Carleman estimates for semi-discrete parabolic operators and application to the controllability of semi-linear semi-discrete parabolic equations

Franck Boyer, Jérôme Le Rousseau

CARLEMAN ESTIMATES FOR SEMI-DISCRETE PARABOLIC OPERATORS AND APPLICATION TO THE CONTROLLABILITY OF SEMI-LINEAR SEMI-DISCRETE PARABOLIC EQUATIONS

FRANCK BOYER† AND JÉRÔME LE ROUSSEAU‡

Abstract. In arbitrary dimension, in the discrete setting of finite-differences we prove a Carleman estimate for a semi-discrete parabolic operator, in which the large parameter is connected to the mesh size. This estimate is applied for the derivation of a (relaxed) observability estimate, that yield some controlability results for semi-linear semi-discrete parabilic equations. Sub-linear and super-linear cases are considered.

Key words. Parabolic operator – semi-discrete Carleman estimates – observability – null controllability – semilinear equations

AMS subject classifications. 35K10; 35K58; 65M06; 93B05; 93B07.

1. Introduction and notation. Let $d \geq 1$, $L_1, \dots, L_d$ be positive real numbers, and $\Omega = \prod_{1 \leq i \leq d} (0, L_i)$. We set $\boldsymbol{x} = (x_1, \dots, x_d) \in \Omega$. With $\omega \Subset \Omega$ we consider the following parabolic problem in $(0,T) \times \Omega$, with $T > 0$,

$$\partial_t y - \nabla_{\boldsymbol{x}} \cdot (\Gamma \nabla_{\boldsymbol{x}} y) = \mathbf{1}_\omega v \text{ in } (0,T) \times \Omega, \quad y_{|\partial\Omega} = 0, \quad \text{and } \ y_{|t=0} = y_0, \tag{1.1}$$

where the diagonal diffusion tensor $\Gamma(\boldsymbol{x}) = \mathrm{Diag}(\gamma_1(\boldsymbol{x}), \dots, \gamma_d(\boldsymbol{x}))$ with $\gamma_i(\boldsymbol{x}) > 0$ satisfies

$$\operatorname{reg}(\Gamma) \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\substack{\boldsymbol{x} \in \Omega \\ i=1,\dots,d}} \Big(\gamma_i(\boldsymbol{x}) + \frac{1}{\gamma_i(\boldsymbol{x})} + |\nabla_{\boldsymbol{x}} \gamma_i(\boldsymbol{x})| \Big) < +\infty. \tag{1.2}$$

The distributed null-controllability problem consists in finding $v \in L^2((0,T)\times\Omega)$ such that $y(T) = 0$. This problem was solved in the 90's by G. Lebeau and L. Robbiano [LR95] and A. Fursikov and O. Yu. Imanuvilov [FI96]. By a duality argument the null-controllability result for (1.1) is equivalent to having the following observability inequality

$$|q(0)|^2_{L^2(\Omega)} \leq C_{\mathrm{obs}} \|q\|^2_{L^2((0,T)\times\omega)}, \tag{1.3}$$

for q solution to $\big(\partial_t + \nabla_{\boldsymbol{x}} \cdot (\Gamma \nabla_{\boldsymbol{x}} y)\big) q = 0$ and $q_{|\partial\Omega} = 0$.

Let us consider the elliptic operator on Ω given by

$$\mathcal{A} = -\nabla_{\boldsymbol{x}} \cdot (\Gamma \nabla_{\boldsymbol{x}}) = - \sum_{1 \leq i \leq d} \partial_{x_i} (\gamma_i \partial_{x_i})$$

with homogeneous Dirichlet boundary conditions on $\partial\Omega$. We shall introduce a finite-difference approximation of the operator $\mathcal{A}$. For a mesh $\mathfrak{M}$ that we shall describe below, associated with a discretization step h, the discrete operator will be denoted

Date: August 22, 2012

†Aix-Marseille Université, Laboratoire d'Analyse Topologie Probabilités (LATP), CNRS UMR 7353, 39 rue F. Joliot-Curie, 13453 Marseille cedex 13, France. (fboyer@latp.univ-mrs.fr).

‡Université d'Orléans, Laboratoire de Mathématiques - Analyse, Probabilités, Modélisation - Orléans (MAPMO), CNRS UMR 7349, Fédération Denis Poisson, CNRS FR 2964, B.P. 6759, 45067 Orléans cedex 2. (jlr@univ-orleans.fr).

by $\mathcal{A}^{\mathfrak{M}}$. It will act on a finite dimensional space $\mathbb{R}^{\mathfrak{M}}$, of dimension $|\mathfrak{M}|$, and will be selfadjoint for the standard inner product in $\mathbb{R}^{\mathfrak{M}}$. Our main result is the derivation of a Carleman estimate for the operators $\partial_t \pm \mathcal{A}^{\mathfrak{M}}$, *i.e.*, a weighted energy estimate with a localized observation term, which is uniform with respect to the discretization parameter h. The weight function is of exponential type.

There is a vast literature on Carleman estimates going back from the original work of T. Carleman [Car39] and the seminal work of L. Hörmander [Hör58] (see also [Hör63, Chapter 8] and [Hör85, Chapter 28]). These estimates were first introduced for the purpose of proving and quantifying unique continuation (see [Zui83] for manifold references). In more recent years, the field of applications of Carleman estimates has gone beyond the original domain they had been introduced for. They are also used in the study of inverse problems and control theory for PDEs. For an introduction to Carleman estimates and their applications to controllability of parabolic equations, as we shall use them here, we refer for instance to [FCG06] and [LL11].

From the semi-discrete Carleman estimates we obtain, we deduce an observation inequality for the operator $\partial_t - \mathcal{A}^{\mathfrak{M}} + a$, where a is a bounded potential function:

$$|q_h(0)|^2_{L^2(\Omega)} \leq C_{\text{obs}} \|q_h\|^2_{L^2((0,T)\times\omega)} + C_h |q_h(T)|^2_{L^2(\Omega)},$$

for q_h (semi-discrete) solution to $(\partial_t - \mathcal{A}^{\mathfrak{M}} + a)q_h = 0$. Special care is placed in the estimation of the observability constant C_{obs} and the constant C_h, in particular in their dependency upon $\|a\|_\infty$. The observability inequality is weak as compared to that one can obtain in the continuous case; compare with (1.3). Here, there is an additional term in the right-hand-side of the inequality. In fact, because of the presence of this term we shall speak of a *relaxed* observability inequality. Earlier work [BHL10a, BHL10b] showed that this term cannot be removed and is connected to an obstruction to the null-controllability of the semi-discrete problem in space dimension greater than two, as pointed out by a counter-example due to O. Kavian (see e.g. the review article [Zua06]). Still, by duality, the relaxed observability estimate we derive is equivalent to a controllability result. Because of the aforementioned counter-example we do not achieve null-controllability, yet we reach a small target, which size goes to zero exponentially as the mesh size $h \to 0$. We speak of a h-null controllability result, a notion that should not be confused with approximate controllability: the size of the neighborhood of zero reached by the solution of the parabolic equation at the final time $t = T$ is not fixed; it is a function of the discretization step.

The dependency of the observability constant with respect to the norm $\|a\|_\infty$ allows one to tackle controllability questions for parabolic equations with semi-linear terms, in particular cases of super-linear terms. In the continuous case, this was achieved in [Bar00, FCZ00]. To our knowledge, in the discrete case this question were only discussed in [MFC12]. Here, we shall consider such questions in the case of semi-discretized equations and we shall be interested in proving h-null controllability results. Some of the results we give are uniform with respect to the discretization parameter: h-null controllability is achieved with a (semi-discrete) control function whose L^2-norm is bounded uniformly in h.

Precise statements of the results we obtain require the introduction of the settings we shall work with.

For $1 \leq i \leq d$, $i \in \mathbb{N}$, we set $\Omega_i = \prod_{\substack{1\leq j\leq d \\ j\neq i}} (0, L_j)$. For $T > 0$ we introduce

$$Q = (0,T)\times\Omega, \quad Q_i = (0,T)\times\Omega_i, \quad 1 \leq i \leq d.$$

We also set boundaries as (see Figure 1)

$$\partial_i^- \Omega = \prod_{1\le j<i} [0, L_j] \times \{0\} \times \prod_{i<j\le d} [0, L_j], \quad \partial_i^+ \Omega = \prod_{1\le j<i} [0, L_j] \times \{L_i\} \times \prod_{i<j\le d} [0, L_j],$$

$$\partial_i \Omega = \partial_i^+ \Omega \cup \partial_i^- \Omega, \quad \partial \Omega = \bigcup_{1\le i\le d} \partial_i \Omega.$$

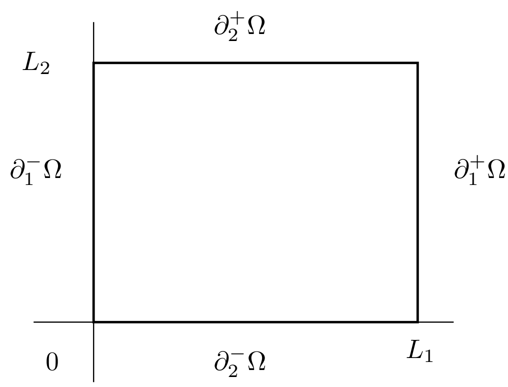

FIG. 1. *Notation for the boundaries in the 2D case*

1.1. Discrete settings. We shall use uniform meshes, *i.e.*, meshes with constant discretization steps in each direction. The introduction of more general meshes is possible. We refer to [BHL10b] for some families of regular non uniform meshes that one can consider.

The notation we introduce will allow us to use a formalism as close as possible to the continuous case, in particular for norms and integrations. *Then most of the computations we carry out can be read in a very intuitive manner, which will ease the reading of the article.* Most of the discrete formalism will then be hidden in the subsequent sections. The notation below is however necessary for a complete and precise reading of the proofs.

We shall use the notation $[\![a, b]\!] = [a, b] \cap \mathbb{N}$.

1.1.1. Primal mesh. For $i \in [\![1, d]\!]$ and $N_i \in \mathbb{N}^*$, we set $h_i = L_i/(N_i + 1)$ and $x_{i,j} = jh_i$, $j \in [\![0, N_i + 1]\!]$, which gives

$$0 = x_{i,0} < x_{i,1} < \cdots < x_{i,N_i} < x_{i,N_i+1} = L_i.$$

We introduce the following set of indices,

$$\mathfrak{N} := \big\{ \boldsymbol{k} = (k_1, \ldots, k_d); \quad k_i \in [\![1, N_i]\!], \ i \in [\![1, d]\!] \big\}.$$

For $\boldsymbol{k} = (k_1, \ldots, k_d) \in \mathfrak{N}$ we set $\boldsymbol{x_k} = (x_{1,k_1}, \ldots, x_{d,k_d}) \in \Omega$. We refer to this discretization as to the primal mesh

$$\mathfrak{M} := \big\{ \boldsymbol{x_k}; \ \boldsymbol{k} \in \mathfrak{N} \big\}, \quad \text{with } |\mathfrak{M}| := \prod_{i\in[\![1,d]\!]} N_i.$$

We set $h = \max_{i\in[\![1,d]\!]} h_i$ and we impose the following condition on the meshes that we consider: there exists $C > 0$ such that

$$C^{-1} h \le h_i \le Ch, \quad i \in [\![1, d]\!]. \tag{1.4}$$

1.1.2. Boundary of the primal mesh. To introduce boundary conditions in the ith direction and related trace operators (see Section 1.1.5) we set $\partial_i\mathfrak{N} = \partial_i^-\mathfrak{N} \cup \partial_i^+\mathfrak{N}$ with

$$\partial_i^-\mathfrak{N} = \big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_j \in [\![1,N_j]\!],\ j \in [\![1,d]\!],\ j \neq i,\ k_i = 0\big\},$$
$$\partial_i^+\mathfrak{N} = \big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_j \in [\![1,N_j]\!],\ j \in [\![1,d]\!],\ j \neq i,\ k_i = N_i+1\big\},$$

and

$$\partial\mathfrak{N} = \underset{i\in[\![1,d]\!]}{\cup} \partial_i\mathfrak{N}, \qquad \partial\mathfrak{M} = \big\{\boldsymbol{x_k};\ \boldsymbol{k} \in \partial\mathfrak{N}\big\}, \quad \partial_i^{\pm}\mathfrak{M} = \big\{\boldsymbol{x_k};\ \boldsymbol{k} \in \partial_i^{\pm}\mathfrak{N}\big\}.$$

1.1.3. Dual meshes. We will need to operate discrete derivatives on functions defined on the primal mesh (see Section 1.1.6). It is easily seen that these derivatives are naturally associated to another set of staggered meshes, called dual meshes. In fact there will be two kinds of such meshes: the ones associated to a first-order discrete derivation and the ones associated to a second-order discrete derivation. Let us define precisely these new meshes (see Figure 2).

For $i \in [\![1,d]\!]$ and $N_i \in \mathbb{N}^*$, we set $x_{i,j} = jh_i$ for $j \in [\![0,N_i]\!] + \frac{1}{2}$, which gives

$$0 = x_{i,0} < x_{i,\frac12} < x_{i,1} < x_{i,1+\frac12} < \cdots < x_{i,N_i} < x_{i,N_i+\frac12} < x_{i,N_i+1} = L_i.$$

For $i \in [\![1,d]\!]$, we introduce a second type of sets of indices

$$\overline{\mathfrak{N}}^{i} := \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_j \in [\![1,N_j]\!]\ j \in [\![1,d]\!],\ j \neq i, \text{and } k_i \in [\![0,N_i]\!] + \frac{1}{2}\Big\}.$$

For $j \in [\![1,d]\!]$, $j \neq i$, we also set $\partial_j\overline{\mathfrak{N}}^{i} = \partial_j^-\overline{\mathfrak{N}}^{i} \cup \partial_j^+\overline{\mathfrak{N}}^{i}$ with

$$\partial_j^-\overline{\mathfrak{N}}^{i} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\ i' \neq j,$$
$$k_i \in [\![0,N_i]\!] + \frac{1}{2},\ \text{and } k_j = 0\Big\},$$
$$\partial_j^+\overline{\mathfrak{N}}^{i} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\ i' \neq j,$$
$$k_i \in [\![0,N_i]\!] + \frac{1}{2},\ \text{and } k_j = N_j+1\Big\},$$

and $\partial\overline{\mathfrak{N}}^{i} = \cup_{\substack{j\in[\![1,d]\!]\\ j\neq i}} \partial_j\overline{\mathfrak{N}}^{i}$. We moreover introduce $\partial_i\overline{\mathfrak{N}}^{i} = \partial_i^-\overline{\mathfrak{N}}^{i} \cup \partial_i^+\overline{\mathfrak{N}}^{i}$ with

$$\partial_i^-\overline{\mathfrak{N}}^{i} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_j \in [\![1,N_j]\!],\ j \in [\![1,d]\!],\ j \neq i,\ k_i = \frac{1}{2}\Big\},$$
$$\partial_i^+\overline{\mathfrak{N}}^{i} = \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_j \in [\![1,N_j]\!],\ j \in [\![1,d]\!],\ j \neq i,\ k_i = N_i + \frac{1}{2}\Big\}.$$

Remark that $\partial_i\overline{\mathfrak{N}}^{i} \subset \overline{\mathfrak{N}}^{i}$ whereas $\partial_j\overline{\mathfrak{N}}^{i} \not\subset \overline{\mathfrak{N}}^{i}$ for $j \neq i$.

For $i,j \in [\![1,d]\!]$, $i \neq j$, we introduce a third type of sets of indices

$$\overline{\mathfrak{N}}^{ij} := \Big\{\boldsymbol{k} = (k_1,\dots,k_d);\ k_{i'} \in [\![1,N_{i'}]\!],\ i' \in [\![1,d]\!],\ i' \neq i,\ \ i' \neq j$$
$$\text{and } k_i \in [\![0,N_i]\!] + \frac{1}{2},\ k_j \in [\![0,N_j]\!] + \frac{1}{2}\Big\}.$$

For $l \in [\![1, d]\!]$, $l \neq i$, $l \neq j$, we also set $\partial_l \overline{\mathfrak{N}}^{ij} = \partial_l^- \overline{\mathfrak{N}}^{ij} \cup \partial_l^+ \overline{\mathfrak{N}}^{ij}$ with

$$\begin{aligned}
\partial_l^- \overline{\mathfrak{N}}^{ij} = \Big\{ & \boldsymbol{k} = (k_1, \dots, k_d);\ k_{i'} \in [\![1, N_{i'}]\!],\ i' \in [\![1, d]\!],\ i' \neq i,\ i' \neq j,\ i' \neq l, \\
& k_i \in [\![0, N_i]\!] + \frac{1}{2},\ k_j \in [\![0, N_j]\!] + \frac{1}{2},\ \text{and } k_l = 0 \Big\}, \\
\partial_l^+ \overline{\mathfrak{N}}^{ij} = \Big\{ & \boldsymbol{k} = (k_1, \dots, k_d);\ k_{i'} \in [\![1, N_{i'}]\!],\ i' \in [\![1, d]\!],\ i' \neq i,\ i' \neq j,\ i' \neq l, \\
& k_i \in [\![0, N_i]\!] + \frac{1}{2},\ k_j \in [\![0, N_j]\!] + \frac{1}{2},\ \text{and } k_l = N_l + 1 \Big\},
\end{aligned}$$

and $\partial \overline{\mathfrak{N}}^{ij} = \cup_{\substack{l \in [\![1,d]\!] \\ l \neq i, l \neq j}} \partial_l \overline{\mathfrak{N}}^{ij}$. Moreover we set $\partial_i \overline{\mathfrak{N}}^{ij} = \partial_i^- \overline{\mathfrak{N}}^{ij} \cup \partial_i^+ \overline{\mathfrak{N}}^{ij}$ with

$$\begin{aligned}
\partial_i^- \overline{\mathfrak{N}}^{ij} = \Big\{ & \boldsymbol{k} = (k_1, \dots, k_d);\ k_{i'} \in [\![1, N_{i'}]\!],\ i' \in [\![1, d]\!],\ i' \neq i,\ i' \neq j, \\
& k_i = \frac{1}{2},\ k_j \in [\![0, N_j]\!] + \frac{1}{2} \Big\}, \\
\partial_i^+ \overline{\mathfrak{N}}^{ij} = \Big\{ & \boldsymbol{k} = (k_1, \dots, k_d);\ k_{i'} \in [\![1, N_{i'}]\!],\ i' \in [\![1, d]\!],\ i' \neq i,\ i' \neq j, \\
& k_i = N_i + \frac{1}{2},\ k_j \in [\![0, N_j]\!] + \frac{1}{2} \Big\}.
\end{aligned}$$

For $\boldsymbol{k} = (k_1, \dots, k_d) \in \overline{\mathfrak{N}}^i$ or $\partial \overline{\mathfrak{N}}^i$ (resp. $\overline{\mathfrak{N}}^{ij}$ or $\partial \overline{\mathfrak{N}}^{ij}$) we also set $\boldsymbol{x_k} = (x_{1,k_1}, \dots, x_{d,k_d})$, which gives the following dual meshes

$$\begin{aligned}
\overline{\mathfrak{M}}^i := \big\{ \boldsymbol{x_k};\ \boldsymbol{k} \in \overline{\mathfrak{N}}^i \big\}, \quad \partial \overline{\mathfrak{M}}^i := \big\{ \boldsymbol{x_k};\ \boldsymbol{k} \in \partial \overline{\mathfrak{N}}^i \big\}, \quad \partial_j^\pm \overline{\mathfrak{M}}^i := \big\{ \boldsymbol{x_k};\ \boldsymbol{k} \in \partial_j^\pm \overline{\mathfrak{N}}^i \big\}, \\
\big(\text{resp. } \overline{\mathfrak{M}}^{ij} := \big\{ \boldsymbol{x_k};\ \boldsymbol{k} \in \overline{\mathfrak{N}}^{ij} \big\}, \quad \partial \overline{\mathfrak{M}}^{ij} := \big\{ \boldsymbol{x_k};\ \boldsymbol{k} \in \partial \overline{\mathfrak{N}}^{ij} \big\}, \\
\partial_l^\pm \overline{\mathfrak{M}}^{ij} := \big\{ \boldsymbol{x_k};\ \boldsymbol{k} \in \partial_l^\pm \overline{\mathfrak{N}}^{ij} \big\}\big).
\end{aligned}$$

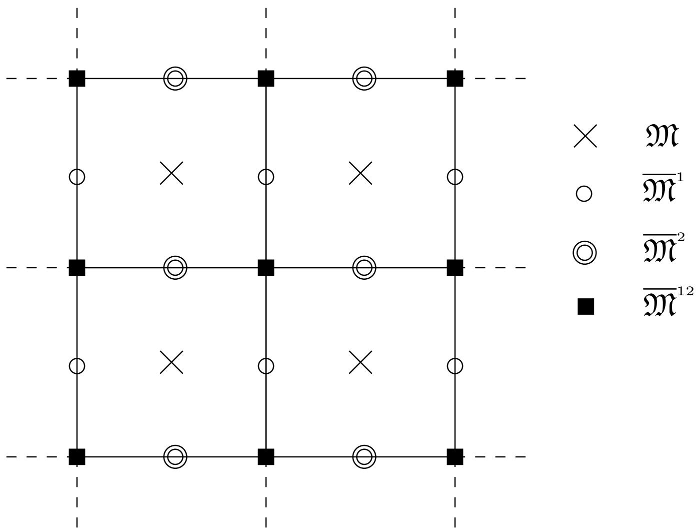

FIG. 2. *Primal and dual meshes in the 2D case.*

1.1.4. Discrete functions. We denote by $\mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^i}$ or $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$) the sets of discrete functions defined on $\mathfrak{M}$ (resp. $\overline{\mathfrak{M}}^i$ or $\overline{\mathfrak{M}}^{ij}$) respectively. If $u \in \mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^i}$ or $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$), we denote by $u_{\boldsymbol{k}}$ its value corresponding to $\boldsymbol{x_k}$ for $\boldsymbol{k} \in \mathfrak{N}$ (resp. $\boldsymbol{k} \in \overline{\mathfrak{N}}^i$ or $\boldsymbol{k} \in \overline{\mathfrak{N}}^{ij}$). For $u \in \mathbb{R}^{\mathfrak{M}}$ we define

$$u^{\mathfrak{M}} = \sum_{\boldsymbol{k} \in \mathfrak{N}} \mathbf{1}_{b_{\boldsymbol{k}}}\, u_{\boldsymbol{k}} \in L^\infty(\Omega), \quad \text{with } b_{\boldsymbol{k}} = \prod_{i \in [\![1,d]\!]} [x_{i,k_i-\frac{1}{2}}, x_{i,k_i+\frac{1}{2}}],\ \boldsymbol{k} \in \mathfrak{N}. \tag{1.5}$$

Since no confusion is possible, by abuse of notation we shall often write u in place of $u^{\mathfrak{M}}$. For $u \in \mathbb{R}^{\mathfrak{M}}$ we define

$$\iint_\Omega u := \iint_\Omega u^{\mathfrak{M}}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\boldsymbol{k} \in \mathfrak{N}} \left|b_{\boldsymbol{k}}\right| u_{\boldsymbol{k}}, \quad \text{where } \left|b_{\boldsymbol{k}}\right| = \prod_{i \in [\![1,d]\!]} h_i.$$

For some $u \in \mathbb{R}^{\mathfrak{M}}$, we shall need to associate boundary values

$$u^{\partial\mathfrak{M}} = \left\{u_{\boldsymbol{k}};\ \boldsymbol{k} \in \partial\mathfrak{N}\right\},$$

i.e., the values of u at the point $\boldsymbol{x_k} \in \partial\mathfrak{M}$. The set of such extended discrete functions is denoted by $\mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$. Homogeneous Dirichlet boundary conditions then consist in the choice $u_{\boldsymbol{k}} = 0$ for $\boldsymbol{k} \in \partial\mathfrak{N}$, in short $u^{\partial\mathfrak{M}} = 0$ or even $u_{|\partial\Omega} = 0$ by abuse of notation (see also Section 1.1.5 below).

Similarly, for $u \in \mathbb{R}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$) we shall associate the following boundary values

$$u^{\partial\overline{\mathfrak{M}}^i} = \left\{u_{\boldsymbol{k}};\ \boldsymbol{k} \in \partial\overline{\mathfrak{N}}^i\right\} \ \left(\text{resp. } u^{\partial\overline{\mathfrak{M}}^{ij}} = \left\{u_{\boldsymbol{k}};\ \boldsymbol{k} \in \partial\overline{\mathfrak{N}}^{ij}\right\}\right).$$

The set of such extended discrete functions is denoted by $\mathbb{R}^{\overline{\mathfrak{M}}^i\cup\partial\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}\cup\partial\overline{\mathfrak{M}}^{ij}}$).

