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► **To cite this version:**

Boujemaa Agrebaoui, Didier Arnal, Abdelkader Ben Hassine. Diamond module for the Lie algebra $so(2n + 1, C)$. 2012. hal-00723844

HAL Id: hal-00723844

<https://hal.science/hal-00723844v1>

Preprint submitted on 14 Aug 2012

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DIAMOND MODULE FOR THE LIE ALGEBRA $\mathfrak{so}(2n + 1, \mathbb{C})$

BOUJEMAÂ AGREBAOUI, DIDIER ARNAL AND ABDELKADER BEN HASSINE

ABSTRACT. The diamond cone is a combinatorial description for a basis of an indecomposable module for the nilpotent factor \mathfrak{n} of a semi simple Lie algebra. After N. J. Wildberger who introduced this notion, this description was achieved for $\mathfrak{sl}(n)$, the rank 2 semi-simple Lie algebras and $\mathfrak{sp}(2n)$.

In the present work, we generalize these constructions to the Lie algebras $\mathfrak{so}(2n + 1)$. The orthogonal semistandard Young tableaux were defined by M. Kashiwara and T. Nakashima, they form a basis for the shape algebra of $\mathfrak{so}(2n + 1)$. Defining the notion of orthogonal quasistandard Young tableaux, we prove these tableaux give a basis for the diamond module for $\mathfrak{so}(2n + 1)$.

1. INTRODUCTION

Let \mathfrak{g} be a semisimple Lie algebra. The simple (finite dimensional) \mathfrak{g} -modules are characterized by their highest weight λ , each of them contains an unique (up to constant) vector v_λ with weight λ , the \mathfrak{g} -action on v_λ generates the corresponding simple module. The direct sum of all these modules is a natural algebra, the shape algebra of \mathfrak{g} .

Consider now the nilpotent factor \mathfrak{n} in the Isawasa decomposition the Lie algebra \mathfrak{g} . It is natural to study nilpotent finite dimensional \mathfrak{n} -modules. They are generally indecomposable, if the module is generated by the action on an unique vector v , we say this module is monogenic. Each of the monogenic nilpotent module is a quotient of a well determined simple \mathfrak{g} -module (viewed as a \mathfrak{n} -module). The natural object corresponding to the shape algebra is now the diamond module, union of all these maximal monogenic modules.

2000 *Mathematics Subject Classification.* 20G05, 05A15, 17B10.

Key words and phrases. Shape algebra, Semistandard Young tableaux, Quasistandard Young tableaux, Jeu de taquin.

B. Agrebaoui, and A. Ben Hassine thank the Institut de Mathématiques de Bourgogne for his hospitality during his stays in Dijon, D. Arnal thanks the University of Sfax for its support and hospitality during his visits in Tunisia.

We report on a program to construct explicit combinatorial model for a basis in the diamond module, called the diamond cone. Such a description is given in the case of $\mathfrak{sl}(n)$ by D. Arnal, N. Bel Barka, and N. J. Wildberger in [ABW], in the case of $\mathfrak{sp}(2n)$, by D. Arnal and O. Khlifi [AK] and, in the case of rank two semisimple Lie algebras, by B. Agrebaoui, D. Arnal and O. Khlifi [AAK].

Let us first recall the $\mathfrak{sl}(n)$ case, which is the simplest one. In this case, the shape algebra (the direct sum of all finite dimensional irreducible representations) admits a well known basis given by semistandard Young tableaux T , if we restrict ourselves to the semistandard tableaux with shape λ , we get a basis for the irreducible module \mathbb{S}^λ , with highest weight characterized by the shape, and still denoted λ . There is a notion of quasistandard tableau. Denote QS^λ the subset of quasistandard tableaux in the set SS^λ of semistandard tableaux with shape λ .

For any tableau T in SS^λ , which is not in QS^λ , there is procedure, based on the usual jeu de taquin (jdt) which transforms T in a new tableau $p(T)$, which is quasistandard, with a shape $\mu < \lambda$. Putting $p(T) = T$ if T is quasistandard, it is possible to prove that the map:

$$p : SS^\lambda \longrightarrow \bigsqcup_{\mu \leq \lambda} QS^\mu,$$

is a bijective map. In other words, we have an indexation of a basis for the module \mathbb{S}^λ , which is well adapted to the description of the \mathfrak{n} indecomposable module $\mathbb{S}^\lambda|_{\mathfrak{n}}$. Indeed, any maximal monogenic \mathfrak{n} submodule in \mathbb{S}^λ is the subspace generated by $\bigsqcup_{\mu \leq \nu} QS^\mu$ for some $\nu \leq \lambda$ (see [ABW] for details).

The situation is very similar for the $\mathfrak{sp}(2n)$ case. A basis for the simple modules $\mathbb{S}^{(\lambda)}$ is given by the set $SS^{(\lambda)}$ of symplectic semistandard Young tableaux with shape λ , in [AK], the notion of symplectic quasistandard Young tableaux is given, let $QS^{(\lambda)}$ be the set of such tableaux with shape λ , using the symplectic jeu de taquin ($sjdt$) defined by J. T. Sheats ([S]), define a bijective map:

$$p : SS^{(\lambda)} \longrightarrow \bigsqcup_{\mu \leq \lambda} QS^{(\mu)}.$$

With this map, we get a basis for the module $\mathbb{S}^{(\lambda)}|_{\mathfrak{n}}$, well adapted with its stratification.

The goal of this paper is to realize the same program for the $\mathfrak{so}(2n+1)$ case. First we recall the definition of semistandard Young tableaux for $\mathfrak{so}(2n+1)$, given by Kashiwara and Nakashima (see [KN], see also the presentation given by Lecouvey in [L]). In this construction, Lecouvey defines the split of an orthogonal Young tableau. An orthogonal semistandard Young tableau with shape λ is a tableau T such that

its split $spl(T)$ is symplectic semistandard: $spl(T) \in SS^{(2\lambda)}$. Unfortunately, this choice is not convenient for our purpose of quasistandardness, therefore we modify the presentation of orthogonal semistandard tableaux, and the splitting procedure, in order to get a new map $dbble$ and say that a tableau T is orthogonal semistandard ($T \in SS^{[\lambda]}$) if and only if $dbble(T) \in SS^{(2\lambda)}$. We prove the equivalence between our construction and the Lecouvey's one, by proving that the spl and the $dbble$ of respective semistandard Young tableaux form the same subset in $SS^{(2\lambda)}$.

We are now able to define orthogonal quasistandard tableau by the same method as for the symplectic case. Denote $QS^{[\lambda]}$ the set of such tableaux, with shape λ , we want to prove that the orthogonal jeu de taquin ($ojdt$), defined by Lecouvey allows to build a bijective map:

$$p: SS^{[\lambda]} \longrightarrow \bigsqcup_{\mu \leq \lambda} QS^{[\mu]}.$$

The orthogonal jeu de taquin on an orthogonal tableau T is defined by using the symplectic jeu de taquin on the split form $spl(T)$ of T , thus it is well defined on our notion of semistandard orthogonal Young tableau. Unfortunately, Lecouvey does not give a rule for this jeu de taquin, directly on the tableau T , therefore we first give such an explicit and direct expression of the action of the jeu de taquin on T itself, at least in the case we consider, *i.e.* when the jeu de taquin motion is horizontal.

Thanks to this expression, we can define the map p , as a 'maximal' (in a sense explained below) action of the orthogonal jeu de taquin $ojdt$, we compute the inverse mapping of $ojdt$ and prove that p is a bijective map.

With this map, we get a basis for the module $\mathbb{S}^{[\lambda]}|_n$, well adapted with its stratification.

2. SEMISTANDARD AND QUASISTANDARD YOUNG TABLEAUX FOR $\mathfrak{sl}(n)$

2.1. Semistandard Young tableaux.

The theory of finite dimensional representations of semisimple Lie algebras is well known and very explicit. In the classical cases, we have a natural representation on a complex space V . For $\mathfrak{sl}(n)$, $V = \mathbb{C}^n$. We first consider simple modules in the tensor product $\otimes^\ell V$. We recover with these modules all the simple $\mathfrak{sl}(n)$ modules. The key to understanding the decomposition of $\otimes^\ell V$ is the Schur-Weyl duality.

Let $\mathbb{C}[S_\ell]$ be the group algebra of the symmetric group S_ℓ . It is a semisimple algebra, its simple components are indexed by the set of partitions $\mathcal{P}(\ell)$ of ℓ (weakly decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$ whose sum is ℓ .)

$$\mathbb{C}[S_\ell] = \bigoplus_{\lambda \in \mathcal{P}} B_\lambda.$$

Let S_ℓ acts on the right side on $\otimes^\ell V$ by permutation:

$$\rho_\ell(\sigma)(v_1 \otimes \cdots \otimes v_\ell) = (v_1 \otimes \cdots \otimes v_\ell)^\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(\ell)}.$$

The Shur-Weyl duality theorem is the fact that the commutant of the natural representation of $\mathfrak{gl}(V)$ in $\otimes^\ell V$ is exactly $\rho_\ell(\mathbb{C}[S_\ell])$ (see [GW]).

Therefore, we have the following decomposition of $\otimes^\ell V$ as $(\mathfrak{gl}(V) - S_\ell)$ simple modules:

$$\otimes^\ell V = \bigoplus_{\lambda \in \mathcal{P}(n, \ell)} \mathbb{S}_\lambda \otimes B_\lambda,$$

where $\mathcal{P}(n, \ell)$ is the set of partitions of ℓ , with length n : $\{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n), \sum \lambda_j = \ell\}$. A Young diagram of shape $\lambda \in \mathcal{P}(n, \ell)$ is a tableau of empty boxes, with $\lambda_j - \lambda_{j+1}$ columns with height j ($1 \leq j \leq n-1$) and λ_n columns with height n .

A standard Young tableau of shape λ is the filling of the corresponding Young diagram, with positive integers in $\{1, \dots, \ell\}$, such that the entries are strictly increasing along rows and columns. The set of standard Young tableaux gives a basis for B_λ . Similarly, a semistandard Young tableau for $\mathfrak{gl}(n)$ with shape λ is the filling of the corresponding Young diagram, with positive integers in $\{1, \dots, n\}$, such that the entries are strictly increasing along columns and weakly increasing along rows. The set of semistandard Young tableaux for $\mathfrak{gl}(n)$ gives a basis for \mathbb{S}_λ . Explicitly, for any pair (T, S) of a semistandard tableau T for $\mathfrak{gl}(n)$ and a standard tableau S , we associate the tensor product:

$$\rho_\lambda(Y_\lambda)(e_{t_1} \otimes \cdots \otimes e_{t_\ell}),$$

where (e_1, \dots, e_n) is the canonical basis of V , the entries t_k of T are indexed by the entry of the corresponding boxes in S , and Y_λ is the Young symmetrizer: the element in $\mathbb{C}[S_\lambda]$ giving the projection on B_λ .

To get a realization of the shape algebra as $\mathfrak{gl}(n)$ -module, we choose for each λ a particular standard tableau, namely the filling of the corresponding Young diagram row by row from the top to the bottom and from the left to the right. With this choice the highest weight vector v_λ in \mathbb{S}^λ is associated to the ‘trivial’ semistandard tableau, for which the boxes in the i^{th} row are filled by the integer i and v_λ is:

$$v_\lambda = (e_1)^{\lambda_1 - \lambda_2} \cdot (e_1 \wedge e_2)^{\lambda_2 - \lambda_3} \cdot \dots \cdot (e_1 \wedge \cdots \wedge e_n)^{\lambda_n}.$$

Finally, the restriction \mathbb{S}^λ of \mathbb{S}_λ to $\mathfrak{sl}(n)$ is simple, and two such restrictions \mathbb{S}^λ and $\mathbb{S}^{\lambda'}$ coincide if and only if $\lambda_j = \lambda'_j$ for any $j < n$. We thus only consider partitions λ with $\lambda_n = 0$. Recall that the usual ordering on weights is then: $\mu \leq \lambda$ if and only if $\mu_j - \mu_{j+1} \leq \lambda_j - \lambda_{j+1}$ for any j , $1 \leq j \leq n-1$.

Since the group $G = SL(n, \mathbb{C})$ is a classic, connected and simply connected Lie group, we can realize the shape algebra as the space of affine functions on the quotient

$\overline{N} \setminus G$, where \overline{N} is the Lie subgroup corresponding to the nilpotent factor $\overline{\mathfrak{n}}$ opposite to \mathfrak{n} (see [GW]).

2.2. Quasistandard Young tableaux and jeu de taquin.

Denote \mathbb{S}_{red}^\bullet the diamond module for $\mathfrak{sl}(n)$. We can realize explicitly this module as the quotient of the shape algebra by the ideal generated by the elements $v_\lambda - 1$ (for any λ), or as the space of polynomial functions on N .

As a \mathfrak{n} -module, \mathbb{S}_{red}^\bullet is indecomposable, this is the union of the modules $\mathbb{S}^\lambda|_{\mathfrak{n}}$, with a natural layering:

$$\mu \leq \lambda \iff \mathbb{S}^\mu|_{\mathfrak{n}} \subset \mathbb{S}^\lambda|_{\mathfrak{n}}.$$

Indeed, in the shape algebra, $v_\nu \cdot v_\mu = v_{\nu+\mu}$, thus in the quotient, the diamond module, $\mathbb{S}^\mu = v_\nu \cdot \mathbb{S}^\mu \subset \mathbb{S}^{\nu+\mu}$.

To get a combinatorial basis in the diamond module, it is necessary to suppress any trivial semistandard tableau, and even any semistandard tableau containing a trivial tableau with a shape $\mu \leq \lambda$. Therefore, we put (see [ABW])

Definition 2.1.

Let $T = (t_{ij})$ be a semistandard tableau, with shape λ . If the top of the first column (the s first boxes) is a trivial tableau, if T contains a column with height s , and if, for all j for which these entries exist, the relation $t_{s(j+1)} < t_{(s+1)j}$ holds, we say that T is not quasistandard at the level s : $T \in NQS_s^\lambda$.

If there is no s for which T is in NQS_s^λ , we say that T is quasistandard. The set of quasistandard tableaux with shape λ is denoted QS^λ .

The principal result in [ABW] is: the quasi-standard tableaux form a basis of the diamond module. This can be proved by using the jeu de taquin (*jdt*). Let us present now this operation due to Schutzenberger.

Let $\mu \leq \lambda$ be two shapes, we let $Y(\mu)$, the Young diagram with shape μ , as a subdiagram placed in the left-top corner of $Y(\lambda)$, the associated Young diagram to λ . An interior corner of $Y(\mu)$ is a box (x, y) of $Y(\mu)$ such that, immediately in the right and immediately below to this box, there is no box of $Y(\mu)$. An exterior corner of $Y(\lambda)$ is an empty box (x', y') which we can add to $Y(\lambda)$ so that $Y(\lambda) \cup \{(x', y')\}$ still is a Young diagram.

