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Description of the minimizers of least squares regularized with ℓ_0 -norm. Uniqueness of the global minimizer

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Abstract. We have an $M \times N$ real-valued arbitrary matrix A (e. g. a dictionary) with $M < N$ and data d describing the sought-after object with the help of A . This work provides an in-depth analysis of the (local and global) minimizers of an objective function \mathcal{F}_d combining a quadratic data-fidelity term and an ℓ_0 penalty applied to each entry of the sought after solution, weighted by a regularization parameter $\beta > 0$. For several decades, this objective focuses a ceaseless effort to conceive algorithms approaching a good minimizer. Our theoretical contributions, summarized below, shed new light on the existing algorithms and can help the conception of innovative numerical schemes.

To solve the normal equation associated with any M -row submatrix of A is equivalent to compute a local minimizer \hat{u} of \mathcal{F}_d . (Local) minimizers \hat{u} of \mathcal{F}_d are strict if and only if the submatrix, composed of those columns of A whose indexes form the support of \hat{u} , has full column rank. An outcome is that strict local minimizers of \mathcal{F}_d are easily computed without knowing the value of β . Each strict local minimizer is linear in data. The global minimizers of \mathcal{F}_d are always strict. They are studied in more details under the (standard) assumption that $\text{rank}(A) = M < N$. The global minimizers with M -length support are seen to be impractical. Given d , critical values β_K for any $K \leq M - 1$ are exhibited such that if $\beta > \beta_K$, all global minimizers of \mathcal{F}_d are K -sparse.

An assumption on A is adopted and proven to be generically true. Then for generically all data d , the objective \mathcal{F}_d has a unique global minimizer and the latter is K -sparse for $K \leq M - 1$.

Instructive small-size (5×10) numerical illustrations confirm the main theoretical results.

Key words: global minimizers, local minimizers, ℓ_0 regularization, nonconvex nonsmooth minimization problems, perturbation analysis, quadratic programming, solution analysis, sparse recovery, strict minimizers, under-determined linear systems, uniqueness of the solution, variational methods.

1 Introduction

Let A be an arbitrary matrix (e. g. a dictionary) such that

$$A \in \mathbb{R}^{M \times N} \quad \text{for} \quad M < N ,$$

where the positive integers M and N are fixed. Given a data vector $d \in \mathbb{R}^M$, we consider an objective function $\mathcal{F}_d : \mathbb{R}^N \rightarrow \mathbb{R}$ of the form

$$\begin{aligned} \mathcal{F}_d(u) &= \|Au - d\|_2^2 + \beta \|u\|_0 , \quad \beta > 0 , \\ \|u\|_0 &= \#\sigma(u) , \end{aligned} \tag{1}$$

where $u \in \mathbb{R}^N$ contains the coefficients describing the sought-after object, β is a regularization parameter, $\#$ stands for cardinality and $\sigma(u)$ is the support of u (i. e. the set of all $i \in \{1, \dots, N\}$ for which the i th entry of u meets $u[i] \neq 0$). By an abuse of language, the penalty in (1) is called the ℓ_0 -norm. Define $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(t) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } t = 0, \\ 1 & \text{if } t \neq 0. \end{cases} \quad (2)$$

Then $\|u\|_0 = \sum_{i=1}^N \phi(u[i]) = \sum_{i \in \sigma(u)} \phi(u[i])$, so \mathcal{F}_d in (1) equivalently reads

$$\mathcal{F}_d(u) = \|Au - d\|_2^2 + \beta \sum_{i=1}^N \phi(u[i]) \quad (3)$$

$$= \|Au - d\|_2^2 + \beta \sum_{i \in \sigma(u)} \phi(u[i]). \quad (4)$$

We focus on all (local and global) minimizers \hat{u} of an objective \mathcal{F}_d of the form (1):

$$\hat{u} \in \mathbb{R}^N \text{ such that } \mathcal{F}_d(\hat{u}) = \min_{u \in \mathcal{O}} \mathcal{F}_d(u), \quad (5)$$

where \mathcal{O} is an open neighborhood of \hat{u} . We note that finding the global minimizer of \mathcal{F}_d must be an *NP-hard* computational problem [9, 37].

The function ϕ in (2) served as a regularizer for a long time. In the context of Markov random fields it was used by Geman and Geman in 1984 [17] and Besag in 1986 [3] as a prior in MAP energies to restore labeled images. The MAP objective reads

$$\mathcal{F}_d(u) = \|Au - d\|_2^2 + \beta \sum_k \phi(D_k u), \quad (6)$$

where D_k is a finite differences operator and ϕ is given by (2). This label-designed form is known as the Potts prior model, or as the multi-level logistic model [4, 22]. Various stochastic and deterministic algorithms were considered to minimize (6). Leclerc [21] proposed in 1989 a deterministic continuation method to restore piecewise constant images. Robini and Magnin [32] introduced the stochastic continuation approach and successfully used it to reconstruct 3D tomographic images. They refined the method and the theory in [33]. Very recently, Robini and Reissman [34] gave theoretical results relating the probability for global convergence and the computation speed.

The problem stated in (1) and (5)—to (locally) minimize \mathcal{F}_d —arises when *sparse* solutions are desired. Typical application fields are signal and image processing, morphologic component analysis, compression, dictionary building, inverse problems, compressive sensing, machine learning, model selection, classification, subset selection, among others. The original hard-thresholding method proposed by Donoho and Johnstone [12] amounts to¹ minimize \mathcal{F}_d where d contains the coefficients of a signal or an image expanded in a wavelet basis ($M = N$). When $M < N$ various (usually strong) restrictions on $\|u\|_0$ (often $\|u\|_0$ is replaced by a less irregular function) and on A (e.g. RIP-like criteria, conditions on $\|A\|$, etc.) are needed to conceive

¹For reminder: if d are some noisy coefficients, the restored coefficients \hat{u} minimize $\|u - d\|^2 + \beta\|u\|_0$ and read $\hat{u}[i] = 0$ if $|d[i]| \leq \sqrt{\beta}$ and $\hat{u}[i] = d[i]$ otherwise.

numerical schemes approximating a minimizer of \mathcal{F}_d , to establish local convergence and derive the asymptotic of the obtained solution. In statistics the problem was widely considered for subset selection and numerous algorithms were designed, with limited theoretical production, as explained in the book of Miller [28]. More recently, Haupt and Nowak [19] investigate the statistical performances of the global minimizer of \mathcal{F}_d and propose an iterative bound-optimization procedure. Fan and Li [14] discuss a variable splitting and penalty decomposition minimization technique for (1), along with other approximations of the ℓ_0 -norm. Liu and Wu [23] mix the ℓ_0 and the ℓ_1 penalties, establish some asymptotic properties of the new estimator and use mixed integer programming aimed at global minimization. For model selection, Lv and Fan [25] approximate the ℓ_0 penalty using functions that are concave on \mathbb{R}_+ and prove a non asymptotic nearly oracle property of the resultant estimator. Thiao *et al.* [36] reformulate the problem so that an approximate solution can be found using difference of convex functions programming. Blumensath and Davies [6] propose an iterative thresholding scheme to approximate a solution and prove convergence to a local minimizer of \mathcal{F}_d . Lu and Zhang [24] suggest a penalty decomposition method to minimize \mathcal{F}_d . Fornasier and Ward [15] propose an iterative thresholding algorithm for finding local minimizers of \mathcal{F}_d where ℓ_0 is replaced by a reasonable sparsity-promoting relaxation given by $\phi(t) = \min\{t, 1\}$; convergence to a local minimizer is established. In a recent report, Chouzenoux *et al.* [8], a mixed $\ell_2 - \ell_0$ regularization is considered: a slightly smoothed version of the objective is analyzed and optimization using a majorize-minimize subspace approach is shown to converge to a local minimizer in a finite number of iterations. Since the submission of our paper, image reconstruction methods were designed where ℓ_0 regularization is applied to the coefficients of the expansion of the sought-after image in a wavelet frame [40, 11]: the provided numerical results outperform ℓ_1 regularization for a reasonable computational cost achieved using penalty decomposition techniques. In a general study on the convergence of descent methods for nonconvex objectives, Attouch *et al.* [1] apply an inexact forward-backward splitting scheme to find a critical point of \mathcal{F}_d . Several other references can be evoked, e.g. [29, 16].

Even though overlooked for several decades, the objective \mathcal{F}_d was essentially considered from a numerical standpoint. The motivation naturally comes from the promising applications and the intrinsic difficulty to minimize \mathcal{F}_d .

The goal of this work is to analyze the (local and global) minimizers \hat{u} of objectives \mathcal{F}_d of the form (1).

We provide detailed results on the minimization problem.

The uniqueness of the global minimizer of \mathcal{F}_d is examined as well.

We do not propose an algorithm. However, our theoretical results raise salient cross-questions to the existing algorithms and can help the conception of innovative numerical schemes.

The minimization of \mathcal{F}_d in (1) might seem close to its constraint variants:

$$\begin{aligned} \text{given } \varepsilon \geq 0, \quad & \text{minimize } \|u\|_0 \quad \text{subject to } \|Au - d\|^2 \leq \varepsilon, \\ \text{given } K \in \mathbb{I}_M, \quad & \text{minimize } \|Au - d\|^2 \quad \text{subject to } \|u\|_0 \leq K. \end{aligned} \tag{7}$$

The latter problems are abundantly studied in the context of sparse recovery in different fields. An excellent account is given in [7], see also the book [26]. For recent achievements, we refer to [38]. It is worth emphasizing that in general, *there is no equivalence between the problems stated in (7) and the minimization of \mathcal{F}_d in (1)* because all these problems are nonconvex.

1.1 Main notations and definitions

We remind that if \hat{u} is a (*local*) *minimizer* of \mathcal{F}_d , the value $\mathcal{F}_d(\hat{u})$ is a (local) minimum² of \mathcal{F}_d reached at (possibly numerous) points \hat{u} . Saying that a (local) minimizer \hat{u} is *strict* means that there is a neighborhood $\mathcal{O} \subset \mathbb{R}^N$, containing \hat{u} , such that $\mathcal{F}_d(\hat{u}) < \mathcal{F}_d(v)$ for any $v \in \mathcal{O} \setminus \{\hat{u}\}$. So \hat{u} is an isolated minimizer.

Let K be any positive integer. The expression $\{u \in \mathbb{R}^K : u \text{ satisfying property } \mathfrak{P}\}$ designates the subset of \mathbb{R}^K formed from all elements u that meet \mathfrak{P} . The identity operator on \mathbb{R}^K is denoted by I_K . The entries of a vector $u \in \mathbb{R}^K$ read $u[i]$, for any i . The i th vector of the canonical basis³ of \mathbb{R}^K is denoted by $e_i \in \mathbb{R}^K$. Given $u \in \mathbb{R}^K$ and $\rho > 0$, the *open ball* at u of radius ρ with respect to the ℓ_p -norm for $1 \leq p \leq \infty$ reads

$$B_p(u, \rho) \stackrel{\text{def}}{=} \{v \in \mathbb{R}^K : \|v - u\|_p < \rho\} .$$

To simplify the notations, the ℓ_2 -norm is *systematically* denoted by

$$\|\cdot\| \stackrel{\text{def}}{=} \|\cdot\|_2 .$$

We denote by \mathbb{I}_K the *totally and strictly ordered* index set

$$\mathbb{I}_K \stackrel{\text{def}}{=} (\{1, \dots, K\}, <) . \quad (8)$$

where the symbol “ $<$ ” stands for the natural order of the positive integers. Accordingly,

any subset $\omega \subseteq \mathbb{I}_N$ inherits the property of being totally and strictly ordered.

We shall often consider the index set \mathbb{I}_N . The complement of $\omega \subseteq \mathbb{I}_N$ in \mathbb{I}_N is denoted by

$$\omega^c = \mathbb{I}_N \setminus \omega \subseteq \mathbb{I}_N .$$

Definition 1 For any $u \in \mathbb{R}^N$, the *support* $\sigma(u)$ of u is defined to be the strictly increasing injective map from \mathbb{I}_N to \mathbb{I}_N , where \mathbb{I}_N is defined according to (8), given by

$$\sigma(u) = \{i \in \mathbb{I}_N : u[i] \neq 0\} \subseteq \mathbb{I}_N .$$

If $u = 0$, clearly $\sigma(u) = \emptyset$.