For $u \in \mathbb{R}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$) we define

$$u^{\overline{\mathfrak{M}}^i} = \sum_{\boldsymbol{k} \in \overline{\mathfrak{N}}^i} \mathbf{1}_{\overline{b}^i_{\boldsymbol{k}}} u_{\boldsymbol{k}} \in L^\infty(\Omega) \quad \text{with } \overline{b}^i_{\boldsymbol{k}} = \prod_{l \in [\![1,d]\!]} [x_{l,k_l-\frac{1}{2}}, x_{l,k_l+\frac{1}{2}}],\ \boldsymbol{k} \in \overline{\mathfrak{N}}^i,$$
$$\Big(\text{resp.} u^{\overline{\mathfrak{M}}^{ij}} = \sum_{\boldsymbol{k} \in \overline{\mathfrak{N}}^{ij}} \mathbf{1}_{\overline{b}^{ij}_{\boldsymbol{k}}} u_{\boldsymbol{k}} \in L^\infty(\Omega) \quad \text{with } \overline{b}^{ij}_{\boldsymbol{k}} = \prod_{l \in [\![1,d]\!]} [x_{l,k_l-\frac{1}{2}}, x_{l,k_l+\frac{1}{2}}],\ \boldsymbol{k} \in \overline{\mathfrak{N}}^{ij}\Big).$$

As above, for $u \in \mathbb{R}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$), we define

$$\iint_\Omega u := \iint_\Omega u^{\overline{\mathfrak{M}}^i}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\boldsymbol{k} \in \overline{\mathfrak{N}}^i} \left|\overline{b}^i_{\boldsymbol{k}}\right| u_{\boldsymbol{k}}, \quad \text{where } \left|\overline{b}^i_{\boldsymbol{k}}\right| = \prod_{l \in [\![1,d]\!]} h_l,$$
$$\Big(\text{resp.} \iint_\Omega u := \iint_\Omega u^{\overline{\mathfrak{M}}^{ij}}(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{\boldsymbol{k} \in \overline{\mathfrak{N}}^{ij}} \left|\overline{b}^{ij}_{\boldsymbol{k}}\right| u_{\boldsymbol{k}}, \quad \text{where } \left|\overline{b}^{ij}_{\boldsymbol{k}}\right| = \prod_{l \in [\![1,d]\!]} h_l\Big).$$

REMARK 1.1. *Above, the definitions of $b_{\boldsymbol{k}}$, $\overline{b}^i_{\boldsymbol{k}}$, and $\overline{b}^{ij}_{\boldsymbol{k}}$ look similar. They are however different as each time the multi-index $\boldsymbol{k} = (k_1, \dots, k_d)$ is chosen in a different set: $\mathfrak{N}$, $\overline{\mathfrak{N}}^i$ and $\overline{\mathfrak{N}}^{ij}$ respectively.*

In particular we define the following L^2-inner product on $\mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^i}$ or $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$)

$$\langle u, v\rangle_{L^2(\Omega)} = \iint_\Omega uv = \iint_\Omega u^{\mathfrak{M}}(\boldsymbol{x}) v^{\mathfrak{M}}(\boldsymbol{x})\, d\boldsymbol{x}, \tag{1.6}$$
$$\Big(\text{resp. } \langle u, v\rangle_{L^2(\Omega)} = \iint_\Omega uv = \iint_\Omega u^{\overline{\mathfrak{M}}^i}(\boldsymbol{x}) v^{\overline{\mathfrak{M}}^i}(\boldsymbol{x})\, d\boldsymbol{x},$$
$$\text{or } \langle u, v\rangle_{L^2(\Omega)} = \iint_\Omega uv = \iint_\Omega u^{\overline{\mathfrak{M}}^{ij}}(\boldsymbol{x}) v^{\overline{\mathfrak{M}}^{ij}}(\boldsymbol{x})\, d\boldsymbol{x}\Big).$$

The associated norms will be denoted by $|u|_{L^2(\Omega)}$.

For semi-discrete function $u(t)$ in $\mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^i}$ or $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$), $t \in (0,T)$, we shall write $\iiint_Q u\ dt = \int_0^T \iint_\Omega u(t)\ dt$, and we define the following L^2-norm

$$\|u(t)\|^2_{L^2(Q)} = \int_0^T \iint_\Omega \big(u(t)\big)^2 dt.$$

Endowing the space of semi-discrete functions $L^2(0,T;\mathbb{R}^{\mathfrak{M}})$ (resp. $L^2(0,T;\mathbb{R}^{\overline{\mathfrak{M}}^i})$ or $L^2(0,T;\mathbb{R}^{\overline{\mathfrak{M}}^{ij}})$) with this norm yields a Hilbert space.

Definition of a space of semi-discrete functions like $L^\infty(0,T,\mathbb{R}^{\mathfrak{M}})$ (resp. $L^\infty(0,T;\mathbb{R}^{\overline{\mathfrak{M}}^i})$ or $L^\infty(0,T;\mathbb{R}^{\overline{\mathfrak{M}}^{ij}})$) can be done similarly with the norm

$$\|u(t)\|_{L^\infty(Q)} = \operatorname*{ess\ sup}_{t\in(0,T)} \sup_{\boldsymbol{k}\in\mathfrak{N}} |u_{\boldsymbol{k}}(t)|.$$

We shall also use mixed norms of the form

$$\|u(t)\|_{L^\infty(0,T;L^2(\Omega))} = \operatorname*{ess\ sup}_{t\in(0,T)} |u(t)|_{L^2(\Omega)}.$$

Similarly we shall use such norms for spaces of semi-discrete functions defined on (or restricted to) $(0,T)\times\omega$.

1.1.5. Traces. Let $i \in [\![1,d]\!]$. For $u \in \mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^j\cup\partial\overline{\mathfrak{M}}^j}$, $j\neq i$), its trace on $\partial_i^+\Omega$, corresponds to $\boldsymbol{k}\in\partial_i^+\mathfrak{N}$ (resp. $\partial_i^+\overline{\mathfrak{N}}^j$), *i.e.*, $k_i = N_i+1$ in our discretization and will be denoted by $u_{|k_i=N_i+1}$ or simply u_{N_i+1}. Similarly its trace on $\partial_i^-\Omega$, corresponds to $\boldsymbol{k}\in\partial_i^-\mathfrak{N}$ (resp. $\partial_i^-\overline{\mathfrak{N}}^j$), *i.e.*, $k_i=0$ and will be denoted by $u_{|k_i=0}$ or simply u_0. The latter notation will be used if no confusion is possible, that is if the context indicates that the trace is taken on $\partial_i^-\Omega$.

By abuse of notation, we shall also use $\partial_i\Omega$, $i\in[\![1,d]\!]$, to denote the boundaries of Ω in the discrete setting. For homogeneous Dirichlet boundary condition we shall write

$$\begin{aligned} v_{|\partial\Omega} = 0 \quad &\Leftrightarrow \quad v_{|\partial_i\Omega} = 0,\ i\in[\![1,d]\!] \\ &\Leftrightarrow \quad v_{|k_i=0} = v_{|k_i=N_i+1} = 0,\ i\in[\![1,d]\!] \\ &\Leftrightarrow \quad v\in\mathbb{R}_0^{\mathfrak{M}\cup\partial\mathfrak{M}}. \end{aligned}$$

For $v\in\mathbb{R}^{\overline{\mathfrak{M}}^i\cup\partial\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}\cup\partial\overline{\mathfrak{M}}^{ij}}$, $j\neq i$), its trace on $\partial_i^+\Omega$, corresponds to $\boldsymbol{k}\in\partial_i^+\overline{\mathfrak{N}}^i$ (resp. $\partial_i^+\overline{\mathfrak{N}}^{ij}$), *i.e.*, $k_i = N_i+\frac12$ in our discretization and will be denoted by $v_{|k_i=N_i+\frac12}$ or simply $v_{N_i+\frac12}$. Similarly its trace on $\partial_i^-\Omega$, corresponds to $\boldsymbol{k}\in\partial_i^-\overline{\mathfrak{N}}^i$ (resp. $\partial_i^-\overline{\mathfrak{N}}^{ij}$), *i.e.*, $k_i=\frac12$ and will be denoted by $v_{|k_i=\frac12}$ or simply $v_{\frac12}$. The latter notation will be used if no confusion is possible, if the context indicates that the trace is taken on $\partial_i^-\Omega$.

For such functions $u\in\mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^j\cup\partial\overline{\mathfrak{M}}^j}$, $j\neq i$) we can then define surface integrals of the type

$$\int_{\partial_i^+\Omega} u_{|\partial_i^+\Omega} = \int_{\Omega_i} u_{|k_i=N_i+1} = \sum_{\substack{\boldsymbol{k}\in\partial_i^+\mathfrak{N} \\ (\text{resp. } \boldsymbol{k}\in\partial_i^+\overline{\mathfrak{N}}^j)}} \big|\partial_i b_{\boldsymbol{k}}\big|\, u_{\boldsymbol{k}},$$

$$\text{where } \big|\partial_i b_{\boldsymbol{k}}\big| = \prod_{\substack{l\in[\![1,d]\!] \\ l\neq i}} h_l,\ \boldsymbol{k}\in\partial_i^+\mathfrak{N} \text{ (resp. } \partial_i^+\overline{\mathfrak{N}}^j),$$

and for $v \in \mathbb{R}^{\overline{\mathfrak{M}}^i \cup \partial \overline{\mathfrak{M}}^i}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^{ij} \cup \partial \overline{\mathfrak{M}}^{ij}}$, $j \neq i$)

$$\int_{\partial_i^+ \Omega} v_{|\partial_i^+ \Omega} = \int_{\Omega_i} v_{|k_i = N_i + \frac{1}{2}} = \sum_{\substack{\boldsymbol{k} \in \partial_i^+ \overline{\mathfrak{M}}^i \\ (\text{resp. } \boldsymbol{k} \in \partial_i^+ \overline{\mathfrak{M}}^{ij})}} \left| \partial_i \overline{b}^i_{\boldsymbol{k}} \right| v_{\boldsymbol{k}},$$

$$\text{where } \left| \partial_i \overline{b}^i_{\boldsymbol{k}} \right| = \prod_{\substack{l \in [\![1,d]\!] \\ l \neq i}} h_l, \ \boldsymbol{k} \in \partial_i^+ \overline{\mathfrak{N}}^i \ (\text{resp. } \partial_i^+ \overline{\mathfrak{N}}^{ij}).$$

Observe that if $\boldsymbol{k} \in \partial_i^+ \mathfrak{N}$ (resp. $\partial_i^+ \overline{\mathfrak{N}}^j$) and $\boldsymbol{k}' \in \partial_i^+ \overline{\mathfrak{N}}^i$ (resp. $\partial_i^+ \overline{\mathfrak{N}}^{ij}$) with $k_l = k'_l$ for $l \neq i$ then $\left| \partial_i b_{\boldsymbol{k}} \right| = \left| \partial_i \overline{b}^i_{\boldsymbol{k}'} \right|$. We thus have

$$\int_{\partial_i^+ \Omega} v_{|\partial_i^+ \Omega} = \int_{\Omega_i} v_{|k_i = N_i + \frac{1}{2}} = \int_{\Omega_i} (\bar{\tau}_i^{\,-} v)_{|k_i = N_i + 1} = \int_{\partial_i^+ \Omega} (\bar{\tau}_i^{\,-} v)_{|\partial_i^+ \Omega}$$

where $\bar{\tau}_i^{\,-} v \in \mathbb{R}^{\mathfrak{M} \cup \partial_i^- \mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^j \cup \partial_i^- \overline{\mathfrak{M}}^j}$) with the translation operator $\bar{\tau}_i^{\,-}$ defined in Section 1.1.6. It is then natural to define the following integrals

$$\int_{\Omega_i} u_{N_i+1} v_{N_i + \frac{1}{2}} = \int_{\Omega_i} u_{|k_i = N_i + 1} v_{|k_i = N_i + \frac{1}{2}} = \int_{\Omega_i} (u \, \bar{\tau}_i^{\,-} v)_{|k_i = N_i + 1} = \int_{\partial_i^+ \Omega} u (\bar{\tau}_i^{\,-} v)_{|\partial_i^+ \Omega}.$$

Such trace integrals will appear when applying discrete integrations by parts in the following sections.

Similar definitions and considerations can be made for integrals over $\partial_i^- \Omega$.

For $u \in \mathbb{R}^{\mathfrak{M} \cup \partial \mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^j \cup \partial \overline{\mathfrak{M}}^j}$, $j \neq i$) we can then introduce the following L^2-norm for the trace on $\partial_i \Omega$:

$$|u|^2_{L^2(\partial_i \Omega)} = |u_{|\partial_i \Omega}|^2_{L^2(\partial_i \Omega)} = \int_{\Omega_i} \left(u_{|k_i = N_i + 1} \right)^2 + \int_{\Omega_i} \left(u_{|k_i = 0} \right)^2.$$

For $v \in \mathbb{R}^{\overline{\mathfrak{M}}^i \cup \partial \overline{\mathfrak{M}}^i}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^{ij} \cup \partial \overline{\mathfrak{M}}^{ij}}$, $j \neq i$) we can then introduce the following L^2-norm for the trace on $\partial_i \Omega$:

$$|v|^2_{L^2(\partial_i \Omega)} = |v_{|\partial_i \Omega}|^2_{L^2(\partial_i \Omega)} = \int_{\Omega_i} \left(u_{|k_i = N_i + \frac{1}{2}} \right)^2 + \int_{\Omega_i} \left(u_{|k_i = \frac{1}{2}} \right)^2.$$

1.1.6. Translation, difference and average operators. Let $i, j \in [\![1, d]\!]$, $j \neq i$. We define the following translations for indices:

$$\begin{aligned} \tau_i^{\pm} : \overline{\mathfrak{N}}^i \ (\text{resp. } \overline{\mathfrak{N}}^{ij}) &\to \mathfrak{N} \cup \partial_i^{\pm} \mathfrak{N} \ (\text{resp. } \overline{\mathfrak{N}}^j \cup \partial_i^{\pm} \overline{\mathfrak{N}}^j), \\ \boldsymbol{k} &\mapsto \tau_i^{\pm} \boldsymbol{k}, \end{aligned}$$

with

$$(\tau_i^{\pm} \boldsymbol{k})_l = \begin{cases} k_l & \text{if } l \neq i, \\ k_l \pm \frac{1}{2} & \text{if } l = i. \end{cases}$$

Translations operators mapping $\mathbb{R}^{\mathfrak{M} \cup \partial \mathfrak{M}} \to \mathbb{R}^{\overline{\mathfrak{M}}^i}$ and $\mathbb{R}^{\overline{\mathfrak{M}}^j \cup \partial \overline{\mathfrak{M}}^j} \to \mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$ are then given by

$$(\tau_i^{\pm} u)_{\boldsymbol{k}} = u_{(\tau_i^{\pm} \boldsymbol{k})}, \quad \boldsymbol{k} \in \overline{\mathfrak{N}}^i \ (\text{resp. } \overline{\mathfrak{N}}^{ij}).$$

A first-order difference operator D_i and an averaging operator A_i are then given by

$$(D_i u)_{\boldsymbol{k}} = (h_i)^{-1}((\tau_i^+ u)_{\boldsymbol{k}} - (\tau_i^- u)_{\boldsymbol{k}}), \quad \boldsymbol{k} \in \overline{\mathfrak{N}}^i \text{ (resp. } \overline{\mathfrak{N}}^{ij}),$$
$$(A_i u)_{\boldsymbol{k}} = \tilde{u}^i_{\boldsymbol{k}} = \frac{1}{2}((\tau_i^+ u)_{\boldsymbol{k}} + (\tau_i^- u)_{\boldsymbol{k}}), \quad \boldsymbol{k} \in \overline{\mathfrak{N}}^i \text{ (resp. } \overline{\mathfrak{N}}^{ij}).$$

Both map $\mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ into $\mathbb{R}^{\overline{\mathfrak{M}}^i}$ and $\mathbb{R}^{\overline{\mathfrak{M}}^j\cup\partial\overline{\mathfrak{M}}^j}$ into $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$.

We also define the following translations for indices:

$$\bar{\tau}_i^{\pm} : \mathfrak{N} \text{ (resp. } \overline{\mathfrak{N}}^j) \to \overline{\mathfrak{N}}^i \text{ (resp. } \overline{\mathfrak{N}}^{ij}),$$
$$\boldsymbol{k} \mapsto \bar{\tau}_i^{\pm}\boldsymbol{k},$$

with

$$(\bar{\tau}_i^{\pm}\boldsymbol{k})_l = \begin{cases} k_l & \text{if } l \neq i, \\ k_l \pm \frac{1}{2} & \text{if } l = i. \end{cases}$$

Translations operators mapping $\mathbb{R}^{\overline{\mathfrak{M}}^i} \to \mathbb{R}^{\mathfrak{M}}$ and $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}} \to \mathbb{R}^{\overline{\mathfrak{M}}^j}$ are then given by

$$(\bar{\tau}_i^{\pm} v)_{\boldsymbol{k}} = v_{(\bar{\tau}_i^{\pm}\boldsymbol{k})}, \quad \boldsymbol{k} \in \mathfrak{N} \text{ (resp. } \overline{\mathfrak{N}}^j).$$

A first-order difference operator $\bar{D}_i$ and an averaging operator $\bar{A}_i$ are then given by

$$(\bar{D}_i v)_{\boldsymbol{k}} = (h_i)^{-1}((\bar{\tau}_i^+ v)_{\boldsymbol{k}} - (\bar{\tau}_i^- v)_{\boldsymbol{k}}), \quad \boldsymbol{k} \in \mathfrak{N} \text{ (resp. } \overline{\mathfrak{N}}^j),$$
$$(\bar{A}_i v)_{\boldsymbol{k}} = \overline{v}^i_{\boldsymbol{k}} = \frac{1}{2}((\bar{\tau}_i^+ v)_{\boldsymbol{k}} + (\bar{\tau}_i^- v)_{\boldsymbol{k}}), \quad \boldsymbol{k} \in \mathfrak{N} \text{ (resp. } \overline{\mathfrak{N}}^j).$$

Both map $\mathbb{R}^{\overline{\mathfrak{M}}^i}$ into $\mathbb{R}^{\mathfrak{M}}$ and $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$ into $\mathbb{R}^{\overline{\mathfrak{M}}^j}$.

1.1.7. Sampling of continuous functions. A continuous function f defined on $\overline{\Omega}$ can be sampled on the primal mesh $f^{\mathfrak{M}} = \{f(\boldsymbol{x_k});\ \boldsymbol{k} \in \mathfrak{N}\}$, which we identify to

$$f^{\mathfrak{M}} = \sum_{\boldsymbol{k}\in\mathfrak{N}} \mathbf{1}_{b_{\boldsymbol{k}}} f_{\boldsymbol{k}}, \quad f_{\boldsymbol{k}} = f(\boldsymbol{x_k}), \quad \boldsymbol{k} \in \mathfrak{N},$$

with $b_{\boldsymbol{k}}$ as defined in (1.5). We also set

$$f^{\partial\mathfrak{M}} = \{f(\boldsymbol{x_k});\ \boldsymbol{k} \in \partial\mathfrak{N}\}, \quad f^{\mathfrak{M}\cup\partial\mathfrak{M}} = \{f(\boldsymbol{x_k});\ \boldsymbol{k} \in \mathfrak{N}\cup\partial\mathfrak{N}\}.$$

The function f can also be sampled on the dual meshes, e.g. $\overline{\mathfrak{M}}^i$, $f^{\overline{\mathfrak{M}}^i} = \{f(\boldsymbol{x_k});\ \boldsymbol{k} \in \overline{\mathfrak{N}}^i\}$ which we identify to

$$f^{\overline{\mathfrak{M}}^i} = \sum_{\boldsymbol{k}\in\overline{\mathfrak{N}}^i} \mathbf{1}_{\bar{b}^i_{\boldsymbol{k}}} f_{\boldsymbol{k}}, \quad f_{\boldsymbol{k}} = f(\boldsymbol{x_k}), \quad \boldsymbol{k} \in \overline{\mathfrak{N}}^i$$

with similar definitions for $f^{\partial\overline{\mathfrak{M}}^i}$, $f^{\overline{\mathfrak{M}}^i\cup\partial\overline{\mathfrak{M}}^i}$ and sampling on the meshes $\overline{\mathfrak{M}}^{ij}$, $\overline{\mathfrak{M}}^{ij} \cup \partial\overline{\mathfrak{M}}^{ij}$.

In the sequel, we shall use the symbol f for both the continuous function and its sampling on the primal or dual meshes. In fact, from the context, one will be able to deduce the appropriate sampling. For example, with u defined on the primal mesh, $\mathfrak{M}$, in the following expression, $\bar{D}_i(\gamma D_i u)$, it is clear that the function γ is sampled on the dual mesh $\overline{\mathfrak{M}}^i$ as $D_i u$ is defined on this mesh and the operator $\bar{D}_i$ acts on functions defined on this mesh.

To evaluate the action of multiple iterations of discrete operators, e.g. $D_i, \bar{D}_i, A_i, \bar{A}_i$ on a continuous function we may require the function to be defined in a neighborhood of $\overline{\Omega}$. This will be the case here of the diffusion coefficients in the elliptic operator and the Carleman weight function we shall introduce. For a function f defined on a neighborhood of $\overline{\Omega}$ we set

$$\tau_i^{\pm} f(\boldsymbol{x}) := f\Big(\boldsymbol{x} \pm \frac{h_i}{2}\boldsymbol{e}_i\Big), \qquad \boldsymbol{e}_i = (\delta_{i1}, \dots, \delta_{id}),$$
$$\mathsf{D}_i f := (h_i)^{-1}(\tau_i^+ - \tau_i^-) f, \quad \mathsf{A}_i f = \hat{f}^i = \frac{1}{2}(\tau_i^+ + \tau_i^-) f.$$

For a function f continuously defined in a neighborhood of $\overline{\Omega}$, the discrete function $D_i f$ is in fact equal to $\mathsf{D}_i f$ sampled on the dual mesh, $\overline{\mathfrak{M}}^i$, and $\bar{D}_i f$ is equal to $\mathsf{D}_i f$ sampled on the primal mesh, $\mathfrak{M}$. We shall use similar meanings for averaging symbols, $\tilde{f}$, $\overline{f}$, and for more general combinations: for instance, if $i \neq j$, $\widetilde{D_j f}^i$, $\overline{D_i \bar{D}_j f}^i$, $\overline{D_i D_j f}^i$ will be respectively the functions $\widehat{\mathsf{D}_j f}^i$ sampled on $\overline{\mathfrak{M}}^{ij}$, $\widehat{\mathsf{D}_i \mathsf{D}_j f}^i$ sampled on $\mathfrak{M}$, and $\widehat{\mathsf{D}_i \mathsf{D}_j f}^i$ sampled on $\overline{\mathfrak{M}}^j$

1.2. Statement of the main results. With the notation we have introduced, the usual consistent finite-difference approximation of the elliptic operator $\mathcal{A}$ with homogeneous Dirichlet boundary conditions is

$$\mathcal{A}^{\mathfrak{M}} u = -\sum_{i \in [\![1,d]\!]} \bar{D}_i(\gamma_i D_i u), \tag{1.7}$$

for $u \in \mathbb{R}^{\mathfrak{M} \cup \partial \mathfrak{M}}$ satisfying $u_{|\partial\Omega} = u^{\partial\mathfrak{M}} = 0$. Recall that, in each term, γ_i is the sampling of the given continuous diffusion coefficient γ_i on the dual mesh $\overline{\mathfrak{M}}^i$, so that for any $u \in \mathbb{R}^{\mathfrak{M} \cup \partial \mathfrak{M}}$ and $\boldsymbol{k} \in \mathfrak{N}$, we have

$$(\mathcal{A}^{\mathfrak{M}} u)_{(\boldsymbol{k})} = -\sum_{i \in [\![1,d]\!]} h_i^{-2}\bigg(\gamma_i\big(\boldsymbol{x}_{\bar{\tau}_i^+(\boldsymbol{k})}\big)\Big(\big(\bar{\tau}_i{}^+\tau_i^+ u\big)_{(\boldsymbol{k})} - u_{(\boldsymbol{k})}\Big) \\ - \gamma_i\big(\boldsymbol{x}_{\bar{\tau}_i^-(\boldsymbol{k})}\big)\Big(u_{(\boldsymbol{k})} - \big(\bar{\tau}_i{}^-\tau_i^- u\big)_{(\boldsymbol{k})}\Big)\bigg).$$

In 2D, this operator is nothing but the standard 5-point discretization. Note however that other consistent choices of discretization of γ_i on the dual meshes are possible, such as the averaging on the dual mesh $\overline{\mathfrak{M}}^i$ of the sampling of γ_i on the primal mesh.