Let us leave $Y(\mu)$ empty inside $Y(\lambda)$ and fill in the skew tableau $Y(\lambda \setminus \mu)$ by integers $t_{ij} \leq n$ in a semistandard way: For all i and all j , $t_{ij} < t_{(i+1)j}$ and $t_{ij} \leq t_{i(j+1)}$, if the boxes are in $Y(\lambda \setminus \mu)$.

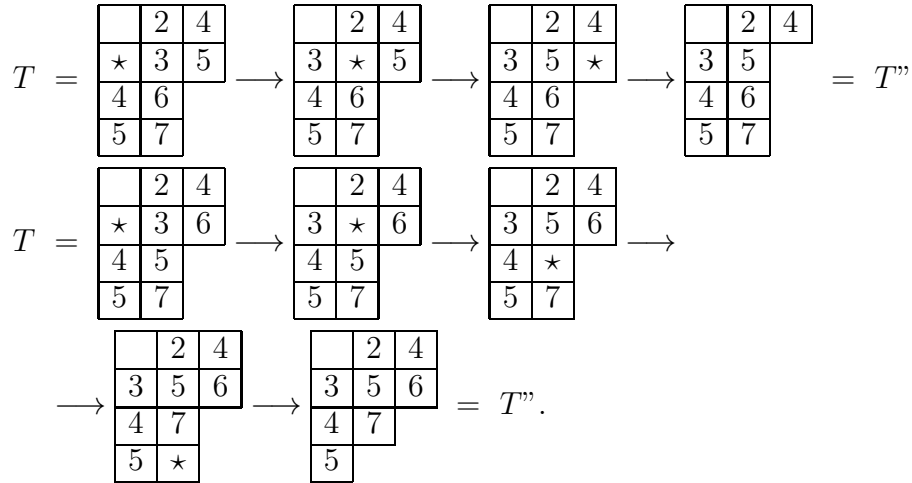
We choose an interior corner of $Y(\mu)$ and we identify it by a star: $\boxed{\star}$. We obtain a pointed skew-tableau $T := T(\lambda \setminus \mu)$. For example

	2	4
\star	3	5
4	6	
5	7	

The jeu de taquin is a way to move the $\boxed{\star}$ in $T(\lambda \setminus \mu)$. After a number of moving, the tableau T becomes a tableau T' in which the \star is in the (i, j) box. The rules of the jeu de taquin is as follows:

- 1- If the box $(i, j + 1)$ exists and either the box $(i + 1, j)$ does not exist or $t_{(i+1)j} > t_{i(j+1)}$, then we push $\boxed{\star}$ to the right, *i.e.*, we replace T' by the tableau T'' where we put $\boxed{t_{i(j+1)}}$ in (i, j) , and $\boxed{\star}$ in $(i, j + 1)$, the other entries in T' being unchanged in T'' .
- 2- If the box $(i + 1, j)$ exists and either the box $(i, j + 1)$ does not exist or $t_{(i+1)j} \leq t_{i(j+1)}$, then we push $\boxed{\star}$ downward, *i.e.*, we replace T' by the tableau T'' where we put $\boxed{t_{(i+1)j}}$ in (i, j) , and $\boxed{\star}$ in $(i + 1, j)$, the other entries in T' being unchanged in T'' .
- 3- If the boxes $(i + 1, j)$ and $(i, j + 1)$ do not exist, we remove the $\boxed{\star}$. The box (i, j) is no longer a box of T'' , but the tableau consisting of boxes of T'' and of the box (i, j) is a Young tableau: the (i, j) box is an exterior corner of T'' .

Example 2.2.



Let us call S'' the empty Young diagram obtained after removing the pointed box and let μ'' be the shape of S'' . The tableau $T'' \setminus S''$ is still semistandard. If (i, j) is

the interior pointed corner of S and (i'', j'') the exterior pointed corner of T'' , we set $(T'' \setminus S'', (i'', j'')) = jdt(T \setminus S, (i, j))$.

Let us explain now the inverse map: $(jdt)^{-1}$. Let $T \setminus S$ be a skew semistandard tableau of shape $\lambda \setminus \mu$, consider the smallest rectangle containing $T \setminus S$, then return this rectangle by performing a central symmetry, and replace each of the entries t_{ij} of the obtained skew tableau by the entries $n+1-t_{ij}$ and \star by \star . The obtained tableau $T' \setminus S' = \sigma(T \setminus S)$ is still a skew semistandard tableau. If we pointed an exterior corner of T , the box $\boxed{\star}$ now is in an interior corner of S' , and reciprocally. Then

$$jdt^{-1}(T'' \setminus S'', (i'', j'')) = \sigma \circ jdt \circ \sigma(T'' \setminus S'', (i'', j'')).$$

For instance, the above applied jeu de taquin is reversed, if $n = 7$, as follows:

$$\begin{array}{ccc} (T, (4, 2)) = \begin{array}{|c|c|c|} \hline & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 4 & 7 & \\ \hline 5 & \star & \\ \hline \end{array} & \sigma(T, (4, 2)) = \begin{array}{|c|c|c|} \hline & \star & 3 \\ \hline & 1 & 4 \\ \hline 2 & 3 & 5 \\ \hline 4 & 6 & \\ \hline \end{array} & \\ \\ jdt \circ \sigma(T, (4, 2)) = \begin{array}{|c|c|c|} \hline & 1 & 3 \\ \hline & 3 & 4 \\ \hline 2 & 5 & \star \\ \hline 4 & 6 & \\ \hline \end{array} & \sigma \circ jdt \circ \sigma(T, (4, 2)) = \begin{array}{|c|c|c|} \hline & 2 & 4 \\ \hline \star & 3 & 6 \\ \hline 4 & 5 & \\ \hline 5 & 7 & \\ \hline \end{array} & \end{array}$$

The jeu de taquin is thus a bijective map:

$$\begin{aligned} jdt : \bigcup_{\lambda \setminus \mu} SS(\lambda \setminus \mu) \times \{\text{interior corners in } \mu\} &\longrightarrow \\ &\longrightarrow \bigcup_{\lambda'' \setminus \mu''} SS(\lambda'' \setminus \mu'') \times \{\text{exterior corners for } \lambda''\}. \end{aligned}$$

Let us now consider a non quasistandard tableau $T = (t_{ij})$ with shape λ and let s be the largest integer such that T is not quasistandard in s . The s top entries of its first column are $1, 2, \dots, s$. We call S the empty tableau with only one column with height s : the shape of S is $\mu = (1, \dots, 1, 0, \dots, 0)$. We consider the pointed skew tableau U whose entries in $\lambda \setminus \mu$ are the entries of T and the pointed box is in the only interior corner in S .

We apply the jeu de taquin. The pointed box moves always to the right and leaves the diagram at the end of the last column of height s .

The row s has just been shifted by one box to the left. We obtain a skew tableau with $s-1$ empty boxes in its first column. We fill in these boxes with $1, \dots, s-1$. If $s > 1$, the obtained tableau T'' is semistandard, not quasistandard in $s-1$, and may be in s , but it is quasistandard in all $t > s$.

This procedure can therefore be repeated and finally we get a quasistandard tableau $T' = p(T)$.

It is easy to check that this procedure realize a bijection between the set of semistandard tableaux SS^λ with shape λ and the union $\sqcup_{\mu \leq \lambda} QS^\mu$ of sets of quasistandard tableaux with shape $\mu \leq \lambda$.

$$SS^\lambda \longleftrightarrow \sqcup_{\mu \leq \lambda} QS^\mu.$$

Considering the quotient map from the shape algebra \mathbb{S}^\bullet to the diamond module \mathbb{S}_{red}^\bullet , we see the restriction of this quotient map to each \mathbb{S}^λ is one-to-one. Thus we can choose only the vectors associated to quasistandard tableaux, to get a basis for the quotient.

Theorem 2.3. ([ABW]) *The set QS^\bullet of quasistandard tableaux is the diamond cone, i.e. a basis of the diamond module \mathbb{S}_{red}^\bullet , which describes the stratification of this indecomposable \mathfrak{n} -module.*

3. THE SYMPLECTIC CASE

3.1. Symplectic Lie algebra and its representations.

We let $V = \mathbb{C}^{2n}$ be the $2n$ dimensional vector space with basis $(e_1, \dots, e_n, e_{\bar{n}}, \dots, e_{\bar{1}})$ and equipped with a symplectic form

$$\Omega = \sum_{i=1}^n e_i^* \wedge e_{\bar{i}}^*.$$

If M is a $n \times n$ matrix, we denote ${}^s M$ the image of M under the symmetry with respect to its second diagonal. Then the symplectic (simple) Lie algebra associated to Ω can be realized as the set of matrices:

$$\mathfrak{sp}(2n) = \left\{ \begin{pmatrix} M & V \\ U & -{}^s M \end{pmatrix}; M, U, V \in Mat(n), U = {}^s U, V = {}^s V \right\}$$

A Cartan subalgebra \mathfrak{h} of $\mathfrak{sp}(2n)$ consists of diagonal matrices

$$H = \text{diag}(\kappa_1, \dots, \kappa_n, -\kappa_n, \dots, -\kappa_1).$$

We let $\theta_j(H) = \kappa_j$ and choose the following simple roots system:

$$\Delta = \{\alpha_i = \theta_i - \theta_{i+1}, i = 1, 2, \dots, n-1, \alpha_n = 2\theta_n\}.$$

For this choice $\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{g}^\alpha$ is the subalgebra of strictly upper triangular matrices in $\mathfrak{sp}(2n)$. Define the fundamental weights as $\omega_k = \theta_1 + \dots + \theta_k$, $1 \leq k \leq n$. The fundamental module $\mathbb{S}^{(\omega_k)}$ is realized as the kernel of the contraction $\varphi_k : \wedge^k V \rightarrow \wedge^{k-2} V$ (with the convention $\wedge^{-1} V = 0$) defined by:

$$\varphi_k(v_1 \wedge \dots \wedge v_k) = \sum_{i < j} \Omega(v_i, v_j) (-1)^{i+j-1} v_1 \wedge \dots \wedge \widehat{v}_i \wedge \dots \wedge \widehat{v}_j \wedge \dots \wedge v_k.$$

Then the set Λ of positive dominant weights is $\Lambda = \{\lambda = \sum_{k=1}^n a_k \omega_k, a_k \in \mathbb{N}\}$. For $\lambda \in \Lambda$, the irreducible module $\mathbb{S}^{(\lambda)}$ is realized as a submodule of $\otimes V$, more precisely as the simple submodule in

$$\text{Sym}^{a_1}(\mathbb{S}^{(\omega_1)}) \otimes \text{Sym}^{a_2}(\mathbb{S}^{(\omega_2)}) \otimes \cdots \otimes \text{Sym}^{a_n}(\mathbb{S}^{(\omega_n)}),$$

with highest weight λ .

Since all these modules are in $\otimes V$, as before, we can describe a combinatorial basis for $\mathbb{S}^{(\lambda)}$, by using semistandard Young tableaux with shape λ . However, we have to select, among the usual semistandard tableaux, some of them, called symplectic semistandard tableaux. The set of such tableaux will be denoted $SS^{(\lambda)}$ (for details see [FH, dC, KN]). We present such a choice in the two next sections.

3.2. Subset in the left or the right side.

Let $X = \{e_1 < e_2 < \cdots < e_p\}$ be a subset of $\{1, \dots, n\}$, denote $Y = \{1, \dots, n\} \setminus X$.

Definition 3.1.

Let J be a subset of X . A subset I of X is said to be in the left (resp. right) side of X if:

- i. $\#I = \#J$,
- ii. $I \cap J = \emptyset$,
- iii. if $J \neq \emptyset$, and $J = \{y_1 < y_2 < \cdots < y_s\}$, then $I = \{x_1 < x_2 < \cdots < x_s\}$ and $x_i < y_i$ (resp. $y_i < x_i$) for all i , $1 \leq i \leq s$.

Denote $L(J)$ the set of all subsets of X in the left side of J . For instance, if $X = [1, 10]$,

$$L(\emptyset) = \{\emptyset\}, \quad L(\{1, 3\}) = \emptyset, \quad L(\{2, 6\}) = \{\{1, 3\}, \{1, 4\}, \{1, 5\}\}.$$

Denote Γ_Y (or Γ_Y^n) the set of subsets J in X , such that $L(J) \neq \emptyset$.

Lemma 3.2.

Let J be in Γ_Y . Then there exists a unique largest subset, denoted $\gamma_Y(J)$, in the left side of J , such that if $\gamma_Y(J) = \{x_1 < \cdots < x_s\}$, then for all $I' = \{x'_1 < \cdots < x'_s\}$, in $L(J)$, the relation $x'_i \leq x_i$ holds for every i ($1 \leq i \leq s$).

Let $J = \{y_1 < \cdots < y_s\}$ be a non empty set in Γ_Y and $I = \gamma_Y(J) = \{x_1, \dots, x_s\}$. Set $Z = X \setminus (I \cup J)$. Let $t \in Z$. If there exists i such that $t < y_i$ then $t < x_i$.

Proof. If $J = \emptyset$, then $L(J) = \{\emptyset\}$, and $\gamma_Y(\emptyset) = \emptyset$.

Let us now suppose $J = \{y_1 < \cdots < y_s\}$ is not empty. We define by induction the elements x_i in X , as follows:

$$\begin{aligned} x_s &= \sup\{t \in X \setminus J, t < y_s\}, \\ x_i &= \sup\{t \in X \setminus J, t < x_{i+1}, t < y_s\} \quad (1 \leq i \leq s-1). \end{aligned}$$

It is easy to prove that the x_i do exist, and if $I' = \{x'_1 < \cdots < x'_s\}$ is in $L(J)$, then, by induction, $x'_i \leq x_i$ for all i .

This implies the unicity of the subset $\gamma_Y(J) = \{x_s > \cdots > x_1\}$.

Suppose the second assertion wrong, and $x_i \leq t$. Since t is not in I , this means $x_i < t$. Let k be the largest index such that $x_k < t$ ($i \leq k$).

If $k = s$, this gives $t < y_i \leq y_s$, and $t \in X \setminus J$, then $x_s < t \leq x_s = \sup\{u \in X \setminus J, u < y_s\}$, which is impossible.

If $k < s$, this gives $t < y_i \leq y_k$ and $t < x_{k+1}$, then $x_k < t \leq \sup\{u \in X \setminus J, u < y_k, u < x_{k+1}\}$, which is also impossible.