The i th column in a matrix $A \in \mathbb{R}^{M \times N}$ is denoted by a_i . It is *systematically* assumed that

$$a_i \neq 0, \quad \forall i \in \mathbb{I}_N .$$

For a matrix $A \in \mathbb{R}^{M \times N}$ and a vector $u \in \mathbb{R}^N$, with any $\omega \subseteq \mathbb{I}_N$, we associate the *submatrix* A_ω and the *subvector* u_ω given by

$$A_\omega \stackrel{\text{def}}{=} (a_{\omega[1]}, \dots, a_{\omega[\#\omega]}) \in \mathbb{R}^{M \times \#\omega} , \quad (9)$$

$$u_\omega \stackrel{\text{def}}{=} (u[\omega[1]], \dots, u[\omega[\#\omega]]) \in \mathbb{R}^{\#\omega} , \quad (10)$$

²These two terms are often assimilated in the literature.

³More precisely, for any $i \in \mathbb{I}_K$, the vector $e_i \in \mathbb{R}^K$ is defined by $e_i[i] = 1$ and $e_i[j] = 0, \forall j \in \mathbb{I}_K \setminus \{i\}$.

respectively, as well as the zero padding operator $Z_\omega : \mathbb{R}^{\#\omega} \rightarrow \mathbb{R}^{\mathbb{N}}$ that inverts (10):

$$u = Z_\omega(u_\omega) \quad u[i] = \begin{cases} 0 & \text{if } i \notin \omega, \\ u_\omega[k] & \text{for the unique } k \text{ such that } \omega[k] = i. \end{cases} \quad (11)$$

Thus for $\omega = \emptyset$ one finds $u_\emptyset = \emptyset$ and $u = Z_\emptyset(u_\emptyset) = 0 \in \mathbb{R}^{\mathbb{N}}$.

Using Definition 1 and the notations in (9)-(10), for any $u \in \mathbb{R}^{\mathbb{N}} \setminus \{0\}$ we have

$$\omega \in \mathbb{I}_{\mathbb{N}} \text{ and } \omega \supseteq \sigma(u) \quad \Rightarrow \quad Au = A_\omega u_\omega. \quad (12)$$

So as to simplify the presentation, we adopt the following *definitions*⁴:

$$\begin{aligned} (a) \quad & A_\emptyset = [] \in \mathbb{R}^{\mathbb{M} \times 0}, \\ (b) \quad & \text{rank}(A_\emptyset) = 0. \end{aligned} \quad (13)$$

In order to avoid possible ambiguities⁵, we set

$$A_\omega^T \stackrel{\text{def}}{=} (A_\omega)^T,$$

where the superscript “ T ” stands for transposed. If A_ω is invertible, similarly $A_\omega^{-1} \stackrel{\text{def}}{=} (A_\omega)^{-1}$.

In the course of this work, we shall recurrently refer to the following quadratic constrained optimization problem:

Given $d \in \mathbb{R}^{\mathbb{M}}$, for a totally and strictly ordered subset $\omega \subseteq \mathbb{I}_{\mathbb{N}}$, problem (\mathcal{P}_ω) reads:

$$(\mathcal{P}_\omega) \quad \boxed{\begin{cases} \min_{u \in \mathbb{R}^{\mathbb{N}}} \|Au - d\|^2, \\ \text{subject to } u[i] = 0, \quad \forall i \in \omega^c. \end{cases}} \quad (14)$$

The definition below will be used to evaluate the extent of some subsets and assumptions.

Definition 2 *Let $\mathcal{S} \subset \mathbb{R}^{\mathbb{K}}$ be a subset.*

- (i) *It is said that \mathcal{S} is negligible in $\mathbb{R}^{\mathbb{K}}$ if there exists $\mathcal{Z} \subset \mathbb{R}^{\mathbb{K}}$ which is closed in $\mathbb{R}^{\mathbb{K}}$, its Lebesgue measure in $\mathbb{R}^{\mathbb{K}}$ is $\mathbb{L}^{\mathbb{K}}(\mathcal{Z}) = 0$ and $\mathcal{S} \subseteq \mathcal{Z}$.*
- (ii) *If a property holds true for all $v \in \mathbb{R}^{\mathbb{K}} \setminus \mathcal{S}$, where \mathcal{S} is negligible in $\mathbb{R}^{\mathbb{K}}$ in the sense of (i), we say that this property is generic on $\mathbb{R}^{\mathbb{K}}$.*

The elements of a negligible subset $\mathcal{S} \subset \mathbb{R}^{\mathbb{K}}$ are highly exceptional⁶ in $\mathbb{R}^{\mathbb{K}}$.

Indeed, the subset $\mathbb{R}^{\mathbb{K}} \setminus \mathcal{S}$ contains an *open and dense* subset of $\mathbb{R}^{\mathbb{K}}$. The chance that a truly random $v \in \mathbb{R}^{\mathbb{K}}$ —i. e. a v following a non singular probability distribution on $\mathbb{R}^{\mathbb{K}}$ —comes across such an \mathcal{S} can be ignored in practice.

⁴Note that (a) corresponds to the zero mapping on \mathbb{R}^0 and that (b) is the usual definition for the rank of an empty matrix.

⁵In the light of (9), A_ω^T could also mean $(A^T)_\omega$.

⁶A property is usually said to hold *almost surely* if it fails only on a subset of null Lebesgue measure. We emphasize that Definition 2 requires in addition that this subset is contained in a *closed* subset.

1.2 Content of the paper

The main result in section 2 tells us that finding a solution of (\mathcal{P}_ω) for $\omega \subset \mathbb{I}_N$ is *equivalent* to computing a (local) minimizer of \mathcal{F}_d . In section 3 we prove that the (local) minimizers \hat{u} of \mathcal{F}_d are *strict* if and only if the submatrix $A_{\sigma(\hat{u})}$ has full column rank. The strict minimizers of \mathcal{F}_d are shown to be linear in data d . The importance of the $(M-1)$ -sparse strict minimizers is emphasized. The global minimizers of \mathcal{F}_d are studied in section 4. They are shown to be strict for any d and for any $\beta > 0$. More details are provided under the standard assumption that $\text{rank}(A) = M < N$. Given $d \in \mathbb{R}^M$, critical values β_K for $K \in \mathbb{I}_{M-1}$ are exhibited such that all global minimizers of \mathcal{F}_d are K -sparse⁷ if $\beta > \beta_K$.

In section 5, a gentle assumption on A is shown to be *generic* for all $M \times N$ real matrices. Under this assumption, for *generically* all data $d \in \mathbb{R}^M$, the objective \mathcal{F}_d for $\beta > \beta_K$, $K \in \mathbb{I}_{M-1}$, has a unique global minimizer and the latter is K -sparse.

Small size (A is 5×10) numerical tests in section 6 illustrate the main theoretical results.

2 All minimizers of \mathcal{F}_d

2.1 Preliminary results

First, we give some basic facts on problem (\mathcal{P}_ω) as defined in (14) needed for later use. If $\omega = \emptyset$, then $\omega^c = \mathbb{R}^N$, so the unique solution of (\mathcal{P}_ω) is $\hat{u} = 0$. For an arbitrary $\omega \subset \mathbb{I}_N$ meeting $\#\omega \geq 1$, (\mathcal{P}_ω) amounts to minimize a quadratic term with respect to only $\#\omega$ components of u , the remaining entries being null. The latter *quadratic* problem (\mathcal{Q}_ω) reads

$$(\mathcal{Q}_\omega) \quad \boxed{\min_{v \in \mathbb{R}^{\#\omega}} \|A_\omega v - d\|^2, \quad \#\omega \geq 1.} \quad (15)$$

Using the zero-padding operator Z_ω in (11), we have

$$\left[\hat{u}_\omega \in \mathbb{R}^{\#\omega} \text{ solves } (\mathcal{Q}_\omega) \text{ and } \hat{u} = Z_\omega(\hat{u}_\omega) \right] \Leftrightarrow \left[\hat{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega), \quad \#\omega \geq 1 \right]$$

The optimality conditions for (\mathcal{Q}_ω) , combined with the definition in (13)(a), give rise to the following equivalence which holds true for any $\omega \subseteq \mathbb{I}_N$:

$$\left[\hat{u} \in \mathbb{R}^N \text{ solves } (\mathcal{P}_\omega) \right] \Leftrightarrow \left[\hat{u}_\omega \in \mathbb{R}^{\#\omega} \text{ solves } A_\omega^T A_\omega v = A_\omega^T d \text{ and } \hat{u} = Z_\omega(\hat{u}_\omega) \right]. \quad (16)$$

Note that $A_\omega^T A_\omega v = A_\omega^T d$ in (16) is the normal equation associated to $A_\omega v = d$.

For $\omega \subseteq \mathbb{I}_N$, let K_ω denote the vector subspace

$$K_\omega \stackrel{\text{def}}{=} \{v \in \mathbb{R}^N : v[i] = 0, \forall i \in \omega^c\}. \quad (17)$$

This notation enables problem (\mathcal{P}_ω) in (14) to be rewritten as

$$\min_{u \in K_\omega} \|Au - d\|^2. \quad (18)$$

The technical lemma below will be used in the sequel. We emphasize that its statement is *independent* of the vector $\hat{u} \in \mathbb{R}^N \setminus \{0\}$.

⁷As usual, a vector u is said to be K -sparse if $\|u\|_0 \leq K$.

Lemma 1 Let $d \in \mathbb{R}^M$, $\beta > 0$ and $\hat{u} \in \mathbb{R}^N \setminus \{0\}$ be arbitrary. For $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u})$, set

$$\rho \stackrel{\text{def}}{=} \min \left\{ \min_{i \in \hat{\sigma}} |\hat{u}[i]|, \frac{\beta}{2(\|A^T(A\hat{u} - d)\|_1 + 1)} \right\}. \quad (19)$$

Then $\rho > 0$.

(i) For ϕ as defined in (2), we have

$$v \in B_\infty(0, \rho) \Rightarrow \sum_{i \in \mathbb{I}_N} \phi(\hat{u}[i] + v[i]) = \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i]) + \sum_{i \in \hat{\sigma}^c} \phi(v[i]) ;$$

(ii) For $K_{\hat{\sigma}}$ defined according to (17), \mathcal{F}_d satisfies

$$v \in B_\infty(0, \rho) \cap (\mathbb{R}^N \setminus K_{\hat{\sigma}}) \Rightarrow \mathcal{F}_d(\hat{u} + v) \geq \mathcal{F}_d(\hat{u}),$$

where the inequality is strict when $\hat{\sigma}^c \neq \emptyset$.

The proof is outlined in Appendix 8.1, p. 26.

2.2 The (local) minimizers of \mathcal{F}_d solve quadratic problems

It is worth emphasizing that no special assumptions on the matrix A are adopted.

We begin with an easy but warning result.

Lemma 2 For any $d \in \mathbb{R}^M$ and $\forall \beta > 0$, \mathcal{F}_d has a strict (local) minimum at $\hat{u} = 0 \in \mathbb{R}^N$.

Proof. Using that $\mathcal{F}_d(0) = \|d\|^2 \geq 0$, we have

$$\mathcal{F}_d(v) = \|Av - d\|^2 + \beta\|v\|_0 = \mathcal{F}_d(0) + \mathcal{R}_d(v), \quad (20)$$

$$\text{where } \mathcal{R}_d(v) = \|Av\|^2 - 2\langle v, A^T d \rangle + \beta\|v\|_0. \quad (21)$$

Noticing that $\beta\|v\|_0 \geq \beta > 0$ for $v \neq 0$ leads to

$$v \in B_2\left(0, \frac{\beta}{2\|A^T d\| + 1}\right) \setminus \{0\} \Rightarrow \mathcal{R}_d(v) \geq -2\|v\| \|A^T d\| + \beta > 0.$$

Inserting the latter implication into (20) entails the statement. \square

For any $\beta > 0$ and $d \in \mathbb{R}^M$, the sparsest strict local minimizer of \mathcal{F}_d reads $\hat{u} = 0$.

Initialization with zero of a suboptimal algorithm should generally be a bad choice.

Indeed, experiments have shown that such an initialization can be harmful, see e. g. [28, 6].

The next proposition states a (might be intuitive) result that is often evoked in this work.

Proposition 1 Let $d \in \mathbb{R}^M$. Given an $\omega \subseteq \mathbb{I}_N$, let \hat{u} solve problem (\mathcal{P}_ω) as formulated in (14). Then for any $\beta > 0$, the objective \mathcal{F}_d in (1) reaches a (local) minimum at \hat{u} and

$$\sigma(\hat{u}) \subseteq \omega, \quad (22)$$

where $\sigma(\hat{u})$ reads according to Definition 1.

Proof. Let \hat{u} solve problem (\mathcal{P}_ω) and $\beta > 0$. The constraint in (\mathcal{P}_ω) entails (22).

