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p -CHAOTICITY AND REGULAR ACTION OF ABELIAN C^1 -DIFFEOMORPHISMS GROUPS OF \mathbb{C}^n FIXING A POINT

YAHYA N'DAO AND ADLENE AYADI

ABSTRACT. In this paper, we introduce the notion of regular action of any subgroup G of $Diff^1(\mathbb{C}^n)$ on \mathbb{C}^n (i.e. the closure of every orbit of G in some open set is a topological sub-manifold of \mathbb{C}^n). We prove that the action of G , can not be p -chaotic for every $0 \leq p \leq n-1$. (i.e. If G has a dense orbit then the set of all regular orbit with order p can not be dense in \mathbb{C}^n). Moreover, we prove that the action of any abelian lie subgroup of $Diff^1(\mathbb{C}^n)$, is regular.

1. Introduction

Denote by $Diff^1(\mathbb{C}^n)$ the group of all C^1 -diffeomorphisms of \mathbb{C}^n . Let G be an abelian subgroup of $Diff^1(\mathbb{C}^n)$ such that $0 \in Fix(G)$ and $\dim(vect(L_G)) = n$, where $vect(L_G)$ is the vector space generated by $L_G = \{Df(0), f \in G\}$ and $Fix(G) = \{x \in \mathbb{C}^n : f(x) = x, \forall f \in G\}$ be the global fixed point set of G . We can assume that $0 \in Fix(G)$, leaving to replace G by $T_a \circ G \circ T_{-a}$ for any translation T_a of any vector $a \in Fix(G)$. There is a natural action $G \times \mathbb{C}^n \rightarrow \mathbb{C}^n$. $(f, x) \mapsto f(x)$. For a point $x \in \mathbb{C}^n$, denote by $G(x) = \{f(x), f \in G\} \subset \mathbb{C}^n$ the orbit of G through x . Denote by \overline{E} (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E . A topological space X is called a topological manifold with dimension $r \geq 0$ over \mathbb{C} if every point has a neighborhood homeomorphic to \mathbb{C}^r . This means that the image of any topological manifold by a homeomorphism is a topological manifold with the same dimension. An orbit γ is called *regular* with order $ord(\gamma) = m$ if for every $y \in \gamma$ there exists an neighborhood O of y such that $\overline{\gamma} \cap O$ is a topological sub-manifold of \mathbb{C}^n with dimension m over \mathbb{C} . In particular, γ is locally dense in \mathbb{C}^n if and only if $m = n$, and it is discrete if and only if $m = 0$. Notice that, the closure of a regular orbit is not necessary a manifold. We say that the action of G is *regular* on \mathbb{C}^n if every orbit of G is regular. The action of G is called *chaotic* if G has a dense orbit and the union of all periodic orbits is dense in \mathbb{C}^n (cf. [11], [13], [5]). We give a generalization of the chaos as follow: The action of G is called *p -chaotic*, $0 \leq p \leq n-1$, if G has a dense orbit and the union of all orbits with

order p is dense in \mathbb{C}^n . See that every chaotic action is 0-chaotic. Here, the question to investigate is the following:

The natural action of any subgroup of $Diff(\mathbb{C}^n)$ can be p -chaotic, $0 \leq p \leq n-1$?

The action of any abelian lie subgroup of $Diff(\mathbb{C}^n)$ can be regular?

The notion of regular orbit is a generalization of non exceptional orbit defined for the action of any group of diffeomorphisms on \mathbb{C}^n . A nonempty subset $E \subset \mathbb{C}^n$ is a minimal set if for every $y \in E$ the orbit of y is dense in E . An orbit with its closure is a *Cantor* set is called an exceptional orbit. Their dynamics were recently initiated for some classes in different point of view, (see for instance, [3],[4],[5],[6],[7],[9]).

The action of G on \mathbb{C}^n is said *proper* if and only if the pre-image of any compact set by the action map, is compact (i.e. for every two compact subsets K_1 and K_2 of \mathbb{C}^n , the subset $\{f \in G, f(K_1) \cap K_2 \neq \emptyset\}$ of G is compact). It is well known, that if the action of a lie group on \mathbb{C}^n is proper then all the orbits are embedded submanifolds in \mathbb{C}^n (see for instance [12] and [15]). Remark that, a proper action of any lie group is regular, this means that the regular action is a generalization of the proper action.