This proves the lemma. □

Of course the same properties are holding on the right side of a subset I in X . Denote $R(I)$ the family of all subsets in the right side of I , say that I is in Δ_Y if $R(I)$ is not empty. Remark that for I in Δ_Y , there exists in $R(I)$ a smallest subset denoted $\delta_Y(I) = \delta(I)$. If moreover $I = \{x_1 < \cdots < x_s\}$ is non empty, denote $\delta_Y(I) = \{y_1 < \cdots < y_s\}$.

For any $J' = \{y'_1 < \cdots < y'_s\}$ in $R(J)$, have $y'_i \geq y_i$ for all i ($1 \leq i \leq s$). Set $Z = X \setminus (I \cup J)$, let $t \in Z$. If there exists i such that $t > x_i$ then $t > y_i$.

3.3. Semistandard and quasistandard symplectic tableaux.

Consider the ordering $1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{1}$ and let A, D subsets in $\{1, \dots, n\}$ such that $k = \#A + \#D \leq n$. Set $I = A \cap D = \{i_1, \dots, i_r\}$. Let us say that the column

$$\frac{A}{D} = \begin{array}{|c|} \hline p_1 \\ \hline \vdots \\ \hline p_s \\ \hline q_t \\ \hline \vdots \\ \hline q_1 \\ \hline \end{array}$$

is a symplectic semistandard column if I is in Δ_{AUD} .

To any symplectic semistandard column, we associate a two columns tableau, the double of this column. Put first:

$$J = \delta_{A \cup D}(I), \quad B = (A \setminus I) \cup J, \quad C = (D \setminus I) \cup J.$$

Remark that, knowing B and C , we have $J = B \cap C$ is in $L(B \cup C)$ and $I = \gamma_{B \cup C}(J)$. Denote the symplectic column:

$$\frac{A}{D} = f(A, D) = g(B, C).$$

The double of $\frac{A}{D}$ is by definition the tableau

$$dble\left(\frac{A}{D}\right) = \frac{A}{C} \frac{B}{D}.$$

It is a semistandard Young tableau for the chosen ordering $1 < 2 < \dots < n < \bar{n} < \dots < \bar{1}$.

Definition 3.3.

Let T be a tableau of shape λ consisting of semistandard columns. The tableau $dble(T)$ is obtained by juxtaposing the doubles of all the columns of T .

We say that T is symplectic semistandard (or semistandard for $\mathfrak{sp}(2n)$) if $dble(T)$ is semistandard (for $\mathfrak{sl}(2n)$).

The set of symplectic semistandard Young tableaux of shape λ is a basis for the simple $\mathfrak{sp}(2n)$ module $\mathbb{S}^{(\lambda)}$. Let us denote by v_λ its highest weight.

In a recent paper by D. Arnal and O. Khlifi (see [AK]) the following two algebras are studied: the shape algebra

$$\mathbb{S}^{(\bullet)} = \bigoplus_{\lambda \in \Lambda} \mathbb{S}^{(\lambda)}.$$

and the reduced shape algebra (the diamond module):

$$\mathbb{S}_{red}^{(\bullet)} = \mathbb{S}^{(\bullet)} / \langle v_\lambda - 1, \lambda \in \Lambda \rangle.$$

The first algebra has for basis the set of symplectic semistandard tableaux $SS^{(\bullet)}$ while the second algebra has for basis the set of symplectic quasistandard tableaux $QS^{(\bullet)}$ defined as follows:

Definition 3.4.

Let T be a symplectic semistandard tableau. We say that T is a symplectic quasistandard tableau if $dble(T)$ is quasistandard (for $\mathfrak{sl}(2n)$).

We note that a symplectic semistandard Young tableau can be quasistandard for $\mathfrak{sl}(2n)$, but not its double. For example

$$T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \bar{2} \\ \hline \bar{2} & \\ \hline \end{array} \implies \text{dble}(T) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & \bar{3} & \bar{2} \\ \hline \bar{3} & \bar{2} & & \\ \hline \end{array}$$

T is quasistandard but $\text{dble}(T)$ is not quasistandard. The set of all symplectic quasistandard tableaux with shape λ is denoted $QS^{(\lambda)}$.

Let us say that a symplectic semistandard tableau T is not quasistandard at the level s and denote $T \in NQS_s$ if $\text{dble}(T)$ is not quasistandard at the level s .

In the paper [S] the symplectic *jeu de taquin* (*sjdt*) is defined on a skew symplectic semistandard tableau, by using its double. Especially, in the case where the \star moves to the right, along the row s in T , the motion is the following:

Suppose the row s contains the entries $\star a'$, then the left column $f(A, D) = g(B, C)$ becomes $g(B \cup \{a'\}, C)$, the right column $f(A', D')$ becomes $f(A' \setminus \{a'\}, D')$,

Suppose the row s contains the entries $\star \bar{d}'$, and the row s in $\text{dble}(T)$ contains $\star \bar{c}'$, then, in T , the left column $f(A, D)$ becomes $f(A, D \cup \{c'\})$, the right column $f(A', D') = g(B', C')$ becomes $g(B', C' \setminus \{c'\})$.

Using this symplectic jeu de taquin, it is possible to prove, like in the $\mathfrak{sl}(n)$ case, that the set of symplectic quasistandard tableaux is a basis for the reduced shape algebra that respect its structure of indecomposable \mathfrak{n} module (see [AK]).

We shall now follow the same strategy in the $\mathfrak{so}(2n+1)$ case.

4. ORTHOGONAL SEMISTANDARD YOUNG TABLEAUX

4.1. $\mathfrak{so}(2n+1)$ and its positive dominant weights.

Let $\mathcal{B}_n = \{i, \bar{i}, 1 \leq i \leq n\} \cup \{0\}$ be an ordered set with the ordering given by:

$$1 < 2 < \dots < n < 0 < \bar{n} < \dots < \bar{2} < \bar{1}.$$

For any a, b in a totally ordered set E , denote $[a, b] = \{x \in E, a \leq x \leq b\}$ for instance, in \mathcal{B}_n , $[1, n] = \{1, 2, \dots, n\}$.

Put $\bar{\bar{i}} = i$ and $\bar{0} = 0$. Let $V = \mathbb{C}^{2n+1}$ with basis $(e_1, \dots, e_n, e_0, e_{\bar{n}}, \dots, e_{\bar{1}})$ indexed by \mathcal{B}_n .

The odd dimensional orthogonal algebra $\mathfrak{g} = \mathfrak{so}(2n+1)$ is the Lie algebra given by the matrices antisymmetric with respect to the non degenerated symmetric bilinear form $Q = \langle \cdot, \cdot \rangle$ defined by

$$\langle e_i, e_{\bar{j}} \rangle = \delta_{ij}, \quad \forall i, j \in \mathcal{B}_n.$$

The matrix of Q is:

$$S = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Denote $A \mapsto {}^sA$ the symmetry with respect to the second diagonal, thus \mathfrak{g} is the set of all $(2n+1) \times (2n+1)$ matrices X so that ${}^sX = -X$, or:

$$X = \begin{pmatrix} A & u & B \\ -{}^s x & 0 & -{}^s u \\ C & x & -{}^s A \end{pmatrix}$$

where A is a $(n \times n)$ -matrix, B and C are $(n \times n)$ -matrices, such that ${}^sB = -B$, ${}^sC = -C$, x and u are $(n \times 1)$ -matrices and, if u is a column matrix, ${}^s u = (u_{n1}, \dots, u_{11})$. The Lie algebra \mathfrak{g} is a simple Lie algebra of type (B_n) .

Denote E_{ij} the usual $n \times n$ matrix with unique non vanishing entry 1 at the row i and the column j , and E_i the column with unique non vanishing entry 1 at the row i , we get the following basis for \mathfrak{g} :

$$\begin{aligned} H_i &= \begin{pmatrix} E_{ii} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -{}^s E_{ii} \end{pmatrix} \quad (1 \leq i \leq n), & X_{ij} &= \begin{pmatrix} E_{ij} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -{}^s E_{ij} \end{pmatrix} \quad (1 \leq i \neq j \leq n), \\ Y_{ij} &= \begin{pmatrix} 0 & 0 & (E_{ij} - {}^s E_{ij}) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (i+j \leq n), & Z_{ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (E_{ij} - {}^s E_{ij}) & 0 & 0 \end{pmatrix} \quad (i+j \leq n), \\ U_i &= \begin{pmatrix} 0 & E_i & 0 \\ 0 & 0 & -{}^s E_i \\ 0 & 0 & 0 \end{pmatrix} \quad (1 \leq i \leq n), & X_i &= \begin{pmatrix} 0 & 0 & 0 \\ -{}^s E_i & 0 & 0 \\ 0 & E_i & 0 \end{pmatrix} \quad (1 \leq i \leq n). \end{aligned}$$

The set of diagonal matrices

$$H = \sum_{i=1}^n \kappa_i H_i,$$

is a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . The dual space \mathfrak{h}^* has for basis the n forms ϵ_j where $\epsilon_j(H) = \kappa_j$.

The roots and the root spaces of \mathfrak{g} are given by the commutation relations:

$$\begin{aligned} [H, X_{ij}] &= (\epsilon_i - \epsilon_j)(H)X_{ij}, \\ [H, Y_{ij}] &= (\epsilon_i + \epsilon_{n+1-j})(H)Y_{ij}, \\ [H, Z_{ij}] &= -(\epsilon_{n+1+i} + \epsilon_j)(H)Z_{ij}, \\ [H, U_i] &= \epsilon_i(H)U_i, \\ [H, X_i] &= -\epsilon_{n+1-i}(H)X_i. \end{aligned}$$

The root system is thus $\pm\epsilon_i \pm \epsilon_j$ ($1 \leq i < j \leq n$) and $\pm\epsilon_i$ ($1 \leq i \leq n$). We choose the simple roots system

$$\Phi = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n, \epsilon_n\}.$$

Then the positive roots are $\epsilon_i - \epsilon_j$, $\epsilon_i + \epsilon_j$ ($1 \leq i < j \leq n$), and ϵ_i , ($1 \leq i \leq n$). The nilpotent factor \mathfrak{n} in the Iwasawa decomposition of \mathfrak{g} is the sum of the corresponding root spaces. It is the set of upper triangular matrices in \mathfrak{g} or the space generated by the matrices X_{ij} , Y_{ij} , for $1 \leq i < j \leq n$, and the U_i , with $1 \leq i \leq n$.

The weight lattice of $\mathfrak{so}(2n+1)$ is generated by $\epsilon_1, \epsilon_2, \dots, \epsilon_{n-1}, \epsilon_n$ together with the further weight $\frac{1}{2}(\epsilon_1 + \dots + \epsilon_n)$.

The Weyl chamber is $\mathcal{W} = \{\sum a_i \epsilon_i, a_1 \geq a_2 \geq \dots \geq a_n \geq 0\}$. The edges of the Weyl chamber are thus the rays generated by the vectors $\epsilon_1, \epsilon_1 + \epsilon_2, \dots, \epsilon_1 + \dots + \epsilon_{n-1}$ and $\epsilon_1 + \dots + \epsilon_n$. For \mathfrak{g} , the intersection of the weight lattice with the closed Weyl cone is the free semigroup generated by the following fundamental weights:

$$\omega_1 = \epsilon_1, \omega_2 = \epsilon_1 + \epsilon_2, \dots, \omega_{n-1} = \epsilon_1 + \dots + \epsilon_{n-1}, \omega_n = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_n).$$

Any weight λ in the Weyl chamber can be written: $\lambda = \sum_{i=1}^n a_i \omega_i$ with a_i a natural number ($a_i \in \mathbb{N}$). Denote $\mathbb{S}^{[\lambda]}$ the corresponding simple module.

4.2. Irreducible representations of $\mathfrak{so}(2n+1)$.

The construction of the fundamental modules $\mathbb{S}^{[\omega_r]}$ is explicitly presented in the excellent book [FH], by W. Fulton and J. Harris.

First, for $r = 1, \dots, n$, the natural antisymmetric tensor representation $\wedge^r V$ is an irreducible highest weight representations of $\mathfrak{so}(2n+1)$, with highest weight ω_r for $r < n$ and $2\omega_n$ for $r = n$. The vectors $e_{i_1} \wedge \dots \wedge e_{i_r}$ ($1 \leq i_1 < \dots < i_r \leq \bar{1}$), form a basis of $\wedge^r V$. Describe now the $\mathfrak{so}(2n+1)$ -action on these vectors.

Recall that the standard action of $\mathfrak{so}(2n+1)$ on V is given by the matrix form of the element $X \in \mathfrak{g}$. Especially, the Chevalley generators act as follows:

$$X_{i,i+1} \cdot e_{i+1} = e_i, \quad X_{i,i+1} \cdot e_{\bar{i}} = -e_{\overline{i+1}} \text{ for } 1 \leq i < n \text{ and } U_n \cdot e_0 = e_n, \quad U_n \cdot e_{\bar{n}} = -e_0,$$

(the other relations vanish) and:

$$X_{i+1,i} \cdot e_i = e_{i+1}, \quad X_{i+1,i} \cdot e_{\overline{i+1}} = -e_{\overline{i}} \text{ for } 1 \leq i < n \text{ and } -X_1 \cdot e_n = e_0, \quad -X_1 \cdot e_0 = -e_{\overline{n}}.$$

The action of $\mathfrak{so}(2n+1)$ on $\wedge^r V$ is the canonical one:

$$X \cdot (e_{i_1} \wedge \cdots \wedge e_{i_r}) = (X \cdot e_{i_1}) \wedge \cdots \wedge e_{i_r} + \cdots + e_{i_1} \wedge \cdots \wedge (X \cdot e_{i_r}).$$

In particular, every $H \in \mathfrak{h}$ acts diagonally:

$$\begin{aligned} H \cdot (e_{i_1} \wedge \cdots \wedge e_{i_k} (\wedge e_0) \wedge e_{\overline{j_1}} \wedge \cdots \wedge e_{\overline{j_s}}) &= \\ &= (\epsilon_{i_1} + \cdots + \epsilon_{i_k} - \epsilon_{j_1} - \cdots - \epsilon_{j_{r-k}})(H) e_{i_1} \wedge \cdots \wedge e_{i_k} (\wedge e_0) \wedge e_{\overline{j_1}} \wedge \cdots \wedge e_{\overline{j_{r-k}}}. \end{aligned}$$

Hence the set of weights of the representation is

$$\{(\epsilon_{i_1} + \cdots + \epsilon_{i_k}) - (\epsilon_{j_1} + \cdots + \epsilon_{j_{r-k}}), \quad 1 \leq i_1 < \cdots < i_k \leq n, \quad 1 \leq j_{r-k} < \cdots < j_1 \leq n\}.$$

The highest weight is $\omega_r = \epsilon_1 + \cdots + \epsilon_r$.

There is still one fundamental representation to describe: $\mathbb{S}^{[\omega_n]}$.

Definition 4.1.