Consider that $\hat{u} \neq 0$ in which case for $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u})$ we have $1 \leq \#\hat{\sigma} \leq \#\omega$. Using the equivalent formulation of (\mathcal{P}_ω) given in (17)-(18), p. 6, yields

$$v \in K_\omega \quad \Rightarrow \quad \|A(\hat{u} + v) - d\|^2 \geq \|A\hat{u} - d\|^2. \quad (23)$$

The inclusion in (22) is equivalent to $\omega^c \subseteq \hat{\sigma}^c$. Let $K_{\hat{\sigma}}$ read according to (17) as well. Then

$$\hat{u} \in K_{\hat{\sigma}} \subseteq K_\omega.$$

Combining the latter relation with (23) entails that

$$v \in K_{\hat{\sigma}} \quad \Rightarrow \quad \|A(\hat{u} + v) - d\|^2 \geq \|A\hat{u} - d\|^2. \quad (24)$$

Let ρ read as in (19), Lemma 1. Noticing that by (2) and (17)

$$v \in K_{\hat{\sigma}} \quad \Rightarrow \quad \phi(v[i]) = 0, \quad \forall i \in \hat{\sigma}^c, \quad (25)$$

the following inequality chain is derived:

$$\begin{aligned} v \in B_\infty(0, \rho) \cap K_{\hat{\sigma}} \quad \Rightarrow \quad \mathcal{F}_d(\hat{u} + v) &= \|A(\hat{u} + v) - d\|^2 + \beta \sum_{i \in \mathbb{I}_N} \phi(\hat{u}[i] + v[i]) \\ &\stackrel{\text{[by Lemma 1(i)]}}{=} \|A(\hat{u} + v) - d\|^2 + \beta \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i]) + \beta \sum_{i \in \hat{\sigma}^c} \phi(v[i]) \\ &\stackrel{\text{[by (25)]}}{=} \|A(\hat{u} + v) - d\|^2 + \beta \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i]) \\ &\stackrel{\text{[by (24)]}}{\geq} \|A\hat{u} - d\|^2 + \beta \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i]) \\ &\stackrel{\text{[by (4), p. 2]}}{=} \mathcal{F}_d(\hat{u}). \end{aligned}$$

Combining the latter implication with Lemma 1(ii), p. 7, shows that

$$\mathcal{F}_d(\hat{u} + v) \geq \mathcal{F}_d(\hat{u}), \quad \forall v \in B_\infty(0, \rho).$$

If $\hat{u} = 0$, this is a (local) minimizer of \mathcal{F}_d by Lemma 2. □

Many authors mention that initialization is paramount for the success of approximate algorithms minimizing \mathcal{F}_d . In view of Proposition 1, if one already has a well elaborated initialization, it could be enough to solve the relevant problem (\mathcal{P}_ω) .

The statement reciprocal to Proposition 1 is obvious but it helps our analysis.

Lemma 3 For $d \in \mathbb{R}^M$ and $\beta > 0$, let \mathcal{F}_d have a (local) minimum at \hat{u} . Then \hat{u} solves $(\mathcal{P}_{\hat{\sigma}})$ for $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u})$.

Proof. Let \hat{u} be a (local) minimizer of \mathcal{F}_d . Denote $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u})$. Then \hat{u} solves the problem:

$$\min_{u \in \mathbb{R}^N} \left\{ \|Au - d\|^2 + \beta \#\hat{\sigma} \right\} \quad \text{subject to } u[i] = 0, \quad \forall i \in \hat{\sigma}^c.$$

Since $\#\hat{\sigma}$ is a constant, \hat{u} solves $(\mathcal{P}_{\hat{\sigma}})$. □

By Proposition 1 and Lemma 3, to solve (\mathcal{P}_ω) for some $\omega \subseteq \mathbb{I}_N$ and then to apply the zero padding operator Z_ω in (11) is *equivalent* to find a (local) minimizer of \mathcal{F}_d .

This equivalence underlies most of the theory developed in this work.

Corollary 1 For $d \in \mathbb{R}^M$ and $\beta > 0$, let \hat{u} be a (local) minimizer of \mathcal{F}_d . Set $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u})$. Then

$$\hat{u} = Z_{\hat{\sigma}}(\hat{u}_{\hat{\sigma}}) \quad \text{where } \hat{u}_{\hat{\sigma}} \text{ satisfies } A_{\hat{\sigma}}^T A_{\hat{\sigma}} \hat{u}_{\hat{\sigma}} = A_{\hat{\sigma}}^T d. \quad (26)$$

Conversely, if $\hat{u} \in \mathbb{R}^N$ meets (26) for $\hat{\sigma} = \sigma(\hat{u})$, then \hat{u} is a (local) minimizer of \mathcal{F}_d .

Proof. By Lemma 3, \hat{u} solves $(\mathcal{P}_{\hat{\sigma}})$. The equation in (26) follows directly from (16) on p. 6. The last claim is a straightforward consequence of (16) and Proposition 1. \square

Equation (26) shows that a (local) minimizer \hat{u} of \mathcal{F}_d follows a *pseudo-hard* thresholding scheme⁸: the nonzero part $\hat{u}_{\hat{\sigma}}$ of \hat{u} is the least squares solution with respect to the submatrix $A_{\hat{\sigma}}$ and the whole data vector d is involved in its computation. Unlike the hard thresholding scheme in [12], insignificant or purely noisy data entries can hardly be discarded from \hat{u} and they risk to pollute its nonzero part $\hat{u}_{\hat{\sigma}}$. Noisy data d should degrade $\hat{u}_{\hat{\sigma}}$ and this effect is stronger if $A_{\hat{\sigma}}^T A_{\hat{\sigma}}$ is ill-conditioned [10]. The quality of the outcome critically depends on the selected (local) minimizer and on the pertinence of A . See also Remark 1 on p. 12.

It may be interesting to evoke another consequence of Proposition 1:

Given $d \in \mathbb{R}^M$, for any $\omega \subseteq \mathbb{I}_N$, \mathcal{F}_d has a (local) minimizer \hat{u} such that $\sigma(\hat{u}) \subseteq \omega$.

3 The strict minimizers of \mathcal{F}_d

We remind, yet again, that no special assumptions on $A \in \mathbb{R}^{M \times N}$ are adopted.

Strict minimizers of an objective function enable unambiguous solutions of inverse problems. We focus on the strict minimizers of \mathcal{F}_d . The definition below is practical to characterize them.

Definition 3 Given a matrix $A \in \mathbb{R}^{M \times N}$, for any $r \in \mathbb{I}_M$ we define Ω_r as the subset of all r -length supports⁹ that correspond to full column rank $M \times r$ submatrices of A , i. e.

$$\Omega_r = \left\{ \omega \subset \mathbb{I}_N : \#\omega = r = \text{rank}(A_\omega) \right\}.$$

Set $\Omega_0 = \emptyset$ and define as well

$$\Omega \stackrel{\text{def}}{=} \bigcup_{r=0}^{M-1} \Omega_r \quad \text{and} \quad \Omega_{\max} \stackrel{\text{def}}{=} \Omega \cup \Omega_M.$$

Definition 3 entails that for any $r \in \mathbb{I}_M$,

$$\text{rank}(A) = r \geq 1 \quad \Leftrightarrow \quad \Omega_r \neq \emptyset \quad \text{and} \quad \Omega_t = \emptyset, \quad \forall t \geq r + 1.$$

⁸In a Bayesian setting, the quadratic data fidelity term in \mathcal{F}_d models data corrupted with Gaussian i.i.d. noise.

⁹Remind that any ω in Ω_r is totally and strictly ordered.

3.1 How to recognize a strict minimizer of \mathcal{F}_d ?

The theorem below gives an exhaustive answer to this question.

Theorem 1 *Given $d \in \mathbb{R}^M$ and $\beta > 0$, let \hat{u} be a (local) minimizer of \mathcal{F}_d . Define*

$$\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u}) .$$

The following statements are equivalent:

- (i) *The (local) minimum that \mathcal{F}_d has at \hat{u} is strict;*
- (ii) $\boxed{\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma}}$;
- (iii) $\hat{\sigma} \in \Omega_{\max}$.

If \hat{u} is a strict (local) minimizer of \mathcal{F}_d , then it reads

$$\hat{u} = Z_{\hat{\sigma}}(\hat{u}_{\hat{\sigma}}) \quad \text{for} \quad \hat{u}_{\hat{\sigma}} = (A_{\hat{\sigma}}^T A_{\hat{\sigma}})^{-1} A_{\hat{\sigma}}^T d \quad (27)$$

and satisfies $\#\hat{\sigma} = \|\hat{u}\|_0 \leq M$.

Proof. It is prompt to remember that by the rank-nullity theorem [18, 27] we have

$$\dim \ker(A_{\hat{\sigma}}) = \#\hat{\sigma} - \text{rank}(A_{\hat{\sigma}}) . \quad (28)$$

Proof of (i) \Rightarrow (ii). Let $\hat{u} \neq 0$ be a *strict* (local) minimizer of \mathcal{F}_d . Assume that (ii) fails. Using (28) the latter means that

$$\dim \ker(A_{\hat{\sigma}}) \geq 1 . \quad (29)$$

By Lemma 3, \hat{u} solves $(\mathcal{P}_{\hat{\sigma}})$. Let ρ read as in (19) and $K_{\hat{\sigma}}$ be defined according to (17), p. 6. Noticing that

$$v \in K_{\hat{\sigma}}, \quad \hat{\sigma} \neq \emptyset \quad \Rightarrow \quad Av = A_{\hat{\sigma}} v_{\hat{\sigma}} , \quad (30)$$

Lemma 1(i) shows that

$$\begin{aligned} \left\{ \begin{array}{l} v \in B_{\infty}(0, \rho) \cap K_{\hat{\sigma}} , \\ v_{\hat{\sigma}} \in \ker(A_{\hat{\sigma}}) \end{array} \right. & \Rightarrow \mathcal{F}_d(\hat{u} + v) = \|A_{\hat{\sigma}}(\hat{u}_{\hat{\sigma}} + v_{\hat{\sigma}}) - d\|^2 + \beta \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i] + v[i]) \\ & \left[\text{by Lemma 1(i)} \right] = \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 + \beta \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i]) + \beta \sum_{i \in \hat{\sigma}^c} \phi(v[i]) \\ & \left[\text{by (25), p. 8} \right] = \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 + \beta \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i]) \\ & \left[\text{by (4), p. 2} \right] = \mathcal{F}_d(\hat{u}) , \end{aligned}$$

i. e. that \hat{u} is not a strict minimizer, which contradicts (i). Hence the assumption in (29) is false. Using (28), (ii) holds true.

If $\hat{u} = 0$, then $\hat{\sigma} = \emptyset$, hence $A_{\hat{\sigma}} \in \mathbb{R}^{M \times 0}$ and $\text{rank}(A_{\hat{\sigma}}) = 0 = \#\hat{\sigma}$ after (13).

Proof of (ii) \Rightarrow (i). Let \hat{u} be a minimizer of \mathcal{F}_d that meets (ii). To have $\#\hat{\sigma} = 0$ is equivalent to $\hat{u} = 0$. By Lemma 2, \hat{u} is a strict minimizer. Focus on $\#\hat{\sigma} \geq 1$. Since $\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma} \leq M$ and problem (\mathcal{Q}_ω) in (15), p. 6, is strictly convex for $\omega = \hat{\sigma}$, its unique solution $\hat{u}_{\hat{\sigma}}$ satisfies

$$v \in \mathbb{R}^{\#\hat{\sigma}} \setminus \{0\} \quad \Rightarrow \quad \|A_{\hat{\sigma}}(\hat{u}_{\hat{\sigma}} + v) - d\|^2 > \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 .$$

Using (30), p. 10, the latter is equivalent to

$$v \in K_{\hat{\sigma}} \setminus \{0\} \quad \Rightarrow \quad \|A(\hat{u} + v) - d\|^2 = \|A_{\hat{\sigma}}(\hat{u}_{\hat{\sigma}} + v_{\hat{\sigma}}) - d\|^2 > \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 = \|A\hat{u} - d\|^2 . \quad (31)$$

Lemma 1(i), along with (25), p. 8, yields

$$\begin{aligned} v \in B_\infty(0, \rho) \cap K_{\hat{\sigma}} \setminus \{0\} \quad \Rightarrow \quad \mathcal{F}_d(\hat{u} + v) &= \|A(u + v) - d\|^2 + \beta \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i]) \\ &\stackrel{[\text{by (31)}]}{>} \|A\hat{u} - d\|^2 + \beta \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i]) \\ &= \mathcal{F}_d(\hat{u}) . \end{aligned}$$

Since $\#\hat{\sigma} \leq M \leq N - 1$, we have $\hat{\sigma}^c \neq \emptyset$. So Lemma 1(ii) tells us that

$$v \in B_\infty(0, \rho) \setminus K_{\hat{\sigma}} \quad \Rightarrow \quad \mathcal{F}_d(\hat{u} + v) > \mathcal{F}_d(\hat{u}) .$$

Combining the last two implications proves (i).

