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STRONGLY REINFORCED VERTEX-REINFORCED-RANDOM-WALK ON THE COMPLETE GRAPH

MICHEL BENAÏM, OLIVIER RAIMOND, BRUNO SCHAPIRA

ABSTRACT. We study Vertex-Reinforced-Random-Walk on the complete graph with weights of the form $w(n) = n^\alpha$, with $\alpha > 1$. Unlike for the Edge-Reinforced-Random-Walk, which in this case localizes a.s. on 2 sites, here we observe various phase transitions, and in particular localization on arbitrary large sets is possible, provided α is close enough to 1. Our proof relies on stochastic approximation techniques.

1. INTRODUCTION

This paper considers a Vertex-Reinforced Random Walk (VRRW) on a finite complete graph with weights $w_\alpha(n) := (n+1)^\alpha$ in the strongly reinforced regime $\alpha > 1$.

Such a process is a discrete time random process $(X_n)_{n \geq 0}$ living in $E = \{1, \dots, N\}$ and such that for all $n \geq 0$ and $j \in E$,

$$\mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \frac{w_\alpha(Z_n(j))}{\sum_{k \neq X_n} w_\alpha(Z_n(k))} \mathbf{1}_{\{X_n \neq j\}},$$

where $Z_n(j) := \sum_{\ell=0}^n \mathbf{1}_{\{X_\ell=j\}}$, is the number of jumps to site j before time n , and $\mathcal{F}_n = \sigma(X_k; k \leq n)$.

The linear regime (i.e $\alpha = 1$) has been initially introduced by Pemantle [P] on a finite graph and then extensively studied for different type of graphs by several authors ([PV, T1, V, LV, BT]).

The main result of the present paper is the following:

Theorem 1.1. *Let $N \geq 2$ and $\alpha > 1$ be given. Then the following properties hold.*

- (i) *With probability one there exists $2 \leq \ell \leq N$, such that (X_n) visits exactly ℓ sites infinitely often, and the empirical occupation measure converges towards the uniform measure on these ℓ sites.*
- (ii) *Let $3 \leq k \leq N$. If $\alpha > (k-1)/(k-2)$, then the probability to visit strictly more than $k-1$ sites infinitely often is zero.*

If $\alpha < (k-1)/(k-2)$, then for any $2 \leq \ell \leq k$, the probability that exactly ℓ sites are visited infinitely often is positive.

This result has to be compared with the situation for Edge Reinforced Random Walks (ERRW). Limic [L] (see also [LT]), proved that for any $\alpha > 1$, and for any free loops graph with bounded degree, the ERRW with weight w_α visits only 2 sites infinitely often. It has also to be compared to the situation on the graph \mathbb{Z} , where for any $\alpha > 1$, the VRRW with weights $(w_\alpha(n))$ visits a.s. only 2 sites infinitely often.

It might be interesting to notice also that when we add one loop to each site, i.e. when at each step, independently of the actual position of the walk, the probability to jump to some site i is proportional to $w_\alpha(Z_n(i))$, then Rubin's construction (see [D]) immediately shows that the walk visits a.s. only one site infinitely often. In fact with our techniques we can study a whole family of processes which interpolate between these two examples: for $c \geq 0$, consider the process with transitions probabilities given by

$$P(X_{n+1} = j \mid \mathcal{F}_n) = \frac{w_\alpha(Z_n(j))\mathbf{1}_{\{X_n \neq j\}} + c w_\alpha(Z_n(j))\mathbf{1}_{\{X_n = j\}}}{\sum_{k \neq X_n} w_\alpha(Z_n(k)) + c w_\alpha(Z_n(X_n))},$$

with the same notation as above. The case $c = 0$ corresponds to the VRRW on the free loop complete graph, and the case $c = 1$ corresponds to the VRRW on the complete graph with loops. Then for any $c \geq 1$, the process visits a.s. only 1 site infinitely often, and when $c \in (0, 1)$, various phase transitions occur, exactly as in Theorem 1.1, except that the critical values are this time equal to $[k - (1 - c)]/[k - 2(1 - c)]$, for $2 \leq k \leq N$ and localization on 1 site is always possible and occurs even a.s. when $\alpha > (1 + c)/(2c)$. Since the proofs of these results are exactly similar as those for Theorem 1.1, we will not give further details here.

Finally let us observe that similar phase transitions as in Theorem 1.1 have been observed in some random graphs models, see for instance [CHJ, OS].

2. A GENERAL FORMALISM FOR VRRW

We present here a general and natural framework for studying VRRW based on the formalism and results introduced in [B] and [BR]. Such a formalism heavily relies on stochastic approximation technics and specifically the dynamical system approach developed in [B2].