The finite dimensional irreducible representation with the highest weight ω_n is called the spin representation and denoted by V_{sp} .

This last fundamental representation however is more mysterious. The fundamental weight ω_n cannot be a weight of any tensor power of the standard representation, it cannot be a submodule in $\otimes V$. We first describe directly this representation:

We index a basis for V_{sp} as what we call spin column:

They are the columns \mathfrak{C} of height n with strictly increasing entries in $[1, n] \cup [\overline{n}, \overline{1}]$, such that for all $i \in \mathcal{B}_n$, i and \overline{i} do not appear simultaneously in \mathfrak{C} . Denote $\mathfrak{C} = \frac{A}{D}^{sp}$ such a column, with $A, D \subset [1, n]$, $\#A + \#D = n$, and $A \cap D = \emptyset$ (to simplify notations, we omit to draw the boxes). The number of such columns (the dimension of V_{sp}) is 2^n . The action of $\mathfrak{so}(2n+1)$ on V_{sp} is given in terms of Chevalley generators as follows:

$$X_{i,i+1} \cdot \begin{array}{c} \vdots \\ i+1 \\ \vdots \\ \overline{i} \\ \vdots \\ sp \end{array} = \frac{1}{\sqrt{2}} \begin{array}{c} \vdots \\ i \\ \vdots \\ \overline{i+1} \\ \vdots \\ sp \end{array} \quad \text{if } 1 \leq i < n, \quad U_n \cdot \begin{array}{c} \vdots \\ \overline{n} \\ \vdots \\ sp \end{array} = \frac{1}{\sqrt{2}} \begin{array}{c} \vdots \\ n \\ \vdots \\ sp \end{array}$$

(the other actions vanish). And

$$X_{i+1,i} \cdot \begin{array}{c} \vdots \\ i \\ \vdots \\ i+1 \\ \vdots \\ sp \end{array} = \frac{1}{\sqrt{2}} \begin{array}{c} \vdots \\ i+1 \\ \vdots \\ i \\ \vdots \\ sp \end{array} \quad \text{if } 1 \leq i < n, \quad -X_1 \cdot \begin{array}{c} \vdots \\ n \\ \vdots \\ sp \end{array} = \frac{1}{\sqrt{2}} \begin{array}{c} \vdots \\ \bar{n} \\ \vdots \\ sp \end{array}$$

(the other actions vanish).

The weight of each column $\mathfrak{C} = \frac{A}{D}_{sp}$ is $\frac{1}{2}(\sum_{i \in A} \epsilon_i - \sum_{i \in D} \epsilon_i)$. Therefore, $\begin{array}{c} 1 \\ 2 \\ \vdots \\ n \\ sp \end{array}$ is

the highest weight vector with weight $\omega_n = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_n)$.

In fact, it turns out that this last fundamental representation does not come from a representation of the group $SO(2n+1, \mathbb{C})$. The point here is that this group is not simply connected, so there are Lie algebra homomorphism on $\mathfrak{so}(2n+1)$ which do not integrate to group homomorphisms on $SO(2n+1, \mathbb{C})$. Correspondingly, the simply connected group with Lie algebra $\mathfrak{so}(2n+1)$ is called the spin group $Spin(2n+1, \mathbb{C})$. It turns out that this spin group is an extension of $SO(2n+1, \mathbb{C})$ with kernel $Z_2 = \pm 1$, i.e. there is a surjective homomorphism $Spin(2n+1, \mathbb{C}) \rightarrow SO(2n+1, \mathbb{C})$ whose kernel consists of two elements.

One can construct both the spin representation V_{sp} and the spin group $Spin(2n+1, \mathbb{C})$ by using the Clifford algebra $Cl(2n+1, \mathbb{C})$ of V , details can be founded in [FH], Chapter 20. Remark that we have:

$$V_{sp} \otimes V_{sp} = \bigoplus_{k=0}^n \wedge^k V.$$

The term ‘spin’ is coming from the application of this representation and this group to theoretical physics.

Any dominant integral weight λ can be written

$$\lambda = \sum_{i=1}^n a_i \omega_i = \sum_{i=1}^n \lambda_i \epsilon_i,$$

where $a_i \in \mathbb{N}$ and $\lambda_i = a_i + \dots + a_{n-1} + \frac{a_n}{2}$ if $i < n$ and $\lambda_n = \frac{a_n}{2}$.

If a_n is even, the representation

$$Sym^{a_1}(V) \otimes Sym^{a_2}(\wedge^2 V) \otimes \dots \otimes Sym^{a_{n-1}}(\wedge^{n-1} V) \otimes Sym^{\frac{a_n}{2}}(\wedge^n V)$$

will contain an irreducible representation $\mathbb{S}^{[\lambda]}$.

If a_n is odd, the tensor

$$\text{Sym}^{a_1}(V) \otimes \text{Sym}^{a_2}(\wedge^2 V) \otimes \cdots \otimes \text{Sym}^{a_{n-1}}(\wedge^{n-1} V) \otimes \text{Sym}^{\frac{a_n-1}{2}}(\wedge^n V) \otimes V_{sp}$$

will contain a copy of $\mathbb{S}^{[\lambda]}$.

Let us now give another way to build the simple submodule in $\otimes V$, using the Schur-Weyl duality.

For any choice of indices i and j , satisfying $1 \leq i < j \leq k$, define the contraction

$$\begin{aligned} \Phi_{ij} : \otimes^k V &\longrightarrow \otimes^{k-2} V \\ v_1 \otimes \cdots \otimes v_k &\longmapsto Q(v_i, v_j) v_1 \otimes \cdots \hat{v}_i \cdots \hat{v}_j \cdots \otimes v_k. \end{aligned}$$

Let $V^{[0]} = \mathbb{C}$, $V^{[1]} = V$ and define

$$V^{[k]} = \bigcap_{ij} \ker (\Phi_{ij} : \otimes^k V \longrightarrow \otimes^{k-2} V)$$

for $k \geq 2$.

For any partition $\lambda = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{2n+1} \geq 0)$ of k , define the $\mathfrak{so}(2n+1)$ -module $\mathbb{S}^{[\lambda]}$ of $\otimes^k V$, by

$$\mathbb{S}^{[\lambda]} = V^{[k]} \cap \mathbb{S}^\lambda,$$

where \mathbb{S}^λ is the $\mathfrak{sl}(2n+1)$ -irreducible module with highest weight λ .

Theorem 4.2. [FH]

For any $d \in \mathbb{N}$ there is an isomorphism of $(\mathfrak{so}(2n+1), S_k)$ -modules

$$V^{[d]} = \bigoplus_{|\lambda|=d} \mathbb{S}^{[\lambda]} \otimes B_\lambda.$$

For every partition $\lambda = (\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq 0)$ the $\mathfrak{so}(2n+1)$ -module $\mathbb{S}^{[\lambda]}$ is the irreducible module with highest weight $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n$.

4.3. Orthogonal semistandard columns.

The definition of semistandard columns for $\mathfrak{so}(2n+1)$ given in this section is equivalent but not identic to the definition given in [L] by Cedric Lecouvey.

With the ordering $1 < 2 < \cdots < n < 0 < \bar{n} < \cdots < \bar{1}$, a column is said to be semistandard if it satisfies the following properties:

- 1- The entries are increasing from top to bottom and if t is not 0 it appears at most one time,

- 2- Let \mathcal{C} be such a column. We denote it $\mathcal{C} = \begin{matrix} A \\ O \\ \overline{D} \end{matrix}$. In O all entrees are 0 and there is A, D are subsets of $[1, n]$,
- 3- Let $I = A \cap D$, then I is in $\Delta_{A \cup D}$. We put $J = \delta_{A \cup D}(I)$,
- 4- $\#(A \cup D \cup J) + \#O \leq n$.

As for $\mathfrak{sp}(2n)$, we put $B = (A \setminus I) \cup J$, $C = (D \setminus I) \cup J$. Let $k = \#O$, there exists subsets in $[1, n] \setminus (A \cup D \cup J)$ having k elements. We denote K the greatest of these subsets.

We denote such a semistandard column by:

$$\mathcal{C} = \begin{matrix} A \\ O \\ \overline{D} \end{matrix} = f(A, O, D) = g(B, O, C).$$

In addition to the admissible columns we have the spin columns, we denote them:

$$\mathfrak{c} = \begin{matrix} A \\ \overline{D}_{sp} \end{matrix} = f(A, D),$$

where $\#A + \#D = n$, $A \cap D = \emptyset$ and the entries increase strictly.

We will say that a column is admissible if it is semistandard and not spin and it is spin if it is semistandard and spin.

As in the $\mathfrak{sp}(2n)$ case, we define the double of a semistandard column. By definition, it is the two columns tableau:

$$dble(\mathcal{C}) = dble \left(\begin{matrix} A \\ O \\ \overline{D} \end{matrix} \right) = \begin{matrix} A & B \\ K & \overline{K} \\ \overline{C} & \overline{D} \end{matrix}, \quad dble(\mathfrak{c}) = dble \left(\begin{matrix} A \\ \overline{D}_{sp} \end{matrix} \right) = \begin{matrix} 1 & A \\ \vdots & \\ n & \overline{D}_{sp} \end{matrix},$$

where it is understood that $A \cup K$, and $D \cup K$ are reordered to be written in a strictly increasing way.

4.4. Relation with the Lecouvey's admissible columns.

Let us mention that, for the non-spin case, this definition is not the Lecouvey's one. We recall that the admissible, non-spin columns in the sense of Lecouvey are those such that:

1- The entries are increasing from top to bottom and if t is not 0 it appears at most one time,

2- Let \mathcal{C}_L such a column. We denote it $\mathcal{C}_L = \begin{matrix} B \\ O \\ \overline{C} \end{matrix}$. In O all entrees are 0 and there is no zero in B and \overline{C} ,

3- Let $k = \#O$, and J^1 be the set $B \cap C \cup \{n+1, \dots, n+k\}$. We have $J^1 \in \Gamma_{BUC}^{n+k}$.

Then we put $I^1 = \gamma^{n+k}(J^1)$, $A^1 = (B \setminus J^1) \cup I^1$, $D^1 = (C \setminus J^1) \cup I^1$ and define the split of the column \mathcal{C}_L as:

$$split(\mathcal{C}_L) = \begin{matrix} A^1 & B \\ \overline{C} & \overline{D^1} \end{matrix}.$$

To prove the equivalence between the two notions, we define the subsets I and K in $[1, n]$, by:

$$I = \gamma_{[1, n] \setminus (I^1 \cup J^1)}(J), \quad K = I^1 \setminus I.$$

Let us remark that if $I = \{x_1 < \dots < x_s\}$ and $K = \{z_1 < \dots < z_k\}$, we do not have $x_i = x_i^1$ et $z_j = x_{s+j}^1$.

For instance, in $\mathfrak{so}(7)$, the following column is admissible in the Lecouvey sense:

$$\mathcal{C}_L = \begin{matrix} \boxed{3} \\ \boxed{0} \\ \boxed{\overline{3}} \end{matrix}.$$

Indeed, we have $n = 3$, $k = 1$, $B = C = \{3\}$, $J = \{3\}$, $J^1 = \{3 < 4\}$, $I^1 = \{1 < 2\} = \{x_1^1 < x_2^1\}$, and $I^1 \cup J^1 = \{1, 2, 3, 4\}$. Then $I = \{2\} = \{x_1\}$ and $K = \{1\} = \{z_1\}$,

$$z_1 = 1 \neq x_2^1 = 2, \quad x_1 = 2 \neq x_1^1 = 1.$$

Put now $A = A^1 \setminus K$, $D = D^1 \setminus K$, we have $A = (B \setminus J) \cup I$, $D = (C \setminus J) \cup I$, and:

$$split(\mathcal{C}_L) = \begin{matrix} A & B \\ K & \overline{K} \\ \overline{C} & \overline{D} \end{matrix}.$$

It is moreover clear that the column $\mathcal{C} = \begin{matrix} A \\ O \\ \overline{D} \end{matrix}$ is semistandard in the sense of the preceding section. In fact its double is the split of \mathcal{C}_L :

$$dble(\mathcal{C}) = split(\mathcal{C}_L).$$

Indeed, we have:

Lemma 4.3.

With our notations, we have:

$$I = \gamma_{B\Delta C}(J).$$

Proof. Put $I' = \gamma_{B\Delta C}(J) = \{x'_1 < \dots < x'_s\}$. By definition:

$x'_s = \sup\{t \notin (B\Delta C \cup J) = B \cup C, t < y_s\}$ and $x_s \notin B \cup C$ satisfies $x_s < y_s$ thus $x_s \leq x'_s$. If the $x_s < x'_s$ would hold, Lemma 3.2 applied to $I^1 \cup J$ and J would give $y_s < x'_s$ which is impossible, thus $x'_s = x_s$.

Suppose now $x'_s = x_s, \dots, x'_{i+1} = x_{i+1}$, then $x'_i = \sup\{t \notin B \cup C, t < y_i, t < x'_{i+1}\}$ and $x_i \notin B \cup C$ satisfies $x_i < y_i$ and $x_i < x_{i+1} = x'_{i+1}$, then $x_i \leq x'_i$. With the same argument as above, the only possibility is $x'_i = x_i$.

This proves $I' = I$. □

The preceding construction defines a map Φ from the set of admissible, non-spin, column in the sense of Lecouvey to the set of admissible column in our sense.

Conversely, if $\mathcal{C} = f(A, O, D) = g(B, O, C)$ is semistandard in our sense, we verify that the column $\mathcal{C}_L = \Psi(\mathcal{C}) = \frac{B}{\overline{C}}$ is admissible, in the sense of Lecouvey and non-spin.

By construction the mappings Φ and Ψ are inverse one each other.

Proposition 4.4.

A basis for the fundamental module $\mathbb{S}^{[\omega_r]}$ is given by the non-spin semistandard columns with height r if $r < n$, and the spin columns for $r = n$.

The admissible, non-spin column, with height n form a basis for the simple module $\mathbb{S}^{[2\omega_n]}$.

We deduce as Lecouvey, that a column \mathcal{C} (resp. \mathfrak{C}) is semistandard for $\mathfrak{so}(2n+1)$ if and only if $spl(\Psi(\mathcal{C}))$ (resp. $\Psi(\mathfrak{C}) = \mathfrak{C}$) is semistandard for $\mathfrak{sp}(2n)$, if and only if $dble(\mathcal{C})$ is semistandard for $\mathfrak{sp}(2n)$.

4.5. Orthogonal semistandard tableaux and shape algebra.

A tableau T for $\mathfrak{so}(2n+1)$ is a succession of columns with decreasing heights such that, there are at most one spin column and in that case, it is the first starting from the left.