Proof of (ii) \Leftrightarrow (iii). Comparing (iii) with Definitions 1 (p. 4) and 3 shows the statement.

Proof of equation (27). It follows from (26) in Corollary 1 (p. 9) where¹⁰ $A_{\hat{\sigma}}^T A_{\hat{\sigma}}$ is invertible. \square

Theorem 1 provides a simple rule enabling to verify if a numerical scheme has reached or not a strict (local) minimizer of \mathcal{F}_d .

The notations Ω_r , Ω and Ω_{\max} are frequently used in this paper. Their interpretation is obvious in the light of Theorem 1.

For any $d \in \mathbb{R}^M$ and $\forall \beta > 0$, the set Ω_{\max} is composed of the supports of all possible strict (local) minimizers of \mathcal{F}_d , while Ω is the subset of those ones that are $(M - 1)$ -sparse.

An easy and useful corollary is presented next.

Corollary 2 *Let $d \in \mathbb{R}^M$. Given an arbitrary $\omega \in \Omega_{\max}$, let \hat{u} solve (\mathcal{P}_ω) . Then*

(i) \hat{u} reads

$$\hat{u} = Z_\omega(\hat{u}_\omega) \quad \text{where} \quad \hat{u}_\omega = (A_\omega^T A_\omega)^{-1} A_\omega^T d , \quad (32)$$

and meets $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u}) \subseteq \omega$ and $\hat{\sigma} \in \Omega_{\max}$;

(ii) for any $\beta > 0$, \hat{u} is a strict (local) minimizer of \mathcal{F}_d ;

¹⁰For $\hat{\sigma} = \emptyset$, (11) and (13)(a) show that (27) yields $\hat{u} = 0$.

(iii) \hat{u} solves $(\mathcal{P}_{\hat{\sigma}})$.

Proof. Using (16) (see p. 6), \hat{u} meets (i) since $A_{\omega}^T A_{\omega}$ is invertible and $\sigma(\hat{u}) \subseteq \omega$ by the constraint in (\mathcal{P}_{ω}) . If $\hat{\sigma} = \emptyset$, (ii) follows from Lemma 2. For $\#\hat{\sigma} \geq 1$, $A_{\hat{\sigma}}$ is an $M \times \#\hat{\sigma}$ submatrix of A_{ω} . Since $\text{rank}(A_{\omega}) = \#\omega$, we have $\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma}$ and so $\hat{\sigma} \in \Omega_{\max}$. By Proposition 1 \hat{u} is a (local) minimizer of \mathcal{F}_d and Theorem 1 leads to (ii). Lemma 3 and Corollary 2(ii) yield (iii). \square

The following consequence of Corollary 2 might be striking:

One can easily compute a strict (local) minimizer \hat{u} of \mathcal{F}_d without knowing the value of the regularization parameter β . Just data d and an $\omega \in \Omega_{\max}$ are needed.

Clearly, the support $\sigma(\hat{u})$ of a nonstrict local minimizer \bar{u} of \mathcal{F}_d contains some sub-supports yielding strict (local) minimizers of. It is easy to see that among them, there is $\hat{\sigma} \subsetneq \sigma(\bar{u})$ such that the corresponding \hat{u} given by (27) strictly decreases the value of \mathcal{F}_d , i.e. $\mathcal{F}_d(\hat{u}) < \mathcal{F}_d(\bar{u})$.

3.2 Every strict (local) minimizer of \mathcal{F}_d is linear in d

Here we explore the behavior of the strict (local) minimizers of \mathcal{F}_d with respect to variations of d . An interesting sequel of Theorem 1 is presented below.

Corollary 3 For $d \in \mathbb{R}^M$ and $\beta > 0$, let \hat{u} be a (local) minimizer of \mathcal{F}_d satisfying $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u}) \in \Omega$. Define $N_{\hat{\sigma}} = \mathbb{R}^M$ and for $\hat{\sigma} \neq \emptyset$,

$$N_{\hat{\sigma}} \stackrel{\text{def}}{=} \ker(A_{\hat{\sigma}}^T) \subset \mathbb{R}^M.$$

We have $\dim N_{\hat{\sigma}} = M - \#\hat{\sigma} \geq 1$ and

$$d' \in N_{\hat{\sigma}} \Rightarrow \mathcal{F}_{d+d'} \text{ has a strict (local) minimum at } \hat{u}.$$

Proof. Since $\hat{\sigma} \in \Omega$, the minimizer \hat{u} is strict by Theorem 1. If $\hat{u} = 0$, the claim follows from Lemma 2. Let $\hat{\sigma} \neq \emptyset$. By $d' \in \ker(A_{\hat{\sigma}}^T)$ we find $A_{\hat{\sigma}}^T(d + d') = A_{\hat{\sigma}}^T d$ for any $d' \in N_{\hat{\sigma}}$. Inserting this into (27) in Theorem 1 yields the result. \square

All data living in the vector subspace $N_{\hat{\sigma}} \supseteq \{0\}$ yield the same strict (local) minimizer \hat{u} .

Remark 1 If data contain some noise n , it can be decomposed in a unique way as $n = n_{N_{\hat{\sigma}}} + n_{N_{\hat{\sigma}}^{\perp}}$ where $n_{N_{\hat{\sigma}}} \in N_{\hat{\sigma}}$ and $n_{N_{\hat{\sigma}}^{\perp}} \in N_{\hat{\sigma}}^{\perp}$. By Corollary 3, the component $n_{N_{\hat{\sigma}}}$ will be removed.

We shall use the definition given below.

Definition 4 Let $\mathcal{O} \subseteq \mathbb{R}^M$ be an open domain. We say that $\mathcal{U} : \mathcal{O} \rightarrow \mathbb{R}^N$ is a local minimizer function for the family of objectives $\mathfrak{F}_{\mathcal{O}} \stackrel{\text{def}}{=} \{\mathcal{F}_d : d \in \mathcal{O}\}$ if for any $d \in \mathcal{O}$, the function \mathcal{F}_d reaches a strict (local) minimum at $\mathcal{U}(d)$.

Corollary 2 shows that for any $d \in \mathbb{R}^M$, each strict (local) minimizer of \mathcal{F}_d is entirely described by an $\omega \in \Omega_{\max}$ via equation (32) in the same corollary.

Consequently, a local minimizer function \mathcal{U} is associated with every $\omega \in \Omega_{\max}$.

Lemma 4 For some arbitrarily fixed $\omega \in \Omega_{\max}$ and $\beta > 0$, the family of functions $\mathfrak{F}_{\mathbb{R}^M}$ has a linear (local) minimizer function $\mathcal{U} : \mathbb{R}^M \rightarrow \mathbb{R}^N$ that reads

$$\forall d \in \mathbb{R}^M, \quad \mathcal{U}(d) = Z_\omega (U_\omega d) \quad \text{where} \quad U_\omega = (A_\omega^T A_\omega)^{-1} A_\omega^T \in \mathbb{R}^{\#\omega \times M}. \quad (33)$$

Proof. Clearly, the function \mathcal{U} in (33) is linear in d . From Corollary 2 (p. 11), for any $\beta > 0$ and for any $d \in \mathbb{R}^M$, \mathcal{F}_d has a strict (local) minimum at $\hat{u} = \mathcal{U}(d)$. Hence \mathcal{U} fits Definition 4. \square

Even if \mathcal{F}_d has many strict (local) minimizers, each one of them is linear in d .

Next we exhibit a negligible subset of \mathbb{R}^M , associated with a nonempty $\omega \in \Omega_{\max}$, whose elements are data d leading to $\|\mathcal{U}(d)\|_0 < \#\omega$.

Lemma 5 For any $\omega \in \Omega_{\max}$, define the subset $D_\omega \subset \mathbb{R}^M$ by

$$D_\omega \stackrel{\text{def}}{=} \bigcup_{i=1}^{\#\omega} \left\{ g \in \mathbb{R}^M : \left\langle e_i, (A_\omega^T A_\omega)^{-1} A_\omega^T g \right\rangle = 0 \right\}. \quad (34)$$

Then D_ω is closed in \mathbb{R}^M and $\mathbb{L}^M(D_\omega) = 0$.

Proof. If $\omega = \emptyset$ then $D_\omega = \emptyset$, hence the claim. Let $\#\omega \geq 1$. For some $i \in \mathbb{I}_{\#\omega}$, set

$$\begin{aligned} D &\stackrel{\text{def}}{=} \left\{ g \in \mathbb{R}^M : \left\langle e_i, (A_\omega^T A_\omega)^{-1} A_\omega^T g \right\rangle = 0 \right\} \\ &= \left\{ g \in \mathbb{R}^M : \left\langle A_\omega (A_\omega^T A_\omega)^{-1} e_i, g \right\rangle = 0 \right\}. \end{aligned}$$

Since $\text{rank} \left(A_\omega (A_\omega^T A_\omega)^{-1} \right) = \#\omega$, then $\ker \left(A_\omega (A_\omega^T A_\omega)^{-1} \right) = \{0\}$. Therefore $A_\omega (A_\omega^T A_\omega)^{-1} e_i \neq 0$. Hence D is a vector subspace of \mathbb{R}^M of dimension $M - 1$ and so $\mathbb{L}^M(D) = 0$. The conclusion follows from the fact that D_ω in (34) is the union of $\#\omega$ subsets like D (see e. g. [35, 13]). \square

Proposition 2 For some arbitrarily fixed $\omega \in \Omega_{\max}$ and $\beta > 0$, let $\mathcal{U} : \mathbb{R}^M \rightarrow \mathbb{R}^N$ be the relevant (local) minimizer function for $\mathfrak{F}_{\mathbb{R}^M}$ as given in (33) (Lemma 4). Let D_ω read as in (34). Then

- (i) the set $\mathbb{R}^M \setminus D_\omega$ contains an open and dense subset of \mathbb{R}^M ;
- (ii) the function $d \mapsto \mathcal{F}_d(\mathcal{U}(d))$ is \mathcal{C}^∞ on $\mathbb{R}^M \setminus D_\omega$ and

$$d \in \mathbb{R}^M \setminus D_\omega \quad \Rightarrow \quad \sigma(\mathcal{U}(d)) = \omega.$$

Proof. Statement (i) is a straightforward consequence of Lemma 5.

If $\omega = \emptyset$, then $\mathcal{U}(d) = 0$, $\forall d \in \mathbb{R}^M$, so (ii) is trivially satisfied. Consider that $\#\omega \geq 1$. For any $i \in \mathbb{I}_{\#\omega}$, the $\omega[i]$ th component of $\mathcal{U}(d)$ reads (see Lemma 4)

$$\mathcal{U}_{\omega[i]}(d) = \left\langle e_i, (A_\omega^T A_\omega)^{-1} A_\omega^T d \right\rangle.$$

The definition of D_ω shows that

$$d \in \mathbb{R}^M \setminus D_\omega \quad \text{and} \quad i \in \mathbb{I}_{\#\omega} \quad \Rightarrow \quad \mathcal{U}_{\omega[i]}(d) \neq 0,$$

whereas $\mathcal{U}_i(d) = 0$ for all $i \in \omega^c$. Consequently,

$$\omega \in \Omega_{\max} \quad \text{and} \quad d \in \mathbb{R}^M \setminus D_\omega \quad \Rightarrow \quad \sigma(\mathcal{U}(d)) = \omega .$$

Then

$$\omega \in \Omega_{\max} \quad \text{and} \quad d \in \mathbb{R}^M \setminus D_\omega \quad \Rightarrow \quad \mathcal{F}_d(\mathcal{U}(d)) = \|A\mathcal{U}(d) - d\|^2 + \beta \#\omega .$$

The map $d \mapsto \|A\mathcal{U}(d) - d\|^2$ is \mathcal{C}^∞ on \mathbb{R}^M —see (33)—hence (ii). \square

Generically, a local minimizer function produces solutions sharing the same support.

The application $d \mapsto \mathcal{F}_d(\mathcal{U}(d))$ is discontinuous on the *negligible* subset D_ω where the support of $\mathcal{U}(d)$ is shrunk.