In [1], A.C. Naolekar and P. Sankaran construct chaotic actions of certain finitely generated abelian groups on even-dimensional spheres, and of finite index subgroups of $SL(n, \mathbb{Z})$ on tori. They also study chaotic group actions via compactly supported homeomorphisms on open manifolds.

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In [10], P.W.Michor and C.Vizman proved that some groups of diffeomorphisms of a manifold M act n -transitively for each finite n (i.e. for any two ordered sets of n different points (x_1, \dots, x_n) and (y_1, \dots, y_n) in M there is a smooth diffeomorphism f in the group such that $f(x_i) = y_i$ for each i).

In [3], the authors studied the minimality of any abelian diffeomorphisms groups acting on \mathbb{C}^n fixing a point and $\dim(L_G) = n$, whose generalize the structure's theorem given in [2] for abelian linear group. This paper can be viewed as a continuation of these works.

Our principal results can be stated as follows:

Theorem 1.1. *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$ such that $0 \in \text{Fix}(G)$ and $\dim(\text{vect}(L_G)) = n$. If G has a dense orbit then the set of all dense orbit is a G -invariant open set, dense in \mathbb{C}^n .*

Corollary 1.2. *The natural action of any abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$ such that $0 \in \text{Fix}(G)$ and $\dim(\text{vect}(L_G)) = n$, can not be p -chaotic for every $0 \leq p \leq n-1$. In particular, it can not be chaotic.*

Theorem 1.3. *The natural action of any abelian lie subgroup of $\text{Diff}^1(\mathbb{C}^n)$ on \mathbb{C}^n is regular.*

As a directly consequence of Theorems 1.3 and 1.1, we prove the regularity action of any abelian linear group on \mathbb{C}^n .

Corollary 1.4. *The natural action of any abelian subgroup of $GL(n, \mathbb{C})$ on \mathbb{C}^n is regular and not chaotic.*

2. Proof Theorem 1.3 and corollary 1.2

We will cite the definition of the exponential map given in [4].

Denote by:

- \mathfrak{g} be the lie algebra associated to G .
- The exponential map $\exp : \mathfrak{g} \longrightarrow G$ is defined in above.

Lemma 2.1. *Let $x \in \mathbb{C}^n$. Then $G(x)$ is regular with order $r \geq 0$ if and only if there exist an open set O_x containing $G(x)$ such that $\overline{G(x)} \cap O_x$ is a manifold with dimension $r \geq 0$.*

Proof. The directly sens is obvious by definition. Conversely, let $x \in G(u)$ and O be an open neighborhood of x such that $\overline{G(u)} \cap O_x$ is a manifold with dimension $r \geq 0$ over \mathbb{C} . Let $y \in G(u)$, then

$y = f(x)$ for some $f \in G$. So $O'_x = f(O_x)$ is an neighborhood of y and satisfying $\overline{G(u)} \cap O'_x = f(\overline{G(u)} \cap O)$ is a manifold with dimension $r \geq 0$ over \mathbb{C} . It follows that $G(u)$ is regular with order r . \square

2.1. Whitney Topology on $C^0(\mathbb{C}^n, \mathbb{C}^n)$. We will use the definition of Whitney topology given in [14]. For each open subset $U \subset \mathbb{C}^n \times \mathbb{C}^n$ let $\tilde{U} \subset C^0(\mathbb{C}^n, \mathbb{C}^n)$ be the set of continuous functions g , whose graphs $\{(x, g(x)) \in \mathbb{C}^n \times \mathbb{C}^n, x \in \mathbb{C}^n\}$ is contained in U . We want to construct a neighborhood basis of each function $f \in C^0(\mathbb{C}^n, \mathbb{C}^n)$. Let $K_j = \{x \in \mathbb{C}^n, \|x\| \leq j\}$ be a countable family of compact sets (closed balls with center 0) covering \mathbb{C}^n such that K_j is contained in the interior of K_{j+1} . Consider then the compact