The double of this tableau is the tableau of $\mathfrak{sp}(2n)$ obtained by duplicate each column of T , arranged in their order.

$$T = \mathcal{C}_1 \mathcal{C}_2 \dots \mathcal{C}_r \implies \text{dble}(T) = \text{dble}(\mathcal{C}_1) \text{dble}(\mathcal{C}_2) \dots \text{dble}(\mathcal{C}_r),$$

resp.

$$T = \mathfrak{C}_1 \mathfrak{C}_2 \dots \mathfrak{C}_r \implies \text{dble}(T) = \text{dble}(\mathfrak{C}_1) \text{dble}(\mathfrak{C}_2) \dots \text{dble}(\mathfrak{C}_r).$$

We extend naturally Ψ to any tableau and get:

$$\Psi(T) = \Psi(\mathcal{C}_1) \Psi(\mathcal{C}_2) \dots \Psi(\mathcal{C}_r) \implies \text{split}(\Psi(T)) = \text{dble}(T),$$

resp.

$$\Psi(\mathfrak{C}T) = \Psi(\mathfrak{C}) \Psi(\mathcal{C}_1) \dots \Psi(\mathcal{C}_r) \implies \text{dble}(\mathfrak{C}) \text{split}(\Psi(T)) = \text{dble}(\mathfrak{C}T).$$

We deduce the definition of a semistandard tableau for $\mathfrak{so}(2n+1)$:

Definition 4.5.

A tableau T is semistandard for $\mathfrak{so}(2n+1)$ if and only if its double $\text{dble}(T)$ is semistandard for $\mathfrak{sp}(2n)$.

1

Since $\text{dble}(\mathfrak{C}) = \mathfrak{C}_0 \mathfrak{C}$, where \mathfrak{C}_0 is the trivial column $\begin{matrix} \vdots \\ n \end{matrix}$, a tableau T is semistan-

dard if and only if $\Psi(T)$ is semistandard in the meaning of Lecouvey.

A dominant weight λ corresponds now to a shape of tableaux, and the set $SS^{[\lambda]}$ of orthogonal semistandard tableaux with shape λ is a basis for the simple module $\mathbb{S}^{[\lambda]}$. Similarly, the set $SS^{[\bullet]}$ of all orthogonal semistandard tableaux is a basis for the shape algebra $\bigoplus_{\lambda} \mathbb{S}^{[\lambda]}$ for the Lie algebra $\mathfrak{so}(2n+1)$.

Remark 4.6. In fact Kostant associates a notion of shape algebra for any reductive group G . In the algebraic case (see [GW]) this algebra is explicitly realized as the space of affine regular functions on the quotient $\overline{N} \backslash G$, where \overline{N} is the analytic subgroup whose Lie algebra is the opposite of \mathfrak{n} .

If G is connected and simply connected, then then this notion of shape algebra is the geometric form of the shape algebra for $\mathfrak{g} = \text{Lie}(G)$. Thus here the shape algebra for $\mathfrak{so}(2n+1)$ is the geometric shape algebra for the group $\text{Spin}(2n+1, \mathbb{C})$.

If we restrict ourselves to the shape algebra for $SO(2n+1, \mathbb{C})$, which has the same Lie algebra, we should obtain an algebra whose basis is given by the collection of all orthogonal semistandard tableaux without any spin column.

5. ORTHOGONAL QUASISTANDARD TABLEAUX

Let us recall our definitions and notations. We say that a tableau T is orthogonal semistandard ($T \in SS^{[\bullet]}$) if and only if $dbl(T)$ is symplectic semistandard ($dbl(T) \in SS^{(\bullet)}$).

Now, it is clear, due to the structure of $splt(dbl(T))$, that the condition $dbl(T) \in SS^{(\bullet)}$ is in fact equivalent to $dbl(T) \in SS$.

Definition 5.1.

Let T be an orthogonal semistandard tableau, with shape λ and $s \leq n$. Say that T is not quasistandard in s and write $T \in NQS_s^{[\lambda]}$, if and only if $dbl(T)$ is not quasistandard in s , $dbl(T) \in NQS_s^{(2\lambda)}$.

Say T is not quasistandard if and only if it exists s such that T is not quasistandard in s .

If it is not the case, we say that T is quasistandard, and denote $T \in QS^{[\lambda]}$. We note $QS^{[\bullet]}$ the union of all the sets $QS^{[\lambda]}$.

The definition $T \in NQS_s^{[\bullet]}$ is equivalent to the following condition, denoted Hs (hypothesis ins): if $dbl(T) = (dt_{ij})$,

- 1- $dt_{s1} = s$, and there exists a column with height s in T ,
- 2- For all j for which these quantities exist, $dt_{(s+1)j} > dt_{s(j+1)}$,

As in the $\mathfrak{sl}(n)$ and the symplectic case, we shall build a bijective map $p = push$ from $SS^{[\lambda]}$ to $\bigsqcup_{\mu \leq \lambda} QS^{[\mu]}$.

Since T is orthogonal semistandard if and only if $dbl(T)$ is symplectic semistandard, we shall use the ‘pouss’ function defined in [AK] for the symplectic case. This function is defined as a ‘maximal’ use of the symplectic jeu de taquin $sjdt$.

But to ‘push’ one step to the left a row s in the tableau T , we have to push two steps to the left the row s in the tableau $dbl(T)$. So we need to verify, that after the first use of the symplectic jeu de taquin on $doubl(T)$, the result is still symplectic non quasistandard in s .

On the other hand, the orthogonal jeu de taquin ($ojdt$) we shall study in the next sections is defined as a double use of the symplectic jeu de taquin. Therefore it is much more natural to directly use it to define the orthogonal ‘push’ function.

6. DIRECT EXPRESSION FOR THE ORTHOGONAL JEU DE TAQUIN $ojdt$ 6.1. Definition of the $ojdt$.

Since the double of our orthogonal semistandard tableaux coincide with the split of the corresponding orthogonal semistandard tableaux defined by Lecouvey and since the orthogonal jeu de taquin is defined with the only use of the split form, we keep the Lecouvey definition for our setting.

Definition 6.1.

Let T be an orthogonal semistandard tableau, with shape λ , we suppose there is inside T , in the left and top corner, an empty Young diagram S , with shape μ . To apply the $ojdt$ to $T \setminus S$, put a \star in an interior corner of S , write down the double of these tableaux, getting a skew symplectic semistandard tableau $dbl(T \setminus S)$ and two pointed boxes. Apply the symplectic jeu de taquin $sjdt$ successively for the two \star , the result is a symplectic semistandard tableau, which is the double of an orthogonal semistandard tableau $T' \setminus S'$. Put:

$$ojdt(T \setminus S) = T' \setminus S'.$$

Indeed, Lecouvey proved in [L] that the double action of the symplectic jeu de taquin on the double of $T \setminus S$ is the double of an orthogonal tableau.

Remark 6.2. The elementary move in the usual jeu de taquin is only a permutation of two successive boxes inside $T \setminus S$, either horizontally (from left to right) or vertically (from the top to the bottom).

The elementary move in the symplectic jeu de taquin is very similar, except that, in the case of an horizontal move, we have to modify the two concerned columns.

The elementary move in the orthogonal jeu de taquin can be a permutation along a diagonal, followed by a modification of the columns, as the following example shows:

$$\begin{array}{c}
 \begin{array}{|c|c|} \hline & 1 \\ \hline \star & 0 \\ \hline 3 & \bar{3} \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline \star & \star & 2 & \bar{3} \\ \hline 3 & 3 & 3 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline \star & 2 & \star & \bar{3} \\ \hline 3 & 3 & 3 & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline \star & 2 & \bar{3} & \bar{3} \\ \hline 3 & 3 & \star & 2 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline \star & 2 & \bar{3} & \bar{3} \\ \hline 3 & 3 & 2 & \star \\ \hline \end{array} \\
 \\
 \mapsto \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline 2 & \star & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{2} & \star \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline 2 & 3 & \bar{3} & \bar{3} \\ \hline 3 & \star & \bar{2} & \star \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline 2 & 3 & \bar{3} & \bar{3} \\ \hline 3 & \bar{2} & \star & \star \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline & 1 \\ \hline 3 & \bar{3} \\ \hline 0 & \star \\ \hline \end{array}
 \end{array}$$

From now on, we consider only the ‘horizontal situation’ (HS hypothesis):

- 1- In the tableau $dbl(T \setminus S)$, the double star are in the row s ,

- 2- $dbl(T \setminus S) = (dt_{ij})$ has two columns with height s ,
- 3- $dt_{s(j+1)} < dt_{(s+1)j}$ for each j where these two entries exist.

In this situation we can describe the elementary move.

Theorem 6.3.

Suppose the skew tableau $T \setminus S = (t_{ij})$ and the star are in the *HS* situation, then

- 1- The situation $t_{sj} = 0$ and $t_{(s+1)j} = 0$ is impossible, for any j .
- 2- The move is always horizontal,
- 3- For each elementary move on $T \setminus S$, with our notation, the move is exactly like for the horizontal move in the *sjdt*, with the addition that if $t_{s(j+1)} = 0$ and \star is in the (s, j) box, then the move is simply a permutation of these two entries.

The assertions of the theorem mean there is only horizontal elementary moves, each of them being:

Move 1- if $(t_{sj}, t_{s(j+1)}) = (\star, a)$ with a unbarred, the move is:

$$g(B_j \cup \{\star\}, O_j, C_j) f(A_{j+1} \cup \{a\}, O_{j+1}, D_{j+1}) \mapsto g(B_j \cup \{a\}, O_j, C_j) f(A_{j+1} \cup \{\star\}, O_{j+1}, D_{j+1}),$$

Move 2- if $(t_{sj}, t_{s(j+1)}) = (\star, \bar{c})$ with \bar{c} , the move is:

$$f(A_j, O_j, D_j \cup \{\star\}) g(B_{j+1}, O_{j+1}, C_{j+1} \cup \{c\}) \mapsto f(A_j, O_j, D_j \cup \{c\}) g(B_{j+1}, O_{j+1}, C_{j+1} \cup \{\star\}),$$

Move 3- if $(t_{sj}, t_{s(j+1)}) = (\star, 0)$, the move is:

$$f(A_j, O_j \cup \{\star\}, D_j) g(B_{j+1}, O_{j+1} \cup \{0\}, C_{j+1}) \mapsto f(A_j, O_j \cup \{0\}, D_j) g(B_{j+1}, O_{j+1} \cup \{\star\}, C_{j+1}),$$

6.2. Proof of the theorem.

Let $\mathcal{C} = f(A, O, D) = g(B, O, C)$ be a column in $T \setminus S$, we note its double:

$$dbl(\mathcal{C}) = \begin{array}{cc} A & B \\ \hline K & \bar{K} \\ \hline C & \bar{C} \end{array} = \begin{array}{cc} E & B \\ \hline C & \bar{F} \end{array},$$

and recall the definition of $I = A \cup D = \{x_1 < \dots < x_r\}$, $J = B \cap C = \{y_1 < \dots < y_r\}$, $K = \{z_1 < \dots < z_k\}$. Suppose the s row in this double is: $\boxed{u_s} \boxed{v_s}$. In this section, like in the *HS* situation, we assume:

$$v_s < u_{s+1}.$$

The proof of the theorem needs the following technical propositions.

Proposition 6.4.

1. If $u_s = e_s$ and $v_s = b_s$ are unbarred, then u_s is in A , and: either $e_s = b_s$ are in $A \setminus I$ or $e_s \in I$, $e_s = x_i$ and $b_s = y_i$.

2. If $u_s = e_s$ is unbarred and $v_s = \overline{f_s}$ is barred, then e_s and f_s are in K , $f_s = z_1$ and $e_s = z_k$.
3. If $u_s = \overline{c_s}$ and $v_s = \overline{f_s}$ are barred, then $f_s \in D$ and, either $c_s = f_s \in D \setminus I$ or $f_s \in I$, $f_s = x_i$ and $c_s = y_i$.

Proof.

Case 1 If $e_s = b_s$, then $b_s \notin K$, $e_s \notin K$, $e_s \in A$, and $e_s \in A \setminus B = A \setminus I$.

If $e_s < b_s$, let us assume $e_s \in K$, since K is the largest possible subset in $[1, n] \setminus (A \cup C)$, this implies $[e_s, n] \subset A \cup C \cup K = E \cup C$ but $b_s \notin E$ since $e_s < b_s < e_{s+1}$, if b_s was in C , then $b_s \in C \setminus D = C \setminus J = D \setminus I$, thus $b_s \in D$, which is impossible, therefore $u_s = e_s$ is in A .

Now $b_s \in B \setminus A = J$, there exists i such that $b_s = y_i$, let us consider x_i . We put $I^{\leq w} = I \cap [1, w]$. Then Lemme 5.1 in [AK] implies $x_i \in I^{\leq b_s} = I^{\leq e_s}$, then $x_i \leq e_s$. Now, if $x_i < e_s$, then $x_i \in I^{\leq e_s-1}$, but b_s is not in $J^{\leq e_s-1}$, and this is a contradiction with Lemme 5.1 in [AK].

Case 2 By the assumption $v_s < u_{s+1}$, $e_s = \sup(A \cup K)$ and $f_s > \sup(C)$. Therefore f_s is not in C , thus $f_s \notin B$, suppose $f_s \in D$, then $f_s \in D \setminus C = D \setminus I$, this means $f_s \in A$, $f_s \leq e_s$, now if f_s is in K , $f_s \leq e_s$ too.

Let $w \in D$, if $w \in D \setminus I$, then $w \in C$, $w < f_s$, if $w \in I$, then $w = x_i < w' = y_i$ in $J \subset C$, thus $w < w' < f_s$. The relation $w < f_s$ holds in any case. But there is as much entries strictly below f_s and in C , thus $D = F \cap [1, f_s - 1]$ and $K = \{z_1 < \dots < z_k\} = F \cap [f_s, n]$. Especially, $f_s = z_1$.

On the other hand, if e_s is in A , $e_s > z_k$, therefore $e_s \in A \setminus C = I$, $e_s \in D$, and $e_s < z_k$, which is impossible, then $e_s \in K$, $e_s = z_k$.

Case 3 This case is the symmetric of case 1. The proof is the same *mutatis mutandis*. \square

Let us now define the element v'_s as follows:

- 1- If $v_s = b_s$ is in B , we put: $v'_s = v_s$ if $v_s \notin C$ and $v'_s = \gamma_{E \cup (C \setminus \{v_s\})}(v_s)$ if $v_s \in C$.
2. If $v_s = \overline{f_s}$ with $f_s \in F$, and $u_s = e_s \in E$, we put $v'_s = \delta_{(E \setminus \{f_s\}) \cup C}(f_s)$.
3. If $v_s = \overline{f_s}$ with $f_s \in F$, and $u_s = \overline{c_s}$, with $c_s \in C$, we put $v'_s = \overline{f_s}$ if $f_s \notin E$ and $v'_s = \delta_{(E \setminus \{f_s\}) \cup C}(f_s)$ if $f_s \in E$.

