



HAL
open science

REGULARITY CRITERIA FOR STRONG SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS

Abdelhafid Younsi

► **To cite this version:**

Abdelhafid Younsi. REGULARITY CRITERIA FOR STRONG SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS. 2014. hal-00724028v4

HAL Id: hal-00724028

<https://hal.science/hal-00724028v4>

Preprint submitted on 15 Mar 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

REGULARITY CRITERIA FOR STRONG SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS

ABSTRACT. In this paper, we study the regularity problem of the 3D incompressible Navier–Stokes equations. We prove that the strong solution exists globally for new regularity criteria. For negligible forces, we give an improvement of the known time interval of regularity obtained in [9].

1. INTRODUCTION

Two of the profound open problems in the theory of three dimensional viscous flows are the unique solvability theorem for all time and the regularity of solutions. For the three-dimensional Navier-Stokes system weak solutions are known to exist by a basic result by Leray from 1934 [10], but the uniqueness is still open problem [1]-[3] and [8]. Furthermore, the strong solutions for the 3D Navier-Stokes equations are unique and can be shown to exist on a certain finite time interval for small initial data and small forcing term, but the global regularity for the 3D Navier-Stokes is still open problems (see [4]-[8], [12]-[14] and references therein). In 1933 [9], Leray showed that in the absence of forcing ($f = 0$), all solutions of Navier-Stokes equations are eventually smooth (i.e. after some $T^* > 0$ depending on the data). Kato and Fujita [7] showed that a smooth solution to the three-dimensional Navier-Stokes equations exists for all time if f is small in some sense and u_0 is small in $H^{\frac{1}{2}}$.

In this paper, we give a new condition for global existence in time for strong solution for 3D Navier-Stokes equations with external force. We show that no singularity can occur in finite time for a large class of forcing term. We also give an extension of the time interval of regularity to the 3D Navier-Stokes equations with negligible forces [3, 5, 11, 13]. This result means that the solution does not blow up at T^* .

2. NOTATIONS AND PRELIMINARIES

In this section we introduce notations and the definitions of standard functional spaces that will be used throughout the paper. We denote by $H_{per}^m(\Omega)$, the Sobolev space of periodic functions. These spaces are endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)} \text{ and the norm } \|u\|_m = \sum_{|\beta| \leq m} (\|D^\beta u\|_{L^2(\Omega)}^2)^{\frac{1}{2}}.$$

We define the spaces V_m as completions of smooth, divergence-free, periodic, zero-average functions with respect to the H_{per}^m norms. V'_m denote the dual space of V_m .

Let P be the orthogonal projection in $L^2_{per}(\mathbb{R}^3)^3$ with the range V_0 . Let $A = -P\Delta$ the Stokes operator. It is easy to check that $Au = -\Delta u$ for every

2010 *Mathematics Subject Classification.* 35Q30, 35D35, 35B44 .

Key words and phrases. Navier-Stokes equations. Strong solutions. Blow-up.

$u \in D(A)$. We recall that the operator A is a closed positive self-adjoint unbounded operator.

The eigenvalues of A are $\{\lambda_j\}_{j=1}^{j=\infty}$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and the corresponding orthonormal set of eigenfunctions $\{w_j\}_{j=1}^{j=\infty}$ is complete in V_0

$$Aw_j = \lambda_j w_j, \quad w_j \in D(A), \forall j. \quad (2.1)$$

Let us now define the trilinear form $b(.,.,.)$ associated with the inertia terms

$$b(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j dx. \quad (2.2)$$

The continuity property of the trilinear form enables us to define (using Riesz representation Theorem) a bilinear continuous operator $B(u, v); V_2 \times V_2 \rightarrow V_2'$ will be defined by

$$\langle B(u, v), w \rangle = b(u, v, w), \quad \forall w \in V_2. \quad (2.3)$$

Recall that for u satisfying $\nabla \cdot u = 0$ we have

$$b(u, u, u) = 0 \text{ and } b(u, v, w) = -b(u, w, v). \quad (2.4)$$

We recall some well known inequalities that we will be using in what follows.

Young's inequality

$$ab \leq \frac{\sigma}{p} a^p + \frac{1}{q\sigma^{\frac{q}{p}}} b^q, \quad a, b, \sigma > 0, \quad p > 1, \quad q = \frac{p}{p-1}. \quad (2.5)$$

Poincaré's inequality

$$\lambda_1 \|u\|^2 \leq \|A^{\frac{1}{2}} u\|^2 \text{ for all } u \in V_0, \quad (2.6)$$

λ_1 is the first eigenvalue of the Stokes operator.