Let $A = (A_{i,j})_{i,j \leq N}$ be a $N \times N$ symmetric matrix with nonnegative entries. We assume that $A_{i,j} > 0$ for $i \neq j$, and that $\sum_j A_{i,j}$ does not depend on i . Let $\alpha > 1$ be given. We consider the process $(X_n)_{n \geq 0}$

living in $E = \{1, \dots, N\}$, with transition probabilities given by

$$\mathbf{P}(X_{n+1} = j \mid \mathcal{F}_n) = \frac{A_{X_n, j}(1 + Z_n(j))^\alpha}{\sum_{k \leq N} A_{X_n, k}(1 + Z_n(k))^\alpha},$$

where Z_n is defined like in the introduction. The case of the VRRW on the complete graph is obtained by taking

$$(1) \quad A_{i, j} = 1 - \delta_{ij}.$$

For $i \leq N$, set $v_n(i) = Z_n(i)/(n+1)$. Note that if $A_{i, i} = 0$ and $X_k = i$ then $X_{k+1} \neq i$. In particular for any such i , and $n \geq 1$, $v_n(i) \leq (1/2 + 1/(n+1)) \leq 3/4$. In other words, for all $n \geq 1$, v_n belongs to the *reduced simplex*

$$\Delta := \left\{ v \in \mathbb{R}_+^N : v_i \leq 3/4 \text{ if } A_{ii} = 0 \text{ and } \sum_i v_i = 1 \right\}.$$

In the following, we might sometimes view an element $f = (f_i)_{i \leq N} \in \mathbb{R}^N$ as a function on E , and so we will also use the notation $f(i)$ for f_i .

Now for $\epsilon \in [0, 1]$ and $v \in \Delta$ we let $K(\epsilon, v)$ denote the transition matrix defined by

$$(2) \quad K_{i, j}(\epsilon, v) := \frac{A_{i, j}(\epsilon + v_j)^\alpha}{\sum_k A_{i, k}(\epsilon + v_k)^\alpha},$$

for all $i, j \leq N$. To shorten notation we let

$$(3) \quad K(v) := K(0, v).$$

Two obvious, but key, observations are that

$$\mathbf{P}(X_{n+1} = j \mid \mathcal{F}_n) = K_{X_n, j}((n+1)^{-1}, v_n),$$

and

$$\lim_{n \rightarrow \infty} K((n+1)^{-1}, v) = K(v).$$

Hence, relying on [B] and [BR], the behavior of (v_n) can be analyzed through the ordinary differential equation $\dot{v} = -v + \pi(v)$, where $\pi(v)$ is the invariant probability measure of the Markov chain with transition matrix $K(v)$.

3. THE LIMIT SET THEOREM

3.1. The limiting differential equation and its equilibria. For $v \in \Delta$, set $v^\alpha = (v_1^\alpha, \dots, v_N^\alpha)$, and

$$(4) \quad H(v) := \sum_{i, j} A_{i, j} v_i^\alpha v_j^\alpha = \langle Av^\alpha, v^\alpha \rangle.$$

Note that H is positive on Δ . Hence one can define

$$\pi_i(v) = \frac{v_i^\alpha (Av^\alpha)_i}{H(v)}, \quad i = 1, \dots, N.$$

From the relation

$$v_j^\alpha (Av^\alpha)_i K_{i,j}(v) = A_{i,j} v_i^\alpha v_j^\alpha,$$

it follows that $K(v)$ is reversible with respect to $\pi(v)$.

For $r = 0, 1$ set $T_r\Delta = \{v \in \mathbb{R}^N : \sum_i v_i = r\}$ and let $\iota : T_1\Delta \rightarrow \Delta$ be the map defined by $\iota(v) = \operatorname{argmin}\{\|y - v\| : y \in \Delta\}$. Since Δ is convex ι is a Lipschitz retraction from $T_1\Delta$ onto Δ .

Let now $F : T_1\Delta \rightarrow T_0\Delta$ be the vector field defined as

$$(5) \quad F(v) = -v + \pi(\iota(v)).$$

Note that F is Lipschitz. Thus by standard results, F induces a global flow $\Phi : \mathbb{R} \times T_1\Delta \rightarrow T_1\Delta$ where for all $x \in T_1\Delta$, $t \mapsto \Phi(t, x) := \Phi_t(x)$ is the solution to $\dot{v} = F(v)$ with initial condition $v(0) = x$.

For any subset $I \subset \{1, \dots, N\}$, we let

$$\Delta_I := \{v \in \Delta : v_i = 0 \quad \forall i \in E \setminus I\},$$

denote the I -face of Δ . We let $\operatorname{int}\Delta_I = \{v \in \Delta_I : v_i > 0 \quad \forall i \in I\}$ denote the (relative) interior of Δ_I . For $v \in \Delta$ we let $\operatorname{supp}(v) = \{i \in E : v_i \neq 0\}$, so that v always lies in the $\operatorname{supp}(v)$ -face of Δ .

Lemma 3.1. *The flow Φ leaves Δ positively invariant: $\forall t \geq 0$, $\Phi_t(\Delta) \subset \Delta$; and*

for each $I \subset E$, the face Δ_I is locally invariant: $\forall v \in \Delta_I$, $\forall t \in \mathbb{R}$, $\Phi_t(v) \in \Delta \Leftrightarrow \Phi_t(v) \in \Delta_I$.

