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***Abstract* — Hybridized discontinuous Galerkin methods preserve the advantages of classical discontinuous Galerkin methods and in addition enable to circumvent the issue of the number of degrees of freedom. The principles of these numerical methods are summed up for 3d time-harmonic Maxwell's equations and basic examples are proposed to assess their efficiency.**

I. INTRODUCTION

Discontinuous Galerkin (DG) methods are currently widespread for the discretization of time-transient Maxwell's equations [1, 2]. This success can be explained by several advantages compared to other approaches such as the flexibility for hp-adaptivity or a natural parallelism [3]. Nonetheless for stationary problems as time-harmonic Maxwell's equations, the number of degrees of freedom is several times larger than when using a classical conforming finite element method and this aspect is cumbersome. Hybridized DG (HDG) methods have been proposed recently to circumvent this problem [4]: preserving the discontinuous approximations of both electric and magnetic fields, the only global linear system to solve determines the degrees of freedom of a new *hybrid variable* "living" on the interface of each element of the mesh. Thus, a HDG method leads to the solution of a reduced-size linear system without altering the nice properties of a DG method. We consider here such a HDG method for solving the 3d time-harmonic Maxwell's equations and illustrate the accuracy and the computing cost of the approach.

II. HDG DISCRETIZATION

Electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields are approximated by discontinuous vector fields $\mathbf{E}_h$ and $\mathbf{H}_h$ whose components are polynomials in each element, *i.e.* $\mathbf{E}_h$ and $\mathbf{H}_h$ belong to $(\mathbb{P}_p(K))^3$ in each element K of the mesh with $\mathbb{P}_p(K)$ the space of polynomials of degree p on K. The fields $\mathbf{E}_h$ and $\mathbf{H}_h$ are computed by weakly verifying time-harmonic Maxwell's equations in each element K of the mesh; it means that for all $\mathbf{V}$ in $(\mathbb{P}_p(K))^3$, $\mathbf{E}_h$ and $\mathbf{H}_h$ satisfy[1]

$$\begin{aligned}(\mathrm{i}\omega\varepsilon_r\mathbf{E}_h,\mathbf{V})-(\mathbf{H}_h,\mathbf{curl}\,\mathbf{V})+\langle\widehat{\mathbf{H}}_h^t,\mathbf{n}\times\mathbf{V}\rangle=0,\\(\mathrm{i}\omega\mu_r\mathbf{H}_h,\mathbf{V})+(\mathbf{E}_h,\mathbf{curl}\,\mathbf{V})-\langle\widehat{\mathbf{E}}_h^t,\mathbf{n}\times\mathbf{V}\rangle=0.\end{aligned}\tag{1}$$

Here i is the imaginary unit, ω the angular frequency, ε_r and μ_r the relative permittivity and permeability. Symbols $(\cdot\,,\,\cdot)$ and $\langle\cdot\,,\,\cdot\rangle$ denote the scalar product of complex-valued vectors integrated over K and ∂K, the boundary of K; $\mathbf{n}$ is the outward unit normal vector on ∂K with $\times$ the cross-product. Quantities $\widehat{\mathbf{H}}_h^t$ and $\widehat{\mathbf{E}}_h^t$ are only defined on ∂K and assumed to be approximations of the tangential component of the magnetic and electric fields on ∂K.

In the classical DG approach, both quantities $\widehat{\mathbf{H}}_h^t$ and $\widehat{\mathbf{E}}_h^t$ are defined as linear combinations of the electric and magnetic fields of the element K and of its neighboring elements. This strategy leads to the solution of a linear system which can be several times larger than the system obtained by a more classical conforming finite element method.