3.3 Strict minimizers with an M-length support

Proposition 3 *Let $\text{rank}(A) = M$, $\beta > 0$ and*

$$U_M \stackrel{\text{def}}{=} \{ \hat{u} \in \mathbb{R}^N : \hat{u} \text{ is a strict (local) minimizer of } \mathcal{F}_d \text{ meeting } \|\hat{u}\|_0 = M \} .$$

Put

$$Q_M \stackrel{\text{def}}{=} \bigcup_{\omega \in \Omega_M} \bigcup_{i \in \mathbb{I}_M} \{ g \in \mathbb{R}^M : \langle e_i, A_\omega^{-1} g \rangle = 0 \} . \quad (35)$$

Then $\mathbb{R}^M \setminus Q_M$ contains an open and dense subset of \mathbb{R}^M and

$$d \in \mathbb{R}^M \setminus Q_M \quad \Rightarrow \quad \#\mathcal{U}_M = \#\Omega_M \quad \text{and} \quad \mathcal{F}_d(\hat{u}) = \beta M, \quad \forall \hat{u} \in U_M .$$

Proof. Note that

$$Q_M = \bigcup_{\omega \in \Omega_M} D_\omega ,$$

where D_ω is given in (34). Proposition 2(i) entails that $\mathbb{R}^M \setminus Q_M$ contains an open and dense subset of \mathbb{R}^M . Since $\text{rank}(A) = M$, we have $\#\Omega_M \geq 1$. By Proposition 2(ii)

$$d \in \mathbb{R}^M \setminus Q_M \quad \text{and} \quad \omega \in \Omega_M \quad \Rightarrow \quad \mathcal{F}_d \text{ has a strict (local) minimizer } \hat{u} \text{ obeying } \sigma(\hat{u}) = \omega .$$

Hence $\hat{u} \in U_M$ and $\hat{u}_\omega = A_\omega^{-1} d$, so $\|A\hat{u} - d\|^2 = 0$ and $\mathcal{F}_d(\hat{u}) = \beta M$.

If $\#\Omega_M = 1$, the proof is done. Let $\#\Omega_M \geq 2$. If $d \in \mathbb{R}^M \setminus Q_M$ and $\varpi \in \Omega_M \setminus \{\omega\}$, then \mathcal{F}_d has a strict (local) minimizer \hat{u}' meeting $\sigma(\hat{u}') = \varpi$, hence $\hat{u}' \neq \hat{u}$, $\hat{u}' \in U_M$ and $\mathcal{F}_d(\hat{u}') = \beta M$. \square

For any $\beta > 0$, \mathcal{F}_d generically has $\#\Omega_M$ strict minimizers \hat{u} obeying $\|\hat{u}\|_0 = M$ and $\mathcal{F}_d(\hat{u}) = \beta M$. This disables us to discriminate between all these strict minimizers.

Hence the interest in minimizers with supports living in Ω , i. e. strict $(M - 1)$ -sparse minimizers of \mathcal{F}_d .

4 On the global minimizers of \mathcal{F}_d

The statement below gives a *necessary condition* for a global minimizer of \mathcal{F}_d . It follows directly from [30, Proposition 3.4] where¹¹ the regularization term is $\|Du\|_0$ for a full row rank matrix D . For \mathcal{F}_d in (1) with $\|a_i\|_2 = 1, \forall i \in \mathbb{I}_N$, a simpler condition was derived later on in [37, Theorem 12], using different tools. For completeness, the proof for a general A is outlined in Appendix 8.2 on p. 27.

Proposition 4 For $d \in \mathbb{R}^M$ and $\beta > 0$, let \mathcal{F}_d have a global minimum at \hat{u} . Then¹²

$$i \in \sigma(\hat{u}) \quad \Rightarrow \quad |\hat{u}[i]| \geq \frac{\sqrt{\beta}}{\|a_i\|}. \quad (36)$$

Observe that the lower bound on $\{|\hat{u}[i]| : i \in \sigma(\hat{u})\}$ given in (36) is independent of d . This means that in general, (36) provides a pessimistic bound.

The proof of the proposition shows that (36) is met also by all (local) minimizers of \mathcal{F}_d satisfying

$$\mathcal{F}_d(\hat{u}) \leq \mathcal{F}_d(\hat{u} + \rho e_i), \quad \forall \rho \in \mathbb{R}, \quad \forall i \in \mathbb{I}_N.$$

4.1 The global minimizers of \mathcal{F}_d are strict

Remark 2 Let $d \in \mathbb{R}^M$ and $\beta > \|d\|^2$. Then \mathcal{F}_d has a strict global minimum at $\hat{u} = 0$. Indeed,

$$v \neq 0 \quad \Rightarrow \quad \|v\|_0 \geq 1 \quad \Rightarrow \quad \mathcal{F}_d(0) = \|d\|^2 < \beta \leq \|Av - d\|^2 + \beta\|v\|_0.$$

For least-squares regularized with a more regular ϕ , one usually gets $\hat{u} = 0$ asymptotically as $\beta \rightarrow +\infty$ but $\hat{u} \neq 0$ for finite values of β . The latter does not hold for \mathcal{F}_d (Remark 2).

Some theoretical results on the global minimizers of \mathcal{F}_d were obtained by the past [30, 19, 37, 6]. May be their existence seemed “offered” and the question has never been examined.

Theorem 2 Let $d \in \mathbb{R}^M$ and $\beta > 0$. Then

(i) the set of all global minimizers of \mathcal{F}_d

$$\hat{U} \stackrel{\text{def}}{=} \left\{ \hat{u} \in \mathbb{R}^N : \hat{u} = \min_{u \in \mathbb{R}^N} \mathcal{F}_d(u) \right\}$$

is nonempty;

(ii) every $\hat{u} \in \hat{U}$ is a strict minimizer of \mathcal{F}_d , i. e.

$$\sigma(\hat{u}) \in \Omega_{\max},$$

hence $\|\hat{u}\|_0 \leq M$.

The proof of the theorem is outlined in Appendix 8.3, p. 29.

One can note that if $\text{rank}(A) = M$, any global minimizer \hat{u} of \mathcal{F}_d obeys

$$\mathcal{F}_d(\hat{u}) \leq \beta M.$$

According to Theorem 2, the global minimizers of \mathcal{F}_d are strict and their number is finite: this is a nice property that fails for many convex nonsmooth optimization problems.

¹¹Just set $g_i = e_i, P = I_M$ and $H = I_N$ in [30, Proposition 3.4].

¹²Remember that $a_i \neq 0$ for all $i \in \mathbb{I}_N$ and that $\|\cdot\| = \|\cdot\|_2$.

4.2 Can we get K -sparse global minimizers for $K \leq M - 1$?

In order to simplify the presentation, in what follows we consider that

$$\text{rank}(A) = M < N .$$

Since \mathcal{F}_d has a large number (typically equal to $\#\Omega_M$) of strict minimizers with $\|\hat{u}\|_0 = M$ yielding the same value $\mathcal{F}_d(\hat{u}) = \beta M$ (see Proposition 3 on p. 14 and the comments next to it), it is important to get sure that the global minimizers of \mathcal{F}_d are $(M - 1)$ -sparse.

We introduce a notation which is used in the rest of this article. For any $K \in \mathbb{I}_{M-1}$, put

$$\bar{\Omega}_K \stackrel{\text{def}}{=} \bigcup_{r=0}^K \Omega_r , \quad (37)$$

where Ω_r read according to Definition 3 (p. 9). Theorem 1 gives a clear meaning of the sets¹³ $\bar{\Omega}_K$.

For any $d \in \mathbb{R}^M$ and any $\beta > 0$, for any fixed $K \in \mathbb{I}_{M-1}$, the set $\bar{\Omega}_K$ is composed of the supports of all possible K -sparse strict (local) minimizers of \mathcal{F}_d .

The next propositions checks the existence of $\beta > 0$ ensuring that all the global minimizers of \mathcal{F}_d are K -sparse, for some $K \in \mathbb{I}_{M-1}$.

Proposition 5 *Let $d \in \mathbb{R}^M$. For any $K \in \mathbb{I}_{M-1}$, there exists $\beta_K \geq 0$ such that if $\beta > \beta_K$, then each global minimizer \hat{u} of \mathcal{F}_d meets*

$$\|\hat{u}\|_0 \leq K \quad \text{and} \quad \sigma(\hat{u}) \in \bar{\Omega}_K . \quad (38)$$

One can choose $\beta_K = \|A\tilde{u} - d\|^2$ where \tilde{u} solves (\mathcal{P}_ω) for some $\omega \in \Omega_K$.

The proof is given in Appendix 8.4 (p. 30). The value of β_K in the statement is easy to compute but it is not sharp¹⁴.

5 Uniqueness of the global minimizer of \mathcal{F}_d

The presentation is simplified using the notation introduced next.

Given a matrix $A \in \mathbb{R}^{M \times N}$, with any $\omega \in \Omega$ (see Definition 3), we associate the $M \times M$ matrix Π_ω that yields the orthogonal projection¹⁵ onto the subspace spanned by the columns of A_ω :

$$\Pi_\omega \stackrel{\text{def}}{=} A_\omega (A_\omega^T A_\omega)^{-1} A_\omega^T . \quad (39)$$

¹³Clearly, $\bar{\Omega}_{M-1} = \Omega$.

¹⁴For $\beta \gtrsim \beta_K$ the global minimizers of \mathcal{F}_d might be k -sparse for $k \ll K$. A sharp β_K can be obtained as

$$\beta_K = \min_{\omega \in \Omega_K} \{ \|A\tilde{u} - d\|^2 : \tilde{u} \text{ solve } (\mathcal{P}_\omega) \text{ for } \omega \in \Omega_K \} .$$

¹⁵If $\omega = \emptyset$, we have $A_\omega \in \mathbb{R}^{M \times 0}$ and so Π_\emptyset is an $M \times M$ null matrix.

Checking if a global minimizer \hat{u} of \mathcal{F}_d is unique requires to compare its value $\mathcal{F}_d(\hat{u})$ with the values $\mathcal{F}_d(\bar{u})$ of the concurrent strict minimizers \bar{u} . Let \hat{u} be an $(M-1)$ -sparse strict (local) minimizer of \mathcal{F}_d . Then $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u}) \in \Omega$. Using (27) in Theorem 1 (p. 10) and (39) shows that

$$\begin{aligned} \mathcal{F}_d(\hat{u}) &= \|A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} - d\|^2 + \beta \# \hat{\sigma} \\ &= \|A_{\hat{\sigma}} (A_{\hat{\sigma}}^T A_{\hat{\sigma}})^{-1} A_{\hat{\sigma}}^T d - d\|^2 + \beta \# \hat{\sigma} \\ &= \|(\mathbb{I}_M - \Pi_{\hat{\sigma}})d\|^2 + \beta \# \hat{\sigma} \\ &= d^T (I_M - \Pi_{\hat{\sigma}}) d + \beta \# \hat{\sigma} . \end{aligned} \quad (40)$$

Let \bar{u} be another $(M-1)$ -sparse strict minimizer of \mathcal{F}_d , $\bar{u} \neq \hat{u}$, and set $\bar{\sigma} \stackrel{\text{def}}{=} \sigma(\bar{u})$. Then

$$\mathcal{F}_d(\hat{u}) - \mathcal{F}_d(\bar{u}) = d^T (\Pi_{\bar{\sigma}} - \Pi_{\hat{\sigma}}) d + \beta (\# \hat{\sigma} - \# \bar{\sigma}) .$$

If both \hat{u} and \bar{u} are global minimizers of \mathcal{F}_d , the equality given above yields

$$\mathcal{F}_d(\hat{u}) = \mathcal{F}_d(\bar{u}) \quad \Leftrightarrow \quad d^T (\Pi_{\bar{\sigma}} - \Pi_{\hat{\sigma}}) d = -\beta (\# \hat{\sigma} - \# \bar{\sigma}) . \quad (41)$$

Equation (41) reveals that the uniqueness of the global minimizer of \mathcal{F}_d cannot be guaranteed without suitable assumptions on A and on d .

5.1 A generically true assumption on A

We adopt an assumption on the matrix A in \mathcal{F}_d in order to restrict the cases when (41) takes place for some supports $\hat{\sigma} \neq \bar{\sigma}$ meeting $\# \hat{\sigma} = \# \bar{\sigma}$.

H1 *The matrix $A \in \mathbb{R}^{M \times N}$ is such that for some given $K \in \mathbb{I}_{M-1}$,*

$$r \in \mathbb{I}_K \quad \text{and} \quad (\omega, \varpi) \in (\Omega_r \times \Omega_r), \quad \omega \neq \varpi \quad \Rightarrow \quad \Pi_\omega \neq \Pi_\varpi . \quad (42)$$

Assumption H1 means that the angle (or the gap) between the equidimensional subspaces $\text{Range}(A_\omega)$ and $\text{Range}(A_\varpi)$ must be nonzero [27]. For instance, if $(i, j) \in \mathbb{I}_N \times \mathbb{I}_N$ meet $i \neq j$, H1 implies that $a_i \neq \kappa a_j$ for any $\kappa \in \mathbb{R} \setminus \{0\}$ since $\Pi_{\{i\}} = a_i a_i^T / \|a_i\|^2$.