Proof. For all $v \in \Delta$, $\pi(v)$ lies in Δ . Indeed for the Markov chain having transition matrix $K(v)$ the empirical occupation measure lies in Δ and by the ergodic theorem (for finite Markov chains) the same is true for $\pi(v)$. Hence $F(v)$ points inward Δ for all $v \in \Delta$, proving that Δ is positively invariant. Since $F_i(v) = 0$ when $v_i = 0$, each face is locally invariant. \square

Let

$$\mathcal{C} = \{v \in \Delta : F(v) = 0\},$$

denote the *equilibria set* of F . Relying on stochastic approximation theory [B, B2, BR] it will be shown below (Proposition 3.5) that $(v_n, n \geq 1)$ converges almost surely to \mathcal{C} .

The next result is similar to the case $\alpha = 1$ (see for instance [P]):

Lemma 3.2. *The map $H : \Delta \rightarrow \mathbb{R}$ is a strict Lyapunov function, meaning that $\langle \nabla H(v), F(v) \rangle$ is positive for all $v \in \Delta \setminus \mathcal{C}$.*

Proof. One has $\partial_i H(v) = 2\alpha v_i^{\alpha-1}(Av^\alpha)_i$. Thus

$$\begin{aligned} \langle \nabla H(v), F(v) \rangle &= \sum_i 2\alpha v_i^{\alpha-1}(Av^\alpha)_i \left(-v_i + \frac{v_i^\alpha (Av^\alpha)_i}{H(v)} \right) \\ &= \frac{2\alpha}{H(v)} \left(-\left(\sum_i v_i^{\alpha-1}(Av^\alpha)_i v_i \right)^2 + \sum_i \left(v_i^{\alpha-1}(Av^\alpha)_i \right)^2 v_i \right) \\ &\geq 0, \end{aligned}$$

with equality only when $v_i^{\alpha-1}(Av^\alpha)_i$ does not depend on $i \in \text{supp}(v)$, i.e. only when v is an equilibrium. \square

Remark 3.3. The barycenter $v = (1/N, \dots, 1/N)$ is always an equilibrium.

Lemma 3.4. $H(\mathcal{C})$ has empty interior.

Proof. The computation of $\partial_i H(v)$ shows that

$$(\partial_i - \partial_j)H(v) = 2\alpha(v_i^{\alpha-1}(Av^\alpha)_i - v_j^{\alpha-1}(Av^\alpha)_j).$$

Hence, for all v in the relative interior of Δ , $F(v) = 0 \Leftrightarrow \nabla H(v) = 0$. In other word $\mathcal{C} \cap \text{int}\Delta = \nabla H^{-1}(0) \cap \text{int}\Delta$. By Sard's theorem, it follows that $H(\mathcal{C} \cap \text{int}\Delta)$ has measure zero, hence empty interior. Similarly, for each face I , $H(\mathcal{C} \cap \text{int}\Delta_I)$ has empty interior. This proves the lemma. \square

Proposition 3.5. The set of limit points of (v_n) is a connected subset of \mathcal{C} .

Proof. By proposition 3.3 and Theorem 3.4 in [B] or Proposition 4.6 in [BR], we get that the limit set of $(v(n))$ is an internally chain transitive set for Φ . Since H is a strict Lyapunov function and $H(\mathcal{C})$ has empty interior, it follows from Proposition 6.4 in [B2] that such a limit set is contained in \mathcal{C} . \square

In particular, when all the equilibria of F are isolated, then v_n converges a.s. toward one of them, as $n \rightarrow \infty$.

Remark 3.6. When A is not symmetric, the convergence result given by Proposition 3.5 fails to hold. Indeed, an example is constructed in [B] with $N = 3$ and $\alpha = 1$ for which the limit set of $(v_n)_{n \geq 1}$ equals $\partial\Delta$. This behavior persists for $\alpha = 1 + \epsilon$ and $\epsilon > 0$ small enough.

3.2. Stable and unstable equilibria. An equilibrium v is called *linearly stable* provided all the eigenvalues of $DF(v)$, the differential of F at point v , have negative real parts. It is called *linearly unstable* if one of its eigenvalues has a positive real part.

Now we will see with the next result that to study the stability or instability of the equilibria, it suffices in fact to consider only those which belong to the interior of Δ . In the following we let (e_1, \dots, e_N) denote the canonical basis of \mathbb{R}^N .

Lemma 3.7. *Let v be an equilibrium. Then, for $i, j \in \text{supp}(v)$, we have*

$$D_{e_i - e_j} F(v) = (\alpha - 1)(e_i - e_j) + \alpha \sum_{\ell} \frac{v_{\ell}^{\alpha} (A_{\ell, i} v_i^{\alpha-1} - A_{\ell, j} v_j^{\alpha-1})}{H(v)} e_{\ell},$$

and for $i \notin \text{supp}(v)$,

$$D_{e_i - v} F(v) = -(e_i - v).$$

Furthermore, the eigenvalues of $DF(v)$ are all reals.

