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REVIEW ARTICLE

Inhomogeneous cosmological models: exact solutions and their applications

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Abstract. Recently, inhomogeneous generalisations of the Friedmann – Lemaître – Robertson – Walker cosmological models have gained interest in the astrophysical community and are more often employed to study cosmological phenomena. However, in many papers the inhomogeneous cosmological models are treated as an alternative to the FLRW models. In fact, they are not an alternative, but an exact perturbation of the latter, and are gradually becoming a necessity in modern cosmology. The assumption of homogeneity is just a first approximation introduced to simplify equations. So far this assumption is commonly believed to have worked well, but future and more precise observations will not be properly analysed unless inhomogeneities are taken into account. This paper reviews recent developments in the field and shows the importance of an inhomogeneous framework in the analysis of cosmological observations.

1. Introduction

For this article, we define inhomogeneous cosmological models as follows: they are those exact solutions of Einstein's equations that contain at least a subclass of nonvacuum and nonstatic Friedmann – Lemaître – Robertson – Walker (FLRW) solutions as a limit. The reason for this choice is that such FLRW models are universally recognised as a good first approximation to a realistic description of our actual Universe, so it makes sense to consider only those other models that have a chance to be a still better approximation. Models that do not include an FLRW limit would not easily fulfil this condition.

Among the models so defined we chose for a more detailed description only the Lemaître [103] – Tolman [159] (LT) and Szekeres [154] models because they were the basis for the greatest number of papers aimed at physical and astrophysical interpretation. The other inhomogeneous models are only partly listed and some of them are briefly described.

The LT and Szekeres models describe the evolution of the Universe in the post-recombination era, in which only gravitational interactions play a role. The matter source in them is dust, i.e. a perfect fluid with zero pressure (the generalisation to nonzero cosmological constant is known, but less frequently used). Thus, they should not be considered for application to pre-recombination epochs, in which the pressure cannot be neglected. In particular, they should not be applied to the inflationary epoch. Also, they are not suitable for including "dark energy" in any other form than cosmological constant. On the contrary, these models are sometimes used to explain observational results attributed to the "dark energy" by effects of inhomogeneities in ordinary matter. They are meant to be a replacement for the linearised perturbations of the FLRW models and for methods describing backreactions with the help of averaged quantities. Because of their symmetries (LT) and quasi-symmetries (Szekeres) they apply to less general situations than the perturbative calculations, but their advantage is that they fulfil the Einstein equations exactly. Therefore, as long as we believe that general relativity is the correct theory of gravitation, the inhomogeneous models can be extrapolated arbitrarily far into the future and are not constrained by any "regimes".

The main body of this review is devoted to a description of those observed effects that can be explained using the LT and Szekeres models. The penultimate section deals with misuses, errors and misconceptions existing in the literature on the inhomogeneous models.

2. Exact solutions of Einstein's equations that can be applied as inhomogeneous cosmological models

The total number of papers in which such solutions were derived or discussed was approx. 750 until 1994 [89]. No-one has updated that statistic. No generalisations of models known until 1994 have been reported in later years. However, the Lemaître [103] – Tolman [159] (LT) models have become popular as a basis for astrophysical

considerations, and the same is happening recently with the Szekeres [154] models. The current number may well be over 1000. We begin by recalling the general classification scheme [89].

(1) The Szekeres - Szafron (S–S) family [154, 152]

These models are invariantly defined by the following properties [153]:

1. They obey the Einstein equations with a perfect fluid source.
2. The flow-lines of the perfect fluid are geodesic and nonrotating.
3. The hypersurfaces orthogonal to the flow-lines are conformally flat.
4. The Ricci tensor of those hypersurfaces has two of its eigenvalues equal.
5. The shear tensor has two of its eigenvalues equal.

Because of property 2, in comoving coordinates the pressure depends only on time. Thus the barotropic equation of state, the most popular one in the astrophysics community, reduces the Szafron metric to the Friedmann – Lemaître – Robertson – Walker (FLRW) class. The only nontrivial solutions in the S–S family that can be reasonably applied in cosmology are the Szekeres metrics [154], in which the source is dust (a perfect fluid with zero pressure). This is a good model for the later phases of the evolution of the Universe, in which gravitation plays a dominant role and large-scale hydrodynamical processes have come to an end.

The metric of the Szekeres solutions is, in units in which $c = 1$,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2\alpha(t,x,y,r)}\mathrm{d}r^2 - \mathrm{e}^{2\beta(t,x,y,r)}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right). \tag{1}$$

The coordinates of (1) are comoving so that $u^\mu = \delta^\mu{}_0$. There are two families of Szekeres solutions, depending on whether $\beta_{,r} = 0$ or $\beta_{,r} \neq 0$. The first family is a simultaneous generalisation of the Friedmann and Kantowski–Sachs models [84]. So far it has found no application in astrophysical cosmology, and we shall not discuss it here. The metric functions in the second family are

$$\begin{aligned} \mathrm{e}^{\beta} &= \Phi(t,r)\mathrm{e}^{\nu(r,x,y)}, \\ \mathrm{e}^{\alpha} &= h(r)\Phi(t,r)\beta_{,r} \equiv h(r)\left(\Phi_{,r} + \Phi\nu_{,r}\right), \\ \mathrm{e}^{-\nu} &= A(r)\left(x^2+y^2\right) + 2B_1(r)x + 2B_2(r)y + C(r), \end{aligned} \tag{2}$$

where $\Phi(t,r)$ is a solution of

$$\Phi_{,t}{}^2 = -k(r) + \frac{2M(r)}{\Phi} + \frac{1}{3}\Lambda\Phi^2; \tag{3}$$

Λ is the cosmological constant, while $h(r)$, $k(r)$, $M(r)$, $A(r)$, $B_1(r)$, $B_2(r)$ and $C(r)$ are arbitrary functions obeying

$$g(r) \stackrel{\mathrm{def}}{=} 4\left(AC - B_1{}^2 - B_2{}^2\right) = 1/h^2(r) + k(r), \tag{4}$$

where $g(r)$ is another arbitrary function of the coordinate r defined as above. The mass density in energy units is

$$\kappa\rho = \frac{\left(2M\mathrm{e}^{3\nu}\right)_{,r}}{\mathrm{e}^{2\beta}\left(\mathrm{e}^{\beta}\right)_{,r}}; \qquad \kappa = 8\pi G/c^4. \tag{5}$$

The bang time function $t_B(r)$ follows from (3):

$$\int_0^{\Phi} \frac{\mathrm{d}\widetilde{\Phi}}{\sqrt{-k + 2M/\widetilde{\Phi} + \frac{1}{3}\Lambda\widetilde{\Phi}^2}} = t - t_B(r). \tag{6}$$

The Szekeres metric has in general no symmetry, but acquires a 3-dimensional symmetry group with 2-dimensional orbits when A, B_1, B_2 and C are all constant.

The sign of $g(r)$ determines the geometry of the (constant t, constant r) 2-surfaces. The geometry is spherical, planar or hyperbolic (pseudo-spherical) when $g > 0$, $g = 0$ or $g < 0$, respectively. With A, B_1, B_2 and C being functions of r, the surfaces $r =$ const within a single space $t =$ const may have different geometries, i.e. they can be spheres in one part of the space and surfaces of constant negative curvature elsewhere, the curvature being zero at the boundary.

The sign of $k(r)$ determines the type of evolution when $\Lambda = 0$; with $k > 0 = \Lambda$ the model expands away from an initial singularity and then recollapses to a final singularity; with $k < 0 = \Lambda$ the model is either ever-expanding or ever-collapsing; $k = 0 = \Lambda$ is the intermediate case corresponding to the 'flat' Friedmann model. Similarly to g, k can have different signs in different regions of the same space. The sign of $k(r)$ influences the sign of $g(r)$. Since $1/h^2$ in (4) must be non-negative, we have the following: with $g > 0$ (spherical geometry), all three types of evolution are allowed; with $g = 0$ (plane geometry), k must be non-positive (only parabolic or hyperbolic evolutions are allowed); and with $g < 0$ (hyperbolic geometry), k must be strictly negative, so only the hyperbolic evolution is allowed. The geometry of the latter two classes is poorly understood [77, 90], and therefore not explored for cosmological applications. Only the quasi-spherical model has been well investigated, and has found applications in the study of the early Universe [73, 119], structure formation [20, 21], supernova [27] and cosmic microwave background (CMB) [23] observations, light propagation [93]. In [93] it was shown that two rays sent from the same source at different times to the same observer pass through different sequences of intermediate matter particles. The change of object position in the sky, due to this effect, should be observable in the future.