Checking if H1 holds for a given matrix A requires a combinatorial search over all possible couples $(\varpi, \omega) \in (\Omega_r \times \Omega_r)$ meeting $\varpi \neq \omega, \forall r \in \mathbb{I}_K$. This is hard to do. Instead, we wish to know whether or not H1 is a *practical* limitation. Using some auxiliary claims, we shall show that H1 *fails to hold only for a negligible subset of matrices* in the space of all $M \times N$ real matrices.

Lemma 6 *Given $r \in \mathbb{I}_{M-1}$ and $\varpi \in \Omega_r$, define the following set of submatrices of A :*

$$\mathcal{H}_\varpi = \left\{ A_\omega : \omega \in \Omega_r \quad \text{and} \quad \Pi_\omega = \Pi_\varpi \right\} .$$

Then \mathcal{H}_ϖ belongs to an $(r \times r)$ -dimensional subspace of the space of all $M \times r$ matrices.

Proof. Using that $\varpi \in \Omega_r$ and $\omega \in \Omega_r$, we have¹⁶

$$\Pi_\omega = \Pi_\varpi \quad \Leftrightarrow \quad A_\omega = \Pi_\varpi A_\omega . \quad (43)$$

¹⁶Using (39) (p. 16), as well as the fact that $A_\omega = \Pi_\varpi A_\omega, \forall \omega \in \Omega$, one easily derives that

$$\Pi_\omega = \Pi_\varpi \quad \Leftrightarrow \quad \begin{cases} A_\omega (A_\omega^T A_\omega)^{-1} A_\omega^T = \Pi_\varpi \\ A_\varpi (A_\varpi^T A_\varpi)^{-1} A_\varpi^T = \Pi_\omega \end{cases} \Rightarrow \begin{cases} A_\omega = \Pi_\varpi A_\omega \\ A_\varpi = \Pi_\omega A_\varpi \end{cases} \Rightarrow \begin{cases} \Pi_\omega = \Pi_\varpi \Pi_\omega \\ \Pi_\varpi = \Pi_\omega \Pi_\varpi \end{cases} \Rightarrow \Pi_\omega = \Pi_\varpi .$$

Therefore \mathcal{H}_ϖ equivalently reads

$$\mathcal{H}_\varpi = \left\{ A_\omega : \omega \in \Omega_r \text{ and } A_\omega = \Pi_\varpi A_\omega \right\}. \quad (44)$$

Let $A_\omega \in \mathcal{H}_\varpi$. Denote the columns of A_ω by \tilde{a}_i for $i \in \mathbb{I}_r$. Then (44) yields

$$\Pi_\varpi \tilde{a}_i = \tilde{a}_i, \quad \forall i \in \mathbb{I}_r \quad \Rightarrow \quad \tilde{a}_i \in \text{Range}(A_\varpi), \quad \forall i \in \mathbb{I}_r.$$

Hence all \tilde{a}_i , $i \in \mathbb{I}_r$, live in the r -dimensional vector subspace $\text{Range}(A_\varpi)$. All the columns of each matrix $A_\omega \in \mathcal{H}_\varpi$ belong to $\text{Range}(A_\varpi)$ as well. It follows that \mathcal{H}_ϖ belongs to a (closed) subspace of dimension $r \times r$ in the space of all $M \times r$ matrices, where $r \leq M - 1$. \square

More precisions on the submatrices of A living in \mathcal{H}_ϖ are given next.

Remark 3 *The negligible subset \mathcal{H}_ϖ in Lemma 6 is formed from all the submatrices of A that are column equivalent to A_ϖ (see [27, p. 171]), that is*

$$A_\omega \in \mathcal{H}_\varpi \quad \Leftrightarrow \quad \exists P \in \mathbb{R}^{r \times r} \text{ meeting } \text{rank}(P) = r \text{ and } A_\omega = A_\varpi P. \quad (45)$$

Observe that P has r^2 unknowns that must satisfy Mr equations and that P must be invertible. Indeed, it should be quite exceptional that such a matrix P does exist.

The latter remark can help to discern structured dictionaries that meet or not H1.

Let us mention a possibly constructive point. If A is such that for a couple $(\omega, \varpi) \in \Omega_r \times \Omega_r$, $\omega \neq \varpi$, $r \in \mathbb{I}_{M-1}$, we have $\Pi_\varpi = \Pi_\omega$, the *non-equality* required in H1 can be reached by choosing an $i \in \omega \setminus (\omega \cap \varpi)$ and replacing the i th column a_i of A by \tilde{a}_i given by

$$\tilde{a}_i = a_i + v \quad \text{for } v \in \ker(A_\varpi^T) \setminus \{0\}.$$

Indeed, using that $\ker(\Pi_\varpi) = \ker(A_\varpi^T)$, one obtains

$$\Pi_\varpi \tilde{a}_i = \Pi_\varpi(a_i + v) = \Pi_\varpi a_i = a_i \neq \tilde{a}_i.$$

By (43), the new matrix (with \tilde{a}_i in place of a_i) meets $\Pi_\varpi \neq \Pi_\omega$.

Next we inspect the set of all matrices A failing assumption H1.

Lemma 7 *Consider the set \mathcal{H} formed from $M \times N$ real matrices described next:*

$$\mathcal{H} \stackrel{\text{def}}{=} \left\{ A \in \mathbb{R}^{M \times N} : \exists r \in \mathbb{I}_{M-1}, \exists (\varpi, \omega) \in \Omega_r \times \Omega_r, \varpi \neq \omega \text{ and } \Pi_\varpi = \Pi_\omega \right\}.$$

Then \mathcal{H} belongs to a finite union vector subspaces in $\mathbb{R}^{M \times N}$ whose Lebesgue measure in the latter space is null.

Proof. Let $A \in \mathcal{H}$. Then there exists at least one integer $r \in \mathbb{I}_{M-1}$ and at least one pair $(\varpi, \omega) \in \Omega_r \times \Omega_r$ such that $\varpi \neq \omega$ and $\Pi_\varpi = \Pi_\omega$. Using Lemma 6, A contains (at least) one $M \times r$ submatrix A_ϖ belonging to an $r \times r$ vector subspace in the space of all $M \times r$ real matrices. Identifying A with an MN -length vector, its entries are included in a vector subspace of \mathbb{R}^{MN} of dimension no larger than $MN - 1$. The claim of the lemma is straightforward. \square

We can now pleasantly clarify assumption H1.

Theorem 3 Given an arbitrary $K \in \mathbb{I}_{M-1}$, consider the set of $M \times N$ real matrices below

$$\mathcal{A}_K \stackrel{\text{def}}{=} \left\{ A \in \mathbb{R}^{M \times N} : A \text{ satisfies H1 for } K \right\} .$$

Then \mathcal{A}_K contains an open and dense subset in the space of all $M \times N$ real-valued matrices.

Proof. The complement of \mathcal{A}_K in the space of all $M \times N$ real matrices reads

$$\mathcal{A}_K^c = \left\{ A \in \mathbb{R}^{M \times N} : \text{H1 fails for } A \text{ and } K \right\} .$$

It is clear that $\mathcal{A}_K^c \subset \mathcal{H}$, where \mathcal{H} is described in Lemma 7. By the same lemma, \mathcal{A}_K^c is included in a closed subset of vector subspaces in $\mathbb{R}^{M \times N}$ whose Lebesgue measure in the latter space is null. Consequently, \mathcal{A}_K fits the statement. \square

For any $K \in \mathbb{I}_{M-1}$, H1 is a generic property of all $M \times N$ real matrices meeting $M < N$.

We can note that

$$\mathcal{A}_{K+1} \subseteq \mathcal{A}_K, \quad \forall K \in \mathbb{I}_{M-2} .$$

One can hence presume that H1 is weakened as K decreases. This issue is illustrated in section 6.

5.2 A generically true assumption on d

A preliminary result, proven in Appendix 8.5 (p. 31), is stated next.

Lemma 8 Let $(\omega, \varpi) \in \overline{\Omega}_K \times \overline{\Omega}_K$ for $\omega \neq \varpi$ and H1 hold for $K \in \mathbb{I}_{M-1}$. Given $\kappa \in \mathbb{R}$, define

$$\mathbb{T}_\kappa \stackrel{\text{def}}{=} \left\{ g \in \mathbb{R}^M : g^T (\Pi_\omega - \Pi_\varpi) g = \kappa \right\} .$$

Then \mathbb{T}_κ is a closed subset of \mathbb{R}^M and $\mathbb{L}^M(\mathbb{T}_\kappa) = 0$.

More precisely, it is shown that \mathbb{T}_κ are elliptical hyperboloid or conical (for $\kappa = 0$) surfaces of dimension $\leq M - 1$. The proof of the lemma includes Lemma 9 devoted to the eigenvalues of a difference of projection matrices, stated and proven on p. 32.

We exhibit a negligible subset of data in \mathbb{R}^M that can still satisfy the equality in (41), p. 17.

Proposition 6 For $\beta > 0$ and $K \in \mathbb{I}_{M-1}$, put

$$\Sigma_K \stackrel{\text{def}}{=} \bigcup_{n=-K}^K \bigcup_{\omega \in \overline{\Omega}_K} \bigcup_{\varpi \in \overline{\Omega}_K} \left\{ g \in \mathbb{R}^M : \omega \neq \varpi \text{ and } g^T (\Pi_\omega - \Pi_\varpi) g = n\beta \right\} , \quad (46)$$

where $\overline{\Omega}_K$ is given in (37), p. 16. Let H1 hold for K . Then Σ_K is closed in \mathbb{R}^M and $\mathbb{L}^M(\Sigma_K) = 0$.

Proof. For some $n \in \{-K, \dots, K\}$ and $(\omega, \varpi) \in (\overline{\Omega}_K \times \overline{\Omega}_K)$ meeting $\omega \neq \varpi$, put

$$\Sigma \stackrel{\text{def}}{=} \left\{ g \in \mathbb{R}^M : g^T (\Pi_\omega - \Pi_\varpi) g = n\beta \right\} .$$

If $\# \omega \neq \# \varpi$, then $\text{rank}(\Pi_\omega - \Pi_\varpi) \geq 1$. If $\# \omega = \# \varpi$, H1 guarantees that $\text{rank}(\Pi_\omega - \Pi_\varpi) \geq 1$, yet again. The number $n\beta \in \mathbb{R}$ is given. According to Lemma 8, Σ is a closed subset of \mathbb{R}^M and $\mathbb{L}^M(\Sigma) = 0$. The conclusion follows from the fact that Σ_K is a finite union of subsets like Σ . \square

We assume thereafter that if H1 holds for some $K \in \mathbb{I}_{M-1}$, data d satisfy

$$d \in \{g \in \mathbb{R}^M : g \notin \Sigma_K\} = \mathbb{R}^M \setminus \Sigma_K .$$

5.3 The unique global minimizer of \mathcal{F}_d is K -sparse for $K \leq (M - 1)$

We are looking for guarantees that \mathcal{F}_d has a *unique* global minimizer \hat{u} obeying

$$\|\hat{u}\|_0 \leq K \text{ for some fixed } K \in \mathbb{I}_{M-1} .$$

This is the aim of the next theorem.

Theorem 4 *Given $K \in \mathbb{I}_{M-1}$, let H1 hold for K , $\beta > \beta_K$ where β_K meets Proposition 5 (p. 16) and $\Sigma_K \subset \mathbb{R}^M$ read as in (46). Consider that*

$$d \in \mathbb{R}^M \setminus \Sigma_K .$$

Then

- (i) data d belong to an open and dense subset of \mathbb{R}^M ;
- (ii) \mathcal{F}_d has a unique global minimizer \hat{u} satisfying $\|\hat{u}\|_0 \leq K$.

Proof. The set Σ_K being negligible in \mathbb{R}^M (Proposition 6) entails (ii).