Proof. For any $i, j \leq N$, and $v \in \Delta$,

$$\partial_j (Av^{\alpha})_i = \alpha A_{i, j} v_j^{\alpha-1},$$

and then by using that A is symmetric, we get

$$\partial_j H(v) = 2\alpha v_j^{\alpha-1} (Av^{\alpha})_j.$$

Thus

$$\begin{aligned} \partial_j \pi_i(v) &= \delta_{i, j} \alpha \frac{v_i^{\alpha-1} (Av^{\alpha})_i}{H(v)} + \alpha \frac{v_i^{\alpha} A_{i, j} v_j^{\alpha-1}}{H(v)} \\ &\quad - 2\alpha \frac{v_i^{\alpha} (Av^{\alpha})_i}{H^2(v)} v_j^{\alpha-1} (Av^{\alpha})_j. \end{aligned}$$

Now assume that v is an equilibrium, and let $i, j \in \text{supp}(v)$. We get with Lemma 3.2

$$\partial_j \pi_i(v) = \delta_{i, j} \alpha + \alpha \frac{v_i^{\alpha} A_{i, j} v_j^{\alpha-1}}{H(v)} - 2\alpha v_i,$$

and then

$$\partial_j F_i(v) = \delta_{i, j} (\alpha - 1) + \alpha \frac{v_i^{\alpha} A_{i, j} v_j^{\alpha-1}}{H(v)} - 2\alpha v_i.$$

On the other hand if $v_i = 0$ or $v_j = 0$, then

$$\partial_j \pi_i(v) = 0,$$

and thus

$$\partial_j F_i(v) = -\delta_{i, j}.$$

The first part of the lemma follows. To see that eigenvalues are real, we may assume without loss of generality that $v \in \text{int}(\Delta)$. Note that $\partial_j F_i(v) v_j = \partial_i F_j(v) v_i$. Therefore, the transpose of $DF(v)$ is self adjoint

with respect to the dot product $(x, y) = \sum_i v_i x_i y_i$, and this concludes the proof of the lemma. \square

As announced above we deduce

Corollary 3.8. *An equilibrium on a face is linearly stable (respectively unstable), if and only if, it is so for the restriction of F to this face.*

Proof. Indeed, assume that v is an equilibrium on a face Δ_I associated to some subset I . Then the previous lemma shows that for any $i \notin I$, $e_i - v$ is a stable direction. So the result of the corollary follows from our definitions of stable and unstable equilibria. \square

In other words to study the stability or instability of equilibria, and we will see in the next two subsections why this question is important, it suffices to consider those belonging to the interior of Δ .

3.3. Non convergence towards unstable equilibria. The purpose of this section is to prove the following result.

Theorem 3.9. *Let v^* be a linearly unstable equilibrium. Then the probability that v_n converges towards v^* is equal to 0.*

Proof. Let us recall now that for $g \in \mathbb{R}^N$ and $i \in E$, we use the notation $g(i) = g_i$. For $u, v \in \mathbb{R}^N$, we also set $uv := \sum_i u_i v_i$, and $\|u\| = \sup_i |u_i|$. Furthermore, C will denote a non-random constant that may vary from lines to lines.

For $v \in \Delta$, let $Q(v)$ be the pseudo-inverse of $K(v)$ defined by:

$$(I - K(v))Q(v)g = Q(v)(I - K(v))g = g - (\pi(v)g)\mathbf{1},$$

for all $g \in \mathbb{R}^N$, with I is the identity matrix and $\mathbf{1}(i) = 1$ for all $i \in E$. Then by a direct application of the implicit function theorems (see Lemma 5.1 in [B]) one has

Lemma 3.10. *For any $v \in \Delta$ and $i \in E$, $Q(v)$, $K(v)Q(v)$, and $(\partial/\partial v_i)(K(v)Q(v))$, are bounded operators on $\ell^\infty(E)$, the space of bounded functions on E , and their norms are uniformly bounded in $v \in \Delta$.*

Now for all $n \geq 1$ and $i \in E$, we can write

$$v_{n+1}(i) - v_n(i) = \frac{1}{n+1}(-v_n(i) + e_i(X_{n+1})).$$

Note that $K((n+1)^{-1}, v_n) = K(\tilde{v}_n)$, where

$$\tilde{v}_n(i) = \frac{v_n(i) + 1/(n+1)}{1 + N/(n+1)} \quad \text{for } 1 \leq i \leq N.$$

Note also that $\|\tilde{v}_n - v_n\| \leq C/n$. Let next z_n be defined by

$$z_n(i) = v_n(i) + \frac{K(\tilde{v}_n)Q(\tilde{v}_n)e_i(X_n)}{n} \quad \text{for } 1 \leq i \leq N.$$

Then Lemma 3.10 implies that $\|z_n - v_n\| \leq C/n$. Moreover, we can write:

$$(6) \quad z_{n+1} - z_n = \frac{F(z_n)}{n+1} + \frac{\epsilon_{n+1}}{n+1} + \frac{r_{n+1}}{n+1},$$

where ϵ_{n+1} and r_{n+1} are such that for all $1 \leq i \leq N$,

$$\epsilon_{n+1}(i) := Q(\tilde{v}_n)e_i(X_{n+1}) - K(\tilde{v}_n)Q(\tilde{v}_n)e_i(X_n),$$

and $r_{n+1} = \sum_{k=1}^4 r_{n+1,k}$, with

$$\begin{aligned} r_{n+1,1} &= F(v_n) - F(z_n) \\ r_{n+1,2} &= \pi(\tilde{v}_n) - \pi(v_n) \\ r_{n+1,3}(i) &= K(\tilde{v}_n)Q(\tilde{v}_n)e_i(X_n) \left(1 - \frac{n+1}{n}\right) \\ r_{n+1,4}(i) &= K(\tilde{v}_{n+1})Q(\tilde{v}_{n+1})e_i(X_{n+1}) - K(\tilde{v}_n)Q(\tilde{v}_n)e_i(X_{n+1}), \end{aligned}$$

for $1 \leq i \leq N$.