Corollary 6.5.

We have $u_s = v'_s$, or the s row in $\text{dble}(C)$ is $\boxed{v'_s | v_s}$.

Proof. In the case 1, we saw that either $e_s = b_s = v'_s$ or e_s is the greatest element in $I^{\leq b_s}$, that means $u_s = e_s = \sup\{t \notin (E \cup C)^{\leq b_s}, t < b_s\} = v'_s$.

In the case 2, we saw that $e_s \leq f_s$, and $e_s = z_k, f_s = z_1$. Then $[z_1, n] \subset E \cup C$, and $\delta_{(E \setminus \{e_s\}) \cup C}(f_s) = e_s$.

The case 3 is similar to the case 1. □

Let us now prove the theorem. We suppose that the orthogonal jeu de taquin was well defined and was always moving horizontally until some point, where the star is in the row s and some column.

Lemma 6.6.

In the tableau T , it is impossible to have one of the following disposition, for any j :

$$T = \begin{array}{ccccc} \dots & \vdots & \dots & & \\ \dots & 0 & & & \\ \hline \dots & 0 & & & \\ \vdots & & & & \\ & (j) & & & \end{array} \quad \text{or} \quad T = \begin{array}{ccccc} \dots & \vdots & \vdots & \dots & \\ \dots & & 0 & & \\ \hline \dots & a & & & \\ \vdots & & & & \\ & (j-1) & (j) & & \end{array}.$$

with a unbarred.

Proof. We assume that T has the above first form then the tableau $db\ell e(T)$ has the following disposition in its columns $2j$ and $2j+1$,

$$db\ell e(T) = \begin{array}{ccccc} \dots & \vdots & \vdots & \dots & \\ \dots & a'_s & \overline{d'_s} & & \\ \hline \dots & a'_{s+1} & \overline{d'_{s+1}} & & \\ \vdots & & & & \\ & (2j) & (2j+1) & & \end{array}.$$

Since $\overline{d'_s} > a'_{s+1}$, this is in contradiction with our assumption HS .

Similarly, suppose the tableau has the second form, then $db\ell e(T)$ is

$$db\ell e(T) = \begin{array}{cccccc} \dots & \vdots & \vdots & \vdots & \vdots & \dots \\ \dots & & & e_s & \overline{f_s} & \\ \hline \dots & e'_{s+1} & b_{s+1} & & & \\ \vdots & & & & & \\ & (2j-2) & (2j-1) & (2j) & (2j+1) & \end{array}.$$

□

The proposition said that in the columns $(2j)(2j+1)$, $K_j = \{z_1 < \dots < z_r\}$, $f_s = z_1$, $e_s = z_r$. But, in these columns, $f > \sup(C_j)$, this implies $[f, n] \subset K_j \cup C_j \cup A_j$, thus $n \in K_j \cup C_j \cup A_j$, and $n \geq f > \sup C_j$, therefore, $n \in K_j \cup A_j$. On the other hand, we have $e_s < b_{s+1} \leq n$, thus $e_s = \sup(A_j \cup K_j) < n$. This is impossible.

6.2.1. Study of the case $\boxed{\star|0}$.

We now assume that there exists, on the s row, a star and a zero to the right of the star. Then doubling the two concerned columns is the following:

$$\begin{array}{c} \begin{array}{cc} A_1 & A_2 \\ \star & 0 \\ \hline O_1 & D_2 \\ D_1 & \end{array} & \mapsto & \begin{array}{cccc} E_{11} & B_1 & E_2 & \frac{B_2}{F_{21}} \\ \star & \star & e & \bar{f} \\ \hline \frac{E_{12}}{C_1} & \bar{F}_1 & \bar{C}_2 & \bar{F}_{22} \end{array} \end{array}$$

We therefore have $e = \sup(E_2 \cup \{e\})$, and since the tableau is symplectic semistandard, $\sup(E_2) \geq \sup(B_1)$, $e > \sup(B_1)$.

Lemma 6.7.

If $e \in F_1$, we set $x = \gamma_{B_1 \cup F_1}(e)$, then $\inf(E_{12}) > x > \sup(B_1)$.

Proof. Either we are at the starting point, there is no B_1 , no A_1 and $E_{12} = K_1$, $F_1 = K_1 \cup C_1$. By the definition of K_1 , since x is not in F_1 , then $x < \inf(K_1)$, this proves the lemma in this case.

Or there was a preceding step, where we had:

$$\begin{array}{c} \begin{array}{cccc} \tilde{E}_{11} & B_1 & E_2 & \frac{B_2}{F_{21}} \\ \star & \tilde{e} & \tilde{b} & e \\ \hline \tilde{E}_{12} & \tilde{C}_1 & \bar{F}_1 & \bar{C}_2 & \bar{F}_{22} \end{array} \end{array}$$

Our induction hypothesis says that the move is horizontal, the element \tilde{e} leaves its column. Since, in the first column, all the entries are distinct, in the $sjdt$, this column does not change. Then in the next symplectic step in $sjdt$, the star moves still horizontally, the element \tilde{b} leaves the second column, which as above does not change. \tilde{b} goes inside the first column, under the name $y = \gamma_{B_1 \cup F_1}(\tilde{b})$, since $\tilde{b} < e_{12} = \inf(\tilde{E}_{12})$, the first column becomes:

$$g(\tilde{E}_{11} \cup \tilde{E}_{12} \cup \{\tilde{b}\}, \tilde{C}_1) = f(\tilde{E}_{11} \cup \tilde{E}_{12} \cup \{y\}, (\tilde{C}_1 \setminus \{\tilde{b}\}) \cup \{y\}).$$

Suppose $\tilde{b} = \tilde{e}$, then $y = \tilde{b}$, $\tilde{E}_{11} = E_{11}$, $\tilde{E}_{12} = E_{12}$, $\tilde{C}_1 = C_1$.

Suppose now $\tilde{b} > \tilde{e}$, then $e_{12} > \tilde{b} \geq y \geq \tilde{e} > e_{11}$, then $\tilde{E}_{11} = E_{11}$, $\tilde{E}_{12} = E_{12}$, $\tilde{C}_1 = C_1 \setminus \{y\} \cup \{\tilde{b}\}$.

So in every case, we get:

$$\begin{array}{cccccc} & & E_{11} & B_1 & E_2 & \frac{B_2}{F_{21}} \\ \begin{array}{c} s \\ \star \end{array} & & y & \star & e & \bar{f} \\ \hline & & \frac{E_{12}}{C_1} & \bar{F}_1 & \bar{C}_2 & \bar{F}_{22} \end{array}$$

Now $b_1 = \sup(B_1) < \tilde{b} \leq e$. Therefore:

either $e = \tilde{b} \notin B_1 \cup F_1$ thus $e_{12} > x = e = \tilde{b} > b_1$,

or $e > \tilde{b} \notin B_1 \cup F_1$ and by the definition of x , $x \geq \tilde{b} > b_1 = \sup(B_1)$.

That means $x > \sup(A_1)$, $E_{11} = A_1$, $x < \inf(K_1) = e_{12} = \inf(E_{12})$. □

In the next step of the symplectic jeu de taquin $sjdt$, e leaves the third column and becomes x . We put $F'_1 = (F_1 \setminus \{e\}) \cup \{x\}$, and get, with the lemma, the following sequence of steps:

$$\mapsto_s \begin{array}{cccccc} & & E_{11} & B_1 & E_2 & \frac{B_2}{F_{21}} \\ & & \star & x & \star & \bar{f} \\ \hline & & \frac{E_{12}}{C_1} & \bar{F}'_1 & \bar{C}_2 & \bar{F}_{22} \end{array}$$

$$\mapsto_s \begin{array}{cccccc} & & E_{11} & B_1 & E_2 & \frac{B_2}{F_{21}} \\ & & \star & x & \bar{f} & \star \\ \hline & & \frac{E_{12}}{C_1} & \bar{F}'_1 & \bar{C}_2 & \bar{F}_{22} \end{array}.$$

Now we saw $\sup(E_{11}) < x < \inf(E_{12})$. Since $x \notin F_1$, then $x \notin D_1$. Suppose that $x \in C_1 \setminus D_1 = J_1 = B_1 \cap C_1$, then x is in B_1 and according to the Lemma this is wrong. So x is not in C_1 and the next step is:

$$\mapsto_s \begin{array}{cccccc} & & E_{11} & B_1 & E_2 & \frac{B_2}{F_{21}} \\ & & x & \star & \bar{f} & \star \\ \hline & & \frac{E_{12}}{C_1} & \bar{F}'_1 & \bar{C}_2 & \bar{F}_{22} \end{array}$$

Now, if $k_2 = \#O_2$, f is the k_2^{st} element of F_2 starting from the top. We know that $\#[1, n] \setminus (A_2 \cup D_2 \cup J_2) = \#[1, n] \setminus (B_2 \cup D_2) \geq k_2$. Denote X the greatest subset having k_2 elements in F_2 and Y the greatest subset having k_2 elements in $[1, n] \setminus B_2$. We have $X = \{x_1 < x_2 < \cdots < x_{k_2}\}$ and $Y = \{y_1 < \cdots < y_{k_2}\}$.

By construction, if $K_2 = \{z_1 < \cdots < z_{k_2}\}$, then $z_i \leq y_i$.

Consider an element y_i . If $y_i \in D_2$, then $y_i \in F_2$. If $y_i \notin D_2$, then y_i is in $[1, n] \setminus (D_2 \cup B_2)$, so y_i is a z_j , for some $j \leq i$, $y_i \in K_2$, and $y_i \in F_2$: this proves $Y \subset F_2$.

Since X is the greatest of the subsets in F_2 , $y_i \leq x_i$ but since $X \subset F_2 \subset [1, n] \setminus B_2$ and Y is the greatest such subset, $x_i \leq y_i$.

That means $X = Y$, and especially $f = x_1 = y_1$.

On the other hand, if $e \in A_2$, Since $e \notin C_2$, then $e \in A_2 \setminus C_2 = I_2 = A_2 \cap D_2$ so $e \in F_2 = K_2 \cup D_2$. But if $e \notin A_2$, then $e \in K_2$, so $e \in F_2$.

In all cases e is the greatest element in F_2 .

Indeed, if there exists t in F_2 such that $t > e$, then either $t \in K_2$, then $t \in E_2 \cup \{e\}$, which is impossible, or $t \in D_2$ and $t \notin C_2$, then $t \in D_2 \setminus C_2 = I_2 \subset A_2 \subset E_2 \cup \{e\}$, which is still impossible.

So $e = x_{k_2} = y_{k_2}$.

Let $t > f$ and $t \neq e$. Then either $t \in B_2$ or $t \in Y \subset F_2$, in all cases

$$t \in (F_2 \cup B_2) \setminus \{e\} = ((E_2 \cup \{e\}) \cup C_2) \setminus \{e\} = E_2 \cup C_2.$$

So $e = \delta_{E_2 \cup C_2}(f)$.

Then, in the next step, we set $E'_2 = (E_2 \setminus \{f\}) \cup \{e\}$, and get, as \bar{f} leaves the column 3,

$$\begin{array}{cccc} E_{11} & B_1 & E_2 & \frac{B_2}{F_{21}} \\ \mapsto_s & x & \star & \bar{f} & \star \\ \hline & \frac{E_{12}}{C_1} & \bar{F}'_1 & \bar{C}_2 & \bar{F}_{22} \end{array}$$

$$\begin{array}{cccc} E_{11} & B_1 & E'_2 & \frac{B_2}{F_{21}} \\ \mapsto_s & x & \bar{e} & \star & \star \\ \hline & \frac{E_{12}}{C_1} & \bar{F}'_1 & \bar{C}_2 & \bar{F}_{22} \end{array}$$

Indeed, our assumption HS gives: $f < \inf(C_2) \leq \inf(F_1)$, and $e > f$, thus $f < \inf(F'_1)$, on the other side, we saw that $e > \sup(B_1)$.

Now, the two first columns give a \mathfrak{so} column with one 0 more. More precisely, the corresponding first \mathfrak{so} column becomes:

$$f(A_1, O_1 \cup \{\star\}, D_1) = \frac{\begin{array}{c} A_1 \\ \star \\ O_1 \\ \hline D_1 \end{array}}{\quad} \mapsto g(B_1, O_1 \cup \{0\}, C_1) = \frac{\begin{array}{c} A_1 \\ 0 \\ O_1 \\ \hline D_1 \end{array}}{\quad} = f(A_1, O_1 \cup \{0\}, D_1).$$

The two last columns is also the double of a \mathfrak{so} column, in fact it gives:

$$f(A_2, O_2, D_2) = \frac{\begin{array}{c} A_2 \\ O'_2 \\ 0 \\ \hline D_2 \end{array}}{\quad} \mapsto g(B_2, O_2 \setminus \{0\} \cup \{\star\}, C_2) = \frac{\begin{array}{c} A_2 \\ O'_2 \\ \star \\ \hline D_2 \end{array}}{\quad} = f((A_2, O_2 \setminus \{0\}) \cup \{\star\}, D_2)$$

This is exactly the Move 3 case described after the theorem.

6.2.2. Study of the case $\boxed{\star|a}$.

Let us now study the case:

$$\begin{array}{c} \begin{array}{cc} A_{11} & A_{21} \\ \star & a \\ \hline A_{12} & A_{22} \\ O_1 & O_2 \\ \hline D_1 & D_2 \end{array} \\ \sim \begin{array}{c} \begin{array}{cccc} E_{11} & B_{11} & E_{21} & B_{21} \\ \star & \star & e & b \\ \hline E_{12} & B_{12} & E_{22} & B_{22} \\ C_1 & F_1 & C_2 & F_2 \end{array} \end{array} \end{array}$$

Step 1 We move the first \star for 2 steps, the second one for one step.

Then the entry e enters into the column 2 which becomes $g(B_1 \cup \{e\}, F_1)$ or, if $x = \gamma_{B_{11} \cup B_{12} \cup F_1}(e)$, and $F'_1 = (F_1 \setminus \{e\}) \cup \{x\}$,

$$\mapsto \begin{array}{c} \begin{array}{cccc} E_{11} & B_{11} & E_{21} & B_{21} \\ \star & x & \star & b \\ \hline E_{12} & B_{12} & E_{22} & B_{22} \\ C_1 & F'_1 & C_2 & F_2 \end{array} \end{array}$$

The argument of Lemma 6.7 tell us that $x > \sup(B_{11}) = b_{11}$ and $x \leq e < b_{12} = \inf(B_{12})$. Then x is not in C_1 , as above, it enters in the first column:

$$\mapsto \begin{array}{c} \begin{array}{cccc} E_{11} & B_{11} & E_{21} & B_{21} \\ x & \star & \star & b \\ \hline E_{12} & B_{12} & E_{22} & B_{22} \\ C_1 & F'_1 & C_2 & F_2 \end{array} \end{array}$$

Similarly, b goes in the column 3, under the name $y = \gamma_{E_{21} \cup E_{22} \cup C_2}(b)$:

$$\mapsto \frac{s \quad \begin{array}{cccc} E_{11} & B_{11} & E_{21} & B_{21} \\ x & \star & y & \star \\ \hline E_{12} & B_{12} & E_{22} & B_{22} \\ C_1 & F_1' & C_2' & F_2' \end{array}}{\quad}$$

As before, y is in this row, $F_1' = (F_1 \setminus \{e\}) \cup \{x\}$.