In the HDG approach, $\widehat{\mathbf{H}}_h^t$ is considered as a new *hybrid variable*, which is single-valued on each interface of the mesh. On the boundary of each K, the variable $\widehat{\mathbf{E}}_h^t$ is then defined from the hybrid variable and the local field by

$$\widehat{\mathbf{E}}_h^t=\mathbf{E}_h^t+\tau\mathbf{n}\times(\widehat{\mathbf{H}}_h^t-\mathbf{H}_h^t)\text{ on }\partial K,\tag{2}$$

where τ is a (often strictly) positive parameter. Contrary to $\widehat{\mathbf{H}}_h^t$, $\widehat{\mathbf{E}}_h^t$ has *a priori* two distinct values on each side of an interface between two elements. We thus need to weakly enforce that $\widehat{\mathbf{E}}_h^t$ is single-valued on each interface

$$\langle[\![\widehat{\mathbf{E}}_h^t]\!],\boldsymbol{\eta}\rangle_F=0,\ \forall F\in\mathcal{F}_h^i,\tag{3a}$$

and that the scheme is consistent with the boundary conditions (Silver-Müller conditions for instance)

$$\langle\widehat{\mathbf{E}}_h^t\times\mathbf{n}+\widehat{\mathbf{H}}_h^t-\mathbf{G}^{\mathrm{i}},\boldsymbol{\eta}\rangle_F=0,\ \forall F\in\mathcal{F}_h^b.\tag{3b}$$

Here $\mathcal{F}_h^i$ and $\mathcal{F}_h^b$ are the set of the interior and boundary faces of the mesh ($\mathcal{F}_h=\mathcal{F}_h^i\cup\mathcal{F}_h^b$), $[\![\mathbf{V}]\!]$ denotes the jump of $\mathbf{V}$ on any face, $\mathbf{G}^{\mathrm{i}}$ is a source term provided by an incident electromagnetic wave on the boundary and the meaning of $\langle\cdot\,,\,\cdot\rangle_F$ comes from (1) with F replacing ∂K. The test functions $\boldsymbol{\eta}$ belongs to the tangential trace space $\mathbf{M}_h^p$

$$\begin{aligned}\mathbf{M}_h^p=\{\boldsymbol{\eta}\in(L^2(\mathcal{F}_h))^3\mid\boldsymbol{\eta}|_F\in(\mathbb{P}_p(F))^3,\\(\boldsymbol{\eta}\cdot\mathbf{n})|_F=0,\ \forall F\in\mathcal{F}_h\},\end{aligned}\tag{4}$$

with $L^2(\mathcal{F}_h)$ the space of square-integrable functions on $\mathcal{F}_h$. The hybrid variable $\widehat{\mathbf{H}}_h^t$ belongs to this space $\mathbf{M}_h^p$.

Thus, from (1), we obtain disconnected local problems in each element and a reduced-size linear system can be assembled only for the hybrid variable in a classical finite element way starting from the interface conditions (3); see for instance [5] for more details. Once the linear system for the unknowns associated to the hybrid variable has been solved, the local electromagnetic field within each K can be recovered locally by solving (1).

III. PRELIMINARY NUMERICAL RESULTS

A. Accuracy of the method

We consider the propagation of a plane wave in vacuum. The computational domain is chosen to be the unit cube

[1]For sake of simplicity, we omit the volume source term but it can be straightforwardly added.

$\Omega = (-0.5; 0.5)^3$, and a Silver-Müller absorbing boundary condition is imposed on the whole boundary. Parameters ε_r, μ_r and τ are set to be 1, and ω to be 2π.

A sequence of regular tetrahedral meshes is employed. Table I displays the error in L^2-norm and the convergence orders estimated between two consecutive meshes, *i.e.* an estimate of the exponent α in the asymptotic convergence rate h^α, for the HDG method and $p = 1$. Table II yields the same characteristics for $p = 2$. In Table I and II, "mesh size" denotes the edge length of the tetrahedrons on the edge of the unit cube and N_{dof} the number of degrees of freedom for the hybrid variable. We observe that the asymptotic convergence orders of the approximate solutions for both $\mathbf{E}$ and $\mathbf{H}$ are *optimal*, *i.e.* of order $p+1$ for both $\mathbf{E}$ and $\mathbf{H}$ when using polynomial order p. This convergence rate has been proved for the Helmholtz equation in 2d [6] but no theoretical result is currently available for Maxwell's equations in 3d.