By using the facts that F and π are Lipschitz functions on Δ , $\|v_n - \tilde{v}_n\| + \|v_n - z_n\| \leq C/n$, and by applying Lemma 3.10, we deduce that

$$\|r_{n+1}\| \leq C/n.$$

Moreover, we have

$$\mathbb{E}[\epsilon_{n+1} \mid \mathcal{F}_n] = 0.$$

Since v^* is linearly unstable there exists, by Lemma 3.7, $f \in T_0\Delta$ and $\lambda > 0$ such that

- (i): $D_f F(v) = \lambda f$ and,
- (ii): $f_i = 0$ for $i \in E \setminus \text{supp}(v^*)$.

Such an f being fixed, we claim that on the event $\{v_n \rightarrow v^*\}$,

$$(7) \quad \liminf_{n \rightarrow \infty} \left\{ \mathbb{E}[(\epsilon_{n+2}f)^2 \mid \mathcal{F}_{n+1}] + \mathbb{E}[(\epsilon_{n+1}f)^2 \mid \mathcal{F}_n] \right\} > 0.$$

To prove this claim, we first introduce some notation: for μ a probability measure on E , and $g \in \mathbb{R}^N$, denote by $V_\mu(g)$ the variance of g with respect to μ

$$V_\mu(g) := \frac{1}{2} \sum_{1 \leq j, k \leq N} \mu(j)\mu(k)(g(j) - g(k))^2.$$

Then for any $n \geq 0$ and $i \leq N$, let $\mu_{n,i}$ be the probability measure defined by $\mu_{n,i}(j) = K_{i,j}(\tilde{v}_n)$. Set also $V_n(i) := \mathbf{V}_{\mu_{n,i}}(Q(\tilde{v}_n)f)$. Then we have that

$$\mathbf{E}[(\epsilon_{n+1}f)^2 \mid \mathcal{F}_n] = V_n(X_n).$$

Furthermore, when v_n converges toward v^* , $K(\tilde{v}_n)$ and $Q(\tilde{v}_n)$ converge respectively toward $K(v^*)$ and $Q(v^*)$. Thus, for any $i \in E$,

$$\liminf_{n \rightarrow \infty} V_n(i) \geq V^*(i) := \mathbf{V}_{\mu_i^*}(Q(v^*)f),$$

where $\mu_i^*(j) = K_{i,j}(v^*)$, for all $j \leq N$. Next by using the fact that when $X_n = i$ and $A_{i,i} = 0$, then $X_{n+1} \neq X_n$, we get that

$$\liminf \left\{ \mathbf{E}[(\epsilon_{n+2}f)^2 \mid \mathcal{F}_{n+1}] + \mathbf{E}[(\epsilon_{n+1}f)^2 \mid \mathcal{F}_n] \right\} \geq \min_i c^*(i),$$

where

$$c^*(i) := \min_{j \in \mathcal{A}_i} (V^*(i) + V^*(j)),$$

and

$$\mathcal{A}_i := \begin{cases} E & \text{if } A_{i,i} \neq 0 \\ E \setminus \{i\} & \text{if } A_{i,i} = 0. \end{cases}$$

Now by using that

$$Q(v^*)f - K(v^*)Q(v^*)f = f - (v^*f)\mathbf{1}$$

(recall that $\pi(v^*) = v^*$), we see that $Q(v^*)f$ has constant coordinates on $\text{supp}(v^*)$, if and only if, f has constant coordinates on $\text{supp}(v^*)$. But since $f \in T_0\Delta$ and $f_i = 0$ for $i \notin \text{supp}(v^*)$; this cannot be the case. Since $\mu_i^*(j) > 0$, when $j \neq i$ and $j \in \text{supp}(v^*)$, it follows already that $c^*(i) > 0$, for all $i \notin \text{supp}(v^*)$. Now let $i \in \text{supp}(v^*)$ be given. If $A_{i,i} \neq 0$, then again we have $V^*(i) > 0$, and thus $c^*(i) > 0$. Now assume that $A_{i,i} = 0$. If $\#\text{supp}(v^*) \geq 3$, then there can be at most one value of i , for which $V^*(i) = 0$, and thus in this case we have $c^*(i) > 0$ as well. Let us then consider the case when $\#\text{supp}(v^*) = 2$, and say $\text{supp}(v^*) = \{i, j\}$. Recall that if $A_{i,i} \neq 0$, then $V^*(i) > 0$. However, we cannot have $A_{i,i} = A_{j,j} = 0$, since otherwise $v_i^* = v_j^* = 1/2$ and by lemma 3.7 v^* is linearly stable. Finally we have proved that in any case $\min_i c^*(i) > 0$. Theorem 3.9 is then a consequence of (7) and Corollary 3.IV.15 p.126 in [Du]. \square