Step 2 Let us now prove e is not in K_2 :

Assume $e \in K_2$. Then $b \neq e$, with our hypothesis HS , b is not in E_2 , so $b \notin A_2$, and then $b \in J_2$. Set $I_{21} = \{t \in I_2, t < e\}$, $A_{21} = \{t \in A_2, t < e\}$, same definition for K_{21} , B_{21} . Then

$$E_{21} \setminus I_{21} = (A_{21} \setminus I_{21}) \cup K_{21}, \quad A_{21} \setminus I_{21} \subset B_{21}, \quad K_{21} \cap B_{21} = \emptyset.$$

Since $b \in J_2$, there exists $t \in I_{21}$, $t = x_i$ such that $b = y_i$, so $\#(J_2 \cap B_{21}) < \#I_{21}$, or $\#(B_{21} \setminus J_2) > \#(A_{21} \setminus I_{21})$. Then there exists $a \in B_{21} \setminus J_2 \subset A_2 \setminus I_2$ such that $a \notin A_{21}$, $b > a > e$, so we have

$$e_{22} = \inf(E_{22}) \leq a < b,$$

which is in contradiction with our hypothesis HS .

Step 3 Let us show that $y = e$.

If $e = b$, then $b \in A_2 \setminus I_2 = B_2 \setminus J_2$, $b \notin C_2$, so $y = b = e$.

If $e < b$, since $e \notin E_{21} \cup E_{22} \cup C_2$, then $e \in \{t \notin E_{21} \cup E_{22} \cup C_2, t < b\}$, so $e \leq y = \sup\{t \notin E_{21} \cup E_{22} \cup C_2, t < b\}$.

Assume now $e < y$.

Since $b \in B_2$ and $b \notin E_2$, $b \in B_2 \setminus A_2 = J_2$. Let $J_2 = \{y_1 < \dots < y_i = b < \dots < y_r\}$ and $I_2 = \{x_1 < \dots < x_r\}$.

Put $A \Delta D = (A \cup D) \setminus (A \cap D)$, then:

$$x_i = \sup\{t \notin A_2 \Delta D_2, t < x_{i+1}, t < y_i = b\},$$

Let us prove that $y \geq x_{i+1}$. In fact if $y < x_{i+1}$, then $y \in \{t \notin E_{21} \cup E_{22} \cup C_2, t < b\}$, so $y \leq x_i$ and

$$e < y \leq x_i < b < e_{22} = \inf(E_{22}).$$

But this is impossible because there is no elements in E between e and b_{22} . So $y \geq x_{i+1}$. Since $x_{i+1} \in E_2$ et $y \notin E_2$, we have $y > x_{i+1}$.

Now $x_{i+1} = \sup\{t \notin A_2 \Delta D_2, t < x_{i+2}, t < y_{i+1}\}$.

Following the same argument we prove $y \geq x_{i+2}$:

As before: if $y < x_{i+2}$, then $e < y \leq x_{i+1}$, which is wrong, so $y \geq x_{i+2}$, $y > x_{i+2}$, and so on... Finally:

$$y > x_r = \sup\{t \notin A_2 \triangle D_2, t < y_r\},$$

But $y \notin A_2 \triangle D_2$, $y < b \leq y_r$, so $y \leq x_r$, which is contradiction.

So the hypothesis $e < y$ is wrong: $e = y$.

Step 4 End of the move.

Then we obtain $e \notin F'_1$ and :

$$\begin{array}{c} \begin{array}{cccc} E_{11} & B_{11} & E_{21} & B_{21} \\ x & e & \star & \star \\ \hline E_{12} & B_{12} & E_{22} & B_{22} \\ C_1 & F'_1 & C'_2 & F_2 \end{array} \\ \sim g(B_1 \cup \{e\}, O_1, C_1) f((A_2 \setminus \{e\}) \cup \{\star\}, O_2, D_2). \end{array}$$

This is exactly the Move 1 case described after the theorem.

6.2.3. Study of the case $\boxed{\star|\bar{d}}$.

Let us now study the last case:

$$\begin{array}{cc} A_1 & A_2 \\ O_1 & O_2 \\ \hline D_{11} & D_{21} \\ \star & \bar{d} \\ \hline D_{12} & D_{22} \end{array}$$

Doubling the tableau, we get:

$$\begin{array}{c} \begin{array}{cccc} E_1 & B_1 & E_2 & B_2 \\ \hline C_{11} & F_{11} & C_{21} & F_{21} \\ \star & \star & \bar{c} & \bar{f} \\ \hline C_{12} & F_{12} & C_{22} & F_{22} \end{array} \\ \sim \end{array}$$

Step 1 We move the first \star for 2 steps, the second one for one step.

We get:

$$\begin{array}{c} \begin{array}{cccc} E_1 & B_1 & E_2 & B_2 \\ \hline E_1 & F_{11} & E_2 & F_{21} \\ C_{11} & F_{11} & C_{21} & F_{21} \\ \star & \bar{c} & \bar{f} & \star \\ \hline C_{12} & F_{12} & C_{22} & F_{22} \end{array} \\ \mapsto \end{array} .$$

Now \bar{c} enters the first column under the name $x = \delta_{B_1 \cup F_{11} \cup F_{12}}(c)$. We put $B'_1 = (B_1 \setminus \{c\}) \cup \{x\}$.

Remark that $x \notin C_{11} \cup C_{12}$ and $x \notin B_1 \cup F_1 = B_1 \cup K_1 \cup D_1$ by construction. Therefore x is not in E_1 . In fact, if $x \in E_1 = A_1 \cup K_1$, then since $x \notin K_1$, $x \in A_1$, so

$$x \in A_1 \setminus D_1 = A_1 \setminus I_1 = B_1 \setminus J_1 \subset B_1,$$

which is impossible, so $x \notin E_1$ and we have $c_{11} = \inf(C_{11}) > x > c_{12} = \sup(C_{12})$, then x still remains on the same row:

$$\mapsto \begin{array}{cccc} E_1 & B'_1 & E_2 & B_2 \\ \hline C_{11} & F_{11} & C_{21} & F_{21} \\ \hline s & \bar{x} & \star & \bar{f} \\ \hline C_{12} & F_{12} & C_{22} & F_{22} \end{array}.$$

Let us be more precise: since $\sup(C_{22}) < f < \inf(C_{21})$, we have $f \notin C_2 \setminus \{c\}$, in other words, \bar{f} is on the row s . But now the double of the column 3 is the semistandard Young tableau with 2 columns:

$$\begin{array}{cc} E_2 & E'_2 \\ \hline C'_{21} & C_{21} \\ \hline s & \bar{y}' & \bar{f} \\ \hline C_{22} & C_{22} \end{array}$$

where we set as usual, $y = \delta_{(E_2 \setminus \{f\}) \cup (C_2 \setminus \{c\})}(f)$, $E'_2 = E_2 \setminus \{f\} \cup \{y\}$, and we put $y' = \inf(C_{21} \cup \{y\})$, and $C'_{21} = C_{21} \setminus \{y'\} \cup \{y\}$ if $y' = c_{21} > y$, $C'_{21} = C_{21}$ otherwise.

In the next step, it is \bar{y}' which will enter in column 1.

Step 2 Let us prove: $y = y' = c$.

- Assume first $f = c$:

If $f = c$, $f \notin E_2$, so $y = f = c$ and since $y = c < c_{11}$, $y' = y = c$.

- Assume now $f < c$ and $f \in K_2$:

Since K_2 is the greatest subset with k_2 elements included in $[1, n] \setminus (A_2 \cup D_2 \cup J_2) = [1, n] \setminus (A_2 \cup C_2)$, for all t , $t > f$ implies $t \in K_2 \cup A_2 \cup C_2 = E_2 \cup C_2$. On the other hand $f \in K_2$ implies $f \in E_2$, and the the only $t > f$ not in $(E_2 \setminus \{f\}) \cup (C_2 \setminus \{c\})$ is c , that is $y = \delta_{(E_2 \setminus \{f\}) \cup (C_2 \setminus \{c\})}(f) = c$, and as above $y' = c$.

- Assume finally $f < c$ et $f \notin K_2$

Since f is in $F_2 = K_2 \cup D_2$, $f \in D_2$. Since $c_{22} = \sup(C_{22}) < f < c < c_{21} = \inf(C_{21})$, $f \notin C_2$. Then $f \in D_2 \setminus C_2 = I_2$. Especially, $f \in E_2$.

Write $I_2 = \{x_1 < \dots < x_i = f < \dots < x_r\}$. Set $J_2 = \{y_1 < \dots < y_r\}$. We have:

$$y_i = \inf\{t \notin A_2 \triangle D_2, t > x_i = f, t > y_{i-1}\} \quad \text{et} \quad y = \inf\{t \notin E_2 \cup C_{21} \cup C_{22}, t > f\}.$$

Therefore $c \notin E_2 \cup C_{21} \cup C_{22}$ et $c > f$, so $c \geq y$.

Assume, by contradiction, $c > y$.

In this case, $c_{22} < f = x_i < y < c$, so $y \notin C_2$. Since $y \notin E_2$, y is not in A_2 . If y belonged to D_2 , it would be in $D_2 \setminus A_2 = C_2 \setminus B_2$, which is impossible, so $y \notin A_2 \cup D_2$, especially $y \notin A_2 \triangle D_2$.

If $y > y_{i-1}$, then y belongs to $\{t \notin A_2 \triangle D_2, t > x_i = f, t > y_{i-1}\}$, so $y \geq y_i$, but y_i is in C_2 , so $y > y_i$. And

$$f = x_i < y_i < y < c.$$

But $y_i \in C_2$ and there is no element in C_2 between f and c , so this is impossible and $y \leq y_{i-1}$, we even have $y < y_{i-1}$ because $y \notin C_2$,

$$x_{i-1} < x_i = f < y < y_{i-1}.$$

Repeating the same argument, we prove that $x_1 \leq f < y < y_1$, but

$$y_1 = \inf\{t \notin A_2 \triangle D_2, t > x_1\}$$

and since $y \notin A_2 \triangle D_2$, therefore $y \geq y_1$, which is impossible, so $y = c$ and $y' = c$ as above.

Step 3 End of the move.

By pushing $y' = c$, We finally obtain:

$$\mapsto \begin{array}{cccc} E_1 & B'_1 & E'_2 & B_2 \\ \hline C_{11} & F_{11} & C_{21} & F_{21} \\ \hline s & \bar{x} & \bar{c} & \star \\ \hline C_{12} & F_{12} & C_{22} & F_{22} \end{array}.$$

Which is the double of the orthogonal semistandard tableau:

$$f(A_1, O_1, D_1 \cup \{c\})g(B_2, O_2, (C_2 \setminus \{c\}) \cup \{\star\}).$$

This is exactly the Move 2 case described after the theorem.

In all the considered cases, the entries situated under the star in the right, are still unchanged after the moving. So the condition HS remains true in the tableau $dbl(T)$, for all the columns coming after the column containing the star.

6.3. Starting with a spin column.

In this section, we suppose the first column of the tableau T is a spin column $f(A, D)$, with a trivial top (the entry in $(s, 1)$ is s). Then we remove the s top boxes in this column, put a \star on the row s , and we look at the first move, under the hypothesis HS . If $s = n$, the first column is trivial and the jeu de taquin does not really consider the spin column. We suppose now $s < n$.

We consider the tableau $dbl e(T)$, we remove the s boxes in the top of the two first columns, we put two stars in $(s, 1)$ and $(s, 2)$. Remark that our hypothesis impose that the entry on the right of the star is unbarred. Indeed, if it was barred, in $dbl e(T)$, there is a \bar{c} in the box $(s, 3)$, hypothesis HS implies that all the entries in the second column in $dbl e(T)$, below s are barred, in other word, the spin column is $f([1, s], [s+1, n])$, and $\bar{c} \leq \bar{n}$, which is in contradiction with hypothesis HS .

Similarly, if it was 0, we saw there is no unbarred entry before \star in column 2, that means the spin column is still $f([1, s], [s+1, n])$, the entry in the box $(s+1, 3)$ is barred, it can only be \bar{n} , this implies \bar{f} is larger than this entry, which is impossible.

The only possible case is $\boxed{\star|a}$. We just give the succession of tableaux (we do not put the index sp):

$$\begin{array}{c} \begin{array}{ccccc} & & & E_{21} & B_{21} \\ & & & e & b \\ \hline s & \star & \star & & \\ & s+1 & A_{12} & E_{22} & B_{22} \\ & \vdots & \overline{D_1} & \overline{C_2} & \overline{F_2} \\ & n & & & \end{array} & \mapsto & \begin{array}{ccccc} & & & E_{21} & B_{21} \\ & & & \star & b \\ \hline s & \star & x & \star & \\ & s+1 & A_{12} & E_{22} & B_{22} \\ & \vdots & \overline{D'_1} & \overline{C_2} & \overline{F_2} \\ & n & & & \end{array}, \end{array}$$

with the same meaning for x and D'_1 , as above. By definition of $x = \gamma_{A_1 \cup D_1}(e)$, we have $x = s$ (even if $e = s$). The next step is

$$\begin{array}{c} \begin{array}{ccccc} & & & E_{21} & B_{21} \\ & & & y & \star \\ \hline s & s & \star & & \\ & s+1 & A_{12} & E_{22} & B_{22} \\ & \vdots & \overline{D'_1} & \overline{C'_2} & \overline{F_2} \\ & n & & & \end{array} & \mapsto & \begin{array}{ccccc} & & & E_{21} & B_{21} \\ & & & \star & \star \\ \hline s & s & e & \star & \star \\ & s+1 & A_{12} & E_{22} & B_{22} \\ & \vdots & \overline{D'_1} & \overline{C'_2} & \overline{F_2} \\ & n & & & \end{array}. \end{array}$$

We see that the two first columns is the double of the spin column

$$f(\{1, \dots, s-1, e\}, (D_1 \setminus \{e\}) \cup \{s\})$$

from whose we remove the $s-1$ first boxes.