The semi-discrete forward and backward parabolic operators are then given by $\mathcal{P}_{\pm}^{\mathfrak{M}} = \partial_t \pm \mathcal{A}^{\mathfrak{M}}$.

1.2.1. Carleman estimate. For the Carleman estimate and the observation/control results we choose here to treat the case of a distributed observation in $\omega \Subset \Omega$. The weight function is of the form $r = e^{s\varphi}$ with $\varphi = e^{\lambda\psi}$, with ψ fulfilling the following assumption. Construction of such a weight function is classical (see e.g. [FI96]).

Assumption 1.2. *Let $\omega_0 \Subset \omega$ be an open set. Let $\tilde{\Omega}$ be a smooth open and connected neighborhood of $\overline{\Omega}$ in $\mathbb{R}^d$. The function $\psi = \psi(x)$ is in $\mathscr{C}^p(\overline{\tilde{\Omega}}, \mathbb{R})$, p sufficiently large, and satisfies, for some $c > 0$,*

$$\psi > 0 \text{ in } \tilde{\Omega}, \quad |\nabla\psi| \geq c \text{ in } \tilde{\Omega} \setminus \omega_0,$$
$$\partial_{n_i}\psi(\boldsymbol{x}) \leq -c < 0 \text{ and } \partial_{x_i}^2\psi(\boldsymbol{x}) \geq 0 \text{ in } V_{\partial_i\Omega}.$$

where $V_{\partial_i\Omega}$ is a sufficiently small neighborhood of $\partial_i\Omega$ in $\tilde{\Omega}$, in which the outward unit normal n_i to Ω is extended from $\partial_i\Omega$.

Let $K > \|\psi\|_\infty$ and set

$$\begin{aligned} &\varphi(x) = e^{\lambda\psi(x)} - e^{\lambda K} < 0, \quad \phi(x) = e^{\lambda\psi(x)}, \\ &r(t,x) = e^{s(t)\varphi(x)}, \quad \rho(t,x) = \big(r(t,x)\big)^{-1}, \end{aligned} \tag{1.8}$$

with

$$s(t) = \tau\theta(t), \quad \tau > 0, \quad \theta(t) = \big((t+\delta T)(T+\delta T - t)\big)^{-1},$$

for $0 < \delta < \frac{1}{2}$. The parameter δ is introduced to avoid singularities at time $t = 0$ and $t = T$. Further comments are provided in Remark 1.4 below.

We have

$$T^{-2} \leq \min_{[0,T]} \theta, \qquad \frac{1}{T^2\delta} \underset{\delta\to 0}{\sim} \max_{[0,T]} \theta = \theta(0) = \theta(T) = \frac{1}{T^2\delta(1+\delta)} \leq \frac{1}{T^2\delta}. \tag{1.9}$$

We note that

$$\partial_t\theta = (2t - T)\theta^2. \tag{1.10}$$

To state the Carleman estimate for the semi-discrete operator $\mathcal{P}^{\mathfrak{M}}_{\pm}$, we introduce the following discrete gradient $\Upsilon = (D_1, \ldots, D_d)^t$ and the following notation

$$\nabla_\gamma f = \Big(\sqrt{\gamma_1}\partial_{x_1} f, \ldots, \sqrt{\gamma_d}\partial_{x_d} f\Big)^t, \qquad \Delta_\gamma f = \sum_{i\in[\![1,d]\!]} \gamma_i \partial^2_{x_i} f.$$

In the discrete setting we also introduce $D_{i,\gamma} f = \sqrt{\gamma_i} D_i f$, $i \in [\![1,d]\!]$, and

$$\Upsilon_\gamma f = \Big(\sqrt{\gamma_1} D_1 f, \ldots, \sqrt{\gamma_d} D_d f\Big)^t = \big(D_{1,\gamma} f, \ldots, D_{d,\gamma} f\big)^t.$$

The enlarged neighborhood $\tilde{\Omega}$ of Ω introduced in Assumption 1.2 allows us to apply multiple discrete operators such as D_i and A_i on the weight functions. In particular, this then yields on $\partial_i\Omega$

$$(r\overline{D_i\rho}^i)_{|k_i=0} \leq 0, \quad (r\overline{D_i\rho}^i)_{|k_i=N_i+1} \geq 0, \quad i \in [\![1,d]\!]. \tag{1.11}$$

We now state our first result, a uniform Carleman estimate for the semi-discrete parabolic operators $P^{\mathfrak{M}}_{\pm} = \partial_t \pm \mathcal{A}^{\mathfrak{M}}$.

THEOREM 1.3. *Let* $\mathrm{reg}^0 > 0$ *be given and let a function* ψ *satisfy Assumption 1.2. We then define the function* φ *according to* (1.8). *For the parameter* $\lambda \geq 1$ *sufficiently large, there exist* C, $\tau_0 \geq 1$, $h_0 > 0$, $\varepsilon_0 > 0$, *depending on* ω, ω_0, T, reg^0, *such that for any* Γ, *with* $\mathrm{reg}(\Gamma) \leq \mathrm{reg}^0$ *we have*

$$\begin{aligned} \tau^{-1}\|\theta^{-\frac{1}{2}} e^{\tau\theta\varphi}\partial_t u\|^2_{L^2(Q)} &+ \tau \sum_{i\in[\![1,d]\!]} \big(\|\theta e^{\tau\theta\varphi} D_i u\|^2_{L^2(Q)} + \|\theta e^{\tau\theta\varphi}\overline{D_i u}^i\|^2_{L^2(Q)}\big) \\ &+ \tau^3\|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\|^2_{L^2(Q)} \leq C\big(\|e^{\tau\theta\varphi} P^{\mathfrak{M}} u\|^2_{L^2(Q)} + \tau^3\|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\|^2_{L^2((0,T)\times\omega)}\big) \\ &\qquad + Ch^{-2}\big(|e^{\tau\theta\varphi} u_{|t=0}|^2_{L^2(\Omega)} + |e^{\tau\theta\varphi} u_{|t=T}|^2_{L^2(\Omega)}\big), \end{aligned} \tag{1.12}$$

for all $\tau \geq \tau_0(T+T^2)$, $0 < h \leq h_0$, $0 < \delta \leq 1/2$, $\tau h(\delta T^2)^{-1} \leq \varepsilon_0$, *and* $u \in \mathscr{C}^1([0,T], \mathbb{R}^{\mathfrak{M} \cup \partial\mathfrak{M}})$, *satisfying* $u_{|(0,T)\times\partial\Omega} = 0$.

Remark 1.4 (Choice of the parameter δ).
In the present Carleman estimate the parameter δ is introduced to avoid the singularity of the weight function at times $t = 0$ and $t = T$. Such singularities, corresponding to the case $\delta = 0$, are exploited in the continuous case as originally introduced in [FI96]. Here the parameter δ is taken different from 0 and yet connected to the other parameters: $\tau h(\delta T^2)^{-1} \leq \varepsilon_0$. Many choices are possible for δ. For the controllability results we shall choose δ proportional to the discretization parameter h.

1.2.2. Relaxed observability estimate. The adjoint system associated with the controlled system with potential

$$\partial_t y + \mathcal{A}^{\mathfrak{M}} y + ay = \mathbf{1}_\omega v, \quad t \in (0,T), \quad y_{|\partial\Omega} = 0, \tag{1.13}$$

is given by

$$-\partial_t q + \mathcal{A}^{\mathfrak{M}} q + aq = 0, \quad t \in (0,T), \quad q_{|\partial\Omega} = 0. \tag{1.14}$$

With the Carleman estimate we proved in Theorem 1.3 we have following relaxed observability estimate for the solutions to (1.14):

$$|q(0)|^2_{L^2(\Omega)} \leq C_{\text{obs}} \|q\|^2_{L^2((0,T)\times\omega)} + e^{-\frac{C_1}{h} + T\|a\|_\infty} |q(T)|^2_{L^2(\Omega)},$$

with $C_{\text{obs}} = e^{C_2(1+\frac{1}{T}+T\|a\|_\infty + \|a\|_\infty^{\frac{2}{3}})}$, if the discretization parameter is chosen sufficiently small. A precise statement and a proof are given in Section 4.1.

1.2.3. Controllability results. From the relaxed observability estimate given above we obtain a h-null controllability result for the linear operator $\mathcal{P}^{\mathfrak{M}}$. This result can be extended to classes of semi-linear equations:

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + \mathcal{G}(y) = \mathbf{1}_\omega v, \quad t \in (0,T), \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0,$$

with $G(x) = xg(x)$. The equation is linearized yielding a bounded potential and a control can be built. Then a fixed-point argument allows one to obtain a control function for the non-linear equation.

First we consider the sublinear case, *i.e.*, we assume that g is bounded. We then prove a h-null controllability result with a control that satisfies

$$\|v\|_{L^2(Q)} \leq C|y_0|_{L^2(\Omega)},$$

were the constant C is uniform with respect to the discretization parameter h; see Section 5.1 for a precise statement and a proof.

Second we consider classes of superlinear equations. Following [FCZ00] we assume that we have

$$|g(x)| \leq K \ln^r(e + |x|), \quad x \in \mathbb{R}, \quad \text{with } 0 \leq r < \frac{3}{2}.$$

Here the precise dependency of the observability constant upon the norm of the potential $\|a\|_\infty$ allows one tackle such nonlinearities.

In arbitrary dimension we obtain a h-null controllability result; see Section 5.2.2 for a precise statement and a proof. However, the size of the control function is not proven uniform with respect to the discretization parameter h:

$$\|v\|_{L^2(Q)} \leq C_h |y_0|_{L^2(\Omega)},$$

In fact a boundedness argument is needed and here we exploit the finite-dimensional structure to achieve it. The constants are however not uniform. A refined treatment of this question require further analysis of the semi-discrete heat kernel; see remark 5.6. In one space dimension, this difficulty can be circumvented and the uniformity of the control function is recovered; see Section 5.2.3

1.3. Outline. In Section 2 we present discrete calculus results and estimates for the Carleman weight function in preparation for the proof of Theorem 1.3. Section 3 is devoted to the proof of Theorem 1.3. In Section 4 we prove the relaxed observability estimate and a h-null controllability results in the linear case. In Section 5 we study h-null controllability in the semi-linear case. Some technical proofs are gathered in appendices.

2. Some preliminary discrete calculus results. This section aims to provide calculus rules for discrete operators such as D_i, $\bar{D}_i$ and also to provide estimates for the successive applications of such operators on the weight functions.

2.1. Discrete calculus formulae. We present calculus results for the finite-difference operators that were defined in the introductory section. Proofs are similar to that given in the one-dimension case in [BHL10a].

LEMMA 2.1. *Let the functions f_1 and f_2 be continuously defined in a neighborhood of $\overline{\Omega}$. For $i \in [\![1, d]\!]$, we have*

$$\mathsf{D}_i(f_1 f_2) = \mathsf{D}_i(f_1)\, \hat{f}_2^{\,i} + \hat{f}_1^{\,i}\, \mathsf{D}_i(f_2).$$

Note that the immediate translation of the proposition to discrete functions $f_1, f_2 \in \mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^j}$, $j \neq i$), and $g_1, g_2 \in \mathbb{R}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$, $j \neq i$)

$$D_i(f_1 f_2) = D_i(f_1)\, \tilde{f}_2^{\,i} + \tilde{f}_1^{\,i}\, D_i(f_2), \qquad \bar{D}_i(g_1 g_2) = \bar{D}_i(g_1)\, \overline{g}_2^{\,i} + \overline{g}_1^{\,i}\, \bar{D}_i(g_2).$$

LEMMA 2.2. *Let the functions f_1 and f_2 be continuously defined in a neighborhood of $\overline{\Omega}$. For $i \in [\![1, d]\!]$, we have*

$$\widehat{f_1 f_2}^{\,i} = \hat{f}_1^{\,i} \hat{f}_2^{\,i} + \frac{h_i^2}{4} \mathsf{D}_i(f_1) \mathsf{D}_i(f_2).$$

Note that the immediate translation of the proposition to discrete functions $f_1, f_2 \in \mathbb{R}^{\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^j}$, $j \neq i$), and $g_1, g_2 \in \mathbb{R}^{\overline{\mathfrak{M}}^i}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^{ij}}$, $j \neq i$)

$$\widetilde{f_1 f_2}^{\,i} = \tilde{f}_1^{\,i} \tilde{f}_2^{\,i} + \frac{h_i^2}{4} D_i(f_1) D_i(f_2), \qquad \overline{g_1 g_2}^{\,i} = \overline{g}_1^{\,i} \overline{g}_2^{\,i} + \frac{h_i^2}{4} \bar{D}_i(g_1) \bar{D}_i(g_2).$$

Some of the following properties can be extended in such a manner to discrete functions. We shall not always write it explicitly.

Averaging a function twice gives the following formula.

LEMMA 2.3. *Let the function f be continuously defined over $\mathbb{R}$. For $i \in [\![1,d]\!]$ we have*

$$\mathsf{A}_i^2 f := \widehat{\widehat{f}}^{\,i} = f + \frac{h_i^2}{4}\mathsf{D}_i\mathsf{D}_i f.$$

The following proposition covers discrete integrations by parts and related formulae.

PROPOSITION 2.4. *Let $f \in \mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ and $g \in \mathbb{R}^{\overline{\mathfrak{M}}^i}$. For $i \in [\![1,d]\!]$ we have*

$$\iint_\Omega f(\bar{D}_i g) = -\iint_\Omega (D_i f) g + \int_{\Omega_i} (f_{N_i+1} g_{N_i+\frac12} - f_0 g_{\frac12}),$$

$$\iint_\Omega f\overline{g}^i = \iint_\Omega \tilde{f}^i g - \frac{h_i}{2}\int_{\Omega_i} (f_{N_i+1} g_{N_i+\frac12} + f_0 g_{\frac12}).$$

LEMMA 2.5. *Let $i \in [\![1,d]\!]$ and $v \in \mathbb{R}^{\mathfrak{M}\cup\partial\mathfrak{M}}$ (resp. $\mathbb{R}^{\overline{\mathfrak{M}}^j\cup\partial\overline{\mathfrak{M}}^j}$ for $j \neq i$) be such that $v_{|\partial_i\Omega} = 0$. Then $\iint_\Omega v = \iint_\Omega \tilde{v}^i$.*

LEMMA 2.6. *Let f be a smooth function defined in a neighborhood of $\overline{\Omega}$. For $i \in [\![1,d]\!]$ we have*

$$\tau_i^\pm f = f \pm \frac{h_i}{2}\int_0^1 \partial_i f(.\pm\sigma \boldsymbol{h}_i/2)\, d\sigma, \quad \mathsf{A}_i^\ell f = f + C_\ell h_i^2 \int_{-1}^1 (1-|\sigma|)\, \partial_i^2 f(.+l_\ell \sigma \boldsymbol{h}_i)\, d\sigma,$$

$$\mathsf{D}_i^\ell f = \partial_i^\ell f + C'_\ell h_i^2 \int_{-1}^1 (1-|\sigma|)^{\ell+1}\, \partial_i^{\ell+2} f(.+l_\ell\sigma\boldsymbol{h}_i)\, d\sigma, \quad \ell = 1,2, \quad l_1 = \frac12,\ l_2 = 1,$$

with $\boldsymbol{h}_i = h_i \boldsymbol{e}_i$.

For $i,j \in [\![1,d]\!]$, $i \neq j$, we have

$$\begin{aligned}\mathsf{D}_i\mathsf{D}_j f = \partial_{ij}^2 f &+ C''\frac{|\boldsymbol{h}_{ij}^+|^4}{h_i h_j}\int_{-1}^1 (1-|\sigma|)^3 f^{(4)}(.+\sigma\boldsymbol{h}_{ij}^+/2;\boldsymbol{\eta}^+,\dots,\boldsymbol{\eta}^+)\, d\sigma\\ &+ C'''\frac{|\boldsymbol{h}_{ij}^+|^4}{h_i h_j}\int_{-1}^1 (1-|\sigma|)^3 f^{(4)}(\boldsymbol{x}+\sigma\boldsymbol{h}_{ij}^-/2;\boldsymbol{\eta}^-,\dots,\boldsymbol{\eta}^-)\, d\sigma,\end{aligned}$$

with $\boldsymbol{h}_{ij}^\pm = h_i\boldsymbol{e}_i \pm h_j\boldsymbol{e}_j$ and $\boldsymbol{\eta}^\pm = \frac{1}{|\boldsymbol{h}_{ij}^\pm|}(\boldsymbol{h}_{ij}^\pm)$.

Note that $\frac{|\boldsymbol{h}_{ij}^+|^4}{h_i h_j} = \mathcal{O}(h^2)$ by (1.4), for $i,j \in [\![1,d]\!]$, $j \neq i$.

2.2. Calculus results related to the weight functions. We now present some technical lemmata related to discrete operations performed on the Carleman weight functions ρ and $r = \rho^{-1}$, as defined in Section 1.2.1. The positive parameters τ and h will be large and small respectively and we are particularly interested in the dependence on τ, h and λ in the following basic estimates.

We assume $\tau \geq 1$ and $\lambda \geq 1$.

LEMMA 2.7. *Let α and β be multi-indices in the x variable. We have*

$$\begin{aligned}\partial^\beta(r\partial^\alpha\rho) =&|\alpha|^{|\beta|}(-s\phi)^{|\alpha|}\lambda^{|\alpha+\beta|}(\nabla\psi)^{\alpha+\beta}\\ &+|\alpha||\beta|(s\phi)^{|\alpha|}\lambda^{|\alpha+\beta|-1}\mathcal{O}(1) + s^{|\alpha|-1}|\alpha|(|\alpha|-1)\mathcal{O}_\lambda(1) = \mathcal{O}_\lambda(s^{|\alpha|}).\end{aligned} \tag{2.1}$$

Let $\sigma \in [-1,1]$ and $i \in [\![1,d]\!]$. We have

$$\partial^\beta (r(t,.)(\partial^\alpha \rho)(t,.+\sigma \boldsymbol{h}_i)) = \mathcal{O}_\lambda(s^{|\alpha|}(1+(sh)^{|\beta|}))\ e^{\mathcal{O}_\lambda(sh)}. \tag{2.2}$$

Provided $\tau h(\max_{[0,T]}\theta) \leq \mathfrak{K}$ we have $\partial^\beta(r(t,.)(\partial^\alpha\rho)(t,.+\sigma\boldsymbol{h}_i)) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|})$. The same expressions hold with r and ρ interchanged and with s changed into $-s$.

A proof is given in [BHL10a, proof of Lemma 3.7] in the time independent case. This proof applies to the time-dependent case by noting that the condition $\tau h(\max_{[0,T]}\theta) \leq \mathfrak{K}$ implies that $s(t)h \leq \mathfrak{K}$ for all $t \in [0,T]$.

Lemma 2.8. *Let α be a multi-index in the x variable. We have*

$$\partial_t(r\partial^\alpha\rho) = s^{|\alpha|}T\theta\mathcal{O}_\lambda(1).$$

Proof. We proceed by induction on $|\alpha|$. The result holds for $|\alpha| = 0$, and we assume it also holds in the case $|\alpha| = n$. In the case $|\alpha| = n+1$, with $|\alpha| \geq 1$, we write $\alpha = \alpha' + \alpha''$ with $|\alpha''| = 1$ and we have

$$r\partial^\alpha\rho = -sr\partial^{\alpha'}((\partial^{\alpha''}\varphi)\,\rho)) = -\Big(\sum_{\delta'+\delta''=\alpha'} \tbinom{\alpha'}{\delta'}\,(\partial^{\delta''+\alpha''}\varphi)\, sr\partial^{\delta'}\rho\Big).$$

Next we write

$$|\partial_t(sr\partial^{\delta'}\rho)| \leq |(\partial_t s)r\partial^{\delta'}\rho| + |s\partial_t(r\partial^{\delta'}\rho)| \leq sT\theta s^{|\delta'|}$$

by (1.10) and Lemma 2.7 for the estimation of the first term and by the inductive hypothesis for the second term. We then conclude as $|\delta'|+1 \leq |\alpha'|+1 = |\alpha| = n+1$. ∎

With the Leibniz formula we have the following estimate.

Corollary 2.9. *Let α, α', and β be multi-indices in the x variable. We have*

$$\begin{aligned}\partial^\beta(r^2(\partial^\alpha\rho)\partial^{\alpha'}\rho) =& |\alpha+\alpha'|^{|\beta|}(-s\phi)^{|\alpha+\alpha'|}\lambda^{|\alpha+\alpha'+\beta|}(\nabla\psi)^{\alpha+\alpha'+\beta}\\ &+ |\beta||\alpha+\alpha'|(s\phi)^{|\alpha+\alpha'|}\lambda^{|\alpha+\alpha'+\beta|-1}\mathcal{O}(1)\\ &+ s^{|\alpha+\alpha'|-1}(|\alpha|(|\alpha|-1)+|\alpha'|(|\alpha'|-1))\mathcal{O}_\lambda(1) = \mathcal{O}_\lambda(s^{|\alpha+\alpha'|}).\end{aligned}$$

The proofs of the following properties can be found in Appendix A of [BHL10b] (except the one of Proposition 2.14 which is specific to the parabolic case).

Proposition 2.10. *Let α be a multi-index in the x variable. Let $i,j \in [\![1,d]\!]$, provided $\tau h \max_{[0,T]}\theta \leq \mathfrak{K}$, we have*

$$\begin{aligned} r\tau_i^{\pm}\partial^\alpha\rho &= r\partial^\alpha\rho + s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}(sh) = s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\ r\mathsf{A}_i^k\partial^\alpha\rho &= r\partial^\alpha\rho + s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad k=1,2,\\ r\mathsf{A}_i^k\mathsf{D}_i\rho &= r\partial_x\rho + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad k=0,1,\\ r\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\rho &= r\partial_i^{k_i}\partial_j^{k_j}\rho + s^2\,\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad k_i+k_j \leq 2.\end{aligned}$$

The same estimates hold with ρ and r interchanged.