The quasi-spherical Szekeres models can be imagined as deformations of the spherically symmetric models after which the spheres (still identifiable in the Szekeres geometry) are no longer concentric. The mass-density distribution may be interpreted as a superposition of a mass monopole and a mass dipole [56, 134].‡

(2) The Lemaître and Lemaître – Tolman models

(2.a) The Lemaître model

The Lemaître metric [103] describes a spherically symmetric inhomogeneous fluid

‡ A special case of this may be interpreted as a pure mass dipole, but then the density is necessarily negative over approx. half of each sphere, and the physical interpretation of such an object is unknown.

with anisotropic pressure.§ In comoving coordinates it has the following form

$$\mathrm{d}s^2 = \mathrm{e}^{A(t,r)}\mathrm{d}t^2 - \mathrm{e}^{B(t,r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right). \tag{7}$$

The Einstein equations reduce to:

$$\kappa R^2 R_{,r}\,\rho = 2M_{,r}, \tag{8}$$

$$\kappa R^2 R_{,t} p = -2M_{,t}, \tag{9}$$

where $(R_{,t}\,, R_{,r}\,) \stackrel{\text{def}}{=} (\partial R/\partial t, \partial R/\partial r)$, p is the pressure, ρ is the mass density in energy units, and $M(t,r)$ is defined by:

$$2M = R + \mathrm{e}^{-A} R {R_{,t}}^2 - \mathrm{e}^{-B} R {R_{,r}}^2 - \frac{1}{3}\Lambda R^3. \tag{10}$$

In the Newtonian limit, Mc^2/G is equal to the mass inside the shell of radial coordinate r. However, in curved space it is not an integrated rest mass, but the active gravitational mass that generates the gravitational field. As can be seen from (9), in the expanding Universe the mass decreases with time. The function B can be written in the following form [19]

$$\mathrm{e}^{B(t,r)} = \frac{{R_{,r}}^2(t,r)}{1+2E(r)} \exp\left(\int_{t_0}^{t} \mathrm{d}\widetilde{t} \frac{2R_{,t}(\widetilde{t},r)}{\left[\rho(\widetilde{t},r) + p(\widetilde{t},r)\right] R_{,r}\,(\widetilde{t},r)} p_{,r}\,(\widetilde{t},r)\right), \tag{11}$$

where $E(r)$ is an arbitrary function. The equations of motion ${T^{\alpha\beta}}_{;\beta} = 0$ reduce to

$${T^{0\alpha}}_{;\alpha} = 0 \qquad \Rightarrow \quad B_{,t} + 4\frac{R_{,t}}{R} = -\frac{2\rho_{,t}}{\rho + p}, \tag{12}$$

$${T^{1\alpha}}_{;\alpha} = 0 \qquad \Rightarrow \quad A_{,r} = -\frac{2p_{,r}}{\rho + p}, \tag{13}$$

$${T^{2\alpha}}_{;\alpha} = {T^{3\alpha}}_{;\alpha} = 0 \quad \Rightarrow \quad \frac{\partial p}{\partial \theta} = 0 = \frac{\partial p}{\partial \phi}. \tag{14}$$

The Lemaître model has been employed to study the conditions of the early Universe [72], the mass of the Universe [3], supernova observations [102], structure formation and the impact of pressure gradients on shell crossing singularities [33].

(2.b) The Lemaître-Tolman model

In the special case of dust with the cosmological constant, the above equations reproduce the Lemaître–Tolman (LT) model [103, 159]. When $p_{,r} = 0$, equation (13) implies $A_{,r} = 0$, which means that the component g_{00} can be scaled to 1, and, using (11), the metric (7) becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{R_{,r}^2}{1+2E}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right). \tag{15}$$

Equation (10) becomes then identical to (3):

$${R_{,t}}^2 = 2E + \frac{2M}{R} + \frac{\Lambda}{3}R^2. \tag{16}$$

§ The subclass of isotropic pressure is usually credited to Misner and Sharp [118], and occasionally to Podurets [135].

Because the pressure is zero, the mass does not depend on time. The mass density follows from (8), and the bang time function $t_B(r)$ is given by (6), with $(-k, \Phi)$ replaced by $(2E, R)$.

For reviews of applications of the LT models see [89, 134, 32]. Selected examples: formation of black holes [96, 68], of galaxy clusters [94, 95], superclusters [29], cosmic voids [31], interpretation of supernova observations [45, 79, 5, 47, 4, 63, 16, 41, 42, 112, 22, 111, 40, 69, 70, 53, 62, 15, 35, 64], CMB [2, 176, 165, 137, 51, 120], redshift drift [163, 9], and averaging (see the contributions by Buchert, Räsänen, and Wiltshire in this issue).

Some of these applications will be discussed in Sec. 4.

(3) The Stephani – Barnes (S–B) family

This is the family of perfect fluid solutions with zero shear, zero rotation and nonzero expansion. It consists of two collections of solutions:

(3a) The conformally flat solution:

$$ds^2 = D^2 dt^2 - V^{-2}(t,x,y,z)(dx^2 + dy^2 + dz^2), \tag{17}$$

where:

$$D = F(t) V_{,t} / V, \tag{18}$$

$$V = \frac{1}{R}\left\{1 + \frac{1}{4}k(t)\left[(x - x_0(t))^2 + (y - y_0(t))^2 + (z - z_0(t))^2\right]\right\}, \tag{19}$$

$F(t), R(t), k(t), x_0(t), y_0(t)$ and $z_0(t)$ are arbitrary functions of time, F is related to the expansion scalar θ by $\theta = 3/F$, and $k(t)$ is a generalisation of the FLRW curvature index k, it can change sign during evolution. The matter density and pressure are:

$$\kappa\rho = 3kR^2 + 3/F^2 \stackrel{\text{def}}{=} 3C^2(t), \tag{20}$$

$$\kappa p = -3C^2(t) + 2CC_{,t}\, V/V_{,t}\,. \tag{21}$$

This solution was found by Stephani [147]; it is the most general conformally flat solution with a perfect fluid source and nonzero expansion. As seen from (20) – (21), the matter density in it depends only on the comoving time, while the pressure depends on all the coordinates. In general, the solution has no symmetry. In Refs. [36, 37, 54, 98] it was shown that the source has the thermodynamics of a single-component perfect fluid only if the metric (20) – (21) is specialised so that it acquires an at least 3-dimensional symmetry group acting on at least 2-dimensional orbits. The FLRW limit follows when the functions k, x_0, y_0 and z_0 are all constant.

The arbitrary functions of time cause that the evolution of the spacetime is not determined. This is because no equation of state was imposed on (20) – (21). Unfortunately, the two types of equations of state that are most often used in cosmology and astrophysics (dust, $p = 0$, and a barotropic equation of state, $f(p, \rho) = 0$) both reduce (20) – (21) to a FLRW model.

(3b) The Petrov type D solutions

Equations (20) and (21) still apply here, but now $V(t,x,y,z)$ is determined by the following equation (resulting from the Einstein equations):

$$w_{uu}/w^2 = f(u), \tag{22}$$

where $f(u)$ is an arbitrary function. The variable u and the function w are related to the coordinates x, y, z, and to the function $V(t,x,y,z)$ as follows:

$$(u,w) = \begin{cases} (r^2, V) & \text{for spherically symmetric models;} \\ (z, V) & \text{for plane symmetric models;} \\ (x/y, V/y) & \text{for hyperbolically symmetric models,} \end{cases} \tag{23}$$

where $r^2 = x^2 + y^2 + z^2$. These three classes of models were found by Barnes [12], but the spherically symmetric case was known much earlier (and rediscovered many times over, see [89] for a full list). The Einstein equations were reduced to the form (22) by Kustaanheimo and Qvist [100]. With $f(u) = 0$, the Barnes models all become conformally flat and are then subcases of the Stephani solution.

Many papers discussed methods of solving (22) and examples of particular solutions, but with no relation to cosmology. An interesting application was a counterexample to the Ehlers – Geren – Sachs (EGS) theorem [48, 13] – it was shown that almost isotropic CMB is also possible in an inhomogeneous universe.

One member of the Barnes family of solutions, found by McVittie [116], is worth noting here:

$$\mathrm{d}s^2 = \left[\frac{1-\mu(t,r)}{1+\mu(t,r)}\right]^2 \mathrm{d}t^2 - R^2(t)\frac{[1+\mu(t,r)]^4}{\left(1+\frac{1}{4}\,kr^2\right)^2}\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)\right], \tag{24}$$

$$\text{where :} \qquad \mu(t,r) = \frac{m}{2rR}\sqrt{1+\frac{1}{4}\,kr^2},$$

m and k being arbitrary constants and $R(t)$ being an arbitrary function. In the case $m = 0$ this solution reproduces the whole FLRW class, and when $(k,R) = (0,1)$ it reproduces the Schwarzschild solution. So, it is an exact superposition of the FLRW and Schwarzschild metrics, with a perfect fluid source. It was published in 1933 (!).

A few authors attempted to apply this solution to observational cosmology. However, all those attempts were fallacious. McVittie's discussion of the influence of cosmic expansion on planetary orbits was coordinate-dependent. Järnefelt's perturbative discussions of the same problem [89] produced no conclusive result because the author did not define a length unit that would be unchanging in time. Later, McVittie [117] applied his solution to a discussion of stellar collapse, but the case $k = 0$ that he discussed has a spatially homogeneous density, so is unrealistic. Noerdlinger and Petrosian [123] considered the problem of whether clusters or superclusters of galaxies participate in the cosmological expansion. Their discussion was mostly Newtonian; they used the McVittie solution only to estimate the relativistic correction to the result.