TABLE I. CONVERGENCE OF THE HDG METHOD WITH $p = 1$.

mesh size	N_{dof}	$\|\mathbf{E}-\mathbf{E}_h\|_{L^2}$		$\|\mathbf{H}-\mathbf{H}_h\|_{L^2}$	
		value	order	value	order
1/2	720	2.27e-1	-	2.35e-1	-
1/4	5184	6.02e-2	1.9	6.68e-2	1.8
1/8	39168	1.54e-2	2.0	1.78e-2	1.9

TABLE II. CONVERGENCE OF THE HDG METHOD WITH $p = 2$.

mesh size	N_{dof}	$\|\mathbf{E}-\mathbf{E}_h\|_{L^2}$		$\|\mathbf{H}-\mathbf{H}_h\|_{L^2}$	
		value	order	value	order
1/2	1440	3.13e-2	-	3.36e-2	-
1/4	10368	4.00e-3	3.0	4.44e-3	2.9
1/8	78336	4.93e-4	3.0	5.53e-4	3.0

B. Computational costs

We add a perfectly conducting cube of side length $1/3$ centered at the origin in the previous problem and ω is set to be 4π. We can link the HDG method with a Upwind Flux (UF) DG discretization [7] and, because both approaches provide us the same solution, we can "fairly" compare their numerical costs. For each method the solution of the problem is based on a Schwarz-type domain decomposition method as described in [8]: a global iterative solver is coupled with local solvers in each subdomain of the decomposition. Several kinds of local solvers are studied but they all benefit of a single-precision LU factorization of the local problems. The timing measures for this factorization phase are given in Table IV where N_s denotes the number of subdomains (one per processor). Three local solvers are considered: two Krylov subspace solvers preconditioned by the LU factorization, GMRes (DD-gmres) or BiCGStab(l) (DD-bicgl), and one direct solver using iterative refinement (DD-itref) to get the double-precision accuracy. The corresponding CPU times are summarized in Table V where #it denotes the number of iterations for the Schwarz algorithm.

TABLE III. MESHES FOR THE SCATTERING BY A METALLIC CUBE.

Mesh	# Vertices	# Tetrahedra	# Faces
M1	29,062	156,000	318,000
M2	67,590	82,944	757,728
M3	131,922	744,000	1,503,600

IV. CONCLUSION AND PERSPECTIVES

HDG methods lead to the same optimal accuracy as some classical DG methods but greatly reduce the computational burden in comparison. They make DG methods more attractive for stationary problems. We are currently working on efficient strategies for solving the linear systems for the hybrid variable unknowns, as well as on the the further development and application of the proposed HDG method to more challenging propagation problems. Other questions need to be addressed, especially the use of a variable order p and non-conforming meshes, as it can be naturally dealt with the HDG strategy [4]. Moreover, it should also facilitate the rigorous coupling with other discretizations because the HDG approach clearly separate the local volume problems (1) and the interface conditions (3).

TABLE IV. COMPUTING TIMES AND MEMORY FOR L AND U FACTORS.

Mesh	N_s	CPU time (max)		RAM (max)	
		HDG	UF	HDG	UF
M1	32	26.57s	270.18s	218MB	895MB
M2	64	44.17s	450.10s	288MB	1166MB
M3	128	51.85s	474.57s	288MB	1186MB

TABLE V. COMPUTING TIMES (SCHWARZ ALGORITHM).

Mesh	Strategy	#it	CPU time (min/max)	
			HDG	UF
M1	DD-bicgl	10	83.97s/90.20s	301.22s/330.76s
	DD-gmres	10	43.15s/46.22s	152.21s/167.03s
	DD-itref	10	32.26s/34.64s	107.25s/117.47s
M2	DD-bicgl	13	148.74s/158.89s	512.75s/564.21s
	DD-gmres	13	77.15s/82.05s	262.20s/287.40s
	DD-itref	13	54.18s/59.55s	168.55s/185.22s
M3	DD-bicgl	16	181.29s/213.25s	625.02s/723.77s
	DD-gmres	18	105.36s/125.05s	340.34s/397.49s
	DD-itref	17	72.34s/84.80s	205.33s/246.02s

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