked products. As a piece of evidence for the quality of this approximation, we

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show in the present note that, for fixed $t > 0$, the distribution of $F_n^{\text{fd}}(t)$ has roughly the same tail heaviness as the distribution of $F_n(t)$.

2. MAIN RESULT

If X is any log-normal random variable, so that $\log X \sim \mathcal{N}(\mu, \sigma^2)$ for some real μ and positive σ , then

$$(1) \quad \sup\{v : \mathbf{E}[e^{v \log^2 X}] < \infty\} = \frac{1}{2\sigma^2}.$$

This follows from

$$\mathbf{E}[e^{v \log^2 X}] = \frac{1}{\sqrt{1 - 2\sigma^2 v}} \exp\left(\frac{\mu^2 v}{1 - 2\sigma^2 v}\right), \quad v < \frac{1}{2\sigma^2}.$$

Our main result shows that $F_n(t)$ has approximately log-normal tails, in the sense that the left-hand side of (1) is finite and positive if X is replaced by $F_n(t)$. Furthermore, this “critical moment” is the same for $F_n(t)$ and the frozen drift approximation $F_n^{\text{fd}}(t)$. We write \mathbf{E}^n for the expectation w.r.t. the T_n -forward measure \mathbb{Q}^n .

Theorem 1. *In the log-normal LIBOR market model, we have for all $t > 0$ and all $1 \leq n \leq M$*

$$\begin{aligned} \sup\{v : \mathbf{E}^M[e^{v \log^2 F_n(t)}] < \infty\} &= \sup\{v : \mathbf{E}^M[e^{v \log^2 F_n^{\text{fd}}(t)}] < \infty\} \\ &= \frac{1}{2 \int_0^t \sigma_n(s)^2 ds}. \end{aligned}$$

Proof. Note that the latter equality is obvious from (1), since $F_n^{\text{fd}}(t)$ is log-normal with log-variance parameter $\sigma^2 = \int_0^t \sigma_n(s)^2 ds$. We now show the first equality. Recall that the measure change from the T_n -forward measure to the T_{n-1} -forward measure is effected by the likelihood process [1]

$$\frac{d\mathbb{Q}^n}{d\mathbb{Q}^{n-1}} \Big|_{\mathcal{F}_t} = \frac{1 + \tau_n F_n(0)}{1 + \tau_n F_n(t)}.$$

Therefore, putting $\phi(x) = \exp(\log^2 x)$, we obtain

$$\begin{aligned} \mathbf{E}^M[\phi(F_n(t))^v] &= \mathbf{E}^{M-1} \left[\phi(F_n(t))^v \times \frac{1 + \tau_M F_M(0)}{1 + \tau_M F_M(t)} \right] \\ &= \dots = \\ &= \mathbf{E}^n \left[\phi(F_n(t))^v \prod_{i=n+1}^M \frac{1 + \tau_i F_i(0)}{1 + \tau_i F_i(t)} \right] \\ &\leq \mathbf{E}^n[\phi(F_n(t))^v] \prod_{i=n+1}^M (1 + \tau_i F_i(0)), \end{aligned}$$

hence

$$\sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} \leq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}.$$

On the other hand, for $1 < k \leq M$ and $v \in \mathbb{R}$ we have

$$\begin{aligned} \mathbf{E}^{k-1}[\phi(F_n(t))^v] &= \mathbf{E}^k \left[\phi(F_n(t))^v \times \frac{1 + \tau_k F_k(t)}{1 + \tau_k F_k(0)} \right] \\ &= \frac{1}{1 + \tau_k F_k(0)} (\mathbf{E}^k[\phi(F_n(t))^v] + \tau_k \mathbf{E}^k[F_k(t) \phi(F_n(t))^v]). \end{aligned}$$

Now let $\varepsilon > 0$ be arbitrary, and define q by $\frac{1}{q} + \frac{1}{1+\varepsilon} = 1$. Then Hölder’s inequality yields

$$\mathbf{E}^k[F_k(t) \phi(F_n(t))^v] \leq \mathbf{E}^k[F_k(t)^q]^{1/q} \times \mathbf{E}^k[\phi(F_n(t))^{v(1+\varepsilon)}]^{1/(1+\varepsilon)}.$$

By the finite moment assumption, we obtain the implication

$$\mathbf{E}^k[\phi(F_n(t))^{v(1+\varepsilon)}] < \infty \implies \mathbf{E}^{k-1}[\phi(F_n(t))^v] < \infty, \quad v \in \mathbb{R}.$$

(Note that the left-hand side implies $\mathbf{E}^k[\phi(F_n(t))^v] < \infty$.)

Inductively, this leads to the implication

$$\mathbf{E}^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty \implies \mathbf{E}^n[\phi(F_n(t))^v] < \infty, \quad v \in \mathbb{R}.$$

Therefore, we find

$$\begin{aligned} \sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} &\geq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty\} \\ &= \frac{1}{(1+\varepsilon)^{M-n}} \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}. \end{aligned}$$

Since ε was arbitrary,

$$\sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} \geq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}$$

follows, which finishes the proof. \square

The following corollary explicitly shows the implication of Theorem 1 on the distribution tails of $F_n(t)$: At least one of the two tails is log-normal, and neither tail is heavier than log-normal. We are talking here about the dominating factor of the tail asymptotics; a lower order factor (a power of x , e.g.) could make the tails (slightly) heavier or lighter than log-normal. These refined asymptotics are left for future research. Note also that our result does so far not exclude the possibility that *one* of the two tails is (significantly) lighter than log-normal.

Corollary 2. *Under the assumptions of Theorem 1, the supremum of all real v that satisfy the condition*

$$\mathbf{P}[F_n(t) > x] + \mathbf{P}[F_n(t) < \frac{1}{x}] = O(e^{-v \log^2 x}), \quad x \rightarrow \infty,$$

is given by

$$v^* = \frac{1}{2 \int_0^t \sigma_n(s)^2 ds}.$$

The same holds for $F_n^{\text{fd}}(t)$ instead of $F_n(t)$.

Proof. From Theorem 1 we obtain

$$v^* = \sup\{v : \mathbf{P}[e^{\log^2 F_n(t)} > x] = O(x^{-v})\}.$$

Now for $x > 1$ we have

$$\mathbf{P}[e^{\log^2 F_n(t)} > x] = \mathbf{P}[F_n(t) > e^{\sqrt{\log x}}] + \mathbf{P}[F_n(t) < e^{-\sqrt{\log x}}],$$

and the claim follows after replacing x by $\exp(\log^2 x)$. The same reasoning works for the frozen drift rate $F_n^{\text{fd}}(t)$. \square

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