The disadvantage of this solution, shared with the whole S–B family, is that it contains an arbitrary function of time, and so does not define any evolution law for the

Universe. One way out of this is to impose an equation of state – but so far no-one had a workable idea on what this equation should be. A barotropic equation $f(\rho, p) = 0$ reduces the McVittie solution to pure FLRW.

The subcase $k = 0$ is of little interest for cosmology because, similarly to the Stephani [147] solution, it has spatially homogeneous mass density, and so its whole inhomogeneity is hidden in pressure gradients. So far, nobody has provided a physical interpretation of this situation.

Global properties of the McVittie solution were discussed by Sussman [151]. He showed that the seemingly self-evident interpretation (a point particle in an expanding Universe) is not consistent with the global geometry. The set $r = 0$ is a null boundary, and its intersection with any t = constant hypersurface H is at an infinite geodesic distance from any other point of H.

More recently, there appeared a collection of papers discussing global properties of the McVittie solution, and some generalisations of it, for example by Nolan [124, 125], Carrera and Giulini [44] and Kaloper *et al.* [82]. However, from the point of view of cosmology, the results of these papers would require clarification, for which there is not enough space in this review, for the following reasons:

1. They discuss the physically uninteresting subcase $k = 0$.

2. They treat the McVittie solution as if it were the only existing candidate for a model of a black hole embedded in a FLRW Universe. They overlook the fact that it is a member of the large Barnes family that might be surveyed for more such examples. They also overlook the fact that the LT and Szekeres models do contain subcases describing black holes in a cosmological background, which are physically much better understood, see for example Ref. [96].

3. They are involved in a tangle of polemics, the later authors pointing out alleged errors in the earlier papers. Consequently, an extended re-analysis would be necessary in order to sort out who is right.

4. Some errors in the most recent papers are evident. Examples:

(4a) Carrera and Giulini [44] cite Sussman [151] and Gautreau [71] as examples of a "confusion" about interpreting the McVittie solution as a point particle in a FLRW Universe. In truth, Sussman was the first to point out *and resolve* this confusion, while Gautreau's paper has nothing to do with McVittie (it used the LT model to discuss the influence of cosmic expansion on planetary orbits).

(4b) Kaloper *et al.* [82] claim that there is some kind of singularity where invariants built of *second derivatives* of the Riemann tensor diverge; they call it a "very soft" singularity. In this, they revive the infamous "weak singularity" concept of Vanderveld *et al.* [166], which was proven in Refs. [32] and [97] to be no singularity at all; see also sec. 7 of the present text.

(4) Generalisations of the LT and Barnes models

For the LT model and for the whole Barnes class generalisations were found in which the matter source is a charged dust, or, respectively, a charged perfect fluid obeying

the Einstein – Maxwell equations. These do not seem to have a direct application in cosmology, so we do not review them here; see Refs. [89] and [134] for overviews. Nevertheless, the charged LT solution has interesting physical properties [134, 91, 92]. In addition, several generalisations of the LT and Barnes models were found, in which the source has nonzero viscosity or heat conduction. The physical interpretation of these in a cosmological context is less clear; see Ref. [89] for a review.

(5) Other models

The list that might be given here depends on how one defines a cosmological model. In Ref. [89] it was proposed that the term "cosmological model" may denote only such a solution of Einstein's equations that contains a nontrivial member of the FLRW class as a limiting case. We shall stick to this terminology here, thereby eliminating more than 1500 papers [89] whose authors used the term "cosmological model" for their results. In Ref. [89] this definition was used in a strict formal way, which resulted in listing a large number of metrics, most of which do not seem to have any relation to observational cosmology because, for example, they contain fields of unclear interpretation, often coupled together in ways that are difficult to interpret. Examples of those are briefly listed here to give the reader an idea about the wealth of the existing material.

(5a) Models with null radiation

These are superpositions of the FLRW models with various vacuum solutions, like those of Schwarzschild, Kerr, Kerr – Newman, etc. The superpositions are not perfect fluid solutions, and their energy-momentum tensors were interpreted ex post as mixtures of perfect fluid with null radiation (whose energy-momentum tensor is $T_{\mu\nu} = \tau k_\mu k_\nu$ with $k^\mu k_\mu = 0$), sometimes also with electromagnetic field. The solutions were in fact guessed in the course of exercises in metric-building and interpreting. As a result, the different contributions to the source are coupled through common constants so that, for example, the null radiation can in some cases vanish only if either the perfect fluid component or the inhomogeneity on the FLRW background go away. In particular, the superposition of the Schwarzschild and FLRW solutions in this family is different from the McVittie solution [116]. This activity was started by Vaidya [164], who found a superposition of the Kerr and FLRW solutions, and the probably most sophisticated composite was found by Patel and Koppar [132]; it is an infinite sequence of perturbations of the flat FLRW background whose first-order term is the Kerr solution.

(5b) The "stiff-fluid" models

These are solutions of the Einstein equations with a 2-dimensional Abelian symmetry group acting on spacelike orbits, in which the perfect fluid source obeys the "stiff equation of state", energy density = pressure (the source can be alternatively interpreted as a massless scalar field). It was claimed that these models apply to the early Universe, but the real reason behind the popularity of this activity was that such solutions can be relatively simply generated from vacuum solutions with the same symmetry, of which many are known. This activity began with the paper by Tabensky

and Taub [157], and the probably most sophisticated example of an explicit solution was given by Belinskii [14]. See [89] for an extended review.

(5c) Examples of other solutions (see [89] for a full list)

1. The Petrov type N perfect fluid solutions of Oleson [129].

2. A few simple examples of spherically symmetric perfect fluid solutions with shear, expansion and acceleration being all nonzero, see Ref. [89].

3. Examples of algebraically special solutions defined by requirements imposed on the degenerate principal null congruence of the Weyl tensor [89].

4. Anisotropic soliton-like perturbations propagating on the flat FLRW background (the pressure has different values for different directions). The most elaborate example of an explicit solution was given by Diaz, Gleiser and Pullin [55].

In this article we will discuss only those exact inhomogeneous cosmological models that allow for testable observational predictions. For more exhaustive discussions the reader is referred to [89, 134, 32].

3. Distance measurements

The concept of distance lies at the root of almost all cosmological observations whose interpretation strongly depends on this quantity. The distance however depends on the assumed model of the universe and on the matter distribution in it. The effect of inhomogeneities on the measured distance has been addressed frequently after the papers by Kristian & Sachs [99] and Dyer & Roeder [57] were published (see also [136] and references therein). For example, Partovi & Mashhoon [130] showed that the inhomogeneities affect the second order coefficient in the series expansion of the luminosity distance, i.e. the deceleration parameter. Using the same line of calculations Pascual-Sánchez argued that in such a case the deceleration parameter can be negative just due to the presence of inhomogeneities [131]. However, cosmologists often disregard the effect of inhomogeneities and just apply the FLRW relation. The 'justification' is: 1) even if density variations are large, the fluctuations of the gravitational potential are small and therefore the perturbation scheme can be applied, and 2) since the perturbations are Gaussian, they vanish after averaging, and therefore they should have little impact on observations. However, as shown by Sachs [140], the equation for the angular diameter distance D_A is

$$\frac{\mathrm{d}^2 D_A}{\mathrm{d}s^2} = -(|\sigma|^2 + \frac{1}{2} R_{\alpha\beta} k^\alpha k^\beta) D_A, \tag{25}$$

where σ is the shear, $R_{\alpha\beta}$ is the Ricci tensor and $R_{\alpha\beta}k^\alpha k^\beta = \kappa T_{\alpha\beta} k^\alpha k^\beta$. In the case of dust ($p = 0$), in the comoving and synchronous coordinates, $R_{\alpha\beta}k^\alpha k^\beta = \kappa\rho k^0 k^0$. As seen, the distance *does* depend on density fluctuations (not on the gravitational potential), and secondly, even if the perturbations vanish after averaging (i.e. $\langle\rho\rangle = \langle\rho_b + \delta\rho\rangle = \rho_b$, where ρ_b is the background density), they do modify the distance and the final result deviates from the homogeneous solution $\rho = \rho(t)$. This is a consequence of (25). This

means that one needs to know an exact model to calculate the distance – a statistical information about the density distribution, like the matter power spectrum, is not sufficient to calculate it. The matter power spectrum can only be used (within the linear regime) to estimate fluctuations around the mean distance-redshift relation, to be precise $\langle\Delta_D^2\rangle$ (where Δ_D is given by (26)). Thus, this method does not provide any information about the change of Δ_D, which as mentioned above, very often is assumed to be zero. Apart from the above mentioned 2 arguments sometimes people quote Weinberg's argument [169] that although for a single case the distance is modified by the inhomogeneities, but due to photon conservation, when averaged over large enough angular scales the overall effect is zero. A detailed discussion why this kind of reasoning should not apply is presented in [60].

To show how matter inhomogeneities affect the distance let us consider the following examples:

(i) A large-scale inhomogeneous matter distribution (Gpc-scale) whose volume average does not vanish, $\langle\delta\rho\rangle \neq 0$. This is the giant void model with best fit parameters as presented in [35] (the reader is referred there for more details).