3.4. Convergence towards stable equilibria and localization.

Theorem 3.11. *Let v^* be a linearly stable equilibrium. Then the probability that v_n converges towards v^* is positive.*

Proof. follows from Corollary 6.5 in [B] since any linearly stable equilibrium is a minimal attractor. \square

Theorem 3.12. *Let $v^* \in \Delta$ be a linearly stable equilibrium. Then a.s. on the event $\{\lim_{n \rightarrow \infty} v_n = v^*\}$, the set $E \setminus \text{supp}(v^*)$ is visited only finitely many times.*

The proof follows directly from the next two lemmas:

Lemma 3.13. *There exists $\nu > 0$ such that on the event $\{\lim_{n \rightarrow \infty} v_n = v^*\}$,*

$$\lim_{n \rightarrow \infty} n^\nu \|v_n - v^*\| = 0.$$

Proof. This is similar to Lemma 8 in [BT]. We give here an alternative and more direct proof relying on [B2]. Since v^* is a linearly stable equilibrium there exists a neighborhood B of v^* , a constant $C > 0$ and $\lambda > 0$, such that $\|\Phi_t(v) - v^*\| \leq Ce^{-\lambda t}$, for all $v \in B$ (see e.g [R], Theorem 5.1). Let $\tau_n = \sum_{k=1}^n 1/k$ and let $V : \mathbb{R}^+ \rightarrow \Delta$ denote the continuous time piecewise affine process defined by a) $V(\tau_n) = z_n$ and b) V is affine on $[\tau_n, \tau_{n+1}]$. By (6) and Doob's inequalities, the interpolated process V is almost surely a $-1/2$ asymptotic pseudo trajectory of Φ , meaning that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \left(\sup_{0 \leq h \leq T} \|\Phi_h(V(t) - V(t+h))\| \right) \leq -1/2$$

for all $T > 0$. For a proof of this later assertion see [B2], Proposition 8.3. Now, by Lemma 8.7 in [B2]

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\|V(t) - v^*\|) \leq -\min(1/2, \lambda)$$

on the event $\{v_n \rightarrow v^*\}$. This proves that $\|z_n - v^*\| = O(n^{-\min(1/2, \lambda)})$, which concludes the proof of the lemma. \square

Lemma 3.14. *For any $I \subseteq \{1, \dots, N\}$, and $\nu \in (0, 1)$, a.s. on the event*

$$E_\nu(I) := \left\{ \lim_{n \rightarrow \infty} v_n(i) n^\nu = 0 \quad \forall i \in I \right\},$$

the set I is visited only finitely many times.

Proof. For $m \geq 1$, set

$$E_{m,\nu}(I) := \{ |v_k(i)| \leq k^{-\nu} \quad \forall k \geq m \quad \forall i \in I \}.$$

Note that on $E_{m,\nu}(I)$, at each time $k \geq m$, the probability to jump to some vertex $i \in I$, is bounded above by $p_k := N^{1+\alpha} k^{-\alpha\nu}$. Let now $(\xi_k)_{k \geq m}$ denotes some sequence of independent Bernoulli random variables with respective parameters $(p_k)_{k \geq m}$. Then for any $n \geq m$, on

$E_{m,\nu}(I)$ the number of jumps on I between time m and n is stochastically dominated by

$$Z_n := \sum_{k=m}^n \xi_k.$$

However, it is well known that a.s. $\limsup Z_n/n^{1-\nu'} < \infty$, for any $\nu' < \alpha\nu \wedge 1$. We deduce that a.s. for any $\nu' < \alpha\nu \wedge 1$,

$$E_{m,\nu}(I) \subseteq E_{\nu'}(I).$$

Since $E_\nu(I) \subseteq \cup_m E_{m,\nu}(I)$, we deduce that a.s. for any $\nu' < \alpha\nu \wedge 1$,

$$E_\nu(I) \subseteq E_{\nu'}(I).$$

Since $\alpha > 1$, it follows by induction that a.s.

$$E_\nu(I) \subseteq E_\beta(I),$$

for any $\beta \in (1/\alpha, 1)$. But a simple application of the Borel-Cantelli lemma shows that for any such β , a.s. on $E_\beta(I)$, the set I is visited only finitely many times. This concludes the proof of the lemma. \square

4. THE CASE OF THE VRRW ON THE COMPLETE GRAPH

In this section we study in detail the case of the VRRW on the complete graph described in the introduction. In other words, A is given by (1).

Since the case $N = 2$ is trivial, we assume in all this section that $N \geq 3$.

We first study the stability of the centers of the faces. As already explained, this reduces to analyze the center of Δ .

Lemma 4.1. *Let $v = (1/N, \dots, 1/N)$ be the center of Δ . Then v is a linearly stable (respectively unstable) equilibrium if $\alpha < (N-1)/(N-2)$, (respectively $\alpha > (N-1)/(N-2)$).*

Proof. Lemma 3.7 shows that for all $i \neq j$,

$$D_{e_i - e_j} F(v) = \left(-1 + \alpha \left(\frac{N-2}{N-1} \right) \right) (e_i - e_j).$$

The lemma follows immediately. \square

By combining this lemma with Corollary 3.8, we get

Lemma 4.2. *Let v be the center of the face Δ_I associated with some subset I with cardinality $k \leq N$. Then v is a linearly stable (respectively unstable) equilibrium if $\alpha < (k-1)/(k-2)$, (respectively $\alpha > (k-1)/(k-2)$).*

It remains to study the stability of the other equilibria. We will see that they are all unstable, which will conclude the proof of Theorem 1.1.