subsets $L_j = K_j \setminus \overset{\circ}{K_{j-1}}$, which are compact sets, too. Let $\epsilon = (\epsilon_j)_j$ be a sequence of positive numbers and then define $V_{(f; \epsilon)} = \{f \in C^0(\mathbb{C}^n, \mathbb{C}^n) : \|f(x) - g(x)\| < \epsilon_j, \text{ for any } x \in L_j, \forall j\}$. We claim this is a neighborhood system of the function f in $C^0(\mathbb{C}^n, \mathbb{C}^n)$. Since L_i is compact, the set $U = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n : \|f(x) - y\| < \epsilon_j, \text{ if } x \in L_j\}$ is open. Thus, $V_{(f; \epsilon)} = \tilde{U}$ is an open neighborhood of f . On the other hand, if O is an open subset of $\mathbb{C}^n \times \mathbb{C}^n$ which contains the graph of f , then since L_j is compact, it follows that there exists $\epsilon_j > 0$ such that if $x \in L_j$ and $\|y - f(x)\| < \epsilon_j$, then $(x, y) \in O$. Thus, taking $\tilde{\epsilon} = (\epsilon_j)_j$ we have $V_{(f; \tilde{\epsilon})} \subset \tilde{O}$, so we have obtained the family $V_{(f; \epsilon)}$ is a neighborhood system of f . Moreover, for each given $\epsilon = (\epsilon_j)_j$, we can find a C^∞ -function $\epsilon : \mathbb{C}^n \longrightarrow \mathbb{R}_+$, such that $\epsilon(x) < \epsilon_j$ for any $x \in L_j$. It follows that the family $V_{(f; \epsilon)} = \{g \in C^0(\mathbb{C}^n, \mathbb{C}^n) : \|f(x) - g(x)\| < \epsilon(x), \text{ for every } x \in \mathbb{C}^n\}$ is also a neighborhood system.

2.2. Linear map. For a subset $E \subset \mathbb{C}^n$, denote by $\text{vect}(E)$ the vector subspace of \mathbb{C}^n generated by all elements of E . E is called G -invariant if $f(E) \subset E$ for any $f \in G$; that is E is a union of orbits. Set $\mathcal{A}(G)$ be the algebra generated by G . For a fixed vector $x \in \mathbb{C}^n \setminus \{0\}$, denote by:

- $\Phi_x : \mathcal{A}(G) \longrightarrow \Phi_x(\mathcal{A}(G)) \subset \mathbb{C}^n$ the linear map given by $\Phi_x(f) = f(x)$.
- $E(x) = \Phi_x(\mathcal{A}(G))$.

Lemma 2.2. *The linear map $\Phi_x : \mathcal{A}(G) \longrightarrow E(x)$ is continuous.*

Proof. Firstly, we take the restriction of the Whitney topology to $\mathcal{A}(G)$. Secondly, let $f \in \mathcal{A}(G)$ and $\varepsilon > 0$. Then for $\epsilon = (\varepsilon_j)_j$ with $\varepsilon_j = \varepsilon$ and for $V_{(f;\epsilon)}$ be a neighborhood system of f , we obtain: for every $g \in V_{(f;\epsilon)} \cap \mathcal{A}(G)$ and for every $y \in L_j$, $\|f(y) - g(y)\| < \varepsilon$, $\forall j$. In particular for $j = j_0$ in which $x \in L_{j_0}$, we have $\|f(x) - g(x)\| < \varepsilon$, so $\|\Phi_x(f) - \Phi_x(g)\| < \varepsilon$. It follows that Φ_x is continuous. \square

Denote by:

- $r(x) = \dim(E(x))$.
- $U_j = \{y \in \mathbb{C}^n, r(y) \geq j\}$.

Proposition 2.3. *Let G be a subgroup of $\text{Diff}^1(\mathbb{C}^n)$. Suppose that G has a dense orbit. Then U_n is a G -invariant open subset of \mathbb{C}^n .*

Proof. Let $x \in \mathbb{C}^n$ such that $\overline{G(x)} = \mathbb{C}^n$, then $x \in U_n$ and so $U_n \neq \emptyset$. Let $y \in U_n$, then $E(y) = \mathbb{C}^n$ and so, there exist $f_1, \dots, f_n \in F_y$ such that the n vectors $f_1(y), \dots, f_n(y)$ are linearly independent in \mathbb{C}^n . For all $z \in \mathbb{C}^n$, we consider the Gram's determinant

$$\Delta(z) = \det(\langle f_i(z) | f_j(z) \rangle)_{1 \leq i, j \leq n}$$

of the vectors $f_1(z), \dots, f_n(z)$ where $\langle \cdot | \cdot \rangle$ denotes the scalar product in \mathbb{C}^n . It is well known that these vectors are independent if and only if $\Delta(z) \neq 0$, in particular $\Delta(y) \neq 0$. Let

$$V_y = \{z \in \mathbb{C}^n, \Delta(z) \neq 0\}$$

The set V_y is open in \mathbb{C}^n , because the map $z \mapsto \Delta(z)$ is continuous. Now $\Delta(y) \neq 0$, and so $y \in V_y \subset U_n$. The proof is completed. \square

The construction of the open U given in [3], is the same of U_n if G has a dense orbit.