3. NAVIER-STOKES EQUATIONS

The conventional Navier-Stokes system can be written in the evolution form

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u &= f, \quad t > 0, \\ \operatorname{div} u &= 0, \quad \text{in } \Omega \times (0, \infty) \text{ and } u(x, 0) = u_0, \quad \text{in } \Omega. \end{aligned} \quad (3.1)$$

We recall that a Leray weak solution of the Navier-Stokes equations is a solution which is bounded and weakly continuous in the space of periodic divergence-free L^2 functions, whose gradient is square-integrable in space and time and which satisfies the energy inequality. The proof of the following theorem is given in [13].

Theorem 3.1. *Assume that $f \in L^2(0, T; V_1')$ and $u_0 \in V_0$ are given. Then there exists at least one solution u of (3.1) such that $u \in L^2(0, T; V_1) \cap L^\infty(0, T; V_0)$, $\forall T > 0$.*

For strong solutions, we have the following result [13].

Theorem 3.2. *Assume that $u_0 \in V_1$ and $f \in V_0$ are given. then there exists a $T > 0$ depending on $\|u_0\|_1$, ν and $\|f\|$, such that there exists a unique strong solution $u \in L^\infty(0, T; V_1) \cap L^2(0, T; V_2)$.*

This result was obtained for a type of inequality similar to

$$\|u(\cdot, t)\|_1^2 \leq \frac{1 + \|u_0\|_1^2}{\sqrt{1 - Kt(1 + \|u_0\|_1)^2}}, \quad (3.2)$$

where $K = (2\frac{\|f\|^2}{\nu} + \frac{c_1}{\nu^3})$. Hereafter, $c_i \in \mathbb{N}$, will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. The bound in (3.2) is only finite while

$$Kt(1 + \|u_0\|_1)^2 < 1; \quad (3.3)$$

if we choose T satisfying

$$T < \frac{1}{K(1 + \|u_0\|_1)^2}. \quad (3.4)$$

The main result of this paper is given in the following theorem.

Theorem 3.3. *Assume that $u_0 \in V_1$ and u is the corresponding strong solution to (3.1) on $[0, T]$, then*

i) If $f \in V_0$ and

$$c_8 T \|f\|^2 + c_9 \|u(0)\|^2 + \arctan \|u(0)\|_1^2 < \frac{\pi}{2}, \quad (3.5)$$

then u exists for each finite time T and remains smooth.

ii) If $f \in L^2(0, T, V_0)$ and

$$c_{10} \int_0^T \|f\|^2 ds + c_{11} \|u(0)\|^2 + \arctan \|u(0)\|_1^2 + \arctan \|u(0)\|_1^2 < \frac{\pi}{2}, \quad (3.6)$$

then u exists globally (T can be ∞) and remains smooth.

Proof. Multiplying (3.1) by Δu , we have

$$\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|_1^2 + \nu \|\Delta u\|^2 - \int_{\Omega} (u \cdot \nabla u) \cdot \Delta u \, dx = (f, \Delta u). \quad (3.7)$$

Using schwartz and Young inequality we get

$$\begin{aligned} |(f, \Delta u)| &\leq \|f\|_{L^2} \|\Delta u\|_{L^2} \\ &\leq c_3 \|f\|_{L^2}^2 + \frac{\nu}{2} \|\Delta u\|_{L^2}^2. \end{aligned} \quad (3.8)$$

For the nonlinear term, we use the Hölder's inequality

$$\begin{aligned} \left| \int_{\Omega} (u \cdot \nabla u) \cdot \Delta u \, dx \right| &\leq c_4 \|u\|_{L^6} \|\nabla u\|_{L^3} \|\Delta u\|_{L^2} \\ &\leq c_5 \|\nabla u\|_{L^3}^{\frac{3}{2}} \|\Delta u\|_{L^2}^{\frac{3}{2}}. \end{aligned} \quad (3.9)$$

However, an application of Young's inequality to (3.9) yields

$$\left| \int_{\Omega} (u \cdot \nabla u) \cdot \Delta u \, dx \right| \leq c_6 \|\nabla u\|^6 + \frac{\nu}{2} \|\Delta u\|^2. \quad (3.10)$$

Combining (3.7), (3.8) and (3.10), we have that

$$\frac{d}{dt} \|u(\cdot, t)\|_1^2 \leq c_3 \|f\|^2 + c_6 \|u\|_1^6. \quad (3.11)$$