(ii) Small-scale inhomogeneities (Mpc-scale) whose volume average vanishes, $\langle\delta\rho\rangle = 0$. However, the average of the density fluctuations along the line of sight is not zero $\langle\delta\rho\rangle_{1D} \neq 0$. The model is based on the Swiss-Cheese model presented in [24] and the reader is referred there for more details.

(iii) As above but now the distance is calculated within the weak lensing approximation, and is based on the model presented and discussed in [25].

(iv) Small-scale inhomogeneities (Mpc-scale) whose volume average vanishes, $\langle\delta\rho\rangle = 0$. Also, the average of density fluctuations along the line of sight is zero, $\langle\delta\rho\rangle_{1D} = 0$. The model is based on the Swiss-Cheese model presented in [26].

The results in terms of the distance corrections, δ_D, are presented in Fig. 1. The distance correction δ_D is defined by the following relation

$$D_A = \bar{D}_A(1 + \Delta_D), \tag{26}$$

where $\bar{D}_A$ is the distance in the homogeneous (background) model. As seen, the correction is of the order of a few percent, thus, owing to the increasing precision of the observations, the inhomogeneities need to be taken into account.

Below we will discuss several examples showing how inhomogeneities can influence our interpretation of cosmological observations. As we cannot discuss here every single paper that deals with this issue we will just focus on some major developments and refer only to a few (not all) papers dealing with this problem. We will also omit the Stephani models in our review as being less physically motivated.

4. The direct method

The studies of inhomogeneities and their effect on observations can be divided in 3 approaches: direct methods, the inverse problem and the averaging approach. Only

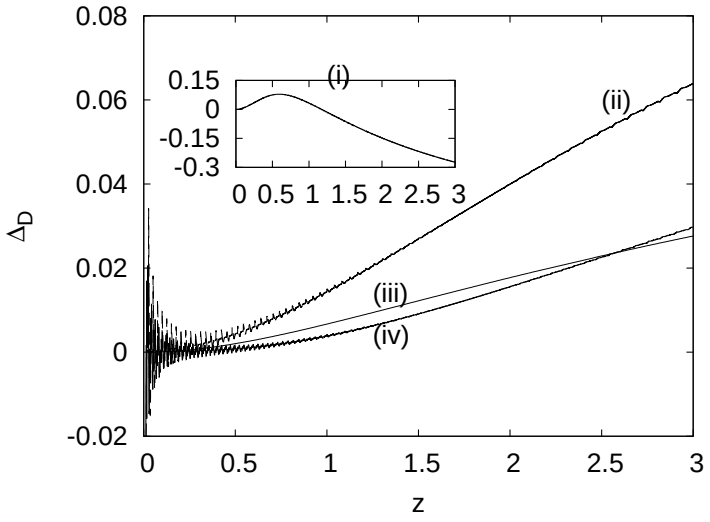

Figure 1. The distance correction. Roman numbers refer to the labels of the models discussed in this section. Models (ii)-(iv) are based on the Swiss-Cheese models with Mpc-scale inhomogeneities hence large fluctuations for low redshifts.

the first two will be discussed in this review – for the averaging approach see the contributions by Buchert, Räsänen, and Wiltshire in this issue. In the first approach a model is specified by a set of a priori chosen parameters and the observational data is used to find the best fit for these parameters. In the second approach the observational data is used to define the model with no a priori constraints imposed on it.

4.1. Giant void models

The giant void configurations are characterised by underdensity profiles increasing with radius on Gpc scales. One of the first and simplest models was the one discussed by Tomita [160, 161, 162]. He considered a model consisting of a low-density inner and a higher density outer homogeneous regions connected at some redshift and showed that such a configuration can explain the supernova dimming. After 2006 the number of papers concerning the giant void models rapidly increased. Assuming a density profile and an expansion rate, or a shape of the bang time function, one analyses cosmological observations to constrain the parameter space of the giant void. However, the particular constraints strongly depend on the assumed parameterisations, and almost every single paper introduced its own. As it is impossible to discuss all of them, we will focus here on three examples:

(i) The GBH void model [69] is defined by the following functions

$$M(r) = \frac{1}{2} H_0(r)^2 \Omega_m(r) R_0^3 \quad \& \quad k(r) = H_0(r)^2 (\Omega_m(r) - 1) R_0^2, \tag{27}$$

where

$$\Omega_m(r) = \Omega_{\rm out} + \left(\Omega_{\rm in} - \Omega_{\rm out}\right) \left(\frac{1 - \tanh[(r - r_0)/2\Delta r]}{1 + \tanh[r_0/2\Delta r]} \right), \tag{28}$$

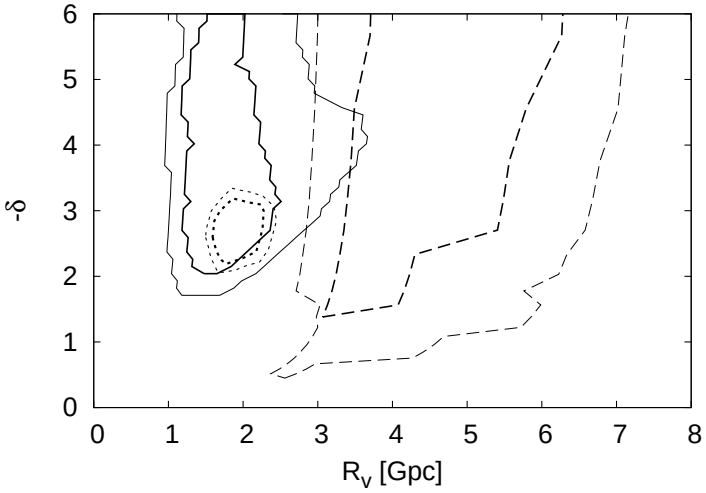

Figure 2. Supernova constraints on the size and depth of giant void models. Dashed line: the GBH model (27), solid line: the BW model (30), and dotted line: the ZMS spline model (31).

$$H_0(r) = H_{\rm out} + \Big(H_{\rm in} - H_{\rm out}\Big)\left(\frac{1-\tanh[(r-r_0)/2\Delta r]}{1+\tanh[r_0/2\Delta r]}\right). \qquad (29)$$

There are 6 parameters here, $\Omega_{\rm out}$ determined by the assumption of asymptotic flatness, $\Omega_{\rm in}$ determined by LSS observations, $H_{\rm out}$ determined by CMB observations, $H_{\rm in}$ determined by HST observations, r_0 characterizing the size of the void, Δr characterizing the transition to uniformity. But in the GBH model it is assumed that $\Omega_{\rm out} = 1$.

(ii) Bolejko and Wyithe class I model [35] is defined by

$$\rho(t_0, r) = \rho_b\left[1 + \delta_\rho - \delta_\rho \exp\left(-\frac{r^2}{\sigma^2}\right)\right], \quad \& \quad t_B = 0. \qquad (30)$$

where $\rho_b = \Omega_m \times (3H_0^2)/(8\pi G)$. It contains 4 parameters: $H_0, \Omega_m, \delta_\rho, \sigma$. In [35] it was assumed that $\Omega_m = 0.3$ but here we will allow this parameter to vary.

(iii) The spline model of Zibin, Moss, and Scott [176, 120] is defined by

$$\rho(t_0, r) = \rho_{EdS}(1+\delta), \quad \& \quad t_B = 0, \qquad (31)$$

where δ is given by a three-point cubic spline to the initial density fluctuation $\delta_j = \delta_{t_i, r_j} =$ and $j = 1, 2, 3$. By construction, $r_1 = 0 = \delta_3$. Thus the model depends on 5 free parameters: $\delta_1, \delta_2, r_2, r_3$ and ρ_{EdS} which just depends on H_0.

Using cosmological observations (like supernovae and CMB, etc.) one can constrain the range of the above parameters. Usually three parameters are most interesting: the size and depth of the void and the local value of H_0. In order to compare the constraints from these 3 parameterisations let us assume that the depth is just the density contrast ($\delta = 1 - \rho_{out}/\rho_{in}$) and the radius of a void is defined as a place (at the current instant)

where the density contrast becomes smaller than -0.1 as we proceed from outside into the void.

The constraints on δ and R_v from the supernova observations‖ are presented in Figure 2. As seen, using different parameterisations one obtains different constraints. For example, a void of size $R_v = 1.5$ Gpc and $\delta = -3$ is consistent with the constraints coming from (30), but is excluded by those coming from (27).