Lemma 2.11. *Let α and β be multi-indices in the x variable and $k \in \mathbb{N}$. Let $i,j \in [\![1,d]\!]$, provided $\tau h\max_{[0,T]}\theta \leq \mathfrak{K}$, we have*

$$\begin{aligned}&\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(\partial^\beta(r\partial^\alpha\rho)) = \partial_i^{k_i}\partial_j^{k_j}\partial^\beta(r\partial^\alpha\rho) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|}), \quad k_i+k_j\leq 2,\\ &\mathsf{A}_i^k\partial^\beta(r\partial^\alpha\rho) = \partial^\beta(r\partial^\alpha\rho) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|}).\end{aligned}$$

Let $\sigma \in [-1,1]$, *we have* $\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta(r(t,.)\partial^\alpha\rho(t,.+\sigma\boldsymbol{h}_i)) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|})$, *for* $k_i+k_j \leq 2$. *The same estimates hold with* r *and* ρ *interchanged.*

LEMMA 2.12. *Let* α, α' *and* β *be multi-indices in the* x *variable and* $k \in \mathbb{N}$. *Let* $i,j \in [\![1,d]\!]$, *provided* $\tau h \max_{[0,T]}\theta \leq \mathfrak{K}$, *we have*

$$\begin{aligned}
&\mathsf{A}_i^k\partial^\beta(r^2(\partial^\alpha\rho)\partial^{\alpha'}\rho) = \partial^\beta(r^2(\partial^\alpha\rho)\partial^{\alpha'}\rho) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\alpha'|}) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\alpha'|}),\\
&\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta(r^2(\partial^\alpha\rho)\partial^{\alpha'}\rho) = \partial_i^{k_i}\partial_j^{k_j}(\partial^\beta(r^2(\partial^\alpha\rho)\partial^{\alpha'}\rho)) + h^2\mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\alpha'|})\\
&\qquad\qquad = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\alpha'|}), \quad k_i+k_j \leq 2.
\end{aligned}$$

Let $\sigma,\sigma' \in [-1,1]$. *We have*

$$\begin{aligned}
&\mathsf{A}_i^k\partial^\beta\Big(r(t,.)^2(\partial^\alpha\rho(t,.+\sigma\boldsymbol{h}_i))\partial^{\alpha'}\rho(y,.+\sigma'\boldsymbol{h}_j)\Big) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\alpha'|}),\\
&\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta\Big(r(t,.)^2(\partial^\alpha\rho(t,.+\sigma\boldsymbol{h}_i))\partial^{\alpha'}\rho(t,.+\sigma'\boldsymbol{h}_j)\Big) = \mathcal{O}_{\lambda,\mathfrak{K}}(s^{|\alpha|+|\alpha'|}), \quad k_i+k_j \leq 2.
\end{aligned}$$

The same estimates hold with r *and* ρ *interchanged.*

PROPOSITION 2.13. *Let* α *be a multi-index in the* x *variable and* $k \in \mathbb{N}$. *Let* $i,j \in [\![1,d]\!]$, *provided* $sh \leq \mathfrak{K}$, *we have*

$$\begin{aligned}
&\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\mathsf{A}_i^k\partial^\alpha(r\widehat{\mathsf{D}_i\rho}^{\,i}) = \partial_i^{k_i}\partial_j^{k_j}\partial^\alpha(r\partial_x\rho) + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
&\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(r\mathsf{D}_i^2\rho) = \partial_i^{k_i}\partial_j^{k_j}(r\partial_i^2\rho) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
&r\mathsf{A}_i^2\rho = 1 + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2, \quad \mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}(r\mathsf{A}_i^2\rho) = \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2).
\end{aligned}$$

The same estimates hold with r *and* ρ *interchanged.*

PROPOSITION 2.14. *Provided* $\tau h \max_{[0,T]}\theta \leq \mathfrak{K}$, *and* σ *is bounded,we have*

$$\begin{aligned}
&\partial_t(r(.,\boldsymbol{x})(\partial^\alpha\rho)(.,\boldsymbol{x}+\sigma\boldsymbol{h}_i)) = Ts^{|\alpha|}\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
&\partial_t(r\mathsf{A}_i^2\rho) = T(sh)^2\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
&\partial_t(r\mathsf{D}_i^2\rho) = Ts^2\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}$$

The same estimates hold with r *and* ρ *interchanged.*

Proof. We set $\nu(t,\boldsymbol{x},\sigma h_i) := r(t,\boldsymbol{x})\rho(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i)$ and simply have $\nu(t,\boldsymbol{x},\sigma h_i) = e^{s(t)(\varphi(\boldsymbol{x})-\varphi(\boldsymbol{x}+\sigma\boldsymbol{h}_i))} = e^{\mathcal{O}_\lambda(s(t)h)} = \mathcal{O}_{\lambda,\mathfrak{K}}(1)$, by a first-order Taylor formula. We have

$$\partial_t\nu(t,\boldsymbol{x},\sigma h_i) = (\partial_t s)\big(\varphi(\boldsymbol{x})-\varphi(\boldsymbol{x}+\sigma\boldsymbol{h}_i)\big)\nu(t,\boldsymbol{x},\sigma h_i) = \tau\theta'(t)h_i\mathcal{O}_{\lambda,\mathfrak{K}}(1) = T\theta(t)\mathcal{O}_{\lambda,\mathfrak{K}}(s(t)h),$$

by (1.10).

Next, we write $r(t,\boldsymbol{x})(\partial^\alpha\rho)(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i) = \nu(t,\boldsymbol{x},\sigma h_i)\mu_\alpha(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i)$, where we have set $\mu_\alpha = r\partial_i^\alpha\rho$. We have

$$\partial_t\mu_\alpha = Ts^{|\alpha|}\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$

by Lemma 2.8. This yields

$$\begin{aligned}
\partial_t(r(t,\boldsymbol{x})(\partial^\alpha\rho)(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i)) &= (\partial_t\nu(t,\boldsymbol{x},\sigma h_i))\,\mu_\alpha(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i) + \nu(t,\boldsymbol{x},\sigma h_i)\partial_t\mu_\alpha(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i)\\
&= Ts^{|\alpha|}\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}$$

Next we write

$$r(t,\boldsymbol{x})\mathsf{A}_i^2\rho(t,\boldsymbol{x}) = 1 + Ch_i^2 \int_{-1}^{1} (1-|\sigma|) r(t,\boldsymbol{x})\partial_i^2\rho(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i)\, d\sigma,$$

which gives

$$\partial_t\big(r(t,\boldsymbol{x})\mathsf{A}_i^2\rho(t,\boldsymbol{x})\big) = Ch_i^2 \int_{-1}^{1} (1-|\sigma|)\partial_t\big(r(t,\boldsymbol{x})\partial_i^2\rho(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i)\big)\, d\sigma,$$

and the second result follows. Similarly, we write

$$r(t,\boldsymbol{x})\mathsf{D}_i^2\rho(t,\boldsymbol{x}) = r(t,\boldsymbol{x})\partial_i^2\rho(t,\boldsymbol{x}) + Ch_i^4 \int_{-1}^{1} (1-|\sigma|)^3 r(t,\boldsymbol{x})\partial_i^4\rho(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i)\, d\sigma,$$

which gives

$$\partial_t\big(r(t,\boldsymbol{x})\mathsf{D}_i^2\rho(t,\boldsymbol{x})\big) = \partial_t\big(r(t,\boldsymbol{x})\partial_i^2\rho(t,\boldsymbol{x})\big) + Ch_i^4 \int_{-1}^{1} (1-|\sigma|)^3\partial_t\big(r(t,\boldsymbol{x})\partial_i^4\rho(t,\boldsymbol{x}+\sigma\boldsymbol{h}_i)\big)\, d\sigma,$$

and the third estimate follows by using Lemma 2.8 and the first result of the present Proposition. ◻

Proposition 2.15. *Let α, β be multi-indices in the x variable, $i,j \in [\![1,d]\!]$ and $k_i, k_i', k_j, k_j' \in \mathbb{N}$. For $k_i + k_j \leq 2$, provided $sh \leq \mathfrak{K}$ we have*

$$\begin{aligned}
\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta\big(r^2(\partial^\alpha\rho)\widehat{\mathsf{D}_i\rho}^{\,i}\big) &= \partial_i^{k_i}\partial_j^{k_j}\partial^\beta\big(r^2(\partial^\alpha\rho)\partial_i\rho\big) + s^{|\alpha|+1}\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)\\
&= s^{|\alpha|+1}\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta\big(r^2(\partial^\alpha\rho)\mathsf{A}_i^2\rho\big) &= \partial_i^{k_i}\partial_j^{k_j}\partial^\beta\big(r(\partial^\alpha\rho)\big) + s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)\\
&= s^{|\alpha|}\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\beta\big(r^2(\partial^\alpha\rho)\mathsf{D}_i^2\rho\big) &= \partial_i^{k_i}\partial_j^{k_j}\partial^\beta\big(r^2(\partial^\alpha\rho)\partial_i^2\rho\big) + s^{|\alpha|+2}\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)\\
&= s^{|\alpha|+2}\mathcal{O}_{\lambda,\mathfrak{K}}(1),
\end{aligned}$$

and we have

$$\begin{aligned}
\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\alpha\big(r^2\widehat{\mathsf{D}_i\rho}^{\,i}\mathsf{D}_j^2\rho\big) &= \partial_i^{k_i}\partial_j^{k_j}\partial^\alpha\big(r^2(\partial_i\rho)\partial_j^2\rho\big) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s^3\mathcal{O}_{\lambda,\mathfrak{K}}(1),\\
\mathsf{A}_i^{k_i'}\mathsf{A}_j^{k_j'}\mathsf{D}_i^{k_i}\mathsf{D}_j^{k_j}\partial^\alpha\big(r^2\widehat{\mathsf{D}_i\rho}^{\,i}\mathsf{A}_j^2\rho\big) &= \partial_i^{k_i}\partial_j^{k_j}\partial^\alpha\big(r\partial_i\rho\big) + s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2) = s\mathcal{O}_{\lambda,\mathfrak{K}}(1).
\end{aligned}$$

3. Proof of the Carleman estimate.

Here we prove the result of Theorem 1.3. We shall carry out the proof for the operator $\mathcal{P}^{\mathfrak{M}} = \mathcal{P}_-^{\mathfrak{M}} = \partial_t - \mathcal{A}^{\mathfrak{M}}$. The proof is the same for $\mathcal{P}_+^{\mathfrak{M}} = \partial_t + \mathcal{A}^{\mathfrak{M}}$ (change t in to $T-t$).

We set $f := P^{\mathfrak{M}}u$. At first, we shall work with the function $v = ru$, *i.e.*, $u = \rho v$, that satisfies

$$r\Big(\partial_t(\rho v) + \sum_{i\in[\![1,d]\!]} \bar{D}_i\gamma_i D_i(\rho v)\Big) = rf. \tag{3.1}$$

We have

$$r\partial_t(\rho v) = \partial_t v + r(\partial_t\rho)v = \partial_t v - \tau(\partial_t\theta)\varphi v.$$

Following [FI96], we write

$$g = Av + Bv,$$

where $Av = A_1v + A_2v + A_3v$, $Bv = B_1v + B_2v + B_3v$ with

$$\begin{aligned} g = rf &- \sum_{i\in[\![1,d]\!]} \frac{h_i}{4} r\overline{D_i\rho}^i (\bar{D}_i\gamma_i)(\bar{\tau_i}^+ D_iv - \bar{\tau_i}^- D_iv) \\ &- \sum_{i\in[\![1,d]\!]} \frac{h_i^2}{4} (\bar{D}_i\gamma_i) r(\bar{D}_iD_i\rho)\overline{D_iv}^i - h_i \sum_{i\in[\![1,d]\!]} \mathcal{O}(1) r\overline{D_i\rho}^i\, \overline{D_iv}^i \\ &- \sum_{i\in[\![1,d]\!]} \Big(r(\bar{D}_i\gamma_i)\overline{D_i\rho}^i + h_i\mathcal{O}(1) r(\bar{D}_iD_i\rho)\Big) \overline{\tilde{v}}^i - 2s(\Delta_\gamma\phi)v, \end{aligned}$$

and

$$\begin{aligned} &A_1v = \sum_{i\in[\![1,d]\!]} r\overline{\tilde{\rho}}^i\, \bar{D}_i(\gamma_i D_iv), \quad A_2v = \sum_{i\in[\![1,d]\!]} \gamma_i r(\bar{D}_iD_i\rho)\,\overline{\tilde{v}}^i, \\ &A_3v = -\tau(\partial_t\theta)\varphi v, \\ &B_1v = 2\sum_{i\in[\![1,d]\!]} \gamma_i r\overline{D_i\rho}^i\, \overline{D_iv}^i, \quad B_2v = -2s(\Delta_\gamma\phi)v, \quad B_3v = \partial_t v. \end{aligned}$$

An explanation for the introduction of this additional term B_2v is provided in [LL11].

Equation (3.1) now reads $Av + Bv = g$ and we write

$$\|Av\|^2_{L^2(Q)} + \|Bv\|^2_{L^2(Q)} + 2\,(Av, Bv)_{L^2(Q)} = \|g\|^2_{L^2(Q)}. \tag{3.2}$$

We shall need the following estimation of $\|g\|_{L^2(Q)}$. The proof can be adapted from Lemma 4.2 and its proof in [BHL10a] (the time dependency of the present weight function does not affect the argument and computations of the proof).

Lemma 3.1 (Estimate of the r.h.s.). *For $\tau h(\max_{[0,T]}\theta) \leq \mathfrak{K}$ we have*

$$\|g\|^2_{L^2(Q)} \leq C_{\lambda,\mathfrak{K}} \Big(\|rf\|^2_{L^2(Q)} + \|sv\|^2_{L^2(Q)} + h^2\|s\Upsilon v\|^2_{L^2(Q)} \Big). \tag{3.3}$$

Most of the remaining of the proof will be dedicated to computing the inner-product $(Av, Bv)_{L^2(Q)}$. Developing the inner-product $(Av, Bv)_{L^2(Q)}$, we set $I_{ij} = (A_iv, B_jv)_{L^2(Q)}$.

Lemma 3.2 (Estimate of I_{11} (Lemma 3.3 in [BHL10b])). *For $\tau h(\max_{[0,T]}\theta) \leq \mathfrak{K}$, the term I_{11} can be estimated from below in the following way*

$$I_{11} \geq -\lambda^2 \|(s\phi)^{\frac{1}{2}} |\nabla_\gamma\psi| \Upsilon_\gamma v\|^2_{L^2(\Omega)} + Y_{11} - X_{11} - W_{11} - J_{11},$$

with

$$\begin{aligned} Y_{11} = \sum_{i\in[\![1,d]\!]} \iint_{Q_i} \Big(&\big(({\gamma_i}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2))\, r\overline{D_i\rho}^i\big)_{N_i+1} \big(D_iv\big)^2_{N_i+\frac{1}{2}} \\ &- \big(({\gamma_i}^2 + \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2))\, r\overline{D_i\rho}^i\big)_0 \big(D_iv\big)^2_{\frac{1}{2}} \Big) dt, \end{aligned}$$

and

$$X_{11} = \sum_{i\in[\![1,d]\!]} \iint_Q \nu_{11,i} \big(D_iv\big)^2\, dt + \sum_{i\in[\![1,d]\!]} \iint_Q \overline{\nu}_{11,i} \big(\overline{D_iv}^i\big)^2\, dt,$$

with $\nu_{11,i}$ *and* $\overline{\nu}_{11,i}$ *of the form* $s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$ *and*

$$W_{11} = \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iint_Q \gamma_{11,ij}\big(D_iD_jv\big)^2\,dt + \sum_{i\in[\![1,d]\!]} \iint_Q \gamma_{11,ii}\big(\bar{D}_iD_iv\big)^2\,dt,$$

with $\gamma_{11,ij}$ *and* $\gamma_{11,ii}$ *of the form* $h^2\big(s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big)$ *and*

$$J_{11} = \sum_{i\in[\![1,d]\!]} \iint_{Q_i} \Big((\delta^{(2)}_{11,i})_{N_i+\frac12}\big(D_iv\big)^2_{N_i+\frac12} + (\delta^{(2)}_{11,i})_{\frac12}\big(D_iv\big)^2_{\frac12} \Big)\,dt,$$

with $\delta^{(2)}_{11,i} = sh_i\lambda\phi\mathcal{O}(1) + sh_i\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$*.*

Lemma 3.3 (Estimate of I_{12} (Lemma 3.4 in [BHL10b])). *For* $\tau h(\max_{[0,T]}\theta) \leq \mathfrak{K}$*, the term* I_{12} *can be estimated from below in the following way*

$$I_{12} \geq 2\lambda^2\|(s\phi)^{\frac12}|\nabla_\gamma\psi|\Upsilon_\gamma v\|^2_{L^2(Q)} - X_{12},$$

with

$$X_{12} = \sum_{i\in[\![1,d]\!]} \iint_Q \nu_{12,i}\big(D_iv\big)^2\,dt + \iint_Q \mu_{12}v^2\,dt,$$

where $\mu_{12} = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1)$*, and* $\nu_{12,i} = s\lambda\phi\mathcal{O}(1) + s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$*.*

Lemma 3.4. *There exists* $\varepsilon_1(\lambda) > 0$ *such that, for* $0 < \tau h(\max_{[0,T]}\theta) \leq \varepsilon_1(\lambda)$*, the term* I_{13} *can be estimated from below in the following way*

$$I_{13} \geq -C_{\lambda,\mathfrak{K}} \int_\Omega |\Upsilon v(T)|^2 - X_{13}$$

with $C > 0$ *and*

$$X_{13} = \iint_Q \big(s(sh) + T(sh)^2\theta\big)\mathcal{O}_{\lambda,\mathfrak{K}}(1)|\Upsilon v|^2\,dt + \iint_Q s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big(\partial_t v\big)^2\,dt.$$

For a proof see Appendix A.

Lemma 3.5 (Estimate of I_{21} (Lemma 3.5 in [BHL10b])). *For* $\tau h(\max_{[0,T]}\theta) \leq \mathfrak{K}$*, the term* I_{21} *can be estimated from below in the following way*

$$I_{21} \geq 3\lambda^4\|(s\phi)^{\frac32}\big(\nabla_\gamma\psi\big)^2 v\|^2_{L^2(Q)} + Y_{21} - W_{21} - X_{21},$$

with

$$\begin{aligned}
Y_{21} = &\sum_{i\in[\![1,d]\!]} \int_{Q_i} \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)(r\overline{D_i\rho}^{\,i})_0\big(D_iv\big)^2_{\frac12}\,dt\\
&+ \sum_{i\in[\![1,d]\!]} \int_{Q_i} \mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2)(r\overline{D_i\rho}^{\,i})_{N_x+1}\big(D_iv\big)^2_{N_x+\frac12}\,dt,\\
W_{21} = &\sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iint_Q \gamma_{21,ij}\big(D_iD_jv\big)^2\,dt, \qquad X_{21} = \iint_Q \mu_{21}v^2\,dt + \sum_{i\in[\![1,d]\!]} \iint_Q \nu_{21,i}\big(D_iv\big)^2\,dt,
\end{aligned}$$

where

$$\begin{aligned}
&\gamma_{21,ij} = h\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2), \quad \mu_{21} = (s\lambda\phi)^3\mathcal{O}(1) + s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda,\mathfrak{K}}(sh),\\
&\nu_{21,i} = s\mathcal{O}_{\lambda,\mathfrak{K}}((sh)^2).
\end{aligned}$$

LEMMA 3.6 (Estimate of I_{22} (Lemma 3.6 in [BHL10b])). *For $\tau h(\max_{[0,T]} \theta) \leq \mathfrak{K}$, the term I_{22} is of the following form*

$$I_{22} = -2\lambda^4 \|(s\phi)^{\frac{3}{2}} |\nabla_\gamma \psi|^2 v\|^2_{L^2(Q)} - X_{22},$$

with

$$X_{22} = \iint\limits_Q \mu_{22} v^2 \, dt + \sum_{i \in [\![1,d]\!]} \iint\limits_Q \nu_{22,i} \big(D_i v\big)^2 \, dt$$

where $\mu_{22} = (s\lambda\phi)^3 \mathcal{O}(1) + s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1) + s^3 \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, and $\nu_{22,i} = s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$.

LEMMA 3.7. *For $\tau h(\max_{[0,T]} \theta) \leq \mathfrak{K}$, the term I_{23} can be estimated from below in the following way*

$$I_{23} \geq \int\limits_\Omega s^2 \big(\mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2_{|t=T} + \mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2_{|t=0}\big) - X_{23},$$

with

$$\begin{aligned} X_{23} = \iint\limits_Q T s^2 \theta \mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2 \, dt + \iint\limits_Q (sh)^2 s^{-1} \mathcal{O}_{\lambda,\mathfrak{K}}(1) \big(\partial_t v\big)^2 \, dt \\ + \sum_{i \in [\![1,d]\!]} \iint\limits_Q (sh)^2 s \mathcal{O}_{\lambda,\mathfrak{K}}(1) \big(D_i v\big)^2 \, dt. \end{aligned}$$

For a proof see Appendix A.

LEMMA 3.8. *For $\tau h(\max_{[0,T]} \theta) \leq \mathfrak{K}$, the term I_{31} is of the following form*

$$I_{31} = -X_{31} = \iint\limits_Q T s^2 \theta \mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2 \, dt + \sum_{i \in [\![1,d]\!]} \iint\limits_Q T (sh)^2 \theta \mathcal{O}_{\lambda,\mathfrak{K}}(1) \big(D_i v\big)^2 \, dt.$$

For a proof see Appendix A.

With (1.10) we may write

$$I_{32} = -X_{32} = 2 \iint\limits_Q \tau s (\partial_t \theta) \varphi (\Delta_\gamma \phi) v^2 = \iint\limits_Q T s^2 \theta \mathcal{O}_{\lambda,\mathfrak{K}}(1) v^2. \tag{3.4}$$

LEMMA 3.9. *For $\tau h(\max_{[0,T]} \theta) \leq \mathfrak{K}$, the term I_{33} can be estimated from below in the following way*

$$I_{33} \geq -X_{33} = \frac{1}{2} \tau \iint\limits_Q \varphi (\partial_t^2 \theta) v^2 \, dt.$$

For a proof see Appendix A.

Continuation of the proof of Theorem 1.3. Collecting the inequalities we have obtained in the previous lemmata, from (3.2) we obtain, for $0 < \tau h(\max_{[0,T]} \theta) \leq \varepsilon_1(\lambda)$,

$$\begin{aligned} \|Av\|^2_{L^2(Q)} + \|Bv\|^2_{L^2(Q)} + 2\lambda^4 \|(s\phi)^{\frac{3}{2}} |\nabla_\gamma \psi|^2 v\|^2_{L^2(Q)} + 2\lambda^2 \|(s\phi)^{\frac{1}{2}} |\nabla_\gamma \psi| \Upsilon_\gamma v\|^2_{L^2(Q)} + 2Y \\ \leq C_{\lambda,\mathfrak{K}} \Big(\|rf\|^2_{L^2(Q)} + \int\limits_\Omega s^2 \big(v^2_{|t=T} + v^2_{|t=0}\big) + \int\limits_\Omega |\Upsilon v|^2_{|t=T} \Big) + 2X + 2W + 2J, \end{aligned} \tag{3.5}$$

where $C > 0$, $Y = Y_{11} + Y_{21}$,

$$X = X_{11} + X_{12} + X_{13} + X_{21} + X_{22} + X_{23} + X_{31} + X_{32} + X_{33} + C_{\lambda,\mathfrak{K}}\Big(\|sv\|^2_{L^2(Q)} + h^2 \sum_{i\in[\![1,d]\!]} \|sD_i v\|^2_{L^2(Q)}\Big),$$

$W = W_{11} + W_{21}$, and $J = J_{11}$. With the following lemma, we may in fact ignore the term Y. This uses the property (1.11) of the weight functions.

Lemma 3.10 (Lemma 3.7 in [BHL10b]). *For all λ there exists $0 < \varepsilon_2(\lambda) < \varepsilon_1(\lambda)$ such that for $0 < \tau h(\max_{[0,T]} \theta) \le \varepsilon_2(\lambda)$, we have $Y \ge 0$.*

Recalling that $|\nabla_\gamma \psi| \ge C > 0$ in $\Omega \setminus \omega_0$ we may thus write

$$\|Av\|^2_{L^2(Q)} + \|Bv\|^2_{L^2(Q)} + \lambda^4 \|(s\phi)^{\frac{3}{2}} v\|^2_{L^2(Q)} + \lambda^2 \|(s\phi)^{\frac{1}{2}} \Upsilon_\gamma v\|^2_{L^2(Q)} \le C_{\lambda,\mathfrak{K}}\Big(\|rf\|^2_{L^2(Q)} + \lambda^4 \|(s\phi)^{\frac{3}{2}} v\|^2_{L^2((0,T)\times\omega_0)} + \lambda^2 \|(s\phi)^{\frac{1}{2}} \Upsilon_\gamma v\|^2_{L^2((0,T)\times\omega_0)} + \int_\Omega s^2 \big(v^2_{|t=T} + v^2_{|t=0}\big) + \int_\Omega |\Upsilon v|^2_{|t=T}\Big) + 2X + 2W + 2J. \quad (3.6)$$

Lemma 3.11. *We have*

$$\lambda^2 \|(s\phi)^{\frac{1}{2}} \Upsilon_\gamma v\|^2_{L^2(Q)} \ge \nu(h,\lambda) + CH - \tilde{X} - \tilde{W},$$

where $\nu(h,\lambda) \ge 0$ for $0 < h \le h_1(\lambda)$ for some $h_1(\lambda)$ sufficiently small and

$$H = \lambda^2 \sum_{i\in[\![1,d]\!]} \iint_Q s\phi\big(\overline{D_i v}^i\big)^2\, dt + \lambda^2 h^2 \sum_{i\in[\![1,d]\!]} \iint_Q s\phi\big(\bar{D}_i D_i v\big)^2\, dt + \lambda^2 h^2 \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iint_Q s\phi\big(D_i D_j v\big)^2\, dt,$$

$$\tilde{X} = h^2 \Big(\sum_{i\in[\![1,d]\!]} \iint_Q s\mathcal{O}_\lambda(1)\big(D_i v\big)^2\, dt + \sum_{i\in[\![1,d]\!]} \iint_Q s\mathcal{O}_\lambda(1)\big(\overline{D_i v}^i\big)^2\, dt\Big),$$

and

$$\tilde{W} = h^4 \Big(\sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iint_Q s\mathcal{O}_\lambda(1)\big(D_i D_j v\big)^2\, dt + \sum_{i\in[\![1,d]\!]} \iint_Q s\mathcal{O}_\lambda(1)\big(\bar{D}_i D_i v\big)^2\, dt\Big).$$

For a proof see Appendix A.