Suppose first $f \in V_0$. Setting $y(t) = \|u(\cdot, t)\|_1^2$ in (3.11), this gives

$$\frac{d}{dt}y \leq c_3\|f\|^2 + c_6y^3. \quad (3.12)$$

Dividing (3.12) by $1 + y^2$, we have

$$\frac{\frac{d}{dt}y}{1 + y^2} \leq \frac{c_3\|f\|^2}{1 + y^2} + c_6y \frac{y^2}{1 + y^2}. \quad (3.13)$$

Since $\left\|\frac{1}{1+y^2}\right\|_\infty \leq 1$ and $\left\|\frac{y^2}{1+y^2}\right\|_\infty \leq 1$, this yields

$$\frac{y'}{1 + y^2} \leq c_3\|f\|^2 + c_6y. \quad (3.14)$$

Integrate this over $[0, T]$ to get

$$\arctan y(t) - \arctan y(0) \leq c_3T\|f\|^2 + c_6 \int_0^T y(s) ds. \quad (3.15)$$

The function \arctan is defined for each $t \in \mathbb{R}$ and $\frac{-\pi}{2} < \arctan t < \frac{\pi}{2}$. Multiplying (3.1) by u to get

$$\frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|^2 + \nu \|u\|_1^2 = (f, u). \quad (3.16)$$

Using Schwartz and Poincare inequality, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u(\cdot, t)\|^2 + \nu \|u\|_1^2 &\leq \|u\| \|f\| \\ &\leq \frac{\nu}{2} \|u\|_1^2 + c_7 \|f\|^2. \end{aligned} \quad (3.17)$$

Integrate (3.17) over $[0, T]$, to get

$$\|u(\cdot, t)\|^2 + \frac{\nu}{2} \int_0^T \|u(s)\|_1^2 ds \leq c_7 \int_0^T \|f\| ds + \|u(0)\|^2. \quad (3.18)$$

For $f \in V_0$, the inequality above gives

$$\arctan y(t) \leq c_8T\|f\|^2 + c_9 \|u(0)\|^2 + \arctan y(0). \quad (3.19)$$

The function $\tan y$ is increasing and invertible for $\frac{-\pi}{2} < y < \frac{\pi}{2}$ with inverse function $\arctan t$. Thus, for

$$c_8T\|f\|^2 + c_9 \|u(0)\|^2 + \arctan \|u(0)\|_1^2 < \frac{\pi}{2} \quad (3.20)$$

we apply the function \tan on (3.19) to get

$$\|u(t)\|_1^2 \leq \tan \left(c_8T\|f\|^2 + c_9 \|u(0)\|^2 + \arctan \|u(0)\|_1^2 \right). \quad (3.21)$$

The assumption (3.20) on the initial data guarantees that the right hand side of (3.21) is finite.

For $f \in L^2(0, T, V_0)$ and for each $T > 0$, integrate (3.14) to get

$$\arctan y(t) - \arctan y(0) \leq c_3 \int_0^T \|f\|^2 ds + c_6 \int_0^T y(s) ds. \quad (3.22)$$

Therefore, (3.18) implies

$$\arctan y(t) \leq c_{10} \int_0^T \|f\|^2 ds + c_{11} \|u(0)\|^2 + \arctan y(0). \quad (3.23)$$

In particular, for

$$c_{10} \int_0^T \|f\|^2 ds + c_{11} \|u(0)\|^2 + \arctan \|u(0)\|_1^2 < \frac{\pi}{2}, \quad (3.24)$$

we apply the function \tan on (3.23) to get

$$\|u(t)\|_1^2 \leq \tan \left(c_{10} \int_0^T \|f\|^2 ds + c_{11} \|u(0)\|^2 + \arctan \|u(0)\|_1^2 \right). \quad (3.25)$$

It follows that the assumption (3.24) guarantees that $\|u(t)\|_1 < \infty$ for all $t > 0$. \square

Recall that the classical regularity result for $f = 0$ [13, Theorem 3.12.] was obtained for a type of inequality similar to

$$\|u(\cdot, t)\|_1^4 \leq \frac{\|u_0\|_1^4}{1 - c_{12} 2t \|u_0\|_1^4}. \quad (3.26)$$

It follows that if $\|u_0\|_1$ is finite, then $\|u(\cdot, t)\|_1$ is finite, at least for

$$t < \nu^3 / 128 \|u_0\|_1^4. \quad (3.27)$$

Consequently, we get the following result for the negligible forces.