Lemma 2.4. ([3], Corollary 1.2) *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$, such that $0 \in \text{Fix}(G)$ and $\dim(\text{vect}(L_G)) = n$. If G has a dense orbit then every orbit in U_n is dense in \mathbb{C}^n .*

Proof of Theorem 1.1. Suppose that the group G has a dense orbit denoted by $G(x)$, $x \in \mathbb{C}^n$. Let \mathcal{L} be the set of all dense orbits, so $\mathcal{L} \neq \emptyset$ since $x \in \mathcal{L}$. let $y \in \mathcal{L}$ then $y \in U_n$ and $U_n \subset \overline{G(y)}$. By Lemma 2.4, for each $z \in U_n$, one has $\overline{G(z)} = \mathbb{C}^n$, so $y \in U_n \subset \mathcal{L}$. By Proposition 2.3, U_n is a non empty open subset of \mathbb{C}^n . This completes the proof. \square

Remark 2.5. By the proof of Theorem 1.1, $U_n = \mathcal{L}$.

Proof of Corollary 1.2. Suppose that the action of the group G is p -chaotic, then G has a dense orbit denoted by $G(x)$, $x \in \mathbb{C}^n$. By Theorem 1.1, the set \mathcal{L} of all dense orbit is a dense open set in \mathbb{C}^n . This means that if \mathcal{P} is the union of all regular orbits with order p , then $\mathcal{L} \cap \mathcal{P} = \emptyset$, so \mathcal{P} can not be dense in \mathbb{C}^n . The proof is completed. \square

3. Regular action of abelian lie subgroups of $\text{Diff}^1(\mathbb{C}^n)$

We will cite the definition of the exponential map given in [4].

3.1. Exponential map. In this section, we illustrate the theory developed of the group $\text{Diff}(\mathbb{C}^n)$ of diffeomorphisms of \mathbb{C}^n . For simplicity, throughout this section we only consider the case of $\mathbb{C} = \mathbb{R}$; however, all results also hold for complexes case. The group $\text{Diff}(\mathbb{R}^n)$ is not a Lie group (it is infinite-dimensional), but in many way it is similar to Lie groups. For example, it easy to define what a smooth map from some Lie group G to $\text{Diff}(\mathbb{R}^n)$ is: it is the same as an action of G on \mathbb{R}^n by diffeomorphisms. Ignoring the technical problem with infinite-dimensionality for now, let us try to see what is the natural analog of the Lie algebra \mathfrak{g} for the group G . It should be the tangent space at the identity; thus, its elements are derivatives of one-parameter families of diffeomorphisms.

Let $\varphi^t : G \rightarrow G$ be one-parameter family of diffeomorphisms. Then, for every point $a \in G$, $\varphi^t(a)$ is a curve in G and thus $\frac{\partial}{\partial t} \varphi^t(a)_{t=0} = \xi(a) \in T_a G$ is a tangent vector to G at a . In other words, $\frac{\partial}{\partial t} \varphi^t$ is a vector field on G .

The exponential map $\exp : \mathfrak{g} \rightarrow G$ is defined by $\exp(x) = \gamma_x(1)$ where $\gamma_x(t)$ is the one-parameter subgroup with tangent vector at 1 equal to x .

If $\xi \in \mathfrak{g}$ is a vectorfield, then $\exp(t\xi)$ should be one-parameter family of diffeomorphisms whose derivative is vector field ξ . So this is the solution of differential equation

$$\frac{\partial}{\partial t} \varphi^t(a)_{t=0} = \xi(a).$$

In other words, φ^t is the time t flow of the vector field. Thus, it is natural to define the Lie algebra of G to be the space \mathfrak{g} of all smooth vector ξ fields on \mathbb{R}^n such that $\exp(t\xi) \in G$ for every $t \in \mathbb{R}$.