Apart from supernovae it is common to include the CMB constraints. However, up to date no one has performed the full CMB analyses within the LT framework. In the standard approach (the FLRW framework) the CMB data is analysed using the temperature anisotropy power spectrum given by the covariance of the temperature fluctuation expanded in spherical harmonics

$$C_l = 4\pi \int \frac{dk}{k} \mathcal{P}_i |\Delta_l(k, \eta_0)|^2 \tag{32}$$

where $\Delta_l(k, \eta_0, \mu)$ is the transfer function, $\mathcal{P}_i$ is the initial power spectrum, η_0 is the conformal time today and μ is the angle $\mu = \mathbf{k}.\mathbf{n}/k$ (with $\mathbf{n}$ the unit vector in the direction of the emission of the radiation). On large scales the transfer function is of the form $\Delta_l(k, \eta_0) = \Delta_l^{\rm LSS}(k) + \Delta_l^{\rm ISW}(k)$, where $\Delta_l^{\rm LSS}(k)$ is the contribution from the last scattering surface given by the Sachs-Wolfe effect and the temperature anisotropy, and $\Delta_l^{\rm ISW}(k)$ is the contribution due to the change in the gravitational potential along the line of sight, known as the integrated Sachs-Wolfe (ISW) effect. The ISW effect depends on the growth of the perturbations within the considered model. As the perturbative scheme within the LT model is still in its infancy [175, 50] (see also the contribution by Clarkson in this issue) it is impossible to estimate the ISW effect in the conventional way. However, the ISW effect is only important for low ℓ and is expected to be smaller than the cosmic variance $\Delta C_l/C_l = \pm(2/(2\ell+1))^{1/2}$. Another effect that is hard to estimate, but expected to be within the cosmic variance limits, is the effect of the reionisation.

Therefore, the analysis of the CMB within the LT framework is done as follows: it is assumed that the generation of the CMB anisotropies at the last scattering is the same as in the standard model. As the post-decoupling effects (like the ISW effect or reionisation) are expected to be smaller than cosmic variance, one does not estimate these effects within the LT framework, but uses standard codes like CMBFAST [144] or CAMB [104]. Therefore, if one does not change the initial power spectrum then the shape and amplitude of the Doppler peaks is just governed by $\Omega_c h^2, \Omega_b h^2, \Omega_k h^2$ (i.e., the physical densities of cold dark mater, baryonic matter and curvature) [38, 58]. Finally, as the angular diameter distance maps the physical position of the peaks to peaks in the angular power spectrum C_l as a function of the multipole ℓ, one needs to fit the angular diameter distance to the last scattering surface. Thus, the only difference between the standard analysis of the CMB and the analysis within the LT model is the change of distance to last scattering. Other effects have not been taken into account because of

‖ The constraining function is $\chi^2 = \sum_i \frac{(\mu_i - \mu_0)^2}{\sigma_i^2}$, where μ_i and σ_i correspond to the measurements of the 557 supernovae [6], μ_0 is the distance modulus in the considered model.

lack of a fully developed perturbative scheme within the LT framework, though the change of initial power spectrum was considered in [120, 122].

From the above description it is apparent that such an analysis only weakly constrains the giant void [51]. To successfully fit the CMB data one just needs to fit $\Omega_c h^2, \Omega_b h^2, \Omega_k h^2$ (in the region that emitted CMB) and the distance to the last scattering surface. This however, as argued in [35], can be achieved by changing the properties of the model outside the void. The region from which the CMB was emitted is currently approximately 13 Gpc away from us, therefore its $\Omega_c h^2, \Omega_b h^2, \Omega_k h^2$ are not related to a void which has a radius of ~ 3 Gpc. Similarly, the distance to the last scattering instant can be tuned by the properties of the model outside the void.

In most cases, however, one does not consider any modification outside the void and just assumes that the Universe (from our Galaxy up to the last scattering) is described by the chosen parametrisation, such as for example (27) or (31). Such a procedure leads to large systematics. For example, if the background (i.e the model far away from the void) is assumed to be the Einstein-de Sitter model then in order to have a good shape of the CMB power spectrum a low value of the expansion rate is required. This is because the proper shape of the power spectrum requires that $\Omega_m h^2 \approx 0.13$, so if one assumes that $\Omega_m = 1$ then one gets $h \approx 0.4$. This, on the other hand, has strong implications for the void. To fit the supernovae one needs a fluctuation of the expansion rate of amplitude $\delta_H \approx 0.1 - 0.2$ [63, 35], so this implies that the local expansion rate is low, i.e. $H_0 \approx 45$ km s^{-1} Mpc^{-1} [120] or $H_0 \approx 60$ km s^{-1} Mpc^{-1} [69]. This, when combined with local measurements of H_0, seems to rule out the giant void. The assumption of an Einstein-de Sitter background also impairs the BAO analysis like the one in [120]. When the assumption of spatial flatness is relaxed one obtains better results, see for example [17].

The local expansion rate within the LT region is also important for the age considerations. A small H_0 implies a large age of the Universe [120]. On the contrary, a large H_0 and $\Lambda = 0$ imply a shorter age. Thus $H_0 \approx 70$ km s^{-1} Mpc^{-1} gives the age of $11 - 12$ Gyr. This was discussed in [101], where it was shown that the current measurements do not put tight constraints on the model.

Actually, in an LT model, due to the shear, the anisotropy in the expansion increases as the void grows and becomes nonlinear. Hence, physical length scales, and in particular the sound horizon at the drag epoch, which are isotropic in a FLRW model, become more and more different in the radial and transverse direction. This is the reason why the Einstein-de Sitter background is suspected not to be a good approximation for the calculation of the BAO in the case of huge (Gpc) voids.

Different tests have been proposed in the literature to rule out or confirm the giant void proposal. The stronger constraints compared to other observational probes are the kinematic Sunyaev-Zeldovich (kSZ) effects [150] that can be observed on distant galaxy clusters. Since these clusters are off the centre of the LT Universe, there should be a large CMB dipole in their frame of reference which would manifest itself for us as a kSZ effect. Such a test was performed by the authors of [70] who showed that observations

of nine clusters with large error bars can rule out LT models with voids of size greater than ~ 1.5 Gpc. More recently, using the observational data from the the South Pole Telescope and the Atacama Cosmology Telescope, Zhang & Stebbins [174] put tighter constrains on the size of a void ~ 850 Mpc. However, the paper uses the "Hubble bubble" void model (i.e. based on a 2-region FLRW model, with negative curvature inside and spatially flat outside). Also, if one introduces non-adiabatic perturbations then the observational constraints are relaxed [172].

Another test uses the spectral distortion of the CMB black-body spectrum [43]. A large local void causes ionised gas to move outward, in motion relative to the frame of the CMB. This produces a Doppler anisotropy in the frame of the gas. A large void will imply large anisotropies which will be reflected back at us as spectral distortions. The test has been performed in [43], using the particular case of "Hubble bubble" void models. Large voids with large density contrasts are thus ruled out. However, the test has only been applied to the Hubble bubble class of models and other models may evade this test.

The void can also be tested directly by means of galaxy surveys as the one reported in [87]. Here, a deep, wide-field near-infrared survey is presented and explored to provide implications for local large scale structure. The results suggest that local structures may exist on scales up to 300 Mpc.

In the papers cited above, the observer has been assumed to be located at the centre of the LT model. But one can find in the literature some models where he/she is assumed to be off the centre. However, the CMB low multipoles put stringent limits on the distance he/she can be from the centre. This has been studied in [4, 18, 143] using different LT models. Using SN Ia data alone, it can be concluded that the observer can be displaced at most 15% of the void scale radius from the centre [4, 18]. But when one takes into account the induced anisotropies in the CMB temperature, the combination of the CMB dipole measurement and the SNe Ia data imposes very strict constraints on how far from the centre the observer can be located, i. e. no more than 1% of the void scale radius [18].

For more details on observational constraints on giant void models see the contributions by Zibin and Marra & Notari in this issue.

4.2. Non-void models – the effect of expansion

Giant voids are not the only configurations that can be used to fit cosmological observations. There is a group of LT models that are defined by the assumption of a homogeneous density distribution at the present instant and a Gpc-scale inhomogeneous expansion rate. Such a configuration was first considered in [22]. A homogeneous density profile at the current instant does not imply a homogeneous profile at all times. Also the bang time function in such cases is of high negative amplitude, around 1-2 Gyr at 2 Gpc [22]. The influence of inhomogeneous expansion on the luminosity distance was further studied in more detail by Enqvist and Mattsson [63]. In their set of different

LT models the observer is located at the centre and the universes are defined by an inhomogeneous expansion rate and a homogeneous density profile in some cases and in other cases by an inhomogeneous expansion and homogeneous t_B.

The analysis of cosmological observations within this type of models implies that they have a better goodness of fit than giant void models [63, 35]. The amplitude of the fluctuations of the expansion needed to fit the observations was found to be around $\delta_H \approx 0.1 - 0.2$ [63, 35]. This seems to imply that an inhomogeneous expansion rate is very important, since, as shown in [22], models with a homogeneous expansion rate and $\Lambda = 0$ cannot successfully fit supernova data.

This property can be linked to the result of Krasiński and Hellaby [95, 31] that velocity perturbations are more efficient at generating structures than density perturbations.

4.3. Swiss-Cheese models

An alternative way of modelling inhomogeneities is the Swiss-Cheese approach. Instead of assuming that the whole universe is modelled by a single inhomogeneity of Gpc-scale, smaller inhomogeneous patches that are matched with each other are considered.