Since $\beta > \beta_K$, all global minimizers of \mathcal{F}_d have their support in $\bar{\Omega}_K$ (Proposition 5). Using that data meet $d \in \mathbb{R}^M \setminus \Sigma_K$, the definition of Σ_K in (46) shows that

$$-K \leq n \leq K \text{ and } (\omega, \varpi) \in (\bar{\Omega}_K \times \bar{\Omega}_K), \omega \neq \varpi \Rightarrow d^T (\Pi_\omega - \Pi_\varpi) d \neq n\beta . \quad (47)$$

The proof is conducted by contradiction. Let \hat{u} and $\bar{u} \neq \hat{u}$ be two global minimizers of \mathcal{F}_d . Then

$$\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u}) \in \bar{\Omega}_K \text{ and } \bar{\sigma} \stackrel{\text{def}}{=} \sigma(\bar{u}) \in \bar{\Omega}_K ,$$

and $\hat{\sigma} \neq \bar{\sigma}$. By $\mathcal{F}_d(\hat{u}) = \mathcal{F}_d(\bar{u})$, equation (41) (p. 17) reads

$$d^T (\Pi_{\hat{\sigma}} - \Pi_{\bar{\sigma}}) d = -\beta (\#\hat{\sigma} - \#\bar{\sigma}) . \quad (48)$$

An enumeration of all possible values of $\#\hat{\sigma} - \#\bar{\sigma}$ shows that

$$\beta (\#\hat{\sigma} - \#\bar{\sigma}) = n\beta \text{ for some } n \in \{-K, \dots, K\} .$$

Inserting this equation into (48) shows that

$$d^T (\Pi_{\hat{\sigma}} - \Pi_{\bar{\sigma}}) d = n\beta \text{ for some } n \in \{-K, \dots, K\} .$$

The last result contradicts (47), hence it violates the assumptions H1 and that $d \in \mathbb{R}^M \setminus \Sigma_K$. Consequently, \mathcal{F}_d cannot have two global minimizers. Since \mathcal{F}_d has always global minimizers (Theorem 2(i), p. 15), it follows that \mathcal{F}_d has a unique global minimizer. \square

For $\beta > \beta_K$, the objective \mathcal{F}_d in (1) has a unique K -sparse global minimizer where $K \leq M - 1$. For all $K \in \mathbb{I}_{M-1}$, the claim holds true in a generic sense.

The proof of Theorem 4 shows a wider result: if H1 holds for some $K \in \mathbb{I}_{M-1}$, $\beta > \beta_K$ and $d \notin \Sigma_K$, any two *distinct* K -sparse strict (local) minimizers \hat{u} and \bar{u} of \mathcal{F}_d satisfy $\mathcal{F}_d(\hat{u}) \neq \mathcal{F}_d(\bar{u})$.

6 Numerical illustrations

6.1 On assumption H1

Assumption H1 (p. 17) requires that $\Pi_\omega \neq \Pi_\varpi$ when $(\omega, \varpi) \in \Omega_r \times \Omega_r$, $\omega \neq \varpi$ for all $r \leq K \in \mathbb{I}_{M-1}$. From a practical viewpoint, the magnitude of $(\Pi_\omega - \Pi_\varpi)$ should be discernible. One way to assess the viability of H1 for a matrix A and $K \in \mathbb{I}_{M-1}$ is to calculate

$$\xi_K(A) \stackrel{\text{def}}{=} \min_{r \in \mathbb{I}_K} \mu_r(A), \quad (49)$$

where

$$\mu_r(A) = \min_{\substack{(\omega, \varpi) \in \Omega_r \times \Omega_r \\ \omega \neq \varpi}} \|\Pi_\omega - \Pi_\varpi\|_2, \quad \forall r \in \mathbb{I}_K.$$

In fact, $\|\Pi_\omega - \Pi_\varpi\|_2 = \sin(\theta)$ where $\theta \in [0, \pi/2]$ is the maximum angle between $\text{Range}(A_\omega)$ and $\text{Range}(A_\varpi)$, see [27, p. 456]. The latter subspaces have the same dimension and $\Pi_\omega \neq \Pi_\varpi$ when $(\omega, \varpi) \in \Omega_r \times \Omega_r$, $\omega \neq \varpi$ and $r \in \mathbb{I}_K$, hence $\theta \in (0, \pi/2]$. Consequently,

$$\text{H1 holds for } K \in \mathbb{I}_{M-1} \Rightarrow \mu_r(A) \in (0, 1], \quad \forall r \in \mathbb{I}_K \Rightarrow \xi_K(A) \in (0, 1].$$

According to (49), we have $\xi_K \geq \xi_{K+1}$, $\forall K \in \mathbb{I}_{M-2}$. Our guess that *assumption H1 is lightened when K decreases* (see the comments following Theorem 3 on p. 19) means that

$$\xi_1(A) > \dots > \xi_{M-1}(A). \quad (50)$$

We provide numerical tests on two subsets of real-valued random matrices for $M = 5$ and $N = 10$, denoted by \mathcal{A}_{20}^N and \mathcal{A}_{1000}^U . The values of $\xi_K(\cdot)$, $K \in \mathbb{I}_{M-1} = \mathbb{I}_4$, for every matrix in \mathcal{A}_{20}^N and in \mathcal{A}_{1000}^U , were calculated using *exhaustive combinatorial search*. All tested matrices satisfy *assumption H1*, which confirms Theorem 3 (p. 19) and its consequences. In order to evaluate the extent of H1, we computed the *worst* and the *best* values of $\xi_K(\cdot)$ over these sets:

$$\begin{cases} \xi_K^{\text{worst}} = \min_{A \in \mathcal{A}} \xi_K(A), \\ \xi_K^{\text{best}} = \max_{A \in \mathcal{A}} \xi_K(A), \end{cases} \quad \forall K \in \mathbb{I}_{M-1}, \quad \mathcal{A} \in \{\mathcal{A}_{20}^N, \mathcal{A}_{1000}^U\}. \quad (51)$$

Set \mathcal{A}_{20}^N . This set was formed from 20 matrices A^n , $n \in \mathbb{I}_{20}$ of size 5×10 . The components of each matrix A^n were independent and uniformly drawn from the standard normal distribution with mean zero and variance one. The values of $\xi_K(\cdot)$ are depicted in Fig. 1. We have¹⁷ $\xi_1(A^{10}) = \xi_2(A^{10})$ and $\xi_1(A^{17}) = \xi_2(A^{17})$. In all other cases (50) is satisfied. Fig. 1 clearly shows that in general $\xi_K(\cdot)$ increases substantially as K decreases (from $M - 1$ to 1).

The *worst* and the *best* values of $\xi_K(\cdot)$ over the whole set \mathcal{A}_{20}^N are displayed in Table 1.

	$K = 1$	$K = 2$	$K = 3$	$K = 4$
ξ_K^{worst}	0.3519	0.1467	0.0676	0.0072
ξ_K^{best}	0.8666	0.5881	0.3966	0.0785

Table 1: The worst and the best values of $\xi_K(A)$, for $K \in \mathbb{I}_{M-1}$, over the set \mathcal{A}_{20}^N , see (51).

¹⁷This is why on the figure, in columns 10 and 17, the green “o” and the red “◇” overlap.

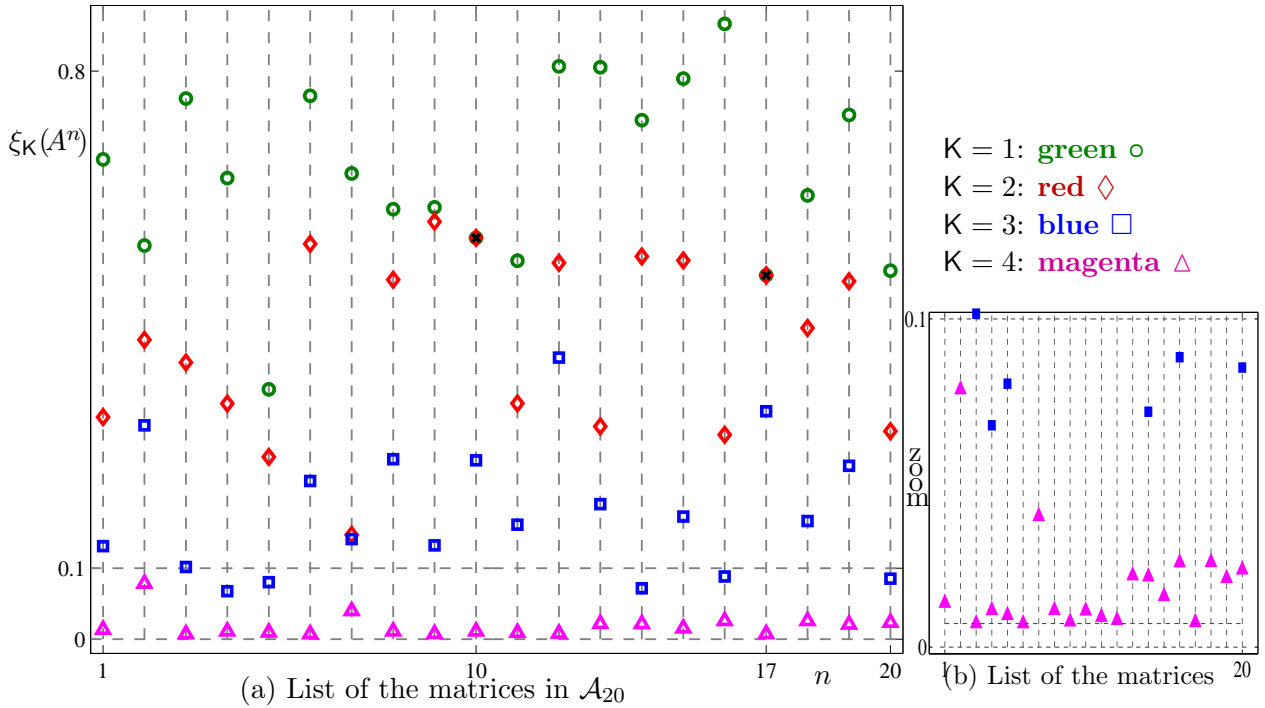


Figure 1: x -axis: the list of the 20 random matrices in \mathcal{A}_{20}^N . (a) y -axis: the value $\xi_K(A^n)$ according to (49), $\forall K \in \mathbb{I}_{M-1}$ and $\forall n \in \mathbb{I}_{20}$. The plot in (b) is a zoom of (a) along the y -axis.

Set \mathcal{A}_{1000}^U . The set \mathcal{A}_{1000}^U was composed of one thousand 5×10 matrices A^n , $n \in \mathbb{I}_{1000}$. The entries of each matrix A^n were independent and uniformly sampled on $[-1, 1]$. The obtained values for ξ_K^{worst} and ξ_K^{best} , calculated according to (51), are shown in Table 2. For $K \in \mathbb{I}_3$, the

	$K=1$	$K=2$	$K=3$	$K=4$
ξ_K^{worst}	0.1085	0.0235	0.0045	0.0001
ξ_K^{best}	0.9526	0.8625	0.5379	0.1152

Table 2: The worst and the best values of $\xi_K(A)$, for $K \in \mathbb{I}_{M-1}$, over the set \mathcal{A}_{1000}^U .

best values of $\xi_K(\cdot)$ were obtained for the same matrix, A^{964} . Note that $\xi_4(A^{964}) = 0.0425 \gg \xi_4^{worst}$. The *worst* values in Table 2 are smaller than those in Table 1 while the *best* values in Table 2 are larger than those in Table 1; one credible reason is that \mathcal{A}_{1000}^U is much larger than \mathcal{A}_{20}^N .

Overall, equation (50) is satisfied on \mathcal{A}_{1000}^U —the percentages in Table 3 are pretty small.

	$\xi_1(A^n) = \xi_2(A^n)$	$\xi_2(A^n) = \xi_3(A^n)$	$\xi_3(A^n) = \xi_4(A^n)$
occurrences $\{n\}$	5 %	1.6 %	0.1 %

Table 3: Percentage of the cases in \mathcal{A}_{1000}^U when (50) fails to hold.

Table 3 and Figure 1 agree with our guess that H1 is more viable for smaller values of K .

Based on the magnitudes for ξ_K^{best} in Tables 1 and 2, one can expect that there are some classes of matrices (random or not) that fit H1 for larger values of $\xi_K(\cdot)$.

6.2 On the global minimizers of \mathcal{F}_d

Here we summarize the outcome of a series of experiments corresponding to several matrices $A \in \mathbb{R}^{M \times N}$ where $M = 5$ and $N = 10$, satisfying H1 for $K = M - 1$, different original vectors $\ddot{u} \in \mathbb{R}^N$ and data samples $d = A\ddot{u} + \text{noise}$, for various values of $\beta > 0$. In each experiment, we computed the complete list of all different strict (local) minimizers of \mathcal{F}_d , say $(\hat{u}^i)_{i=1}^n$. Then the sequence of values $(\mathcal{F}_d(\hat{u}^i))_{i=1}^n$ was sorted in increasing order, $\mathcal{F}_d(\hat{u}^{i_1}) \leq \mathcal{F}_d(\hat{u}^{i_2}) \leq \dots \leq \mathcal{F}_d(\hat{u}^{i_n})$. A global minimizer \hat{u}^{i_1} is unique provided that $\mathcal{F}_d(\hat{u}^{i_1}) < \mathcal{F}_d(\hat{u}^{i_2})$. In order to discard numerical errors, we also checked if $|\mathcal{F}_d(\hat{u}^{i_1}) - \mathcal{F}_d(\hat{u}^{i_2})|$ is well detectable.