Proposition 3.1. ([4], Theorem 3.29) Let G be a Lie group acting on \mathbb{C}^n with lie algebra \mathfrak{g} and let $x \in \mathbb{C}^n$.

(i) The stabilizer $G_x = \{f \in G : f(x) = x\}$ is a closed Lie subgroup in G , with Lie algebra $\mathfrak{h}_x = \{f \in \mathfrak{g} : f(x) = 0\}$.

(ii) The map $G/G_x \rightarrow \mathbb{C}^n$ given by $f.G_x \mapsto f(x)$ is an immersion. Thus, the orbit $G(x)$ is an immersed submanifold in \mathbb{C}^n . In particular $\dim(G(x)) = \dim(\mathfrak{g}) - \dim(\mathfrak{h}_x)$.

Denote by $p = \dim(\mathfrak{g})$. Since G is abelian so is \mathfrak{g} . Set $f_1, \dots, f_p \in \mathfrak{g}$ be the generators of \mathfrak{g} . We let:

- $\exp : \mathfrak{g} \rightarrow G$ the lie exponential map associated to G .

- G_0 be the connected component of G containing the identity map id . So G_0 is generated by $\exp(\mathfrak{g})$ and it is an abelian lie subgroup of G . Since \mathfrak{g} is abelian, $G_0 = \exp(\mathfrak{g})$.

For a fixed point $x \in \mathbb{C}^n$, denote by:

- $G_x = \{f \in G_0, f(x) = x\}$ the stabilizer of G_0 on the point x . It is a lie subgroup of G_0 .

Denote by:

- H be the algebra associated to G_x and F_x is the supplement of H_x in \mathfrak{g} (i.e. $F_x \oplus H_x = \mathfrak{g}$). By Proposition 3.1, we have $H_x = \{f \in \mathfrak{g}, f(x) = 0\}$ and

$$G_0 = \exp(F_x) \circ \exp(H_x).$$

In particular $G_0(x) = \Phi_x(\exp(F_x))$.

- $V = \{\exp(t_1 f_1 + \dots + t_p f_p), |t_k| < 1\}$.

Proposition 3.2. Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$, and $x \in \mathbb{C}^n$. Then:

(i) $G_0(x)$ is the connected component of $G(x)$ containing x .

(ii) The restriction $\Phi_x^{(1)} : \exp(F_x) \cap V \rightarrow \Phi_x(\exp(F_x) \cap V) \subset G_0(x)$ of Φ_x to $\exp(F_x) \cap V$ is an homeomorphism.

Proof. (i) By Lemma 2.2, the map $\Phi_x : \mathcal{A}(G) \rightarrow E(x) \subset \mathbb{C}^n$ is a continuous surjective linear map. The proof follows then from the fact that $G_0(x) = \Phi_x(G_0)$ and G_0 is connected.

(ii) By Lemma 2.2, the map $\Phi_x^{(1)}$ is continuous, surjective.

It is injective: Indded, if $f, g \in \exp(F_x) \cap V$ such that $\Phi_x^{(1)}(f) = \Phi_x^{(1)}(g)$, then $f(x) = g(x)$, so $g^{-1} \circ f(x) = x$. Hence $g^{-1} \circ f \in G_x \cap \exp(F_x) = \{id\}$. It follows that $f = g$.

$(\Phi_x^{(1)})^{-1} : \Phi_x(\exp(F_x) \cap V) \rightarrow \exp(F_x) \cap V$ is continuous; inded, let $y = \exp(f)(x) \in \Phi_x(\exp(F_x) \cap V)$, $f \in F_x$ and $(y_m)_m$ be a sequence