First we need the following lemma, which shows that coordinates of equilibria take at most two different values.

Lemma 4.3. *Let v be an equilibrium in the interior of Δ , which is different from its center. Then $\#\{v_i : i \leq N\} = 2$.*

Proof. Let $a = \sum_i v_i^\alpha$ and $b = \sum_i v_i^{2\alpha}$. Since v is an equilibrium, we have for all i ,

$$v_i = v_i^\alpha \frac{a - v_i^\alpha}{a^2 - b}.$$

Since all coordinates of v are positive by hypothesis, this is equivalent to

$$f(v_i) = a - b/a,$$

for all i , where $f(x) = -x^{2\alpha-1}/a + x^{\alpha-1}$. Now observe that

$$f'(x) = x^{\alpha-2}(- (2\alpha - 1)x^\alpha/a + (\alpha - 1)),$$

does not vanish on $(0, 1)$ if $a \geq (2\alpha - 1)/(\alpha - 1)$, and vanishes in exactly one point otherwise. Thus for any fixed $\lambda \in \mathbb{R}$, the equation $f(x) = \lambda$, has at most one solution in $(0, 1)$, if $a \geq (2\alpha - 1)/(\alpha - 1)$, and at most two otherwise. The lemma follows. \square

Let now v be an equilibrium in the interior of Δ , which is different from its center. The previous lemma shows that its coordinates take exactly two different values, say u_1 and u_2 . Since the action of permutation of the coordinates commutes with F , we can always assume w.l.g. that $v_i = u_1$, for $i \leq k$, and $v_i = u_2$, for $i > k$, for some $k \leq N/2$. Denote now by E_k the set of such equilibria (those in the interior of Δ , not equal to the center of Δ , and having their first k coordinates equal as well as their last $N - k$ coordinates). For $v \in E_k$, we also set $t(v) = v_N/v_1$. We have the

Lemma 4.4. *Assume that $\alpha \geq (N - 1)/(N - 2)$, and let $v \in E_k$, with $k \leq N/2$. If $k > 1$ or if $t(v) < 1$, then v is linearly unstable.*

Proof. It follows from Lemma 3.7 that for any $i < j \leq k$,

$$D_{e_j - e_i} F(v) = \lambda_1(e_j - e_i),$$

and for $k + 1 \leq i < j \leq N$,

$$D_{e_j - e_i} F(v) = \lambda_2(e_j - e_i),$$

with

$$\lambda_m = \left((\alpha - 1) - \alpha \frac{u_m^{2\alpha-1}}{H(v)} \right),$$

for $m = 1, 2$. Then it just suffice to observe that if $u_1 < u_2$, $\lambda_1 > 0$, whereas if $u_1 > u_2$, $\lambda_2 > 0$. Thus if any of the hypotheses of the lemma is satisfied, there $DF(v)$ has at least one positive eigenvalue, and so v is linearly unstable. \square

On the other hand we have the following:

Lemma 4.5. *Let $v \in E_k$ be given. If $k = 1$ and $\alpha \geq (N - 1)/(N - 2)$, then $t(v) < 1$ and v is unstable. Similarly, if $\alpha < (N - 1)/(N - 2)$, then v is linearly unstable.*

Proof. Following our previous notation, set $u_1 := v_1$ and $u_2 := v_N$, and recall that $u_1 \neq u_2$, since v is not equal to the center of Δ . Recall also that $t(v) = u_2/u_1$. Note that since $ku_1 + (N - k)u_2 = 1$, we have $u_1 = 1/(k + (N - k)t)$. Now the fact that v is an equilibrium means that $F(v) = 0$, which is equivalent to say that the function φ defined by

$$\varphi(t) := -(N - k - 1)t^{2\alpha-1} + (N - k)t^\alpha - kt^{\alpha-1} + (k - 1),$$

vanishes at point $t(v)$. We now study the function φ . First $\varphi'(t) = t^{\alpha-2}\psi(t)$, with

$$\psi(t) = -(2\alpha - 1)(N - k - 1)t^\alpha + \alpha(N - k)t - (\alpha - 1)k.$$