6.4. Supression of the trivial top in the first column.

Suppose now T is a semistandard tableau in $NQS_s^{[\bullet]}$, suppress the top s boxes in the first column of T , getting a skew tableau $T \setminus S$. Apply the jeu de taquin to $T \setminus S$ (remark there is only one interior corner). Denote $T' \setminus S'$ the resulting tableau. Suppose $s > 1$, S' is a tableau with one column and $s-1$ boxes. Since the path of stars during two successive application of the jeu de taquin do not cross, if we apply the jeu de taquin to $T' \setminus S'$, the star will move every times horizontally. This is equivalent to say that the tableau T' obtained when we fill up the empty boxes in $T' \setminus S'$ by $1, \dots, s-1$, is in $NQS_{s-1}^{[\bullet]}$. Remark we can also prove this point by using

the preceding computation, and looking case by case the HS condition along the new row $s - 1$.

Then we can repeat the use of the jeu de taquin, getting a tableau $T'' \setminus S''$, where the shape of T'' is the shape of T , where we suppress a column with height s and add a new column with height $s - 2$.

After s repetition of the $ojdt$, we get a tableau $T^{(s)}$, with a shape smaller than the shape of T : the shape of $T^{(s)}$ if the shape of T where we suppress a column with height s .

6.5. The map $ojdt^{-1}$ in the horizontal situation.

We proved that the orthogonal jeu de taquin, in the HS situation is moving only horizontally. This operation is coming from a double action of the symplectic jeu de taquin on $dblc(T)$. Let us now look for the inverse mapping. It is the composition of two inverse of the symplectic jeu de taquin on $dblc(T)$. But we know how to define $(sjdt)^{-1}$ (see [S]).

Let $T \setminus S$ be a skew orthogonal semistandard tableau. Denote $\sigma(T)$ the tableau obtained from T by rotating T half a tour, and replace each barred entry by the corresponding unbarred quantity, each unbarred by the corresponding barred quantity. Keep the 0.

Example 6.8. Consider $\mathfrak{so}(9)$ ($n = 4$) then

$$T = \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline & \bar{3} & 0 \\ \hline 0 & \bar{1} & \\ \hline \bar{1} & & \\ \hline \end{array} \xrightarrow{\sigma} \sigma(T) = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & 0 \\ \hline 0 & \bar{3} & \\ \hline \bar{2} & \bar{1} & \\ \hline \end{array}.$$

Lemma 6.9.

Defining similarly the σ operation for a symplectic tableau, then we have,

$$\sigma(dble(T)) = dble(\sigma(T)).$$

Proof. If T has r columns and the column C_j is $C_j = f(A_j, O_j, D_j)$, then $\sigma(T)$ has r columns and its $r + 1 - j$ column is

$$C'_{r+1-j} = f(D_j, O_j, A_j)$$

Let us consider the double. The columns number $2j, 2j + 1$ in $dblc(T)$ are:

$$f(A_j \cup K_j, C_j)f(B_j, K_j \cup D_j).$$

For the column C'_{r+1-j} , we define the set K'_{r+1-j} , and immediately get $K'_{r+1-j} = K_j$. Therefore in $dblc(\sigma(T))$, the columns $2(r - j), 2(r - j) + 1$ are:

$$f(D_j \cup K_j, B_j)f(C_j, A_j \cup K_j)$$

This proves the lemma. □

Since, for symplectic tableaux, we have (see [S])

$$(sjdt)^{-1} = \sigma \circ sjdt \circ \sigma,$$

and:

$$dble \circ ojdt = sjdt \circ dble,$$

then $ojdt$ is invertible and

$$(ojdt)^{-1} = dble^{-1} \circ (sjdt)^{-1} \circ dble = dble^{-1} \circ \sigma \circ (sjdt) \circ \sigma \circ dble = \sigma \circ (ojdt) \circ \sigma.$$

Lemma 6.10.

If T satisfies the HS condition, along the row s , then $T' = (ojdt)(T)$ is such that $\sigma(T')$ satisfies the HS condition for the row $h+1-s$ if h is the height of T .

Proof. The preceding lemma proves that the motion of the star when we apply $ojdt$ at the row $h+1-s$ of $\sigma(T)$ would be always horizontal.

More precisely, if we look case by case the result of the elementary $ojdt$ move as described above, we directly verify that the tableau $\sigma \circ ojdt(T)$ satisfies the condition HS on the row $h+1-s$. Let us for example look at one case:

Suppose that, in the row s in T , there is $\boxed{\star|a|0}$, then, in the double, there is:

$$\mapsto \begin{array}{ccccccc} & E_{11} & B_{11} & E_{21} & B_2 & E_3 & \frac{B_3}{F_{31}} \\ \mapsto \quad s & \star & \star & e_2 & b_2 & e_3 & \frac{f_3}{f_3} \\ & \frac{E_{12}}{C_1} & \frac{B_{12}}{F_1} & \frac{E_{22}}{C_2} & \overline{F_2} & \overline{C_3} & \overline{F_{32}} \end{array}.$$

This gives successively:

$$\mapsto \begin{array}{ccccccc} & E_{11} & B_{11} & E_{21} & B_2 & E_3 & \frac{B_3}{F_{31}} \\ \mapsto \quad s & x_2 & e_2 & \star & \star & e_3 & \frac{f_3}{f_3} \\ & \frac{E_{12}}{C_1} & \frac{B_{12}}{F_1} & \frac{E_{22}}{C_2} & \overline{F_2} & \overline{C_3} & \overline{F_{32}} \end{array},$$

with $x_2 = \gamma_{B_{11} \cup B_{12} \cup F_1}(e_2) \leq e_2$, and

$$\mapsto \begin{array}{ccccccc} & E_{11} & B_{11} & E_{21} & B_2 & E'_3 & \frac{B_3}{F_{31}} \\ \mapsto \quad s & x_2 & e_2 & x_3 & \overline{e_3} & \star & \star \\ & \frac{E_{12}}{C_1} & \frac{B_{12}}{F_1} & \frac{E_{22}}{C_2} & \overline{F_2} & \overline{C_3} & \overline{F_{32}} \end{array},$$

with $\inf E_{22} > x_3 = \gamma_{B_2 \cup F_2}(e_3) > \sup(B_2)$.

After the action of σ , we get:

$$\mapsto \frac{\begin{array}{cccccc} F_{32} & C_3 & F'_2 & \frac{C'_2}{E_{22}} & \frac{F'_1}{B_{12}} & \frac{C_1}{E_{12}} \\ \hline h+1-s & \star & \star & e_3 & \frac{\bar{x}_3}{\bar{e}_2} & \frac{\bar{x}_2}{\bar{x}_2} \end{array}}{\begin{array}{cccccc} \frac{F_{31}}{B_3} & \frac{\bar{E}'_3}{\bar{B}_2} & \frac{\bar{E}_3}{\bar{B}_2} & \frac{\bar{E}_2}{\bar{E}_{21}} & \frac{\bar{B}_1}{\bar{B}_{11}} & \frac{\bar{E}_1}{\bar{E}_{11}} \end{array}}.$$

We see that the HS condition holds since:

$$x_3 > \sup(B_2), \quad e_2 > \sup(E_{21}), \quad \text{and} \quad x_2 \geq e_2 > \sup(B_{11}).$$

□

7. THE MAP $p = (ojdt)^{max}$ IS BIJECTIVE

We are now in position to define the map ‘push’ or p from $SS^{[\lambda]}$ to $\bigsqcup_{\mu \leq \lambda} QS^{[\mu]}$.

Let T be an orthogonal semistandard tableau, with shape λ .

If T is quasistandard, we put $p(T) = T$.

If T is not quasistandard, we consider the greatest s for which $T \in NQS_s^{[\lambda]}$ (i.e. $T \notin NQS_t^{[\lambda]}$, for each $t > s$). The entry in the box $(s, 1)$ in T is s , then the two entries in the boxes $(s, 1)$ and $(s, 2)$ in $dbl(T)$ are s also, moreover, the HS condition holds on the row s . We remove the s top boxes in the first column of T , getting the skew tableau $T \setminus S$ and apply the $ojdt$ s times to $T \setminus S$.

Lemma 7.1.

Let λ' be the shape of $ojdt^s(T \setminus S)$, we get λ' by removing a column with height s and we have $ojdt(T) \notin NQS_t^{[\lambda']}$, for each $t > s$.

Proof. After the horizontal action by $ojdt^s$ on T , we clearly get a tableau with the announced shape. Now in [AK] it is proved that the action of the $sjdt$ on a symplectic semistandard tableau which is in $NQS_s^{(\mu)}$ but not in $NQS_t^{(\mu)}$, for any $t > s$ is still not in $NQS_t^{(\mu')}$, for any $t > s$.

Applying two times this result on the tableau $dbl(T)$, we get that $dbl(ojdt(T)) \notin NQS_t^{(2\lambda')}$, for any $t > s$, that means the HS condition does not hold on the row t , for any $t > s$, or $ojdt(T) \notin NQS_t^{[\lambda']}$, for any $t > s$. □

Replace now T by $ojdt^s(T \setminus S)$ and repeat the above analyse.

It is clear that, after a finite number of steps, this algorithm gives an orthogonal quasistandard tableau, with a shape smaller than the shape of T . We denote this

tableau by:

$$T' = p(T) = (ojdt)^{max}(T).$$

Remark that if the first column of T was a non trivial spin column, then the first column in T' is a spin column. In all other cases, there is no spin column in T' .

The above algorithm defines the map $p : SS^{[\lambda]} \longrightarrow \bigsqcup_{\mu \leq \lambda} QS^{[\mu]}$. Let us look at the inverse map. We follow the method of [AK].

Start with an orthogonal quasistandard tableau U , with shape μ . Let λ be a shape larger than μ .

Apply σ to U , seeing it as a skew tableau inside a skew tableau with shape $\sigma(\lambda)$, put a \star in the lowest inner corner in $\sigma(\lambda) \setminus \sigma(\mu)$, and apply the $ojdt$, getting a new skew tableau with shape $\sigma(\mu')$. Repeat this operation as far as there is inner corner in $\sigma(\lambda) \setminus \sigma(\mu')$.

At the end of this operation, get a skew tableau T' with shape $\sigma(\lambda \setminus \mu)$, consider $T = \sigma(T')$ where the empty boxes are filled up in the top μ shape by a trivial tableau. By construction T is orthogonal semistandard, with shape λ , and the above algorithm $(ojdt)^{max}$ applied on T is the inverse of this sequence of $\sigma \circ ojdt \circ \sigma$, that means $p(T) = U$, and p is a bijective map.

Remark that if U contains a spin column, this column becomes the last column in $\sigma(U)$, and at the end it becomes a spin column in T , as the first column of T .

Theorem 7.2.

The orthogonal jeu de taquin defines a bijection $p = (ojdt)^{max}$ from the set $SS^{[\lambda]}$ of orthogonal semistandard tableaux with shape λ onto the disjoint union $\bigsqcup_{\mu \leq \lambda} QS^{[\mu]}$ of the set of orthogonal quasistandard tableaux with shape μ ($\mu \leq \lambda$).

Here is an example.

Example 7.3. Suppose $n = 5$, or $\mathfrak{g} = \mathfrak{so}(11)$

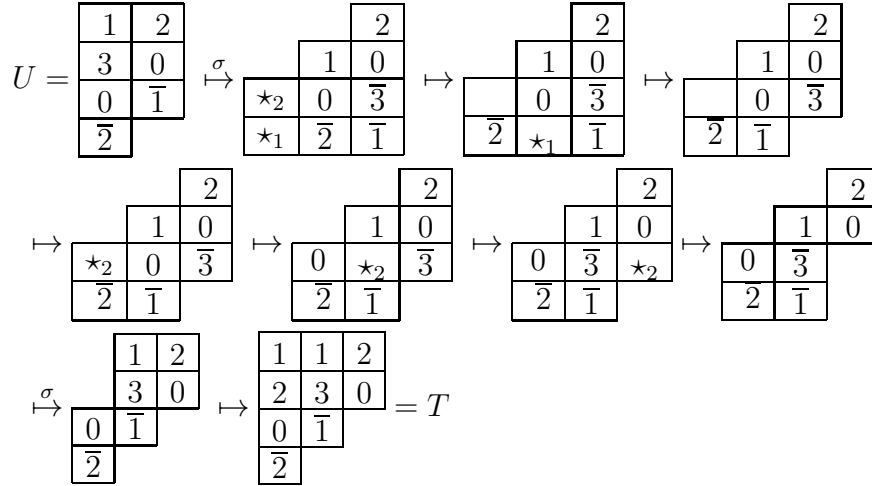
$$T = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & 0 \\ \hline 0 & \bar{1} & \\ \hline \bar{2} & & \\ \hline \end{array} \mapsto dble(T) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 3 & 3 & 3 & 5 & \bar{5} \\ \hline 5 & \bar{5} & \bar{2} & \bar{1} & & \\ \hline \bar{3} & \bar{2} & & & & \\ \hline \end{array}.$$

Then T is in $NQS_2^{(0,1,1,1,0)}$. The $ojdt$ gives successively:

$$\begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline \star & 3 & 0 \\ \hline 0 & \bar{1} & \\ \hline \bar{2} & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 3 & \star & 0 \\ \hline 0 & \bar{1} & \\ \hline \bar{2} & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline & 1 & 2 \\ \hline 3 & 0 & \\ \hline 0 & \bar{1} & \\ \hline \bar{2} & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline \star & 1 & 2 \\ \hline 3 & 0 & \\ \hline 0 & \bar{1} & \\ \hline \bar{2} & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|} \hline \star & 1 & 2 \\ \hline 3 & 0 & \\ \hline 0 & \bar{1} & \\ \hline \bar{2} & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 0 \\ \hline 0 & \bar{1} \\ \hline \bar{2} & \\ \hline \end{array} = U$$

with $U \in QS^{(0,0,1,1,0)}$.

Conversely:



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UNIVERSITÉ DE SFAX, FACULTÉ DES SCIENCES, DÉPARTEMENT DE MATHÉMATIQUES, ROUTE DE SOUKRA, KM 3,5, B.P. 1171, 3000 SFAX, TUNISIE.

E-mail address: B.Agreba@fss.rnu.tn

INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR CNRS 5584, UNIVERSITÉ DE BOURGOGNE, U.F.R. SCIENCES ET TECHNIQUES B.P. 47870, F-21078 DIJON CEDEX, FRANCE

E-mail address: Didier.Arnal@u-bourgogne.fr

UNIVERSITÉ DE SFAX, FACULTÉ DES SCIENCES, DÉPARTEMENT DE MATHÉMATIQUES, ROUTE DE SOUKRA, KM 3,5, B.P. 1171, 3000 SFAX, TUNISIE.

E-mail address: benhassine.abdelkader@yahoo.fr