in $\Phi_x(\exp(F_x) \cap V)$ tending to y . Let (f_1, \dots, f_q) be a basis of F_x and set $y_m = \exp(t_{1,m} f_1 + \dots + t_{q,m} f_q)(x)$ and $y = \exp(t_1 f_1 + \dots + t_q f_q)(x)$, with $|t_k| < 1$ and $|t_{k,m}| < 1$. We can assume (leaving to take a subsequence) that $\lim_{m \rightarrow +\infty} t_{k,m} = s_k$, with $|s_k| \leq 1$ for every $k = 1, \dots, q$. Write $g = \exp(s_1 f_1 + \dots + s_q f_q)$ and $g_m = \exp(t_{1,m} f_1 + \dots + t_{q,m} f_q)$. By continuity of the exponential map we have $(g_m)_m$ tends to g when $m \rightarrow +\infty$. By continuity of Φ_x (Lemma 2.2) we obtain $y_m = \Phi_x(g_m)$ tends to $y = \Phi_x(g)$, so $s_k = t_k$ for every $k = 1, \dots, p$. As $g = (\Phi_x^{(1)})^{-1}(y)$ and $g_m = (\Phi_x^{(1)})^{-1}(y_m)$, it follows that $(\Phi_x^{(1)})^{-1}(y_m)$ tends to $(\Phi_x^{(1)})^{-1}(y)$. This completes the proof. \square

3.2. Wedge, Lie wedge and almost abelian notions.

We will use the notion of wedge and Lie wedge given by K.H. Hofmann in [7] and [8]:

- A *wedge* or a closed convex cone in a finite dimensional vector \mathfrak{g} is a topologically subset ω with $\omega + \omega = \omega$ and $\lambda \cdot \omega \subset \omega$ for every $\lambda \geq 0$. In particular, any vector subspace of \mathfrak{g} is a wedge in \mathfrak{g} .

- $h(\omega) = (-\omega) \cap \omega$ is called the edge of the wedge.

- A *Lie wedge* ω in a Lie algebra \mathfrak{g} is a wedge such that

$$\exp(\text{ad}(x))\omega = \omega, \text{ for all } x \in h(\omega).$$

In particular, any subalgebra of \mathfrak{g} is a Lie wedge in \mathfrak{g} .

- A Lie algebra η is called *almost abelian* if there is a linear form $\alpha : \eta \rightarrow \mathbb{R}$ such that the bracket is given by

$$[X, Y] = \alpha(X)Y - \alpha(Y)X.$$

In particular, any abelian Lie algebra is almost abelian for $\alpha = 0$. If $\alpha \neq 0$ the η is called *truly almost abelian*.

Lemma 3.3. ([8], Theorem 4.3) Let \mathfrak{g} be a Lie algebra, then the following are equivalent:

(i) \mathfrak{g} is almost abelian.

(ii) Every wedge is a Lie wedge.

(iii) For every Lie wedge ω , we have $\overline{\langle \exp(\omega) \rangle} = \exp(\omega)$, where $\langle \exp(\omega) \rangle$ is the group generated by $\exp(\omega)$.

As a consequence of above Lemma, we obtain:

Corollary 3.4. We have $\exp(F_x)$ is a lie subgroup of G_0 .

Proof. By definition, we have F_x is a wedge and by Lemma 3.3, (ii), it is a Lie wedge because \mathfrak{g} is almost abelian (since it is abelian). Then by Lemma 3.3, (iii), we have $\overline{\langle \exp(F_x) \rangle} = \exp(F_x)$, so

$\exp(F_x)$ is closed subgroup of G_0 . It follows that F_x is a Lie group. \square

Corollary 3.5. *(Under notations of Proposition 3.2) The set $B(x) = \Phi_x(\exp(F_x) \cap V)$ is a topological submanifold of \mathbb{C}^n containing x . Moreover, there exists an open subset W of \mathbb{C}^n such that $W \cap G(x) = B(x)$.*

Proof. By Corollary 3.4, $\exp(F_x)$ is a lie subgroup of G_0 , so it is a topological manifold. By Proposition 3.2, $B(x)$ is homoeomorphic to $\exp(F_x) \cap V$ which is an open subset of $\exp(F_x)$. Then $B(x)$ is a topological manifold with dimension equal to $\dim(\exp(F_x))$. On the other hand, by (i), $G_0(x) = \Phi_x(\exp(F_x))$ is a connected component of $G(x)$ containing x , then there exists an open subset O of \mathbb{C}^n such that $O \cap G(x) = G_0(x)$. Since the exponential map \exp is a locally diffeomorphism on a neighborhood of 0 then $\dim(\exp(F_x)) = \dim(F_x)$, so $\dim(B(x)) = \dim(\exp(F_x)) = \dim(F_x)$. By Proposition 3.1, $G_0(x)$ is an immersed submanifold of \mathbb{C}^n with dimension $\dim(F_x) = \dim(\mathfrak{g}) - \dim(H_x)$ because \mathfrak{g} is also the lie algebra of G_0 . Therefore $\dim(B(x)) = \dim(G_0(x))$, so $B(x)$ is an open subset of $G_0(x)$. Then there exists an open subset W of \mathbb{C}^n containing x and contained in O such that $G_0(x) \cap W = B(x)$. It follows that $W \cap G(x) = G_0(x) \cap W = B(x)$. The proof is completed. \square