The Einstein and Straus [65] type Swiss-cheese models were used to study, among other effects, the influence of inhomogeneities on the magnitude-redshift relation [83, 126, 127, 128]. However, since Schwarzschild's is a static solution, any influence of the expansion of the vacuoles remains weak in such models, and the magnitude of the reported effects is very low (as an example, Nottale [127], using a very simplified such model, found an observable amplification by medium density clusters or by superclusters of galaxies of only a tenth of a magnitude).

The recent appearance in the literature of models of Universe in which the inhomogeneities are represented by LT regions within a homogeneous background, where the matter is assumed continuously distributed, with densities both below and above the average, allows us to account for this vacuole expansion. The first authors, to our knowledge, to have considered such LT Swiss-cheese models to deal with the dark energy problem were Kai et al. [81]. However, their aim was to reproduce an accelerated expansion (which is only an artifact of the homogeneity assumption), and not the observed luminosity distance-redshift relation. Therefore, the constraints they found on their model cannot be considered as relevant for cosmological purpose.

Other LT Swiss-cheese models have been proposed to deal with the same dark energy problem [41, 42, 15, 112, 111]. The best results were obtained by Marra et al. [112, 111], who considered a model where holes with radius 350 Mpc are inserted into an Einstein-de Sitter background. Each hole exhibits a low density interior, surrounded by a Gaussian density peak near the boundary that matches smoothly to the exterior Friedmann density, and such that the matter density in the centre is roughly 10^4 times smaller than in the Friedmann background. To have a realistic evolution, it is also demanded that there are no initial peculiar velocities. This implies $0 < E(r) \ll 1$.

Evolving this model from the past to the present day, the inner almost empty region expands faster than the background, and the interpolating overdense region is squeezed by it. The density ratio between the background and the interior of the hole increases by a factor of 2. The evolution is realistic. Matter is falling toward the peaks in density. Overdense regions start contracting and become thin shells, mimicking structures, while underdense regions become larger, mimicking voids, and eventually they occupy most of the volume. The propagation of photons is studied in three cases: the observer is just outside the last hole, in the Friedmann region, looking at photons passing through the holes; the observer is on a high density shell; the observer is in the centre of a hole. The observables calculated are the redshift $z(\lambda)$, the angular diameter distance $D_A(z)$, the luminosity distance $D_L(z)$ and the corresponding distance modulus $\Delta m(z)$. In this model, inhomogeneities are able to mimic at least partly the effects attributed to dark energy.

The last scenario described above has some similarity to the one considered years ago by Sato and coworkers [108, 142, 109, 110, 141]. Maeda, Sasaki and Sato [108] considered a spherical void represented by a low-density FLRW region surrounding the centre of symmetry, itself surrounded by a LT transition region, in turn surrounded by a FLRW background with a higher density which has positive curvature and recollapses. The void has a tendency to expand forever, but it is eventually swallowed up in the final singularity of the background FLRW region. Sato and Maeda [142] have shown that spherical symmetry is a stable property in the expansion of voids, i.e. initially nonspherical voids become more spherical during their expansion. Maeda and Sato [109, 110] investigated the expansion of a shell of zero thickness and finite surface density of matter inside a spatially homogeneous dust medium with different densities on each side of the shell. They derived the equation of motion of the shell and the equation for mass-accumulation in the shell. They solved these equations numerically for the three types of FLRW background. The dependence of the enlargement of the void on the time of its formation was derived. In general, the earlier the formation time, the larger the enlargement. Moreover, the enlargement is increased for higher background density.

When studying the light propagation within Swiss Cheese models an important role is played by the proper randomisations. In some papers, like in the model by Marra et al., structures are lined up. However, as shown in [41, 42, 167, 156, 25] if one allows for randomisation of structures and angles at which light rays enter the structures, then the effect of inhomogeneities on the distance-redshift relation is reduced.

An intriguing result was presented in [52], where the Swiss Cheese model was constructed using the Schwarzschild solution. Their approach is a generalisation of the Lindquist & Wheeler model [105] and aims in describing the matter content of the Universe, which in fact is not a continuous fluid. The result, however, is that the distance to a given redshift in this model is smaller than in a homogeneous perfect fluid model. Thus in order to fit the supernova data, more dark energy is needed. For more details see the contribution by Clifton in this issue.

To escape the spherical symmetry of the vacuoles, a generalisation to the Szekeres

Swiss-cheese models was proposed in [27]. As a first step, and for simplicity, particular classes of axially symmetric quasi-spherical Szekeres holes were used to reproduce the apparent dimming of the supernovae of type Ia. The results were compared with those obtained in the corresponding LT Swiss-cheese models. Although the quantitative picture is different, the qualitative results are comparable, i.e, one cannot fully explain the dimming of the supernovae using small scale ($\sim$ 50 Mpc) inhomogeneities. To fit successfully the data, structures of at least $\sim$ 500 Mpc size are needed. However, this result might be an artifact of axial light rays in axially symmetric models (the model is not fully general). This work is a first step toward using the Szekeres Swiss-cheese models in cosmology.

5. The inverse problem

The inverse problem is conceptually different from the direct approach. Here one does not parametrise a model and look for the best fit values of assumed parameters. Instead, one uses observations to specify the functions that define the model. This idea was pursued by Kristian & Sachs [99], who were the first to consider how to use observations to determine the geometry the Universe. They used series expansions in powers of the diameter distance and focused on such observables like redshift, image distortion, number density, and proper motion. The problem was revived by Ellis et al [61, 149, 107, 8]. They considered the fluid-ray tetrad and focused on the spherically symmetric case and its perturbations. For the review and pedagogical presentation of the fluid-ray tetrad problem see [76].

5.1. Distance

The simplest version of the inverse problem is just to take the distance measurements (angular or luminosity) and use it to define the model. This approach is mostly based on the LT model. However, to define such model one needs 2 functions. This means that using just distance measurements one of the other functions needs to be specified by an ansatz instead of by observations. The simplest ansatz is to assume a spatially flat LT model. An LT model with $E(r) = 0 = \Lambda$ was considered in [45, 79, 166]. The model was fitted to the luminosity distance-redshift relation alone. This implies constraints on $t_B(r)$ which were given either in terms of constraints on the lower order derivatives of $t_B(r)$ taken at the observer as in [45] or in terms of differential equations which were numerically solved as in [79, 166]. Reference [79] also presented an algorithm defining the LT model from distance measurements and the assumption $t_B = 0$.

The aim of this approach was to show that supernova observations alone imply neither dark energy nor accelerated expansion of the universe. However, by imposing additional constraints some tried to argue otherwise. For example if one imposes smoothness conditions, i.e. density profile at the origin with vanishing first derivative, then one obtains that the luminosity distance within the LT model and the FLRW

model are the same up to the second order [158], which implies that the deceleration parameter for pure dust models must be positive. This, however, does not have any serious cosmological implications as, firstly, density does not need to be smooth [139], and, secondly, models with a smooth density profile can also fit the data without dark energy (for example most of the giant void models have a smooth density distribution, see Sec. 4.1).

The inverse problem that uses the angular diameter distances (this relates also to Sec. 5.2 – 5.6) has difficulties at the apparent horizon (AH), where $\mathrm{d}\widehat{R}/\mathrm{d}z = 0$. As this quantity can appear in the denominator (with another quantity vanishing at the AH in the numerator) this can cause problems when numerically solving the equations. This is not a drawback of the model, as improperly claimed in [166], and this can be dealt with in several ways. One of the solutions is to employ the Taylor expansion at the AH [106, 114, 30]. Another approach is based on solving the equations on both sides of the AH and choosing such solutions that approach each other [171]. In [46] the problem was solved by fitting polynomials to the functions $M(r)$ and $E(r)$ and using them as initial conditions for the direct method.

The existence of the AH can in fact be useful as it puts additional constraints that must hold at this location. For example for the Lemaître model (and its subcase the LT model) we have [3]

$$6M = 3R - \Lambda R^3. \tag{33}$$

A generic set of data will not obey the above relation. Also, as discussed in [121, 170], there are some other relations that will not be satisfied by generic data because real observational data is always accompanied by systematics. Thus, these relations can be used to estimate a correction for systematics so that a consistent solution is obtained. The algorithm for such corrections is presented and discussed in [106, 114, 30].

5.2. Distance and galaxy number counts

An algorithm which shows how to define an LT model based on distance and number count data was first presented in [121]. The algorithm was further developed in [106, 114] but no real observational data has been used. In [46] this algorithm was applied to $D(z)$ and $n(z)$ of the same form as in the ΛCDM model. In such a case the model obtained does not exhibit a giant void. The density at the current instant in this case is slowly increasing up to $\delta \approx 0.05$ and then is decreasing with an overall profile more resembling a hump than a void. The bang time function is negative and decreasing to around -2 Gyr at 4 Gpc.

In [88], the same goal of reconstructing a LT model from the luminosity-distance-redshift relationship and the light-cone matter density as a function of redshift that matches the fiducial ΛCDM model was pursued. The results exactly agree with those of [46]. Another result of this paper is that the LT model whose $D_L(z)$ and $\rho(z)$ functions exactly match those of the fiducial ΛCDM model has singular initial conditions for $R_{,r}$,

which means that $R_{,r} \to +\infty$ when the bang time is approached away from the center, i. e., $R_{,r}\left(r \neq 0, t_B(r)\right) = +\infty$.