If we choose $\lambda_1 \ge 1$ sufficiently large, then for $\lambda = \lambda_1$ (fixed for the rest of the proof) and $0 < \tau h(\max_{[0,T]} \theta) \le \varepsilon_3(\lambda) = \min(\varepsilon_1(\lambda_1), \varepsilon_2(\lambda_1))$ and $0 < h \le h_1(\lambda_1)$, from (3.6) and Lemma 3.11, we obtain

$$\|Av\|^2_{L^2(Q)} + \|Bv\|^2_{L^2(Q)} + \|s^{\frac{3}{2}} v\|^2_{L^2(Q)} + \|s^{\frac{1}{2}} \Upsilon v\|^2_{L^2(Q)} + \underline{H} \le C_{\lambda_1,\mathfrak{K}}\Big(\|rf\|^2_{L^2(Q)} + \|s^{\frac{3}{2}} v\|^2_{L^2((0,T)\times\omega_0)} + \|s^{\frac{1}{2}} \Upsilon v\|^2_{L^2((0,T)\times\omega_0)} + \int_\Omega s^2 \big(v^2_{|t=T} + v^2_{|t=0}\big) + \int_\Omega |\Upsilon v|^2_{|t=T}\Big) + \underline{X} + \underline{W} + \underline{J}, \quad (3.7)$$

where

$$\underline{H} = \sum_{i\in[\![1,d]\!]} \|s^{\frac{1}{2}} \overline{D_i v}^i\|^2_{L^2(Q)} + h^2\Big(\sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \|s^{\frac{1}{2}} D_i D_j v\|^2_{L^2(Q)} + \sum_{i\in[\![1,d]\!]} \|s^{\frac{1}{2}} \bar{D}_i D_i v\|^2_{L^2(Q)}\Big), \tag{3.8}$$

$$\begin{aligned}\underline{X} = \iint\limits_Q \mu_1 v^2\, dt + \sum_{i\in[\![1,d]\!]} \iint\limits_Q \nu_{1,i}\big(D_i v\big)^2\, dt + \sum_{i\in[\![1,d]\!]} \iint\limits_Q \overline{\nu}_{1,i}\big(\overline{D_i v}^i\big)^2\, dt\\ + X_{13} + X_{23} + X_{31} + X_{32} + X_{33},\end{aligned}$$

with $\mu_1 = s^2\mathcal{O}_{\lambda_1,\mathfrak{K}}(1) + s^3\mathcal{O}_{\lambda_1,\mathfrak{K}}(sh)$ and $\nu_{1,i}$, $\overline{\nu}_{1,i}$, of the form $s\mathcal{O}_{\lambda_1,\mathfrak{K}}(sh)$, and where

$$\underline{W} = \sum_{\substack{i,j\in[\![1,d]\!]\\ i\neq j}} \iint\limits_Q \gamma_{1,ij}\big(D_i D_j v\big)^2\, dt + \sum_{i\in[\![1,d]\!]} \iint\limits_Q \gamma_{1,ii}\big(\bar{D}_i D_i v\big)^2\, dt,$$

where $\gamma_{1,ij}$ and $\gamma_{1,ii}$ are of the form $sh^2\mathcal{O}_{\lambda_1,\mathfrak{K}}(sh)$, and where

$$\underline{J} = \sum_{i\in[\![1,d]\!]} \iint\limits_{Q_i} \Big((\delta_{1,i})_{N_i+\frac{1}{2}}\big(D_i v\big)^2_{N_i+\frac{1}{2}} + (\delta_{1,i})_{\frac{1}{2}}\big(D_i v\big)^2_{\frac{1}{2}}\Big)\, dt,$$

with $\delta_{1,i} = sh_i\mathcal{O}_{\lambda,\mathfrak{K}}(sh)$. The last term in $\underline{J}$ was obtained by "absorbing" the following term in J_{11}

$$\lambda \sum_{i\in[\![1,d]\!]} \iint\limits_{Q_i} sh_i\Big((\phi)_{N_i+\frac{1}{2}}\mathcal{O}(1)\big(D_i v\big)^2_{N_i+\frac{1}{2}} + (\phi)_{\frac{1}{2}}\mathcal{O}(1)\big(D_i v\big)^2_{\frac{1}{2}}\Big)\, dt,$$

by the volume term

$$\lambda^2 \sum_{i\in[\![1,d]\!]} \iint\limits_Q s\phi\big(D_i v\big)^2\, dt,$$

for λ large.

Observe that

$$1 \le T^2\theta \quad \text{and} \quad |\partial_t^2\theta| \le CT^2\theta^3,$$

We can now choose ε_4 and h_0 sufficiently small, with $0 < \varepsilon_4 \le \varepsilon_3(\lambda_1)$, $0 < h_0 \le h_1(\lambda_1)$, and $\tau_1 \ge 1$ sufficiently large, such that for $\tau \ge \tau_1(T+T^2)$, $0 < h \le h_0$, and $\tau h(\max_{[0,T]}\theta) \le \varepsilon_4$, we obtain

$$\begin{aligned}&\|Av\|^2_{L^2(Q)} + \|Bv\|^2_{L^2(Q)} + \|s^{\frac{3}{2}} v\|^2_{L^2(Q)} + \|s^{\frac{1}{2}} \Upsilon v\|^2_{L^2(Q)} + \underline{H}\\ &\quad \le C_{\lambda_1,\mathfrak{K}}\Big(\|rf\|^2_{L^2(Q)} + \|s^{\frac{3}{2}} v\|^2_{L^2((0,T)\times\omega_0)} + \|s^{\frac{1}{2}}\Upsilon v\|^2_{L^2((0,T)\times\omega_0)}\\ &\qquad + \iint\limits_Q s^{-1}(sh)\big(\partial_t v\big)^2\, dt + h^{-2}\big(\int\limits_\Omega v^2_{|t=0} + \int\limits_\Omega v^2_{|t=T}\big)\Big),\end{aligned} \tag{3.9}$$

where we have used that $\big(D_i v\big)^2 \le Ch^{-2}\big(\big(\tau_i^+ v\big)^2 + \big(\tau_i^- v\big)^2\big)$.

Since $\tau \ge \tau_1(T+T^2)$ then $s(t) \ge \tau_1 > 0$ for any t, we then observe that

$$\begin{aligned}\|s^{-\frac{1}{2}}\partial_t v\|^2_{L^2(Q)} &\le C_{\lambda_1,\mathfrak{K}}\Big(\|s^{-\frac{1}{2}} Bv\|^2_{L^2(Q)} + \|s^{\frac{1}{2}} v\|^2_{L^2(Q)} + \|s^{\frac{1}{2}}\Upsilon v\|^2_{L^2(Q)}\Big)\\ &\le C_{\lambda_1,\tau_1,\mathfrak{K}}\Big(\|Bv\|^2_{L^2(Q)} + \|s^{\frac{3}{2}} v\|^2_{L^2(Q)} + \|s^{\frac{1}{2}}\Upsilon v\|^2_{L^2(Q)}\Big).\end{aligned}$$

With $0 < \varepsilon_0 \leq \varepsilon_4$ sufficiently small and $0 < \tau h(\max_{[0,T]} \theta) \leq \varepsilon_0$ we thus obtain

$$\begin{aligned}\|s^{-\frac{1}{2}} \partial_t v\|^2_{L^2(Q)} + \|s^{\frac{3}{2}} v\|^2_{L^2(Q)} + \|s^{\frac{1}{2}} \Upsilon v\|^2_{L^2(Q)} + \underline{H} \\ \leq C_{\lambda_1,\tau_1,\mathfrak{K}} \Big(\|rf\|^2_{L^2(Q)} + \|s^{\frac{3}{2}} v\|^2_{L^2((0,T)\times\omega_0)} + \|s^{\frac{1}{2}} \Upsilon v\|^2_{L^2((0,T)\times\omega_0)} \\ + h^{-2} \big(\int_\Omega v^2_{|t=0} + \int_\Omega v^2_{|t=T} \big) \Big),\end{aligned}$$

Arguing as at the end of the proof of Theorem 4.1 in [BHL10a] (using Lemma 4.9 therein) for the spatial discrete derivative and as in [FI96] for the time derivative, we obtain

$$\begin{aligned}\tau^3 \|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\|^2_{L^2(Q)} + \tau \sum_{i\in[\![1,d]\!]} \|\theta^{\frac{1}{2}} e^{\tau\theta\varphi} D_i u\|^2_{L^2(Q)} + \tau^{-1} \|\theta^{-\frac{1}{2}} e^{\tau\theta\varphi} \partial_t u\|^2_{L^2(Q)} + \underline{H} \\ \leq C_{\lambda_1,\tau_1,\mathfrak{K}} \Big(\|e^{\tau\theta\varphi} P^{\mathfrak{M}} u\|^2_{L^2(Q)} + \tau^3 \|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\|^2_{L^2((0,T)\times\omega_1)} + \tau \sum_{i\in[\![1,d]\!]} \|\theta^{\frac{1}{2}} e^{\tau\theta\varphi} D_i u\|^2_{L^2((0,T)\times\omega_1)} \\ + h^{-2} \big(|e^{\tau\theta\varphi} u_{|t=0}|^2_{L^2(\Omega)} + |e^{\tau\theta\varphi} u_{|t=T}|^2_{L^2(\Omega)} \big) \Big), \quad (3.10)\end{aligned}$$

with $\omega_0 \Subset \omega_1 \Subset \omega$.

We next remove the volume norms $\tau \|\theta^{\frac{1}{2}} e^{\tau\theta\varphi} D_i u\|^2_{L^2((0,T)\times\omega_1)}$ in the r.h.s. by proceeding as in the proof of Theorem D.2 in [BHL10b]. We obtain

$$\begin{aligned}\tau^3 \|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\|^2_{L^2(Q)} + \tau \sum_{i\in[\![1,d]\!]} \|\theta^{\frac{1}{2}} e^{\tau\theta\varphi} D_i u\|^2_{L^2(Q)} + \tau^{-1} \|\theta^{-\frac{1}{2}} e^{\tau\theta\varphi} \partial_t u\|^2_{L^2(Q)} + \underline{H} \\ \leq C_{\lambda_1,\tau_1,\mathfrak{K}} \Big(\|e^{\tau\theta\varphi} P^{\mathfrak{M}} u\|^2_{L^2(Q)} + \tau^3 \|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\|^2_{L^2((0,T)\times\omega)} \\ + h^{-2} \big(|e^{\tau\theta\varphi} u_{|t=0}|^2_{L^2(\Omega)} + |e^{\tau\theta\varphi} u_{|t=T}|^2_{L^2(\Omega)} \big) \Big), \quad (3.11)\end{aligned}$$

With Lemmata 2.2 and 2.3 and we now write

$$\begin{aligned}\overline{D_i u}^i &= \overline{D_i(\rho v)}^i = \overline{(D_i\rho)\tilde{v}^i}^i + \overline{\tilde{\rho}^i D_i v}^i \\ &= \overline{D_i\rho}^i \overline{\tilde{v}^i}^i + \overline{\tilde{\rho}^i}^i \overline{D_i v}^i + \frac{h_i^2}{4} \big((\bar{D}_i D_i \rho) \overline{D_i v}^i + \overline{D_i \rho}^i \bar{D}_i D_i v \big) \\ &= \overline{D_i\rho}^i v + \overline{\tilde{\rho}^i}^i \overline{D_i v}^i + \frac{h_i^2}{4} \big((\bar{D}_i D_i \rho) \overline{D_i v}^i + 2\overline{D_i \rho}^i \bar{D}_i D_i v \big)\end{aligned}$$

With Proposition 2.10 we then find

$$r\overline{D_i u}^i = sv\mathcal{O}_{\lambda,\mathfrak{K}}(1) + \overline{D_i v}^i \mathcal{O}_{\lambda,\mathfrak{K}}(1) + sh_i^2 (\bar{D}_i D_i v) \mathcal{O}_{\lambda,\mathfrak{K}}(1).$$

With (3.10) and (3.11) and the expression (3.8) of $\underline{H}$ we then obtain

$$\begin{aligned}\tau^3 \|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\|^2_{L^2(Q)} + \tau \sum_{i\in[\![1,d]\!]} \big(\|\theta^{\frac{1}{2}} e^{\tau\theta\varphi} D_i u\|^2_{L^2(Q)} + \|\theta^{\frac{1}{2}} e^{\tau\theta\varphi} \overline{D_i u}^i\|^2_{L^2(Q)} \big) \\ + \tau^{-1} \|\theta^{-\frac{1}{2}} e^{\tau\theta\varphi} \partial_t u\|^2_{L^2(Q)} \leq C_{\lambda_1,\tau_1,\mathfrak{K}} \Big(\|e^{\tau\theta\varphi} P^{\mathfrak{M}} u\|^2_{L^2(Q)} + \tau^3 \|\theta^{\frac{3}{2}} e^{\tau\theta\varphi} u\|^2_{L^2((0,T)\times\omega)} \\ + h^{-2} \big(|e^{\tau\theta\varphi} u_{|t=0}|^2_{L^2(\Omega)} + |e^{\tau\theta\varphi} u_{|t=T}|^2_{L^2(\Omega)} \big) \Big).\end{aligned}$$

Finally, we observe that since $\max_{[0,T]} \theta \leq \frac{1}{\delta T^2(1+\delta)} \leq \frac{1}{T^2\delta}$, a sufficient condition for $\tau h(\max_{[0,T]} \theta) \leq \varepsilon_0$ becomes $\tau h(T^2\delta)^{-1} \leq \varepsilon_0$. This concludes the proof of Theorem 1.3. ∎

4. *h*-null controllability: the linear case. We consider the following semi-discrete parabolic problem with potential

$$\partial_t y + \mathcal{A}^{\mathfrak{M}} y + ay = \mathbf{1}_\omega v, \quad t \in (0,T), \quad y_{|\partial\Omega} = 0, \tag{4.1}$$

To achieve a h-null controllability result for (4.1) we start by proving a relaxed observability estimate.

4.1. A relaxed observability estimate. The adjoint system associated with the controlled system with potential (4.1) is given by

$$-\partial_t q + \mathcal{A}^{\mathfrak{M}} q + aq = 0, \quad t \in (0,T), \quad q_{|\partial\Omega} = 0. \tag{4.2}$$

With the Carleman estimate we proved in Theorem 1.3 we have the following relaxed observability estimate.

PROPOSITION 4.1. *There exists positive constants C_0, C_1 and C_2 such that for all $T > 0$ and all potential function a, under the condition $h \leq \min(h_0, h_1)$ with*

$$h_1 = C_0 (1 + \frac{1}{T} + \|a\|_\infty^{\frac{2}{3}})^{-1}$$

any solution of (1.14) *satisfies*

$$|q(0)|_{L^2(\Omega)} \leq C_{\text{obs}} \|q\|_{L^2((0,T)\times\omega)} + e^{-\frac{C_1}{h} + T\|a\|_\infty} |q(T)|_{L^2(\Omega)}. \tag{4.3}$$

with $C_{\text{obs}} = e^{C_2(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{\frac{2}{3}})}$.

Proof. The change of variable

$$\tilde{q} = e^{\|a\|_\infty (t-T)} q, \tag{4.4}$$

allows us to consider the potential a to be non negative.

With the Carleman estimate we proved in Theorem 1.3 we have

$$\|s^{\frac{3}{2}} e^{s\varphi} q\|^2_{L^2(Q)} \leq C\Big(\|e^{s\varphi} a q\|^2_{L^2(Q)} + \|s^{\frac{3}{2}} e^{s\varphi} q\|^2_{L^2((0,T)\times\omega)} + h^{-2}\big(|e^{s\varphi} q_{|t=0}|^2_{L^2(\Omega)} + |e^{s\varphi} q_{|t=T}|^2_{L^2(\Omega)} \big)\Big),$$

with $s = \tau\theta$ for $\tau \geq \tau_0(T + T^2)$, $0 < h \leq h_0$ and $\tau h (\delta T^2)^{-1} \leq \varepsilon_0$.

As $1 \leq C\theta T^2$ it suffices to have

$$\tau \geq C T^2 \|a\|_\infty^{\frac{2}{3}} \tag{4.5}$$

to obtain

$$\|s^{\frac{3}{2}} e^{s\varphi} q\|^2_{L^2(Q)} \leq C\Big(\|s^{\frac{3}{2}} e^{s\varphi} q\|^2_{L^2((0,T)\times\omega)} + h^{-2}\big(|e^{s\varphi} q_{|t=0}|^2_{L^2(\Omega)} + |e^{s\varphi} q_{|t=T}|^2_{L^2(\Omega)} \big)\Big). \tag{4.6}$$

We thus choose $\tau_1 \geq \tau_0$ sufficiently large to have (4.5) for

$$\tau \geq \tau_1 (T + T^2 + T^2 \|a\|_\infty^{\frac{2}{3}}). \tag{4.7}$$

The positivity of $\mathcal{A}^{\mathfrak{M}} + a$ yields

$$|q(0)|_{L^2} \leq |q(t)|_{L^2}, \quad t \in (0,T). \tag{4.8}$$

Recalling that φ is negative, and independent of time t, we observe that we have

$$\begin{aligned}\int_0^T\int_\Omega s^3e^{2s\varphi}q(t)^2dt &\geq \int_{\frac{T}{4}}^{\frac{3T}{4}}\int_\Omega s^3e^{2s\varphi}q(t)^2\ dt\\ &\geq \int_{\frac{T}{4}}^{\frac{3T}{4}} \tau^3\theta(\tfrac{T}{2})^3e^{\tau\theta(\frac{T}{4})\inf\varphi}|q(0)|^2_{L^2(\Omega)}\ dt\\ &= \frac{1}{2}T\tau^3\theta(\tfrac{T}{2})^3e^{\tau\theta(\frac{T}{4})\inf\varphi}|q(0)|^2_{L^2(\Omega)}\\ &\geq CTe^{-C'\frac{\tau}{T^2}}|q(0)|^2_{L^2(\Omega)}, \qquad (4.9)\end{aligned}$$

as $\tau \geq \tau_0 T^2$.

As $\theta(T) = \theta(0) = \big(T^2(1+\delta)\delta\big)^{-1}$, we have $e^{s\varphi_{|t=0}} = e^{s\varphi_{|t=T}} \leq e^{C\frac{\tau}{\delta T^2}(\sup\varphi)}$ and we find

$$\begin{aligned}&|e^{s\varphi}q_{|t=0}|^2_{L^2(\Omega)} + |e^{s\varphi}q_{|t=T}|^2_{L^2(\Omega)}\\ &\quad\leq C\big(e^{-\frac{C\tau}{\delta T^2}}|q(0)|^2_{L^2(\Omega)} + e^{-\frac{C\tau}{\delta T^2}}|q(T)|^2_{L^2(\Omega)}\big)\\ &\quad\leq C'e^{-\frac{C\tau}{\delta T^2}}|q(T)|^2_{L^2(\Omega)},\end{aligned}$$

as $\sup\varphi < 0$, and using (4.8). We now write

$$\|s^{\frac{3}{2}}e^{s\varphi}q\|^2_{L^2((0,T)\times\omega)} \leq Ce^{\tau\theta(\frac{T}{2})\sup\varphi}\|q\|^2_{L^2((0,T)\times\omega)},$$

Consequently we obtain

$$T|q(0)|^2_{L^2(\Omega)} \leq Ce^{C\frac{\tau}{T^2}}\|q\|^2_{L^2((0,T)\times\omega)} + h^{-2}e^{\frac{\tau}{T^2}\left(C-\frac{C'}{\delta}\right)}|q(T)|^2_{L^2(\Omega)}$$

For $0 < \delta \leq \delta_1 \leq \delta_0$, with δ_1 sufficiently small, we obtain

$$T|q(0)|^2_{L^2(\Omega)} \leq Ce^{C\frac{\tau}{T^2}}\|q\|^2_{L^2((0,T)\times\omega)} + h^{-2}e^{-C'\frac{\tau}{\delta T^2}}|q(T)|^2_{L^2(\Omega)} \qquad (4.10)$$

We recall the conditions of Theorem 1.3:

$$\frac{\tau h}{\delta T^2} \leq \varepsilon_0 \quad\text{and}\quad h \leq h_0.$$

They need to be fulfilled along with $\delta \leq \delta_1$.

We fix $\tau = \tau_0(T + T^2 + T^2\|a\|_\infty^{\frac{2}{3}})$ with τ_0 as chosen in Theorem 1.3. We define h_1 through

$$h_1 = \frac{\varepsilon_0}{\tau_0}\delta_1(1 + \frac{1}{T} + \|a\|_\infty^{\frac{2}{3}})^{-1},$$

which gives

$$\frac{\tau h_1}{\delta_1 T^2} = \varepsilon_0$$

We choose $h \leq \min(h_0, h_1)$, and $\delta = h\delta_1/h_1 \leq \delta_1$ we then find $\frac{\tau h}{\delta T^2} = \varepsilon_0$.

As $\tau/(T^2\delta) = \varepsilon_0/h$, we obtain from (4.10)

$$|q(0)|^2_{L^2(\Omega)} \leq Ce^{C(1+\frac{1}{T}+\|a\|_\infty^{\frac{2}{3}})}\|q\|^2_{L^2((0,T)\times\omega)} + h^{-2}e^{-\frac{C'\varepsilon_0}{h}}|q(T)|^2_{L^2(\Omega)},$$

which gives

$$|q(0)|_{L^2(\Omega)} \leq e^{C_2(1+\frac{1}{T}+\|a\|_\infty^{\frac{2}{3}})}\|q\|_{L^2((0,T)\times\omega)} + e^{-\frac{C_1}{h}}|q(T)|_{L^2(\Omega)}.$$

Recalling that we made the change of variable (4.4) we conclude the proof. ∎

4.2. *h*-null controllability. With the result of Proposition 4.1 we deduce the following h-controllability result for System (4.1).

PROPOSITION 4.2. *There exist positive constants C_1, C_2, C_3 and for $T > 0$ a map $L_{T,a} : \mathbb{R}^{\mathfrak{M}} \to L^2(0,T;\mathbb{R}^{\mathfrak{M}})$, such that if $h \leq \min(h_0, h_2)$ with*

$$h_2 = C_1(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{\frac{2}{3}})^{-1},$$

for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$, there exists a semi-discrete control function v given by $v = L_{T,a}(y_0)$ such that the solution to (4.1) satisfies

$$|y(T)|_{L^2(\Omega)} \leq C_0 e^{-C_2/h}|y_0|_{L^2(\Omega)}, \qquad \text{and} \quad \|v\|_{L^2(Q)} \leq C_0|y_0|_{L^2(\Omega)},$$

with $C_0 = e^{C_3(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{\frac{2}{3}})}$.