Lemma 3.6. *For every neighborhood W of a point $x \in \mathbb{C}^n$, we have $\overline{G(x)} \cap W = \overline{G(x) \cap W} \cap W$.*

Proof. It is clear that $\overline{G(x) \cap W} \cap W \subset \overline{G(x)} \cap W$. Now, let $y \in \overline{G(x)} \cap W$ then there exists a sequence $(y_m)_m$ in $G(x)$ tending to y . So $y_m \in W$ from some row m_0 . Thus $y \in \overline{G(x) \cap W} \cap W$. \square

Proof of Theorem 1.3. Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$. By Corollary 3.5, there exists an open subset W of \mathbb{C}^n such that $W \cap G(x) = B(x)$ is a submanifold of \mathbb{C}^n . So $B(x)$ is locally closed, we can assume that $\overline{B(x)} \cap W = B(x)$. Therefore, by Lemma 3.6 we have $\overline{G(x)} \cap W = \overline{G(x) \cap W} \cap W$, so $B(x) \subset \overline{G(x)} \cap W = \overline{G(x) \cap W} \cap W = \overline{B(x)} \cap W = B(x)$. Hence $\overline{G(x)} \cap W = B(x)$ is a topological manifold. By Lemma 2.1, it follows that $G(x)$ is regular. We conclude that the action of G is regular. \square

Let $M_n(\mathbb{C})$ be the set of all square matrix over \mathbb{C} with order n and $GL(n, \mathbb{C})$ be the group of all reversible matrix of $M_n(\mathbb{C})$. Let L be an abelian subgroup of $GL(n, \mathbb{C})$, denote by:

- $\tilde{L} = \overline{L} \cap GL(n, \mathbb{C})$, where \overline{L} is the closure of L in $M_n(\mathbb{C})$. It is clear that \tilde{L} is a lie subgroup of $GL(n, \mathbb{C})$.

- $\overline{L}(x) = \{Ax, A \in \overline{L}\}$.

We will use the following lemma to prove Corollary 1.4.

Lemma 3.7. *For every $x \in \mathbb{C}^n$. We have $\overline{\tilde{L}(x)} = \overline{\overline{L}(x)}$.*

Proof. We have $\overline{\tilde{L}(x)} \subset \overline{\overline{L}(x)}$. Let $y \in \overline{\tilde{L}(x)}$, so $y = \lim_{m \rightarrow +\infty} A_m(x)$ for some sequence $(A_m)_{m \in \mathbb{N}}$ in \tilde{G} . Therefore, for every $m \in \mathbb{N}$, there exists a sequence $(A_{m,k})_{k \in \mathbb{N}}$ in $G/E(x)$ tending to A_m . Then $\lim_{k \rightarrow +\infty} A_{m,k}x = A_mx$. Thus for every $\varepsilon > 0$, there exists $M > 0$ and for every $m \geq M$, there exists $k_m > 0$, such that for every $k \geq k_m$, we have $\|A_mx - y\| < \frac{\varepsilon}{2}$ and $\|A_{m,k}x - A_mx\| < \frac{\varepsilon}{2}$. Then, for every $m > M$,

$$\|A_{m,k_m}x - y\| \leq \|A_{m,k_m}x - A_mx\| + \|A_mx - y\| < \varepsilon.$$

Hence $\lim_{m \rightarrow +\infty} A_{m,k_m}x = y$, so $y \in \overline{G(x)}$. It follows that $\overline{\tilde{G}(u)} \subset \overline{G(u)}$. The proof is completed. \square

Proof of Corollary 1.4. By Theorem 1.3, the action of \tilde{L} is regular. Then for each $x \in \mathbb{C}^n$, there exists an open subset O of \mathbb{C}^n such that $\tilde{L}(x) \cap O$ is a topological sub-manifold of \mathbb{C}^n . It follows by Lemma 3.7, that $\overline{\tilde{L}(x)} \cap O$ is a topological submanifold of \mathbb{C}^n . The proof is completed. \square

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