5.3. Distance and expansion rate

Reference [46] also described an algorithm for defining a model based on distance and expansion rate observations. Again it was assumed that $D(z)$ and $H(z)$ are the same as in the ΛCDM model. The results suggested a model with a hump rather than a void, with a decreasing bang time function.

5.4. Distance and age of the universe

The first attempt to use real data to define the LT model was presented in [30]. Up to date there are no precise measurements of galaxy number counts, also the measurements of $H(z)$ [145, 148] are based on the assumption that $t_B = 0$ and cannot be used to define a general LT model. Therefore an algorithm for defining a LT model from distance and age measurements is given in [30]. The paper discusses two separate cases with and without the cosmological constant. In the case of $\Lambda = 0$ the results are somewhere between the giant void and hump configurations, i.e. the present-day density profile initially increases as in giant void models, but then decreases. However, due to poor data at high redshift one cannot have confidence in the model at large distances. The constraints on t_B are not tight and are consistent with either increasing or decreasing profiles. When $\Lambda \neq 0$ the results suggest a very slowly increasing profile, but are consistent with a homogeneous configuration.

5.5. Distance and redshift drift

An algorithm defining an LT model based on distance and redshift drift data (both as functions of redshift) can be found in [10]. The model uses the fluid-ray tetrad approach [76].

5.6. Distance, galaxy number counts, and age of the universe

To specify the LT model one needs to know 2 functions and 1 parameter (the cosmological constant, which usually, within the LT framework, is set to be zero). Ref. [28] presented the algorithm how to specify the LT model with the cosmological constant based on the distance, galaxy number counts, and age of the universe data. Using 3 sets of data allows to break the degeneracy (described in Secs. 5.2 and 5.3) between the ΛCDM model and zero-Λ LT model.

5.7. Consistency between observations

Another approach to study observations (instead of directly fitting a model with them) is based on checking the consistency between observations, i.e. to check if the relation

between observations is as given by the cosmological model. Ribeiro and Stoeger considered the consistency between the galaxy luminosity function, and corresponding galaxy number counts [138]. In a follow-up paper they showed that such an analysis strongly depends on the distance definition used [1]. Clarkson, Bassett & Lu [49] studied the relation between $H(z)$ and $D(z)$ data. They found that if the Universe is almost homogeneous on large scales, then the expansion rate and distance are not independent but are related. Thus, by studying relations between observations one can test the large scale homogeneity of the Universe. An additional problem arises when the observed objects evolve. A discussion of a possible distinction between the effect of evolution and inhomogeneity was presented in [74].

The motivation for the consistency checks is that the relation between different sets of observational data does not have to be the same as in the cosmological model that we assume to analyse the data. In [28] it was shown how using 3 different sets of data we can test consistency between observations and the underlying background cosmological models.

6. What if the cosmological constant is not zero?

In most of the literature applying inhomogeneous models to fit the observations the cosmological constant has been set to be zero. Actually, the aim of these works was to get rid of the impenetrable dark energy component. However, if, for some theoretical reason, coming for example from particle physics, a nonzero cosmological constant appeared to be part of the Universe energy budget, the effect of the inhomogeneities observed in the Universe should still be taken into account to build a proper cosmological model. Actually, the studies realised up to now show that their influence is not negligible.

Marra and Paakkonen [113] studied the giant void models with a non-zero cosmological constant. Their conclusion is that if $\Omega_\Lambda \approx 0.7$ then large voids are excluded by cosmological observations. On the other hand, large voids ($R_v \sim 3$ Gpc) with $\Omega_\Lambda \approx 0 - 0.3$ are consistent with the data.

Models with Mpc-scale inhomogeneities and cosmological constant were considered in [27], where it is shown that smaller values of Λ (than when homogeneity is assumed) are sufficient to fit the data. This is because small-scale inhomogeneities lead to an increase of the distance (see also Fig. 1) hence less dark energy is needed [27]. However, if the CMB constraints are taken into account, the opposite is true – in order to have a good fit more dark energy is needed (than when homogeneity is assumed) [26]. Also, as shown in [7], adding inhomogeneities to a model with the cosmological constant can actually improve the fit to the data, compared to purely homogeneous models.

The above mentioned studies are based on the direct approach. The first inverse approach with the pre-assumed cosmological constant was presented in [30]. The full inverse problem that uses the data to derive also the value of the cosmological constant was discussed in Ref. [28].

7. Formation of black holes

When studying black holes it is commonly assumed that these objects can be described using the Schwarzschild or Kerr metrics. This approach has the following caveats: (1) these space-times are asymptotically flat while the real Universe is not; (2) these black holes do not evolve, they exist unchanged from $t = -\infty$ to $t = +\infty$, while real black holes accrete mass.

The solution for the first problem are superpositions of the FLRW models with stationary black holes such as the Swiss cheese Einstein–Straus [65] configuration. Still, such black holes do not evolve, they exist *ab initio* and their masses do not change, whereas in cosmology we are interested in evolving black holes and in their formation.

An LT model can solve both these problems. Its first application to a study of the formation of black holes was presented in [96], and then followed by [68, 67]. Using it, one can study the evolution of primordial black holes or both the formation and evolution. For the most detailed analysis see [80]. The process analysed in detail in Refs. [96] and [80] was predicted by Bondi [39] already in 1947. A black hole is formed because rapidly collapsing matter forces the light rays to also converge toward the final singularity. A black hole with mass comparable to those at the centres of galaxies may form either out of a localised mass-density perturbation, or out of a localised velocity perturbation, or around a pre-existing wormhole [96]. In each case, an apparent horizon is formed because of the rapid collapse, and the collapse is caused either by gravitational attraction of the initial condensation, or by the initial fluctuation of velocity that magnifies itself in the course of collapse.

So far the problem was not considered beyond the LT models. Although the collapse within the Szekeres model was studied [155] it was only within the asymptotically flat models, not within a cosmological background.

8. Observational predictions

There are a number of potentially observable effects that could occur only when inhomogeneities are present and do not exist in the Friedmann models. The best known among them is gravitational lensing (see paragraphs 4 and 5 of Sec. 10). In this section we are not going to discuss the effects that are most often modelled using perturbative methods, like gravitational lensing or the Rees–Sciama effect. Instead, we will focus on the less known effects, in particular those that can potentially be used to distinguish between Gpc-scale inhomogeneous models and homogeneous models with dark energy. Thus, the list below is very selective and does not include all possible observational tests.

- Redshift drift

As the Universe evolves, the redshifts of astronomical objects change with time. For the ΛCDM model $\Delta z > 0$ for $z < 2$. For the giant void models (which are the most popular alternative among the inhomogeneous models) Δz is expected to be negative

for all z. Thus, a detection of a negative redshift drift for all z would be a proof against dark energy. However, the converse is not true, as there are Gpc-scale inhomogeneous models that also have $\Delta z > 0$ for low z [173].

- Galaxy number counts

The galaxy distribution on small scales is very inhomogeneous, with large fluctuations in number counts. However, with the increasing amount of data we should be able to detect an overall trend of $n(z)$. In this case it will be possible to see if the overall behaviour is consistent with the prediction of homogeneous models. Although a detection of a Gpc-scale inhomogeneous trend would be an argument against large-scale homogeneity, the converse argument does not hold as there are inhomogeneous models that can have the same $n(z)$ as homogeneous models [46].

- Kinematic Sunyaev-Zel'dovich effect

The existence of a Gpc-scale inhomogeneity leads to an additional (compared to a homogeneous scenario) peculiar velocity of galaxy clusters. As discussed in Sec. 4.1, the present data already puts tight constraints on the size of such an inhomogeneity. Thus, with new data coming from the Planck mission, the giant void models will be put to the test.

- Lyα observations

Observations of Lyα lines in spectra of distant quasars provide information about the amount of light elements. These observations can be used to constrain cosmological parameters, for example the D/H ratio is very sensitive to $\Omega_b h^2$. The accurate analysis of the observations is difficult as the amount of light elements also depends on astrophysical processes. However, it is believed that low metallicity objects should have the deuterium to hydrogen ratio unchanged from the time of the primordial nucleosynthesis.

Within a homogeneous universe $\Omega_b h^2$ should be everywhere the same. The observations, however, show a large scatter in the data which is also not consistent with the WMAP data [133]. A conventional explanation of this phenomenon is that the errors in the individual measurements of D/H may have been underestimated [146, 133]. As this may be true, in the future, with a large amount of data and more precise observations, it will be possible to detect if the variation of $\Omega_b h^2$ is real.

- Dark flow

In the standard approach, the galaxy velocity field is described using linear perturbations of the Friedmann model. Within this framework, flows of large amplitude on a scale beyond 100 Mpc are exceptional. However, observations show that such a flow exists on scales of at least 150 Mpc [168, 66]. Although such a flow is hard to explain using linear methods, it may still be consistent with the standard cosmological model. But if this flow extends to even larger scales, the ΛCDM model will not be able to account for it.