Corollary 3.4. *Assume that $u_0 \in V_1$ and u is the corresponding strong solution to (3.1) on $[0, T]$, then u exists globally and remains smooth for all $T > 0$ if*

$$c_{11} \|u(0)\|^2 + \arctan \|u(0)\|_1^2 < \frac{\pi}{2}. \quad (3.28)$$

Proof. From (3.11) we get for $f = 0$

$$\frac{d}{dt} \|u\|_1^2 \leq c_6 \|u\|_1^6. \quad (3.29)$$

In particular, we have

$$\frac{d}{dt} y \leq c_6 y (1 + y^2) \quad \text{with } y(t) = \|u(\cdot, t)\|_1^2. \quad (3.30)$$

Dividing (3.30) by $1 + y^2$ yields

$$\frac{\frac{d}{dt} y}{1 + y^2} \leq c_6 y. \quad (3.31)$$

Integrate this over $[0, T]$ to get

$$\arctan y(t) \leq c_6 \int_0^T y(s) ds + \arctan y(0). \quad (3.32)$$

From (3.18), we find

$$\int_0^T y(s) ds \leq c_9 \|u(0)\|^2 \quad (3.33)$$

this implies that (3.32) is equivalent to

$$\arctan y(t) \leq c_{11} \|u(0)\|^2 + \arctan y(0). \quad (3.34)$$

Now, applying the function \tan on (3.34) to get

$$\|u(t)\|_1^2 \leq \tan \left(c_{11} \|u(0)\|^2 + \arctan \|u(0)\|_1^2 \right), \quad (3.35)$$

which is finite thanks to the following assumption

$$c_{11} \|u(0)\|^2 + \arctan \|u(0)\|_1^2 < \frac{\pi}{2} \quad (3.36)$$

and this concludes the proof. \square

Since the condition (3.36) is independent of time we get global estimates for $\|u(\cdot, t)\|_1$ by this method. An important consequence of this result is that for each finite time T^* such that

$$T^* < \nu^3/128 \|u_0\|_1^4, \quad (3.37)$$

there is a u_0 satisfies

$$c_{11} \|u(0)\|^2 + \arctan \sqrt{\nu^3/128T^*} < \frac{\pi}{2}. \quad (3.38)$$

Thus the solution associate to u_0 satisfies (3.37) has a global regularity. But for the same value of $\|u(0)\|_1$ occurs a blow up in finite time T^* by the usual method (3.26). This property follows easily when $\|u(0)\|$ approaches zero.

This result gives a simple condition for global regularity and extends the known corresponding result (3.26), where a blow-up criterion in finite time T depend on u_0 for negligible forces, see [3, 5, 11, 13].

REFERENCES

- [1] Cao, C., Titi, E. S. Regularity criteria for the three dimensional Navier-Stokes equations. *Indiana Univ. Math. J.* 57, 6, (2008)2643-2661.
- [2] Chemin, J. Y. , Gallagher, I., Paicu, M. Global regularity for some classes of large solutions to the Navier-Stokes equations, *Ann. of Math. (2)* 173 , no. 2, 983–1012(2011).
- [3] Doering, C. R. The 3D Navier-Stokes problem. *Annu. Rev. Fluid Mech.* 41, 109-128(2009).
- [4] Deuring, P., Wahl, W. V. Strong solutions of the Navier-Stokes system in Lipschitz bounded domains, *Math. Nachr.* 171, 111–148(1995).
- [5] Foias, C., Manley, O., Rosa, R., Temam, R. *Navier–Stokes Equations and Turbulence.* Cambridge University Press. (2001).
- [6] Iftimie, D., Raugel, G. Some results on the Navier-Stokes equations in thin 3D domains, *J. Differential Equations* 169 , no. 2, 281–331(2001).
- [7] Kato, T., Fujita, H. On the nonstationary Navier-Stokes system., *Rend. Sem. Mat. Univ. Padova* 32 , 243–260(1962).
- [8] Kasyanov, P. O., Toscano, L., Zadoianchuk, N. V. A criterion for the existence of strong solutions for the 3D Navier–Stokes equations, *Applied Mathematics Letters* 26, 15-17(2013).
- [9] Leray, J. Etude de diverses équations intégrales non linéaires et de quelques problèmes que pose l’hydrodynamique, *J. Math. Pures Appl.* 12, 1–82 (1933).
- [10] Leray, J. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.* 63, 193-248(1934).
- [11] Lions, J. L. *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires,* Dunod Gauthier-Villars, Paris. (1969).
- [12] Sell, G. R., You, Y. *Dynamics of Evolutionary Equations,* Applied Math. Sciences 143, Springer, Berlin, (2002).
- [13] Temam, R. *Navier-Stokes equations.* North-Holland Pub. Company, Amsterdam, (1979).
- [14] Wang, K. On global regularity of incompressible Navier-Stokes equations in R^3 , *Commun. Pure Appl. Anal.* 8 , no. 3, 1067-1072(2009).