REMARK 4.3. *Note that the final state is of size $e^{-C/h}|y_0|_{L^2(\Omega)}$. In comparison the result obtained in [BHL10a, BHL10b] based on a Lebeau-Robbiano-type spectral inequality yields a final state of size $e^{-C/h^2}|y_0|_{L^2(\Omega)}$. The method in [BHL10a, BHL10b] does not yield however a precise observability constant as in Proposition 4.1 which is crucial in the study of semi-linear equation as we do below. Questions regarding differences in size of the final state when comparing this two method are of theoretical interest: can one improve the estimate given above? Yet for practical purposes there are very little differences: in fact one is rather interested in a target of size $h^p|y_0|_{L^2(\Omega)}$ in connexion with the consistency of the numerical scheme. We refer to [BHL11] where such questions appear.*

Proof. We use a dual formulation; we consider the adjoint parabolic equation

$$(-\partial_t + \mathcal{A}^{\mathfrak{M}})q + aq = 0, \quad q(T) = q_T. \tag{4.11}$$

The relaxed observability inequality of Proposition 4.1 gives

$$|q(0)|_{L^2(\Omega)} \leq C_{\text{obs}}\|q\|_{L^2((0,T)\times\omega)} + \varepsilon|q_T|_{L^2(\Omega)}, \tag{4.12}$$

with $C_{\text{obs}} = e^{C(1+\frac{1}{T}+T\|a\|_\infty+\|a\|_\infty^{\frac{2}{3}})}$ and $\varepsilon = e^{-\frac{C'}{h}+T\|a\|_\infty}$. We introduce the functional

$$J(q_T) = \frac{1}{2}\int_0^T |q(t)|^2_{L^2(\omega)}\,dt + \frac{\varepsilon}{2}|q_T|^2_{L^2(\Omega)} + \langle y_0, q(0)\rangle. \tag{4.13}$$

The functional J is smooth, strictly convex, and coercive on a finite dimensional space, thus it admits a unique minimizer $q_T = q_T^{opt}$. We denote by $q^{opt}(t)$ the associated solution of the adjoint problem (4.11). The Euler-Lagrange equation associated with this minimization problem reads

$$\int_0^T \langle q^{opt}(t), q(t)\rangle_{L^2(\omega)}\,dt + \varepsilon\langle q_T^{opt}, q_T\rangle_{L^2(\Omega)} = -\langle y_0, q(0)\rangle_{L^2(\Omega)}, \tag{4.14}$$

for any $q_T \in \mathbb{R}^{\mathfrak{M}}$, with the associated solution $q(t)$ of the adjoint problem (4.11). We set the control to $v = L_{T,a}(y_0) = \mathbf{1}_\omega q^{opt}(t)$. We consider now the solution y to the controlled problem

$$\partial_t y + \mathcal{A}^{\mathfrak{M}} y + ay = \mathbf{1}_\omega q^{opt}(t), \quad t \in (0,T), \quad y_{|\partial\Omega} = 0, \quad y(0) = y_0.$$

By multiplying this equation by q and integrating by parts, we deduce

$$\int_0^T \langle q^{opt}(t), q(t)\rangle_{L^2(\omega)}\, dt = \langle y(T), q_T\rangle_{L^2(\Omega)} - \langle y_0, q(0)\rangle_{L^2(\Omega)},$$

for any $q_T \in \mathbb{R}^{\mathfrak{M}}$. With (4.14) we conclude that

$$y(T) = -\varepsilon q_T^{opt}. \tag{4.15}$$

We now take $q_T = q_T^{opt}$ in (4.14) to obtain

$$\|q^{opt}\|^2_{L^2((0,T)\times\omega)} + \varepsilon |q_T^{opt}|^2_{L^2(\Omega)} = -\langle y_0, q^{opt}(0)\rangle_{L^2(\Omega)} \leq |y_0|_{L^2(\Omega)} |q^{opt}(0)|_{L^2(\Omega)},$$

With the observability inequality (4.12) we have

$$|q^{opt}(0)|_{L^2(\Omega)} \leq C_{\text{obs}} \|q^{opt}\|_{L^2((0,T)\times\omega)} + \varepsilon |q_T^{opt}|_{L^2(\Omega)}.$$

With the Young inequality we obtain

$$\varepsilon^{\frac{1}{2}} |q_T^{opt}|_{L^2(\Omega)} \leq (C_{\text{obs}} + \varepsilon^{\frac{1}{2}}) |y_0|_{L^2(\Omega)},.$$

and

$$\|v\|_{L^2((0,T)\times\omega)} = \|q^{opt}\|_{L^2((0,T)\times\omega)} \leq (C_{\text{obs}} + \varepsilon^{\frac{1}{2}}) |y_0|_{L^2(\Omega)}.$$

Hence the *linear* map

$$\begin{aligned} L_{T,a} : L^2(\Omega) &\to L^2((0,T)\times\omega), \\ y_0 &\mapsto v, \end{aligned}$$

is well defined and continuous.

Next we see that $\varepsilon^{\frac{1}{2}} \leq C_{\text{obs}}$ if

$$h \leq C\big(1 + \frac{1}{T} + T\|a\|_\infty + \|a\|_\infty^{\frac{2}{3}}\big)^{-1}.$$

This yields

$$|q_T^{opt}|_{L^2(\Omega)} \leq 2C_{\text{obs}} \varepsilon^{-\frac{1}{2}} |y_0|_{L^2(\Omega)}. \tag{4.16}$$

Moreover, we then have $h \leq C/(T\|a\|_\infty)$ which yields $\varepsilon \leq e^{-C/h}$. We thus find

$$\|v\|_{L^2((0,T)\times\omega)} \leq 2C_{\text{obs}} |y_0|_{L^2(\Omega)},$$

and

$$|y(T)|_{L^2(\Omega)} \leq 2C_{\text{obs}} e^{-C/h} |y_0|_{L^2(\Omega)},$$

which concludes the proof. ∎

5. *h*-null controllability: the semilinear case. We start this section by stating a classical regularity result for the linear equation

$$\partial_t y + \mathcal{A}^{\mathfrak{M}} y + ay = f \in L^2(0,T;\mathbb{R}^{\mathfrak{M}}), \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0 \in \mathbb{R}^{\mathfrak{M}}. \tag{5.1}$$

PROPOSITION 5.1. *For any $a \in L^\infty$ the solution to* (5.1) *satisfies*

$$\|y\|_{L^\infty(0,T,L^2(\Omega))} + \|y\|_{L^2(Q)} + \sum_{i\in[\![1,d]\!]} \|D_i y\|_{L^2(Q)} \leq K_0\big(|y_0|_{L^2(\Omega)} + \|f\|_{L^2(Q)}\big),$$

$$\|\partial_t y\|_{L^2(Q)} + \sum_{i\in[\![1,d]\!]} \|D_i y\|_{L^\infty(0,T,L^2(\Omega))} \leq K_1\big(\sum_{i\in[\![1,d]\!]} |D_i y_0|_{L^2(\Omega)} + \|f\|_{L^2(Q)}\big),$$

with $K_0 = e^{C(1+T+T\|a\|_{L^\infty})}$ and $K_1 = e^{C(1+T+(T^{\frac{1}{2}}+T)\|a\|_{L^\infty})}$

We now consider the semi-linear equation

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + \mathcal{G}(y) = f, \quad t \in (0,T), \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0, \tag{5.2}$$

with $\mathcal{G}$ Lipschitz continuous, since $\mathbb{R}^{\mathfrak{M}}$ is finite dimensional, the Cauchy-Lipschitz theorem applies. For each initial data y_0 and r.h.s. f L^1 w.r.t. t, this yields the existence and uniqueness of maximal solution in $\mathscr{C}^1([0,t_0);\mathbb{R}^{\mathfrak{M}})$ with $0 < t_0 \leq T$. If $t_0 < T$ the solution ceases to exist at $t = t_0$ because of a blow up: $\lim_{t\to t_0^-} |y(t)|_{L^\infty} = +\infty$.

We shall consider the following semilinear semi-discrete control problem.

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + \mathcal{G}(y) = \mathbf{1}_\omega v, \quad t \in (0,T), \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0. \tag{5.3}$$

where $\omega \subset \Omega$. The function $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ is assumed[1] of the form

$$\mathcal{G}(x) = xg(x), \quad x \in \mathbb{R}, \tag{5.4}$$

with g Lipschitz continuous. In Section 5.1 we shall assume that $g \in L^\infty(\mathbb{R})$, *i.e.*, the semi-linearity is sublinear. In Section 5.2, following [FCZ00], we shall consider the more general case of a possibly superlinear semi-linearity:

$$|g(x)| \leq K \ln^r(e + |x|), \quad x \in \mathbb{R}, \quad \text{with } 0 \leq r < \frac{3}{2}. \tag{5.5}$$

5.1. The sublinear case. In the present section we assume that $g \in L^\infty(\mathbb{R})$. The sublinearity of the function $\mathcal{G}$ prevents any blow-up as can be observed by the Gronwall inequality. Solutions to (5.2) are thus defined on $[0,T]$.

We prove the following h-null controllability result.

THEOREM 5.2. *There exist positive constants C_0, C_1 such that for all $T > 0$ and h chosen sufficiently small, for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$, there exists a semi-discrete control function v with*

$$\|v\|_{L^2(Q)} \leq C|y_0|_{L^2(\Omega)},$$

such that the solution to the semi-linear parabolic equation (5.3) *satisfies*

$$|y(T)|_{L^2(\Omega)} \leq Ce^{-C_0/h}|y_0|_{L^2(\Omega)},$$

[1] Regularity as low as locally Lipschitz can be considered. For results with lower regularity in the continuous case we refer to [FCZ00].

with $C = e^{C_1(1+\frac{1}{T}+T\|g\|_\infty+\|g\|_\infty^{\frac{2}{3}})}$.

Observe that the constants are uniform with respect to the discretization parameter h. In particular the L^2-norm of the control function v remains bounded as h varies. Then, up to a subsequence, the semi-discrete controls converge towards a function $v \in L^2((0,T)\times\omega)$ that actually drives the solution of the continuous parabolic problem to zero at time T.

Proof. The proof follows that given by [Ima95] with some particularities due to the discrete case. We set $Z = L^2(0,T;\mathbb{R}^{\mathfrak{M}})$. For $z \in Z$ we consider the linear control problem

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + yg(z) = \mathbf{1}_\omega v, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0. \tag{5.6}$$

We set $a_z = g(z)$. We have $\|a_z\|_{L^\infty(Q)} \le \|g\|_\infty$. If we apply Proposition 4.2 we denote by $v_z = L_{T,a_z}(y_0) \in L^2(0,T;\mathbb{R}^{\mathfrak{M}})$ and y_z the associated control function and controlled solution. We have

$$|y_z(T)|_{L^2(\Omega)} \le Ce^{-C_0/h}|y_0|_{L^2(\Omega)}, \qquad \|v_z\|_{L^2(Q)} \le C|y_0|_{L^2(\Omega)}, \tag{5.7}$$

for $C_0 > 0$ and $C = e^{C_1(1+\frac{1}{T}+T\|g\|_\infty+\|g\|_\infty^{\frac{2}{3}})}$, uniform with respect to z and the discretization parameter h.

With the regularity result of Proposition 5.1 we can define the map

$$\begin{aligned}\Lambda : Z &\to Z,\\ z &\mapsto y_z.\end{aligned}$$

and, as T is fixed and $\|a_z\|_{L^\infty(Q)} \le \|g\|_{L^\infty}$ we have

$$\|\Lambda z\|_{L^2(Q)} = \|y_z\|_{L^2(Q)} \le C\big(|y_0|_{L^2(\Omega)} + \|v_z\|_{L^2(Q)}\big) \le C'|y_0|_{L^2(\Omega)}.$$

Hence, Λ maps the closed ball B of Z of radius $R = C'|y_0|_{L^2(\Omega)}$ into itself.

LEMMA 5.3. *The map* Λ *is continuous on* Z.

The proof of Lemma 5.3 is given below.

Recalling the form of the difference operator D we find

$$|Dy_0|_{L^2(\Omega)} \le Ch^{-1}|y_0|_{L^2(\Omega)}.$$

Additionally from Proposition 5.1 we find that

$$\|\partial_t y_z\|_{L^2(Q)} \le C(|Dy_0|_{L^2(\Omega)} + |v_z|_{L^2(Q)}) \le C\left(h^{-1}+1\right)|y_0|_{L^2(\Omega)}.$$

As $H^1(0,T)$ injects compactly in $L^2(0,T)$ and $\mathbb{R}^{\mathfrak{M}}$ is finite dimensional we get that $\Lambda(B)$ is precompact in Z.

All the previous properties allow us to apply the Schauder topological fixed-point theorem: there exists $y \in Z$ such that $\Lambda(y) = y$. Setting $v = L_{T,a_y}(y_0)$ we obtain

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + yg(y) = \mathbf{1}_\omega v, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0.$$

which concludes the proof as we have found a control v that drives the solution of the semilinear semi-discrete parabolic system to a final state $y(T)$ with the estimates of (5.7). $\square$

Proof of Lemma 5.3. With the continuity and the boundedness of g we have that the map $z \mapsto a_z = g(z)$ is continuous on Z with values in the space $\tilde{Z} = \{a \in Z,\ \ \|a\|_\infty \le \|g\|_\infty\}$.

Let us consider the following controlled parabolic problems

$$\begin{cases}(\partial_t + \mathcal{A}^{\mathfrak{M}})y_1 + a_1 y_1 = \mathbf{1}_\omega v_{a_1},\\ y_{1\,|t=0} = y_0,\end{cases} \qquad \begin{cases}(\partial_t + \mathcal{A}^{\mathfrak{M}})y_2 + a_2 y_2 = \mathbf{1}_\omega v_{a_2},\\ y_{2\,|t=0} = y_0,\end{cases}$$

with $a_1, a_2 \in \tilde{Z}$. The controls v_{a_2} and v_{a_1} are obtained through Proposition 4.2. Setting $Y = y_2 - y_1$ we write

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})Y + a_1 Y = \mathbf{1}_\omega(v_{a_2} - v_{a_1}) + (a_1 - a_2)y_2, \quad Y_{|t=0} = 0$$

From Proposition 5.1 we obtain

$$\|Y\|_{L^\infty(0,T,L^2(\Omega))} \le C\big(\|v_{a_2} - v_{a_1}\|_{L^2(Q)} + \|y_2\|_{L^\infty(Q)}\|a_2 - a_1\|_{L^2(Q)}\big),$$

As we have

$$\|y_2\|_{L^\infty(Q)} \le C_h\|y_2\|_{L^\infty(0,T,L^2(\Omega))} \le C'_h\big(|y_0|_{L^2(\Omega)} + \|v_{a_2}\|_{L^2(Q)}\big)$$

and $\|v_{a_2}\|_{L^2(Q)} \le C|y_0|_{L^2(\Omega)}$ we obtain

$$\|y_2 - y_1\|_{L^\infty(0,T,L^2(\Omega))} \le C_h\big(\|v_{a_2} - v_{a_1}\|_{L^2(Q)} + |y_0|_{L^2(\Omega)}\|a_2 - a_1\|_{L^2(Q)}\big). \tag{5.8}$$

To prove the result of the lemma it thus suffices to prove that the map $a \mapsto v_a$ is continuous on $\tilde{Z}$.

As in the proof of 4.2 we consider the adjoint parabolic equation

$$(-\partial_t + \mathcal{A}^{\mathfrak{M}})q + aq = 0, \quad q(T) = q_T, \tag{5.9}$$

and we denote by $Q(a, q_T)$ its solution. The control v_a of the parabolic system

$$\begin{aligned}(\partial_t + \mathcal{A}^{\mathfrak{M}})y + ay &= \mathbf{1}_\omega v_a,\\ y_{|t=0} &= y_0,\end{aligned}$$

is then given by $v_a = \mathbf{1}_\omega Q(a, q_T^{opt,a})$, with $q_T^{opt,a}$ minimizer of the functional (4.13). We shall thus study the continuity of the map $a \in \tilde{Z} \mapsto \mathbf{1}_\omega Q(a, q_T^{opt,a}) \in Z$.

For the two potentials a_2 and a_1 we can write the associated Euler-Lagrange equations for the two associated minimizers

$$\int_0^T \langle Q(a_1, q_T^{opt,a_1}), Q(a_1, \tilde{q}_T)\rangle_{L^2(\omega)}\, dt + \varepsilon\langle q_T^{opt,a_1}, \tilde{q}_T\rangle_{L^2(\Omega)} + \langle Q(a_1, \tilde{q}_T)(0), y_0\rangle_{L^2(\Omega)} = 0,$$

$$\int_0^T \langle Q(a_2, q_T^{opt,a_2}), Q(a_2, \tilde{q}_T)\rangle_{L^2(\omega)}\, dt + \varepsilon\langle q_T^{opt,a_2}, \tilde{q}_T\rangle_{L^2(\Omega)} + \langle Q(a_2, \tilde{q}_T)(0), y_0\rangle_{L^2(\Omega)} = 0,$$

for any $\tilde{q}_T \in L^2(\Omega)$. Choosing $\tilde{q}_T = q_T^{opt,a_1} - q_T^{opt,a_2}$ and subtracting these two equations yields

$$\begin{aligned}\|Q(a_1, q_T^{opt,a_1}) &- Q(a_2, q_T^{opt,a_2})\|^2_{L^2(0,T,L^2(\omega))} + \varepsilon\|q_T^{opt,a_1} - q_T^{opt,a_2}\|^2_{L^2(\Omega)}\\ &= -\langle Q(a_1, q_T^{opt,a_1} - q_T^{opt,a_2})(0) - Q(a_2, q_T^{opt,a_1} - q_T^{opt,a_2})(0), y_0\rangle_{L^2(\Omega)}\\ &\quad + \int_0^T \langle Q(a_1, q_T^{opt,a_1}), Q(a_1, q_T^{opt,a_2}) - Q(a_2, q_T^{opt,a_2})\rangle_{L^2(\omega)}\, dt\\ &\quad - \int_0^T \langle Q(a_2, q_T^{opt,a_2}), Q(a_1, q_T^{opt,a_1}) - Q(a_2, q_T^{opt,a_1})\rangle_{L^2(\omega)}\, dt.\end{aligned}$$

Applying Proposition 5.1 to the adjoint system (5.9) with $q_T = q_T^{opt,a}$ and using that $a \in \tilde{Z}$, we have

$$\|Q(a, q_T^{opt,a})\|_{L^\infty(Q)} \leq C\|Q(a, q_T^{opt,a})\|_{L^\infty(0,T,L^2(\Omega))} \leq C'|q_T^{opt,a}|_{L^2(\Omega)} \leq C''\varepsilon^{-\frac{1}{2}}|y_0|_{L^2(\Omega)},$$

by (4.16). We thus find

$$\begin{aligned}
&\|Q(a_1, q_T^{opt,a_1}) - Q(a_2, q_T^{opt,a_2})\|^2_{L^2(0,T,L^2(\omega))} + \varepsilon|q_T^{opt,a_1} - q_T^{opt,a_2}|^2_{L^2(\Omega)} \\
&\quad\leq C_{\varepsilon,T}|y_0|_{L^2(\Omega)}\Big(|Q(a_1, q_T^{opt,a_1} - q_T^{opt,a_2})(0) - Q(a_2, q_T^{opt,a_1} - q_T^{opt,a_2})(0)|_{L^2(\Omega)} \\
&\qquad\qquad + \|Q(a_1, q_T^{opt,a_2}) - Q(a_2, q_T^{opt,a_2})\|_{L^2(Q)} \\
&\qquad\qquad + \|Q(a_1, q_T^{opt,a_1}) - Q(a_2, q_T^{opt,a_1})\|_{L^2(Q)}\Big).
\end{aligned}$$

Using now (5.8) for the adjoint system and again (4.16) we obtain

$$\begin{aligned}
&\|Q(a_1, q_T^{opt,a_1}) - Q(a_2, q_T^{opt,a_2})\|^2_{L^2(0,T,L^2(\omega))} + \varepsilon|q_T^{opt,a_1} - q_T^{opt,a_2}|^2_{L^2(\Omega)} \\
&\quad\leq C'_{\varepsilon,T}|y_0|^2_{L^2(\Omega)}\|a_1 - a_2\|_{L^2}.
\end{aligned}$$

This gives the continuity of the map $a \mapsto \mathbf{1}_\omega Q(a, q_T^{opt,a})$ on $\tilde{Z}$ and thus of the map $a \mapsto v_a$ on $\tilde{Z}$. This concludes the proof. □

5.2. The superlinear case. In this section we consider also the semilinear semi-discrete control problem (5.3). The function $\mathcal{G} : \mathbb{R} \to \mathbb{R}$ is assumed of the form

$$\mathcal{G}(x) = xg(x), \quad x \in \mathbb{R}, \tag{5.10}$$

with g Lipschitz continuous and, in agreement with the controllability result of [FCZ00] in the continuous case, we assume that

$$|g(x)| \leq K \ln^r(e + |x|), \quad x \in \mathbb{R}, \quad \text{with } 0 \leq r < \frac{3}{2}. \tag{5.11}$$

To ease the notation we set

$$\zeta(s) = K \ln^r(e + s) \quad \text{for } s \geq 0. \tag{5.12}$$

5.2.1. Preliminary observations. If $T > 0$, for a vanishing r.h.s. f, starting from a sufficiently small initial data ensures the existence of the solution of (5.2) in the time interval $[0, T]$. Moreover the size of the solution at time $t = T$ remains small.

Proposition 5.4. *Let $T > 0$. There exists $M_0 > 0$ and $K_0 > 0$ such that the maximal solution to*

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + \mathcal{G}(y) = 0, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0, \tag{5.13}$$

satisfies

$$|y(t)|_{L^2(\Omega)} \leq |y_0|_{L^2(\Omega)}e^{K_0 t}, \qquad 0 < t < T,$$

if we choose $h^{-d/2}|y_0|_{L^2(\Omega)} \leq M_0$.

This result will be useful for the construction of the control function in the proofs below: if a sufficiently small state is achieved for a time $0 < t_1 < T$ it suffices to set the control function to 0 for time interval (t_1, T) and one still obtains a small solution at the final time T.

Proof. The maximal solution to (5.13) can cease to exist if there is a blow up at some time $t_0 \in (0, T]$. We first prove that this does not occur if either $r \leq 1$ or if the initial condition is chosen sufficiently small.

Taking the L^2 inner-product of the equation with $y(t)$ we have, after a discrete integration by parts,

$$\frac{1}{2}\partial_t |y(t)|^2_{L^2(\Omega)} + \sum_{i \in [\![1,d]\!]} \langle \gamma_i D_i y(t), D_i y(t)\rangle_{L^2(\Omega)} + \langle \mathcal{G}(y(t)), y(t)\rangle_{L^2(\Omega)} = 0,$$

for $0 \leq t < t_0$, which gives

$$\begin{aligned}\frac{1}{2}\partial_t |y(t)|^2_{L^2(\Omega)} &\leq |g(y(t))|_{L^\infty(\Omega)} |y(t)|^2_{L^2(\Omega)}\\ &\leq \zeta\big(|y(t)|_{L^\infty(\Omega)}\big)|y(t)|^2_{L^2(\Omega)}\\ &\leq \zeta\big(C_0 h^{-d/2}|y(t)|_{L^2(\Omega)}\big)|y(t)|^2_{L^2(\Omega)},\end{aligned}$$

using that if $u \in \mathbb{R}^{\mathfrak{M}}$, $|u|_{L^\infty} \leq C h^{-d/2}|u|_{L^2}$. Setting $z(t) = h^{-d}|y(t)|^2_{L^2(\Omega)}$ we obtain

$$z' \leq 2z\zeta(C_0 z^{\frac{1}{2}}).$$

We have $z(t) \geq 0$ and if $z(t_1) = 0$ for some t_1 then z vanishes identically. We may thus assume that $z > 0$ on $[0, t_0)$.

We set $\rho(s) = \big(2s\zeta(C_0 s^{\frac{1}{2}})\big)^{-1}$ for $s \in (0, +\infty)$ and $\mu(s) = \int_1^s \rho(\sigma)d\sigma$. Recall that ζ is defined in (5.12). We have $0 \leq \frac{d}{dt}\mu(z(t)) \leq 1$, which gives

$$\mu(z(t)) - \mu(z(0)) \leq t, \quad \forall 0 \leq t < t_0. \tag{5.14}$$

Notice that ρ is not integrable at 0^+ and that $\mu(1) = 0$. Therefore, there exists a unique $M_0 > 0$ such that $\mu(M_0) = -t_0$.

We now consider two cases:

Case $r \leq 1$. We have $\lim_{z \to +\infty} \mu(s) = +\infty$. Assuming that $\lim_{t \to t_0^-} z(t) = +\infty$, with inequality (5.14) we reach a contradiction. Hence the solution does not blow up in finite time.

Case $r > 1$. In this case the function ρ is integrable at infinity. Assuming that $\lim_{t \to t_0^-} z(t) = +\infty$, with (5.14) we find

$$\lim_{s \to \infty} \mu(s) - \mu(z(0)) \leq t_0.$$

If $z(0) \leq M_0$ then $\mu(z_0) \leq \mu(M_0) = -t_0$ and therefore we get $\lim_{s\to\infty} \mu(s) \leq 0$. This prevents a possible blowup at time t_0.

In both cases, if $z(0)^{\frac{1}{2}} = h^{-d/2}|y_0|_{L^2} \leq M_0$, then the solution exists on $[0, T]$, and moreover we have $\mu(z(t)) \leq t_0 + \mu(z(0)) \leq 0$ which implies that $z(t) \leq 1$ for any $t \in [0, T]$, uniformly w.r.t. h.

There exists $C_1 > 0$ such that $s^{-1} \leq C_1\rho(s)$, for any $s \in (0, 1]$. This yields by integration

$$\ln\Big(\frac{z(t)}{z(0)}\Big) \leq C_1\big(\mu(z(t)) - \mu(z(0))\big) \leq C_1 t.$$

Hence we have $z(t) \leq z(0)e^{C_1 t}$ which gives the result. ∎

5.2.2. Controllability result. We shall prove the following theorem.

THEOREM 5.5. *Let $\mathcal{G}$ satisfy* (5.10)-(5.11). *There exists $C_0 > 0$ such that for $T > 0$ and $M > 0$ there exist positive constants C, $h_3 \le \min(h_0, h_1, h_2)$ and α_0, such that for $0 < h \le h_3$ and all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$, with $|y_0|_{L^2(\Omega)} \le M$, there exists a semi-discrete control function v such that the solution to the semi-linear parabolic equation* (5.3) *satisfies*

$$|y(T)|_{L^2(\Omega)} \le Ce^{-C_0/h}|y_0|_{L^2(\Omega)}, \qquad \text{and} \quad \|v\|_{L^2(Q)} \le C_h|y_0|_{L^2(\Omega)},$$

where $C_h = Ch^{-\alpha_0}$ with $C = C(T, M)$.