Recently, Kashlinsky *et al.* reported the existence of flows on scales over at least 800 Mpc [85, 86]. The result of their analysis is subject to large systematics and so far

has not been confirmed by any other group. However, if such a flow is confirmed, then this will put the ΛCDM model at odds with the data.

- Maximum of the diameter distance

The position of the maximum of the angular distance puts additional constraints on a model, see (33). This relation combines the distance, mass and the cosmological constant [75]. Thus it may serve as a consistency check and may be used to rule out the models that do not meet this criterion. Also, the position itself can be different for different types of models. For example, the giant void models have typically the maximum around $z \approx 1$ while the ΛCDM model has a maximum around $z = 1.6$ [35].

- Non-repeatable light paths (non-RLPs).

In Ref. [93] it was shown that within inhomogeneous models generic light rays do not have repeatable paths: two rays sent from the same source at different times to the same observer pass through different sequences of intermediate matter particles. This effect is a consequence of nonzero shear in the flow of the cosmic medium and does not exist in the Robertson–Walker models. This shows that RLPs are very special and in the real Universe should not exist. As a consequence, cosmological objects should change their positions in the sky. Although the effect is small, in principle it is detectable.

The existence of this effect may also have consequences in applying averaging schemes. Within an averaging scheme, an inhomogeneous distribution is approximated with a uniform (averaged) model. As a fist approximation it is assumed that light propagates along null geodesics of a homogeneous model (the only difference is that the evolution of the model is governed by the Buchert rather than Friedmann equations). However, if geodesics that join the observer and the source proceed at different times though different sequences of intermediate matter particles, then the path of the light ray within an average geometry may not be a geodesic anymore.

9. Pervasive errors and misconceptions

Many astrophysicists tolerate a loose approach to mathematics and physics. Papers written in such a style planted errors and misconceptions in the literature, which were then uncritically cited in other papers and came to be taken as established facts. In this section we present a few most damaging misconceptions (marked by black squares ■) together with their explanations (marked by large asterisks ∗).

■ The LT models that explain away dark energy with matter inhomogeneities contain a "weak singularity" at the centre [166], where the scalar curvature R has the property $g^{\mu\nu}R_{;\mu\nu} \to \infty$.

∗ $g^{\mu\nu}R_{;\mu\nu} \to \infty$ is not a singularity by any accepted criterion in general relativity [97]. It only implies a discontinuity in the derivative of mass density by distance – a thing quite common in Nature (e.g. on the surface of the Earth). At the centre, $g^{\mu\nu}R_{;\mu\nu} \to \infty$ implies a conical profile of density – also a nonsingular configuration.

■ Decelerating inhomogeneous models with $\Lambda = 0$ cannot be fitted to the same distance–redshift relation that implies acceleration in ΛCDM. This is because a certain equation connecting the deceleration parameter q_4 to density, expansion and shear prohibits $q_4 < 0$ [78].

∗ The equation derived in [78] (formally analogous, but inequivalent, to the Raychaudhuri equation) is based on approximations that are not explicitly spelled out [97]. An approximate equation cannot determine the sign of anything. If the approximations are taken as exact constraints imposed on the LT model, they imply zero mass density, i.e. the Schwarzschild limit. Moreover, the q_4 of [78], although it coincides with the deceleration parameter in the Friedmann limit, is not a measure of deceleration in an inhomogeneous model (it is defined by the Taylor expansion of the luminosity–redshift relation). Refs. [30, 46, 79] provide an explicit demonstration that a decelerating LT model with $\Lambda = 0$ *can* be fitted to *exactly the same* distance - redshift relation that holds in the ΛCDM model. This relation is reproduced by a spatially inhomogeneous expansion pattern, without any dark energy.

■ There is a "pathology" in the LT models that causes the redshift-space mass density to become infinite at a certain location (called "critical point") along the past light cone of the central observer [166].

∗ The "critical point" is the apparent horizon (AH), at which the past light cone of the central observer begins to re-converge toward the past. This re-convergence had long been known in the FLRW models [59, 115], and the infinity in density is a purely numerical artifact – a consequence of trying to integrate past AH an expression that becomes $0/0$ at the AH. Ways to handle this problem are known [106, 114, 46].

■ Fitting the LT model to cosmological observations, such as number counts or the Hubble function along the past light cone, results in predicting a huge void, at least several hundred Mpc in radius, around the centre (see discussion in Sec. 4.1).

∗ The implied huge void is a consequence of handpicked constraints imposed on the arbitrary functions of the LT model, for example a constant bang time t_B. With no a priori constraints, the giant void is not implied [46].

■ The bang time function must be constant because $\mathrm{d}t_B/\mathrm{d}r \neq 0$ generates decaying inhomogeneities, which would have to be "huge" in the past, and this would contradict the predictions of the inflationary models (private communication from the referees of [46]).

∗ While it is true that in models with $\mathrm{d}t_B/\mathrm{d}r \neq 0$ the early universe was very inhomogeneous, it does not mean that such models could not be realistic (so far they are consistent with observations after all). Although in the current paradigm the early universe undergoes inflation that is supposed to leave it very homogeneous, the occurrence of inflation is not in any way proven. Inflationary models are just one of hypotheses that compete for observational confirmation. Thus, using them to justify or reject some other hypotheses may sound dogmatic and is in fact un-scientific.

10. Discussion and future prospects

We have seen that one can find in the literature a number of models constructed with exact inhomogeneous solutions of Einstein's equations which fit the available observational data as properly as (and sometimes better than) the standard ΛCDM model.

The LT model with a central observer, which is sometimes criticised as being at odds with the Copernican Principle, must be, in our view, only considered as an intermediate model where the angular inhomogeneities have been smoothed around the observer and only the radial inhomogeneities have been taken into account (an example of such a situation is presented and discussed in [34]). Moreover, the use of oversimplified LT models can create another false idea and false expectation. The false idea is that there is an opposition between the ΛCDM model, belonging to the FLRW class, and the LT model or in general, inhomogeneous models: it is believed that either one or the other could be 'correct', but not both. This putative opposition can then give rise to the expectation that more, and more detailed, observations will be able to tell us which one to reject. In truth, there is no opposition. The inhomogeneous models, like for example the LT model with its two arbitrary functions of one variable, are huge, compared to FLRW, *families* of models that include the Friedmann models as a very simple subcase. The fact, demonstrated in several papers, that even a $\Lambda = 0$ LT model can mimic $\Lambda \neq 0$ in an FLRW model, additionally attests to the flexibility and power of the LT model. Thus, if the Friedmann models, ΛCDM among them, are considered good enough for cosmology, then the LT models can only be better: they constitute an *exact perturbation* of the Friedmann background, and can reproduce the latter as a limit with an arbitrary precision. The right question to ask is not "which model to reject: FLRW or LT?", but "how close to their FLRW limits must the LT arbitrary functions be to satisfy the observational constraints?".

Nature does not create objects that fulfil mathematical assumptions with perfect precision. Objects in mechanics or electrodynamics that are described as spherically symmetric have this symmetry only up to some degree of approximation. An "ideal gas" in thermodynamics is nearly ideal only at sufficiently low pressure. An "incompressible fluid" ..., and so on. Why should the Universe be an exception and be exactly homogeneous in the large (and exactly spatially flat in addition)?

In fact, we already have qualitative evidence that our observed Universe is not FLRW: the gravitational lenses. The FLRW models are conformally flat, so the null geodesics in them are conformal images of the null geodesics from the Minkowski spacetime. In this spacetime, null rays sent from a common origin never intersect again. So, in a conformally flat spacetime null rays issuing from a common source can intersect again only in such points that are singularities of the conformal mapping (and, consequently, of the spacetime itself). Then, however, the positions of the points of second intersection are determined by the geometry of spacetime, and not just by the initial points and directions of the rays, as is the case in a gravitational lens (where, in

addition, there is no spacetime singularity at the intersection point). Hence, a spacetime containing a gravitational lens cannot be conformally flat.

Gravitational lenses are observed in our Universe at the distance scales, at which the FLRW approximation is supposed to already apply, namely the lensing objects and the sources of lensed rays are quasars. So, our Universe does not have the FLRW geometry at large scales.

One more qualitative evidence of our Universe being non-FLRW on large scales may be provided by the effect of non-repeatable light paths, described in Ref. [93].

It is strange that a large part of the astrophysical community is comfortable with the idea of linearised perturbations around homogeneous models, but reacts with strong negative emotions to exact perturbations represented by inhomogeneous models.

In the future, the LT models will be used to extract the cosmic metric from observations. This programme has been initiated in [106, 114, 28]. This is the full inverse problem. To date it has assumed that the metric has the LT form, as a relatively simple case to start from, but the long term intention is to remove the approximation of spatial symmetry.

All known exact solutions of the Einstein equations which can be of cosmological use possess some symmetries or quasisymmetries. The only way to overcome such shortcomings is to obtain a fully operational, exact and inhomogeneous solution of these equations. This can only be achieved using numerical relativity and we suspect that this will be the new way of dealing with cosmology in the years to come.

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