In particular ψ is strictly concave, $\psi(0) < 0$ and $\lim_{t \rightarrow \infty} \psi(t) = -\infty$. Now two cases may appear. Either ψ vanishes in at most one point, in which case φ is nonincreasing, thus vanishes only in 1. But this case is excluded, since $t(v) \neq 1$ by hypothesis. In the other case ψ vanishes in exactly two points, which means that there exist $t_1 < t_2$ such that φ is decreasing in $(0, t_1) \cup (t_2, \infty)$, and increasing in (t_1, t_2) . Now consider first the case when $k = 1$, which implies $\varphi(0) = 0$. If $\alpha \geq (N - 1)/(N - 2)$, then $\varphi'(1) \leq 0$, thus φ has at most one zero in $(0, 1)$ and no zero in $(0, \infty)$. Together with Lemma 4.4 this proves the first part of the lemma. If $\alpha > (N - 1)/(N - 2)$, then $\varphi'(1) > 0$, and thus φ has no zero in $(0, 1)$ and exactly one zero in $(0, \infty)$, in which the derivative of φ is negative. The fact that this zero corresponds to an unstable equilibrium will then follow from (8) below. But before we prove this fact, let us consider now the case $k > 1$ and $\alpha < (N - 1)/(N - 2)$, which imply that $\varphi(0) > 0$ and $\varphi'(1) > 0$. Thus φ vanishes in exactly one point in $(0, 1)$ and another one in $(1, \infty)$, and

again the derivative of φ in these points is negative. So all that remains to do is to prove the following fact

(8) If $v \in E_k$ is such that $\varphi'(t(v)) < 0$, then v is unstable.

Let us prove it now. For $t \in (0, \infty)$, set $u_1(t) = 1/(k + (N - k)t)$ and $u_2(t) = tu_1(t)$. Then let $v(t) \in (0, 1)^N$, be the point whose first k coordinates are equal to $u_1(t)$ and whose last $N - k$ coordinates are equal to $u_2(t)$. Then we have

$$H(v(t)) = k(k-1)u_1(t)^{2\alpha} + 2k(N-k)t^\alpha u_1(t)^{2\alpha} + (N-k)(N-k-1)t^{2\alpha} u_1(t)^{2\alpha},$$

and after some computation we find that

$$F(v(t)) = -\frac{tu_1(t)^{2\alpha+1}}{H(v(t))} \varphi(t) e_k,$$

where e_k is the vector, whose first k coordinates are equal to $-(N - k)$ and whose last $N - k$ coordinates are equal to k . Then notice that $D_{e_k} F(v(t)) = c(d/dt)F(v(t))$, for some constant $c > 0$. But recall that when $v(t)$ is an equilibrium, $\varphi(t) = 0$. Thus

$$\frac{d}{dt} F(v(t)) = -\frac{tu_1(t)^{2\alpha+1}}{H(v(t))} \varphi'(t) e_k.$$

So if $\varphi'(t) < 0$, e_k is an unstable direction, proving (8). This concludes the proof of the lemma. \square

The two above lemmas show that any equilibrium in the interior of Δ , which is not equal to the center of Δ is linearly unstable. Together with Lemma 4.2, and Theorems 3.9, 3.11 and 3.12 this concludes the proof of Theorem 1.1. \square

REFERENCES

- [B] **Benaïm M.:** *Vertex-reinforced random walks and a conjecture of Pemantle*, Ann. Probab. 25, (1997), 361–392.
- [B2] **Benaïm M.:** *Dynamics of Stochastic Approximations*, Sémin. Probab., Lectures Notes in Mathematics, Vol 1709, pp 1–68.
- [BR] **Benaïm M., Raimond O.:** *A class of self-interacting processes with applications to games and reinforced random walks*, SIAM J. Control Optim. 48, (2010), 4707–4730.
- [BT] **Benaïm M., Tarrès P.:** *Dynamics of Vertex-Reinforced Random Walks*, Ann. Probab. 39, (2011), 2178–2223.
- [CHJ] **Chung F., Handjani S., Jungreis D.:** *Generalizations of Polya’s urn problem*, Ann. Comb. 7, (2003), 141–153.
- [D] **Davis B.:** *Reinforced random walk*, Probab. Theory Related Fields 84, (1990), 203–229.
- [Du] **Duflo M.:** *Algorithmes stochastiques*, Mathématiques & Applications (Berlin), 23. Springer-Verlag, Berlin, (1996), xiv+319 pp.

- [L] **Limic V.:** *Attracting edge property for a class of reinforced random walks*, Ann. Probab. 31, (2003), 1615–1654.
- [LT] **Limic V., Tarrès P.:** *Attracting edge and strongly edge reinforced walks*, Ann. Probab. 35, (2007), 1783–1806.
- [LV] **Limic V., Volkov S.:** *VRRW on complete-like graphs: almost sure behavior*, Ann. Appl. Probab. 20, (2010), 2346–2388
- [OS] **Oliveira R., Spencer J.:** *Connectivity transitions in networks with super-linear preferential attachment*, Internet Math. 2, (2005), 121–163.
- [P] **Pemantle R.:** *Vertex-reinforced random walk*, Probab. Theory Related Fields 92 (1992), no. 1, 117–136.
- [PV] **Pemantle R., Volkov S.:** *Vertex-reinforced random walk on \mathbb{Z} has finite range*, Ann. Probab. 27, (1999), 1368–1388.
- [R] **Robinson, C.** Dynamical Systems, CRC Press (1999)
- [T1] **Tarrès P.:** *Vertex-reinforced random walk on \mathbb{Z} eventually gets stuck on five points*, Ann. Probab. 32, (2004), 2650–2701.
- [V] **Volkov S.:** *Phase transition in vertex-reinforced random walks on \mathbb{Z} with non-linear reinforcement*, J. Theoret. Probab. 19, (2006), 691–700.

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