REMARK 5.6. *Note that the constant C_0 that yields the exponential decay of the final state when the discretization is refined is independant of T and M, i.e., the size of the initial condition.*

Observe that the constant C_h in the estimation of the control norm is not uniform with respect to h here. Here we cannot bound the norm of the control if the discretization is refined, i.e., if h decreases to 0. *To achieve a proper estimate one can make use of a control v in $L^\infty(0, T; \mathbb{R}^{\mathfrak{M}})$. This approach was central in the proof of the the controllability of semilinear parabolic equations in [FCZ00]. To that purpose one needs to refine the observability inequality of Proposition 4.1. This is the subject of future work based on the analysis of the semi-discrete heat kernel. Such an estimation will also naturally yield a local controllability result. In dimension $d > 1$ with such an estimation we can replace $h^{-d/2}$ by a constant in* (5.25).

Yet, only using a L^2 control, the result of Theorem 5.5 can be improved if we consider the case of one dimension in space. This is presented in Theorem 5.11 in Section 5.2.3 below. In fact in this case the heat kernel estimation can be replaced by a (discrete) Sobolev inequality.

REMARK 5.7. *Note that the largest discretization step h allowed by the previous theorem is function of the norm of the initial condition of the control problem.*

Proof of Theorem 5.5. We use some of the arguments given by [FCZ00], yet with some particularities due to the discrete case.

Let $R_0 > 0$ be such that $\zeta(R_0) \ge 1$. For $R > R_0$ we introduce

$$S_R(s) = \begin{cases} s & \text{if } -R \le s \le R, \\ \operatorname{sgn}(s)R & \text{otherwise.} \end{cases}$$

Adapting [FCZ00] we introduce the following control time

$$T_R = \min(T, \zeta(R)^{-2/3}).$$

We set $Z_R = L^\infty(0, T_R; \mathbb{R}^{\mathfrak{M}})$ and $Q_R = (0, T_R) \times \mathbb{R}^{\mathfrak{M}}$. We shall denote by $\|.\|_{L^p(Q_R)}$ the natural norm on $L^p(0, T_R; \mathbb{R}^{\mathfrak{M}})$, $p = 2$ or $p = \infty$ (see the end of Section 1.1.4).

For $z \in Z_R$ we set $a_z = g(S_R(z))$. Observe that we have

$$\frac{1}{T_R} \le \frac{1}{T} + \zeta(R)^{2/3}, \qquad \big(T_R^{\frac{1}{2}} + T_R\big)\|a_z\|_\infty \le 2\zeta(R)^{2/3}, \tag{5.15}$$

since

$$|a_z(t, \boldsymbol{k})| = |g(S_R(z(t, \boldsymbol{k}))| \le \zeta(S_R(z(t, \boldsymbol{k})) \le \zeta(R),$$

and $\zeta(R) \geq \zeta(R_0) \geq 1$. We shall choose R in the form $R = R(h) \geq R_0$ to be made precise below.

For $z \in Z_R$ we consider the linear control problem on $[0, T_R]$:

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + ya_z = \mathbf{1}_\omega v, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0. \tag{5.16}$$

If we apply Proposition 4.2 to the control system (5.16), we set

$$v_{R,z} = L_{T_R,a_z}(y_0) \qquad 0 < t \leq T_R.$$

as the associated control function and we denote by $y_{R,z}$ the controlled solution.

We have

$$|y_{R,z}(T_R)|_{L^2(\Omega)} \leq K_2 e^{-\tilde{C}_0/h} |y_0|_{L^2(\Omega)}, \tag{5.17}$$

$$\|v_{R,z}\|_{L^2(0,T_R,L^2(\Omega))} \leq K_2 |y_0|_{L^2(\Omega)}, \tag{5.18}$$

for $C > 0$ uniform with respect to z and the discretization parameter h and with

$$K_2 = e^{C\left(1+\frac{1}{T_R}+T_R\|a_z\|_\infty+\|a_z\|_\infty^{\frac{2}{3}}\right)} \leq e^{C'\left(1+\frac{1}{T}+\zeta(R)^{2/3}\right)}, \tag{5.19}$$

by (5.15).

To apply Proposition 4.2 we require

$$h \leq C\left(1 + \frac{1}{T} + \zeta(R)^{2/3}\right)^{-1} \leq C\left(1 + \frac{1}{T_R}\right)^{-1}, \tag{5.20}$$

using (5.15).

As $y_z \in L^\infty(0, T_R; \mathbb{R}^{\mathfrak{M}})$ by Proposition 5.1 (using that $\mathbb{R}^{\mathfrak{M}}$ is finite dimensional) the following map is well defined

$$\begin{aligned} \Lambda_R : Z_R &\to Z_R, \\ z &\mapsto y_{R,z}. \end{aligned}$$

Lemma 5.8. *The map Λ_R is continuous on $Z_R = L^\infty(0, T_R; \mathbb{R}^{\mathfrak{M}})$.*

We denote by $B_{R,h}$ the ball centered at 0 and of radius $R = R(h)$ in Z_R. Proposition 5.1 gives

$$\|\partial_t y_z\|_{L^\infty(Q_R)} \leq C_h \|\partial_t y_z\|_{L^2(Q_R)} \leq C'_h |Dy_0|_{L^2(\Omega)} \leq C'_h C''_h \|y_0\|_{L^2(\Omega)}.$$

As $H^1(0, T_R)$ injects compactly in $L^\infty(0, T_R)$ and $\mathbb{R}^{\mathfrak{M}}$ is finite dimensional we find that $\Lambda(B_{R,h})$ is precompact in $L^\infty(0, T_R; \mathbb{R}^{\mathfrak{M}})$.

Lemma 5.9. *Let $\alpha > d/2$. For any $M > 0$, there exists $C = C(M, \alpha) > 0$ and $\tilde{h}_3(T, M, \alpha)$, such that for*

$$R = Ch^{-\alpha},$$

the map Λ_R maps $B_{R,h}$ into itself if $0 < h \leq \tilde{h}_3$ and if $|y_0|_{L^2(\Omega)} \leq M$.

All the previous properties allow us to apply the Schauder topological fixed-point theorem if $0 < h \leq \tilde{h}_3$ and $|y_0|_{L^2(\Omega)} \leq M$ and R is chosen according to Lemma 5.9: there exists $y \in B_{R,h}$ such that $\Lambda_R(y) = y$. Setting $v_R = L_{T_R,a_y}(y_0)$ we obtain

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + ya_y = \mathbf{1}_\omega v_R, \quad 0 < t \leq T_R, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0.$$

Since $y \in B_{R,h}$ we have $\|y\|_{L^\infty(Q)} \leq R$. Then we have $a_y = g\big(S_R(y)\big) = g(y)$, which yields

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + \mathcal{G}(y) = \mathbf{1}_\omega v_R, \quad 0 < t \leq T_R, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0.$$

With the value of $R = R(h)$ given by Lemma 5.9 we go back to the estimations (5.17)–(5.18) and find

$$|y(T_R)|_{L^2(\Omega)} \leq K_2 e^{-\tilde{C}_0/h} |y_0|_{L^2(\Omega)}, \qquad \|v_R\|_{L^2(Q_R)} \leq K_2 |y_0|_{L^2(\Omega)},$$

with (use (5.26) in the proof of Lemma 5.9 and that $2r/3 \leq 1$)

$$K_2 \leq e^{C\left(1+\frac{1}{T}+\zeta(R)^{2/3}\right)} \leq e^{2C\zeta(R)^{2/3}} \leq e^{C' \ln(e+R)^{2r/3}} \leq (e+R)^{C''} \leq C''(M) h^{-\alpha_0}, \tag{5.21}$$

with $\alpha_0 > 0$. This yields

$$|y(T_R)|_{L^2(\Omega)} \leq C(M) e^{-C_0/h} |y_0|_{L^2(\Omega)}, \qquad \|v_R\|_{L^2(Q_R)} \leq C(M) h^{-\alpha_0} |y_0|_{L^2(\Omega)}. \tag{5.22}$$

for any $0 < C_0 < \tilde{C}_0$.

We now define v on $[0,T]$ by

$$v = \begin{cases} v_R & \text{if } 0 < t \leq T_R, \\ 0 & \text{if } T_R < t \leq T, \end{cases}$$

We naturally have $\|v\|_{L^2(Q)} \leq C h^{-\alpha_0} |y_0|_{L^2(\Omega)}$.

If we have

$$h^{-d/2} |y(T_R)|_{L^2(\Omega)} \leq M_0, \tag{5.23}$$

we can apply Proposition 5.4 on the time interval[2] $[T_R, T]$, which yields

$$|y(T)|_{L^2(\Omega)} \leq |y(T_R)|_{L^2(\Omega)} e^{K_0(T-T_R)} \leq C(M,T) e^{-C_0/h} |y_0|_{L^2(\Omega)}.$$

With (5.22), choosing $h_3 \leq \min(h_0, h_1, h_2, \tilde{h}_3)$ sufficiently small, condition (5.23) can be fulfilled if $0 < h \leq h_3$, which concludes the proof. □

Proof of Lemma 5.8 (Continuity of the map Λ_R on $L^\infty(0,T_R;\mathbb{R}^{\mathfrak{M}})$). In this proof the values of h and R are kept fixed.

Observe that $z \mapsto S_R(z)$ is continuous on Z_R as S_R is Lipschitz continuous. As g is also Lipschitz continuous we have that the map $z \mapsto a_z$ is continuous on Z_R as well.

Let us consider the following controlled parabolic problems

$$\begin{cases} (\partial_t + \mathcal{A}^{\mathfrak{M}})y_1 + a_1 y_1 = \mathbf{1}_\omega v_{a_1}, \\ y_{1\,|t=0} = y_0, \end{cases} \qquad \begin{cases} (\partial_t + \mathcal{A}^{\mathfrak{M}})y_2 + a_2 y_2 = \mathbf{1}_\omega v_{a_2}, \\ y_{2\,|t=0} = y_0, \end{cases}$$

with $\max\big(\|a_1\|_\infty, \|a_2\|_\infty\big) \leq C \ln^{2r/3}(e+R)$. The controls v_{a_2} and v_{a_1} are obtained through Proposition 4.2. Setting $Y = y_2 - y_1$ we write

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})Y + a_1 Y = \mathbf{1}_\omega (v_{a_2} - v_{a_1}) + (a_1 - a_2) y_2, \quad Y_{|t=0} = 0$$

[2]Here $R = R(h)$. Yet, Proposition 5.4 applies in fact on the interval $[T_R, T_R + T]$ by translation in time.

From Proposition 5.1 we obtain

$$\|Y\|_{L^\infty(Q_R)} \leq C_h \|Y\|_{L^\infty(0,T_R,L^2(\Omega))} \leq C'_h\big(\|v_{a_2}-v_{a_1}\|_{L^2(Q_R)}+\|y_2\|_{L^2(Q_R)}\|a_2-a_1\|_{L^\infty(Q_R)}\big),$$

As we have

$$\|y_2\|_{L^2(Q_R)} \leq C\big(|y_0|_{L^2(\Omega)} + \|v_{a_2}\|_{L^2(Q_R)}\big)$$

and $\|v_{a_2}\|_{L^2(Q_R)} \leq C_R |y_0|_{L^2(\Omega)}$ we obtain

$$\|y_2 - y_1\|_{L^\infty(Q_R)} \leq C_{h,R}\big(\|v_{a_2} - v_{a_1}\|_{L^2(Q_R)} + |y_0|_{L^2(\Omega)}\|a_2 - a_1\|_{L^\infty(Q_R)}\big), \tag{5.24}$$

To prove the result of the lemma it thus suffices to prove that the map $a \mapsto v_a$ is continuous from $L^\infty(0, T_R; \mathbb{R}^{\mathfrak{M}})$ to $L^2(0, T_R; \mathbb{R}^{\mathfrak{M}})$. This is contained in the proof of Lemma 5.3. □

Proof of Lemma 5.9. From Proposition 5.1 and (5.17)–(5.18) we have

$$\begin{aligned}\|y_z\|_{L^\infty(Q_R)} &\leq h^{-\frac{d}{2}}\|y_z\|_{L^\infty(0,T_R,L^2(\Omega))}\\ &\leq h^{-\frac{d}{2}} e^{C_1\left(1+T+\frac{1}{T}+\zeta(R)^{2/3}\right)}|y_0|_{L^2(\Omega)}.\end{aligned} \tag{5.25}$$

We hence find

$$R^{-1}\|y_z\|_{L^\infty(Q_R)} \leq h^{-\frac{d}{2}} e^{C_1\left(1+T+\frac{1}{T}+\ln^{\frac{2r}{3}}(e+R)\right)-\ln(R)}|y_0|_{L^2(\Omega)}.$$

Let $0 < \varepsilon < 1$ be such that $\alpha \geq \frac{d}{2(1-\varepsilon)}$. As $r < 3/2$, there exists $R_1 = R_1(T) > R_0$ such that

$$\zeta(R)^{2/3} = K \ln^{\frac{2r}{3}}(2 + R) \geq 1 + \frac{1}{T}, \tag{5.26}$$

and

$$C_1\big(1 + T + \frac{1}{T} + \ln^{\frac{2r}{3}}(e + R)\big) - \ln(R) \leq -(1 - \varepsilon)\ln(R). \tag{5.27}$$

if $R \geq R_1(T)$, which gives

$$R^{-1}\|y_z\|_{L^\infty(Q_R)} \leq \frac{h^{-\frac{d}{2}}}{R^{1-\varepsilon}}|y_0|_{L^2(\Omega)}.$$

We set

$$R = h^{-\alpha}\ M^{\frac{1}{1-\varepsilon}} \leq h^{-\frac{d}{2(1-\varepsilon)}}\ M^{\frac{1}{1-\varepsilon}}, \tag{5.28}$$

and we have $R \geq R_1(T)$ by taking $0 < h \leq \tilde{\tilde{h}}_3$ with $\tilde{\tilde{h}}_3$ sufficiently small and function of T and M. With the choice for R we then obtain

$$R^{-1}\|y_z\|_{L^\infty(Q_R)} \leq \frac{h^{-\frac{d}{2}}}{R^{1-\varepsilon}}|y_0|_{L^2(\Omega)} \leq \frac{h^{-\frac{d}{2}}}{R^{1-\varepsilon}} M \leq 1.$$

We now recall condition (5.20) that connects R and h:

$$h \leq C\big(1 + \frac{1}{T} + \zeta(R)^{2/3}\big)^{-1}.$$

By (5.26) as $\zeta(R)^{2/3} \geq 1 + \frac{1}{T}$ if $R \geq R_1(T)$ it suffices to have

$$h \leq C\big(2\zeta(R)^{2/3}\big)^{-1}, \quad \textit{i.e.} \quad R \leq e^{\frac{C'}{h^{3/2r}}} - 2.$$

Observe that this last condition is satisfied by R as defined in (5.28) for $0 < h \leq \tilde{h}_3 \leq \tilde{\tilde{h}}_3$ with $\tilde{h}_3$ taken sufficiently small and function of T and M. □

5.2.3. The one-dimensional case. Finally, we study the one-dimensional case for which the result of Theorem 5.5 can be sharpened to yield a control function uniformly bounded with respect to the discretization parameter h. This require a more regular intial condition which can be achieved by simply setting the control function to zero for a arbitrary small time interval according to the following lemma in the case of an inital condition $y_0 \in \mathbb{R}^{\mathfrak{M}}$ that lays in a bounded set for the L^∞-norm.

LEMMA 5.10. *Let $y_0 \in \mathbb{R}^{\mathfrak{M}}$. Consider the* homogeneous *semi-linear equation*

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + \mathcal{G}(y) = 0, \quad t > 0, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0, \tag{5.29}$$

There exists $t_1 > 0$, depending on $|y_0|_{L^\infty(\Omega)}$, such that the solution exists on $[0, t_1]$ and we have

$$|y(t)|_{H^1(\Omega)} \leq C\Big(t^{-\frac{1}{2}}|y_0|_{L^2(\Omega)} + t^{\frac{1}{2}}\beta\big(\Omega, |y_0|_{L^\infty(\Omega)}\big)\Big), \qquad 0 < t \leq t_1,$$

for some continuous function β and some $C > 0$ independent of the discretization parameter h.

Here, we have introduced the following discrete H^1-norm:

$$|u|_{H^1(\Omega)} := |u|_{L^2(\Omega)} + \sum_{i \in [\![1,d]\!]} |D_i u|_{L^2(\Omega)}.$$

Observe below that the proof of Lemma 5.10 holds in arbitrary dimension.

Proof. For any $h > 0$ there exists a unique solution to (5.29) by the Cauchy Lipschitz theorem and we have the Duhamel formula:

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\mathcal{G}(y(s))\, ds,$$

where $S(t) = e^{-t\mathcal{A}^{\mathfrak{M}}}$. For $s \geq 0$, we set $G(s) = \sup_{[-s,s]} |\mathcal{G}|$, which yields a Lipschitz function. We have $|S(t)u|_{L^\infty} \leq |u|_{L^\infty}$ which gives

$$\begin{aligned} |y(t)|_{L^\infty} &\leq |y_0|_{L^\infty} + \int_0^t |\mathcal{G}(y(s))|_{L^\infty}\, ds \\ &\leq |y_0|_{L^\infty} + \int_0^t G\big(|y(s)|_{L^\infty}\big)\, ds. \end{aligned}$$

Take $u_0 > 0$ and define

$$\phi(u) = \int_{u_0}^u \frac{dv}{G(v)}, \qquad u > 0.$$

The function ϕ is increasing and so is its inverse ϕ^{-1}. The Bihari inequality [Bih56] then yields

$$|y(t)|_{L^\infty} \leq \phi^{-1}\big(\phi(|y_0|_{L^\infty}) + t\big), \qquad t \in [0, t_1],$$

with t_1 chosen sufficiently small and function of $|y_0|_{L^\infty}$. This insures the existence of the solution on $[0, t_1]$. Note that t_1 is chosen independently of h. We write

$$|y(t)|_{L^\infty} \leq H(t, |y_0|_{L^\infty}) \leq H(t_1, |y_0|_{L^\infty}) =: \mathcal{H}(|y_0|_{L^\infty}), \qquad t \in [0, t_1],$$

as H increases with respect to t.

We now consider the regularization effect. We have $|S(t)u|_{H^1} \leq Ct^{-\frac{1}{2}}|u|_{L^2}$ as can be derived using an eigenfunction decomposition. From the Duhamel formula we thus obtain

$$|y(t)|_{H^1} \leq Ct^{-\frac{1}{2}}|y_0|_{L^2} + \int_0^t (t-s)^{-\frac{1}{2}}|\mathcal{G}(y(s))|_{L^2}\, ds.$$

As we have $|u|_{L^2} \leq |\Omega|^{\frac{1}{2}}|u|_{L^\infty}$ we obtain

$$\begin{aligned}|\mathcal{G}(y(s))|_{L^2} &\leq |\Omega|^{\frac{1}{2}}|\mathcal{G}(y(s))|_{L^\infty} \leq |\Omega|^{\frac{1}{2}}G\big(|y(s)|_{L^\infty}\big)\\ &\leq |\Omega|^{\frac{1}{2}}G\big(\mathcal{H}(|y_0|_{L^\infty})\big) =: \beta(\Omega, |y_0|_{L^\infty}),\end{aligned}$$

for $0 < s \leq t_1$, which gives

$$\begin{aligned}|y(t)|_{H^1} &\leq Ct^{-\frac{1}{2}}|y_0|_{L^2} + \beta(\Omega, |y_0|_{L^\infty})\int_0^t (t-s)^{-\frac{1}{2}}\, ds\\ &\leq C'\Big(t^{-\frac{1}{2}}|y_0|_{L^2} + t^{\frac{1}{2}}\beta(\Omega, |y_0|_{L^\infty})\Big).\end{aligned}$$

The constants are independent of h. ◻

We can now state the control result.

THEOREM 5.11. *Let $d = 1$ and $\Omega = (0, 1)$ and γ satisfy* (1.2). *There exists C_0 such that, for $T > 0$ and $M > 0$, there exist positive constants C, $h_3 \leq \min(h_0, h_1, h_2)$ such that for $0 < h \leq h_3$ and for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$ satisfying $|y_0|_{H^1(\Omega)} \leq M$, there exists a semi-discrete control function v such that the solution to the semi-linear parabolic equation*

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + \mathcal{G}(y) = \mathbf{1}_\omega v, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0. \tag{5.30}$$

satisfies

$$|y(T)|_{L^2(\Omega)} \leq Ce^{-C_0/h}|y_0|_{L^2(\Omega)}, \qquad \textit{and} \quad \|v\|_{L^2(Q)} \leq C|y_0|_{L^2(\Omega)}.$$

Here $C = C(T, M)$.

Proof. The proof follows that of Theorem 5.5. We set $Z_R = L^\infty(0, T_R; \mathbb{R}^{\mathfrak{M}})$. Denoting by B_R the ball centered at 0 and of radius R in Z_R, the following lemma replaces Lemma 5.9.

LEMMA 5.12. *There exists $R_0 = R_0(T, M)$ such that the map Λ_R maps B_R into itself if $R \geq R_0$ and if $|y_0|_{H^1(\Omega)} \leq M$.*

Here R_0 is not connected to h. We choose $R = R_0$. If we take h sufficiently small, $0 < h \leq \tilde{h}_3$ with $\tilde{h}_3 = \tilde{h}_3(T, M) = \min(h_0, h_1, h_2, C(1 + 1/T + \zeta(R)^{2/3})^{-1})$, then (5.20) is fulfilled.

As Λ_R is also continuous and $\Lambda_R(B_R)$ is precompact this yields the existence of $y \in B_R$ such that $\Lambda_R(y) = y$. Setting $v_R = L_{T_R, a_y}(y_0)$ we obtain

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + ya_y = \mathbf{1}_\omega v, \quad t \in (0, T_R], \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0.$$

Since $y \in B_R$ we have $\|y\|_{L^\infty(Q)} \leq R$. Then we have $a_y = g\big(T_R(y)\big) = g(y)$, which yields

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + \mathcal{G}(y) = \mathbf{1}_\omega v, \quad t \in (0, T_R], \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0.$$

With the estimations (5.17)–(5.18) we have

$$|y(T_R)|_{L^2(\Omega)} \leq C(T,M)e^{-C_0/h}|y_0|_{L^2(\Omega)}, \qquad \|v_R\|_{L^2(Q_R)} \leq C(T,M)|y_0|_{L^2(\Omega)},$$

as $R = R_0$ is chosen independently of h here.

We now define v on $[0,T]$ by

$$v = \begin{cases} v_R & \text{if } 0 < t \leq T_R, \\ 0 & \text{if } T_R < t \leq T, \end{cases}$$

We naturally have $\|v\|_{L^2(Q)} \leq C(T,M)|y_0|_{L^2(\Omega)}$.

If we have

$$h^{-1/2}|y(T_R)|_{L^2(\Omega)} \leq M_0, \tag{5.31}$$

we can apply Proposition 5.4 on the time interval $[T_R, T]$, which yields

$$|y(T)|_{L^2(\Omega)} \leq |y(T_R)|_{L^2(\Omega)} e^{K_0(T-T_R)} \leq C(T,M)e^{-C_0/h}|y_0|_{L^2(\Omega)}.$$

With (5.22), choosing $h_3 \leq \min(h_0, h_1, h_2, \tilde{h}_3)$ sufficiently small, condition (5.23) can be fulfilled if $0 < h \leq h_3$, which concludes the proof. ∎

Proof of Lemma 5.12. From Proposition 5.1 and (5.15) we have

$$\begin{aligned}\|Dy_z\|_{L^\infty(0,T_R,L^2(\Omega))} &\leq e^{C\left(1+T_R+(T_R^{\frac12}+T_R)\|a_z\|_\infty\right)}\left(|Dy_0|_{L^2(\Omega)} + \|v\|_{L^2(Q)}\right) \\ &\leq e^{C\left(1+T_R+\frac{1}{T_R}+(T_R^{\frac12}+T_R)\|a_z\|_\infty+\|a_z\|_\infty^{2/3}\right)}|y_0|_{H^1(\Omega)} \\ &\leq e^{C'\left(1+T+\frac{1}{T}+\zeta(R)^{2/3}\right)}|y_0|_{H^1(\Omega)}.\end{aligned}$$

In the one dimensional case if $f \in \mathbb{R}^{\mathfrak{M}}$ with $f_{|\partial\Omega} = 0$ we have

$$|f|_{L^\infty} \leq |Df|_{L^1(\Omega)} \leq C_\Omega |Df|_{L^2(\Omega)}.$$

We thus obtain

$$\|y_z\|_{L^\infty(Q_R)} \leq e^{C\left(1+T+\frac{1}{T}+\zeta(R)^{2/3}\right)}|y_0|_{H^1(\Omega)}. \tag{5.32}$$

We hence find

$$R^{-1}\|y_z\|_{L^\infty(Q_R)} \leq e^{C\left(1+T+\frac{1}{T}+\ln^{2r/3}(e+R)\right)-\ln(R)}|y_0|_{H^1(\Omega)}.$$

As $r < 3/2$, if $|y_0|_{H^1(\Omega)} \leq M$ there exists $R_0 > 0$, depending on T and M such that

$$\|y_z\|_{L^\infty(Q_R)} \leq R, \qquad \text{if } R \geq R_0.$$

Hence, for $R \geq R_0$ the map Λ_R maps B_R into itself. ∎

Remark 5.13 (local controllability in one space dimension). *Estimate* (5.32) *is used above to prove controllability thanks to the form of the non-linearity. For an arbitrary non-linearity one can also use* (5.32) *and impose a sufficiently small initial condition y_0 in H^1-norm, which yields*

$$\|y_z\|_{L^\infty(Q_R)} \leq R.$$

The rest of the proof remains unchanged and this yields the following local controllability result.

THEOREM 5.14. *Let $d = 1$ and $\Omega = (0, 1)$, γ satisfy (1.2), and the function $\mathcal{G}$ of the form (5.4). There exists C_0 such that, for $T > 0$ there exist positive constants C, $h_3 \leq \min(h_0, h_1, h_2)$ and $\varepsilon > 0$, such that for $0 < h \leq h_3$ and for all initial data $y_0 \in \mathbb{R}^{\mathfrak{M}}$ satisfying $|y_0|_{H^1(\Omega)} \leq \varepsilon$, there exists a semi-discrete control function v such that the solution to the semi-linear parabolic equation*

$$(\partial_t + \mathcal{A}^{\mathfrak{M}})y + \mathcal{G}(y) = \mathbf{1}_\omega v, \qquad y_{|\partial\Omega} = 0, \qquad y(0) = y_0. \tag{5.33}$$

satisfies

$$|y(T)|_{L^2(\Omega)} \leq Ce^{-C_0/h}|y_0|_{L^2(\Omega)}, \qquad \text{and} \quad \|v\|_{L^2(Q)} \leq C|y_0|_{L^2(\Omega)}.$$

REMARK 5.15. *Smallness of the initial condition in H^1-norm can be obtained by setting the control function to zero for a short initial time and starting from a small initial condition in L^2-norm by Lemma 5.10 that also lays in a bounded set of L^∞-norm.*

Appendix A. Proofs of intermediate results in Section 3.

A.1. Proof of Lemma 3.4. We have

$$I_{13} = \sum_{i\in[\![1,d]\!]} \iint_Q r\overline{\tilde{\rho}^i}^i \, \bar{D}_i(\gamma_i D_i v)\partial_t v dt.$$

As $v|_{\partial\Omega} = 0$, with a discrete integration by parts, we have

$$I_{13} = -\sum_{i\in[\![1,d]\!]} \iint_Q D_i\big(r\overline{\tilde{\rho}^i}^i \, \partial_t v\big)\gamma_i D_i v \; dt = Q_1 + Q_2,$$

with

$$\begin{aligned}
Q_1 &= -\sum_{i\in[\![1,d]\!]} \iint_Q D_i\big(r\overline{\tilde{\rho}^i}^i\big)\partial_t \tilde{v}^i \gamma_i D_i v \; dt,\\
Q_2 &= -\sum_{i\in[\![1,d]\!]} \iint_Q \widetilde{r\overline{\tilde{\rho}^i}^i}^i (\partial_t D_i v)\gamma_i D_i v \; dt.
\end{aligned}$$

Proposition 2.13 shows that $D_i\big(r\overline{\tilde{\rho}^i}^i\big) = \mathcal{O}_{\lambda,\mathfrak{K}}(sh)$, it then follows that

$$\begin{aligned}
\big|Q_1\big| &\leq \sum_{i\in[\![1,d]\!]} \iint_Q s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big(\partial_t \tilde{v}^i\big)^2 \; dt + \sum_{i\in[\![1,d]\!]} \iint_Q s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big(D_i v\big)^2 \; dt\\
&\leq \sum_{i\in[\![1,d]\!]} \iint_Q s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\widetilde{\big(\partial_t v\big)^2}^i \; dt + \sum_{i\in[\![1,d]\!]} \iint_Q s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big(D_i v\big)^2 \; dt\\
&= \iint_Q s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big(\partial_t v\big)^2 \; dt + \sum_{i\in[\![1,d]\!]} \iint_Q s\mathcal{O}_{\lambda,\mathfrak{K}}(sh)\big(D_i v\big)^2 \; dt
\end{aligned} \tag{A.1}$$

as $\big(\partial_t \tilde{v}^i\big)^2 \leq \widetilde{\big(\partial_t v\big)^2}^i$, by convexity and as $v|_{\partial\Omega} = 0$.

We write, using that γ_i do not depend on time, that

$$
\begin{aligned}
Q_2 &= -\frac{1}{2} \sum_{i\in[\![1,d]\!]} \iint_Q \widetilde{r\overline{\tilde{\rho}}^i}^{\,i} \gamma_i \partial_t \big(D_i v\big)^2 \, dt \\
&= \frac{1}{2} \sum_{i\in[\![1,d]\!]} \iint_Q \partial_t \Big(\widetilde{r\overline{\tilde{\rho}}^i}^{\,i}\Big) \gamma_i \big(D_i v\big)^2 \, dt - \frac{1}{2} \sum_{i\in[\![1,d]\!]} \int_\Omega \widetilde{r\overline{\tilde{\rho}}^i}^{\,i} \gamma_i \big(D_i v\big)^2 \Big|_{t=0}^{t=T}.
\end{aligned}
$$

We observe that for $0 < sh < \varepsilon_1(\lambda)$ with $\varepsilon_1(\lambda)$ sufficiently small we have $r\overline{\tilde{\rho}}^i > 0$ by Proposition 2.13. The signs of the terms at $t = T$ and $t = 0$ are thus prescribed. Moreover, by Proposition 2.14, we know that $\partial_t(r\overline{\tilde{\rho}}^i) = T(sh)^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)$ so that, for $sh \leq \mathfrak{K}$, we obtained the result.□

A.2. Proof of Lemma 3.7. We have

$$
I_{23} = \sum_{i\in[\![1,d]\!]} \iint_Q \gamma_i r(\bar{D}_i D_i \rho)\, \overline{\tilde{v}^i}^{\,i} \partial_t v \, dt.
$$

As $v|_{\partial\Omega} = 0$ we write

$$
I_{23} = \sum_{i\in[\![1,d]\!]} \iint_Q \widetilde{\gamma_i r(\bar{D}_i D_i \rho)\, \partial_t v}^{\,i}\, \tilde{v}^i \, dt = \quad Q_1 + Q_2,
$$

by Lemma 2.2 with

$$
\begin{aligned}
Q_1 &= \sum_{i\in[\![1,d]\!]} \iint_Q \widetilde{\gamma_i r(\bar{D}_i D_i \rho)}^{\,i}\, \partial_t \tilde{v}^i \tilde{v}^i \, dt, \\
Q_2 &= \sum_{i\in[\![1,d]\!]} \frac{h_i^2}{4} \iint_Q D_i(\gamma_i r(\bar{D}_i D_i \rho))(D_i \partial_t v) \tilde{v}^i \, dt.
\end{aligned}
$$

We have

$$
Q_1 = -\frac{1}{2} \sum_{i\in[\![1,d]\!]} \iint_Q \partial_t \big(\widetilde{\gamma_i r(\bar{D}_i D_i \rho)}^{\,i}\big) \big(\tilde{v}^i\big)^2 \, dt + \frac{1}{2} \sum_{i\in[\![1,d]\!]} \int_\Omega \widetilde{\gamma_i r(\bar{D}_i D_i \rho)}^{\,i} \big(\tilde{v}^i\big)^2 \Big|_{t=0}^{t=T}.
$$

By Proposition 2.13 and Lemma 2.7 we have

$$
\widetilde{\gamma_i r(\bar{D}_i D_i \rho)}^{\,i} = s^2 \mathcal{O}_{\lambda,\mathfrak{K}}(1)
$$

Lemma A.1. *We have*

$$
\partial_t \big(\widetilde{\gamma_i r(\bar{D}_i D_i \rho)}^{\,i}\big) = T s^2 \theta \mathcal{O}_{\lambda,\mathfrak{K}}(1).
$$

Proof. Since γ_i do not depend on time, we have

$$
\partial_t \big(\gamma_i r(\bar{D}_i D_i \rho)\big) = \gamma_i \partial_t \big(r(\bar{D}_i D_i \rho)\big)
$$

which is bounded by $Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)$ thanks to Proposition 2.14. The action of the avering operator $\tilde{\cdot}^i$ does not affect the form of this estimate. □

As $\big(\tilde{v}^i\big)^2 \leq \widetilde{(v)^2}^{\,i}$ and $v|_{\partial\Omega} = 0$, we thus have

$$
Q_1 = \iint_Q T s^2 \theta \mathcal{O}_{\lambda,\mathfrak{K}}(1) \big(v\big)^2 \, dt + \int_\Omega s^2 \Big(\mathcal{O}_{\lambda,\mathfrak{K}}(1) \big(v_{|t=T}\big)^2 + \mathcal{O}_{\lambda,\mathfrak{K}}(1) \big(v_{|t=0}\big)^2\Big). \tag{A.2}
$$

With an integration by parts in time and Lemma 2.1 we obtain $Q_2 = Q_a + Q_b$, with

$$Q_a = -\sum_{i\in[\![1,d]\!]} \frac{h_i^2}{4} \iint_Q \partial_t\big(D_i(\gamma_i r(\bar{D}_i D_i\rho))\tilde{v}^i\big)(D_i v)\, dt$$

$$Q_b = \sum_{i\in[\![1,d]\!]} \frac{h_i^2}{8} \int_\Omega D_i(\gamma_i r(\bar{D}_i D_i\rho))D_i\big(v\big)^2 \big|_{t=0}^{t=T}.$$

With Lemma 2.1 and as $v|_{\partial\Omega} = 0$ with a discrete integration by parts in space (Proposition 2.4) we have

$$Q_a = \sum_{i\in[\![1,d]\!]} \frac{h_i^2}{8} \iint_Q \big(\partial_t \bar{D}_i D_i(\gamma_i r(\bar{D}_i D_i\rho))\big)\big(v\big)^2\, dt$$
$$- \sum_{i\in[\![1,d]\!]} \frac{h_i^2}{4} \iint_Q \big(D_i(\gamma_i r(\bar{D}_i D_i\rho))\big)(\partial_t\tilde{v}^i)(D_i v)\, dt$$

To estimate Q_b we perform a discrete integration by parts using that $v|_{\partial\Omega} = 0$,

$$Q_b = -\sum_{i\in[\![1,d]\!]} \frac{h_i^2}{8} \int_\Omega \bar{D}_i D_i(\gamma_i r(\bar{D}_i D_i\rho))\big(v\big)^2 \big|_{t=0}^{t=T}.$$

Lemma A.2. *We have*

$$D_i(\gamma_i r(\bar{D}_i D_i\rho)) = s^2\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad h_i^2 \bar{D}_i D_i(\gamma_i r(\bar{D}_i D_i\rho)) = s(sh)\mathcal{O}_{\lambda,\mathfrak{K}}(1),$$
$$h_i^2 \partial_t \bar{D}_i D_i(\gamma_i r(\bar{D}_i D_i\rho)) = Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1).$$

Proof. We use Lemma 2.1 to get

$$D_i(\gamma_i r(\bar{D}_i D_i\rho)) = (D_i\gamma_i)\widetilde{r(\bar{D}_i D_i\rho))}^i + \widetilde{\gamma_i}^i D_i\big(r(\bar{D}_i D_i\rho))\big)$$

and the required estimate follows from the Lipschitz regularity of γ_i and Proposition 2.13. The second estimate is deduced from the first one by observing that $h_i D_i = \tau_i^+ - \tau_i^-$. For the third estimate, since γ_i do not depend on the time, we can write

$$h_i^2 \partial_t \bar{D}_i D_i(\gamma_i r(\bar{D}_i D_i\rho)) = (\bar{\tau}_i{}^+ - \bar{\tau}_i{}^-)(\tau_i^+ - \tau_i^-)\big(\gamma_i \partial_t(r(\bar{D}_i D_i\rho))\big).$$

The conclusion comes from Proposition 2.14. ∎

With the Cauchy-Schwarz inequality, using that $\big(\partial_t\tilde{v}^i\big)^2 \leq \widetilde{\big(\partial_t v\big)^2}^i$ and $v|_{\partial\Omega} = 0$ we obtain

$$Q_a \geq \iint_Q Ts^2\theta\mathcal{O}_{\lambda,\mathfrak{K}}(1)\big(v\big)^2\, dt + \iint_Q (sh)^2 s^{-1}\mathcal{O}_{\lambda,\mathfrak{K}}(1)\big(\partial_t v\big)^2\, dt \tag{A.3}$$
$$+ \sum_{i\in[\![1,d]\!]} \iint_Q (sh)^2 s\mathcal{O}_{\lambda,\mathfrak{K}}(1)\big(D_i v\big)^2\, dt.$$

and

$$Q_b = \int_\Omega \mathcal{O}_{\lambda,\mathfrak{K}}(1)\big(s(sh)\big(v\big)^2\big)\big|_{t=T} + \int_\Omega \mathcal{O}_{\lambda,\mathfrak{K}}(1)\big(s(sh)\big(v\big)^2\big)\big|_{t=0}. \tag{A.4}$$

In fine, collecting (A.2), (A.3), (A.4), we obtain the result. ∎

A.3. Proof of Lemma 3.8. We have

$$I_{31} = -2\tau \sum_{i\in[\![1,d]\!]} \iint_Q (\partial_t\theta)\varphi\gamma_i r\overline{D_i\rho}^i\, v\overline{D_i v}^i\, dt.$$

As $v|_{\partial\Omega} = 0$, we write

$$I_{31} = -2\tau \sum_{i\in[\![1,d]\!]} \iint_Q (\partial_t\theta)\widetilde{\varphi\gamma_i r\overline{D_i\rho}^i\, v}^i\, D_i v\, dt$$

We have

$$\widetilde{\varphi\gamma_i r\overline{D_i\rho}^i\, v}^i = \widetilde{\varphi\gamma_i r\overline{D_i\rho}^i}\, \widetilde{v}^i + \frac{h_i^2}{4} D_i\big(\varphi\gamma_i r\overline{D_i\rho}^i\big) D_i v$$

We obtain

$$\begin{aligned} I_{31} &= -\tau \sum_{i\in[\![1,d]\!]} \iint_Q (\partial_t\theta)\widetilde{\varphi\gamma_i r\overline{D_i\rho}^i}^i\, D_i\big(v\big)^2\, dt \\ &\quad - \frac{h_i^2}{2}\tau \sum_{i\in[\![1,d]\!]} \iint_Q (\partial_t\theta) D_i\big(\varphi\gamma_i r\overline{D_i\rho}^i\big)\big(D_i v\big)^2\, dt \\ &= \tau \sum_{i\in[\![1,d]\!]} \iint_Q (\partial_t\theta)\overline{D_i\big(\varphi\gamma_i r\overline{D_i\rho}^i\big)}^i\, \big(v\big)^2\, dt \\ &\quad - \frac{h_i^2}{2}\tau \sum_{i\in[\![1,d]\!]} \iint_Q (\partial_t\theta) D_i\big(\varphi\gamma_i r\overline{D_i\rho}^i\big)\big(D_i v\big)^2\, dt. \end{aligned}$$

with a discrete integration by parts.

By using the Lipschitz continuity of $\varphi\gamma_i$ and Proposition 2.13 we get that

$$D_i\big(\varphi\gamma_i r\overline{D_i\rho}^i\big) = s\mathcal{O}_{\lambda,\mathfrak{K}}(1), \quad \overline{D_i\big(\varphi\gamma_i r\overline{D_i\rho}^i\big)}^i = s\mathcal{O}_{\lambda,\mathfrak{K}}(1).$$

With (1.10), the result follows.□

A.4. Proof of Lemma 3.9. We have

$$\begin{aligned} I_{33} &= -\tau \iint_Q (\partial_t\theta)\varphi\, v\partial_t v\; dt = -\frac{1}{2}\tau \iint_Q (\partial_t\theta)\varphi\, \partial_t\big(v\big)^2\; dt \\ &= \frac{1}{2}\tau \iint_Q (\partial_t^2\theta)\varphi\big(v\big)^2\; dt - \frac{1}{2}\tau \int_\Omega (\partial_t\theta)\varphi\,\big(v\big)^2\Big|_{t=0}^{t=T}. \end{aligned}$$

With (1.10) we have

$$-\partial_t\theta(0) = \partial_t\theta(T) = T\theta^2(T) > 0.$$

As $\varphi < 0$ the result follows. □

A.5. Proof of Lemma 3.11. We choose $i, j \in [\![1, d]\!]$ with $i \neq j$. We have

$$\begin{aligned}
&\iint_Q s\phi\gamma_i \left(D_i v\right)^2 dt \\
&\quad\geq C \iint_Q s\phi \left(D_i v\right)^2 dt = C \iint_Q s\widetilde{\phi \left(D_i v\right)^2}^{\,j} dt \\
&\quad= C \iint_Q s\tilde{\phi}^j \,\widetilde{\left(D_i v\right)^2}^{\,j} dt + C\frac{h^2}{4} \iint_Q s(D_j\phi)\, D_j \left(D_i v\right)^2 dt \\
&\quad= C \iint_Q s\tilde{\phi}^j \left(\widetilde{D_i v}^{\,j}\right)^2 dt + C\frac{h^2}{4} \iint_Q s\tilde{\phi}^j \left(D_j D_i v\right)^2 dt - C\frac{h^2}{4} \iint_Q s(\bar{D}_j D_j\phi) \left(D_i v\right)^2 dt.
\end{aligned}$$

by Proposition 2.4 as $D_i v|_{(0,T)\times\partial_j\Omega} = 0$ and by Lemma 2.2. We thus have

$$\iint_Q s\phi\gamma_i \left(D_i v\right)^2 dt \geq C\frac{h^2}{4} \iint_Q s\tilde{\phi}^j \left(D_j D_i v\right)^2 dt - C\frac{h^2}{4} \iint_Q s(\bar{D}_j D_j\phi) \left(D_i v\right)^2 dt. \quad \text{(A.5)}$$

With Lemma 2.6 we note that

$$\tilde{\phi}^j = \phi + h^2\mathcal{O}_\lambda(1), \qquad \bar{D}_j D_j \phi = \partial_j^2\phi + h^2\mathcal{O}_\lambda(1) = \mathcal{O}_\lambda(1),$$

which justifies the last term in H, and contributes to the first term in $\tilde{X}$ and the first term in $\tilde{W}$.

Similarly for $i \in [\![1, d]\!]$, we also write

$$\begin{aligned}
\iint_Q \gamma_i\phi \left(D_i v\right)^2 dt &\geq C \iint_Q \phi \left(D_i v\right)^2 dt \\
&= \underbrace{\iint_Q \overline{\phi \left(D_i v\right)^2}^{\,i} dt}_{=\mathscr{Q}_i} + \frac{h_i}{2} \iint_{Q_i} \left((\phi \left(D_i v\right)^2)_{\frac{1}{2}} + (\phi \left(D_i v\right)^2)_{N_i+\frac{1}{2}} \right) dt,
\end{aligned}$$

by Proposition 2.4, and Lemma 2.2 yields

$$\begin{aligned}
\mathscr{Q}_i &= \iint_Q \overline{\phi}^{\,i}\, \overline{\left(D_i v\right)^2}^{\,i} dt + \frac{h_i^2}{4} \iint_Q (\bar{D}_i\phi)\, \bar{D}_i \left(D_i v\right)^2 dt \\
&= \iint_Q \overline{\phi}^{\,i} \left(\overline{D_i v}^{\,i}\right)^2 dt + \frac{h_i^2}{4} \iint_Q \overline{\phi}^{\,i} \left(\bar{D}_i D_i v\right)^2 dt - \frac{h_i^2}{4} \iint_Q (D_i\bar{D}_i\phi) \left(D_i v\right)^2 dt \\
&\quad + \frac{h_i^2}{4} \iint_{Q_i} \left((\bar{D}_i\phi)_{N_i+1} \left(D_i v\right)^2_{N_i+\frac{1}{2}} - (\bar{D}_i\phi)_0 \left(D_i v\right)^2_{\frac{1}{2}} \right) dt.
\end{aligned}$$

We observe that

$$\begin{aligned}
\nu(h,\lambda) = \frac{h_i}{2} &\iint_{Q_i} \left((\phi \left(D_i v\right)^2)_{\frac{1}{2}} + (\phi \left(D_i v\right)^2)_{N_i+\frac{1}{2}} \right) dt \\
&+ \frac{h_i^2}{4} \iint_{Q_i} \left((\bar{D}_i\phi)_{N_i+1} \left(D_i v\right)^2_{N_i+\frac{1}{2}} - (\bar{D}_i\phi)_0 \left(D_i v\right)^2_{\frac{1}{2}} \right) dt,
\end{aligned}$$

can be made non-negative for h sufficiently small once λ is fixed, as $\bar{D}_i\phi = \mathcal{O}_\lambda(1)$. With Lemma 2.6 we note that

$$\tilde{\phi}^i = \phi + h\mathcal{O}_\lambda(1), \qquad D_i\bar{D}_i\phi = \partial_i^2\phi + h^2\mathcal{O}_\lambda(1) = \mathcal{O}_\lambda(1),$$

which justifies the first and second term in H, and contributes to the two terms in $\tilde{X}$ and the second term in $\tilde{W}$. ☐

REFERENCES

[Bar00] V. Barbu, *Exact controllability of the superlinear heat equation*, Appl. Math. Optim. **42** (2000), 73–89.

[BHL10a] F. Boyer, F. Hubert, and J. Le Rousseau, *Discrete Carleman estimates and uniform controllability of semi-discrete parabolic equations*, J. Math. Pures Appl. **93** (2010), 240–276.

[BHL10b] ———, *Discrete Carleman estimates for elliptic operators in arbitrary dimension and applications*, SIAM J. Control Optim. **48** (2010), 5357–5397.

[BHL11] ———, *Uniform null-controllability properties for space/time-discretized parabolic equations*, Numer. Math, **118** (2011), 601–661.

[Bih56] I. Bihari, *A generalization of a lemma of bellman and its application to uniqueness problems of differential equations*, Acta Mathematica Hungarica **7** (1956), 81–94.

[Car39] T. Carleman, *Sur une problème d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes*, Ark. Mat. Astr. Fys. **26B** (1939), no. 17, 1–9.

[FCG06] E. Fernández-Cara and S. Guerrero, *Global Carleman inequalities for parabolic systems and application to controllability*, SIAM J. Control Optim. **45** (2006), no. 4, 1395–1446.

[FCZ00] E. Fernández-Cara and E. Zuazua, *Null and approximate controllability for weakly blowing up semilinear heat equations*, Ann. Inst. H. Poincaré, Analyse non lin. **17** (2000), 583–616.

[FI96] A. Fursikov and O. Yu. Imanuvilov, *Controllability of evolution equations*, vol. 34, Seoul National University, Korea, 1996, Lecture notes.

[Hör58] L. Hörmander, *On the uniqueness of the Cauchy problem*, Math. Scand. **6** (1958), 213–225.

[Hör63] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.

[Hör85] ———, *The Analysis of Linear Partial Differential Operators*, vol. IV, Springer-Verlag, 1985.

[Ima95] O. Yu. Imanuvilov, *Controllability of parabolic equations (russian)*, Mat. Sbornik. Novaya Seriya **186** (1995), 109–132.

[LL11] J. Le Rousseau and G. Lebeau, *On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations*, ESAIM Control Optim. Calc. Var. (2011), doi:10.1051/cocv/2011168.

[LR95] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur*, Comm. Partial Differential Equations **20** (1995), 335–356.

[MFC12] A. Münch and E. Fernández-Cara, *Numerical null controllability of semi-linear 1d heat equations : fixed point, least squares and newton methods*, Mathematical Control and Related Fields **to appear** (2012).

[Zua06] E. Zuazua, *Control and numerical approximation of the wave and heat equations*, International Congress of Mathematicians, Madrid, Spain **III** (2006), 1389–1417.

[Zui83] C. Zuily, *Uniqueness and Non Uniqueness in the Cauchy Problem*, Birkhauser, Progress in mathematics, 1983.