In all experiments we carried out, the following facts were observed:

- The global minimizer of \mathcal{F}_d was unique—manifestly data d did never belong to the negligible subset Σ_K in Proposition 6 (p. 19). This confirms Theorem 4 (p. 20).
- The global minimizers of \mathcal{F}_d remained unchanged under large variations of β .
- The necessary condition for a global minimizer in Proposition 4 was met.

Next we present in detail two of these experiments where \mathcal{F}_d is defined using

$$A = \begin{bmatrix} 7 & 2 & 4 & 9 & 0 & 3 & 3 & 6 & 6 & 7 \\ 3 & 4 & 9 & 3 & 3 & 9 & 1 & 3 & 1 & 5 \\ 5 & 4 & 2 & 4 & 0 & 7 & 1 & 9 & 2 & 9 \\ 8 & 4 & 0 & 9 & 6 & 0 & 4 & 2 & 3 & 7 \\ 6 & 3 & 6 & 5 & 0 & 9 & 0 & 0 & 3 & 8 \end{bmatrix} \quad \begin{aligned} d &= A\ddot{u} + n, \\ \text{where } n &\text{ is noise and} \\ \ddot{u} &= (0, 1, 8, 0, 3, 0, 0, 0, 0, 9)^T. \end{aligned} \quad (52)$$

Only integers appear in (52) for better readability. We have $\text{rank}(A) = M = 5$. An exhaustive combinatorial test shows that the arbitrary matrix A in (52) satisfies H1 for $K = M - 1$. The values of $\xi_K(A)$ are seen in Table 4. One notes that $\mu_2(A) > \mu_1(A)$, hence $\xi_1(A) = \xi_2(A)$.

	K = 1	K = 2	K = 3	K = 4
$\xi_K(A)$	0.2737	0.2737	0.2008	0.0564
$\mu_K(A)$	0.2737	0.2799	0.2008	0.0564

Table 4: The values of $\xi_K(A)$ and $\mu_K(A)$, $\forall K \in \mathbb{I}_{M-1}$, for the matrix A in (52).

A way to rate the quality of the solution provided by a global minimizer \hat{u} of \mathcal{F}_d is to compute the mean square error (MSE) of the retained non-zero values $\hat{u}_{\sigma(\hat{u})}$ with respect to $\ddot{u}_{\sigma(\hat{u})}$. So

$$\text{MSE}(\ddot{u}_{\hat{\sigma}}, \hat{u}_{\hat{\sigma}}) = \frac{1}{\#\hat{\sigma}} \sum_{i \in \hat{\sigma}} (\hat{u}[i] - \ddot{u}[i])^2 \quad \text{where } \hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u}). \quad (53)$$

The rationale is that one expects that the global minimizer \hat{u} of \mathcal{F}_d obeys $\hat{\sigma} \subseteq \sigma(\ddot{u})$ where \ddot{u} is the original in (52) and that the vanished entries of \hat{u} correspond to the leas entries of \ddot{u} .

The experiments described hereafter correspond to two data samples relevant to (52)—without and with noise—and to several values of $\beta > 0$.

β	The global minimizer \hat{u} of \mathcal{F}_d (<i>row vector</i>)									$\ \hat{u}\ _0$	$\mathcal{F}_d(\hat{u})$	MSE($\hat{u}_{\hat{\sigma}}, \hat{u}_{\hat{\sigma}}$)	
1	0	1	8	0	3	0	0	0	0	9	4	4	0
10^2	0	0	8.12	0	3.31	0	0	0	0	9.33	3	301.52	0.073
10^3	0	0	0	0	0	12.58	20.28	0	0	0	2	2179.3	284.7
10^4	0	29.95	0	0	0	0	0	0	0	0	1	14144	838.2
$7 \cdot 10^4$	0	0	0	0	0	0	0	0	0	0	0	58864	—
\ddot{u}	= 0	1	8	0	3	0	0	0	0	9			

Table 5: The global minimizer \hat{u} of \mathcal{F}_d and its value $\mathcal{F}_d(\hat{u})$ for the noise-free data d in (54) for different values of β . Last column: the error defined in (53). Last row: the original \ddot{u} in (52).

Noise-free data. The noise-free data in (52) read:

$$d = A\ddot{u} = (97, 130, 101, 85, 123)^T. \quad (54)$$

For different values of β , the global minimizer \hat{u} is given in Table 5. Since $\sigma(\ddot{u}) \in \Omega$ and data are noise-free, \mathcal{F}_d does not have global minimizers with $\|\hat{u}\|_0 = 5$. Actually, applying Proposition 5(i) for $\tilde{u} = \ddot{u}$ yields $\beta_{M-1} = 0$, hence for any $\beta > 0$ all global minimizers of \mathcal{F}_d have a support in $\Omega = \overline{\Omega}_{M-1}$ (see Definition 3 on p. 9 and (37) on p. 16). The global minimizer \hat{u} for $\beta = 1$ meets $\hat{u} = \ddot{u}$. For $\beta = 100$, the global minimizer \hat{u} obeys $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u}) = \{3, 5, 10\} \subsetneq \sigma(\ddot{u})$ and $\|\hat{u}\|_0 = 3$ —the least nonzero entry of the original \ddot{u} is canceled, which is reasonable. The global minimizers corresponding to $\beta \geq 300$ are meaningless. We could not find any value of β giving better 2-sparse global minimizers. Reminding that data are noise-free, this confirms the discussion next to Corollary 1 (p. 9): the global minimizers of \mathcal{F}_d realize an only *pseudo-hard* thresholding. For $\beta \geq 7 \cdot 10^4 > \|d\|^2$, the global minimizer of \mathcal{F}_d is $\hat{u} = 0$ which confirms Remark 2 (p. 15).

Noisy data. Now we consider *noisy data* in (52) for

$$n = (4, -1, 2, -3, 5)^T. \quad (55)$$

This arbitrary noise yields a signal-to-noise ratio¹⁸ (SNR) equal to 14.07 dB. If $\beta \leq 0.04$, \mathcal{F}_d has 252 different strict global minimizers \hat{u} obeying $\|\hat{u}\|_0 = M$ and $\mathcal{F}_d(\hat{u}) = \beta M$ (remind Proposition 3 on p. 14). For $\beta \geq 0.05$, the global minimizer \hat{u} of \mathcal{F}_d is unique and meets $\sigma(\hat{u}) \in \Omega$. For several values of $\beta \geq 0.05$, it is given in Table 6. For $\beta = 1$, the global minimizer is meaningless. We

β	The global minimizer \hat{u} of \mathcal{F}_d (<i>row vector</i>)									$\ \hat{u}\ _0$	$\mathcal{F}_d(\hat{u})$	MSE($\hat{u}_{\hat{\sigma}}, \hat{u}_{\hat{\sigma}}$)	
1	0	6.02	2.66	6.43	0	6.85	0	0	0	0	4	4.0436	35.5
10^2	0	0	8.23	0	2.3	0	0	0	0	9.71	3	301.94	0.35
10^3	0	0	8.14	0	0	0	0	0	0	10.25	2	2174.8	0.79
10^4	0	0	0	0	0	0	0	0	0	14.47	1	14473	29.9
$7 \cdot 10^4$	0	0	0	0	0	0	0	0	0	0	0	60559	—
\ddot{u}	= 0	1	8	0	3	0	0	0	0	9			

Table 6: The global minimizer \hat{u} of \mathcal{F}_d and its value $\mathcal{F}_d(\hat{u})$ for noisy data given by (52) and (55), for different values of β . Last column: the error defined in (53). Last row: the original \ddot{u} .

could not find any value of β yielding a better global minimizer with a 4-length support. For

¹⁸Let us denote $\ddot{d} = A\ddot{u}$ and $d = \ddot{d} + n$. The SNR reads [39] $\text{SNR}(\ddot{d}, d) = 10 \log_{10} \frac{\sum_{i=1}^M (\ddot{d}[i] - \frac{1}{M} \sum_{i=1}^M \ddot{d}[i])^2}{\sum_{i=1}^M (\ddot{d}[i] - \ddot{d}[i])^2}$.

the other values of β , the global minimizer \hat{u} meets $\hat{\sigma} \stackrel{\text{def}}{=} \sigma(\hat{u}) \subsetneq \sigma(\ddot{u})$ and its vanished entries correspond to the least entries in the original \ddot{u} . For $\beta = 100$, the global minimizer seems to furnish a good approximation to \ddot{u} . Observe that the last entry of the global minimizer $\hat{u}[10]$, corresponding to the largest magnitude in \ddot{u} , freely increases when β increases from 10^2 to 10^4 . We tested a tight sequence of intermediate values of β without finding better results. Yet again, $\beta \geq 7 \cdot 10^4 > \|d\|^2$ leads to a unique null global minimizer (see Remark 2).

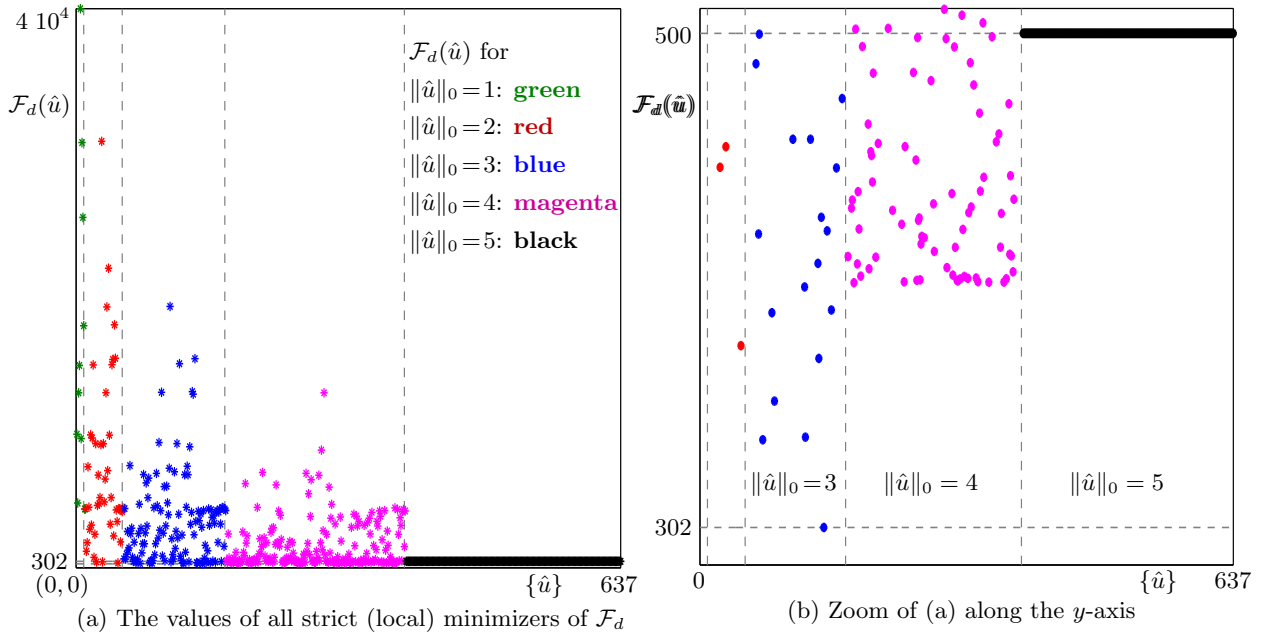


Figure 2: All **638** strict (local) minima of \mathcal{F}_d in (52) for $\beta = 100$ and data d corrupted with the arbitrary noise in (55). The x -axis lists all strict (local) minimizers $\{\hat{u}\}$ of \mathcal{F}_d sorted according to their ℓ_0 -norm $\|\hat{u}\|_0$ in increasing order. (a) The y -axis shows the value $\mathcal{F}_d(\hat{u})$ of these minimizers marked with a star. The value of \mathcal{F}_d for $\hat{u} = 0$ is not shown because it is too large ($\mathcal{F}_d(0) = 60559 = \|d\|^2$). (b) A zoom of (a) along the y -axis. It clearly shows that \mathcal{F}_d has a well recognizable unique global minimizer.

Figure 2 shows the value $\mathcal{F}_d(\hat{u})$ of all the strict local minimizers of \mathcal{F}_d for $\beta = 100$. On the zoom in Figure 2(b) it is well seen that the global minimizer is unique (remember Theorem 4). It meets $\|\hat{u}\|_0 = 3$ and $\mathcal{F}_d(\hat{u}) = 301.94$. One observes that \mathcal{F}_d has $252 = \#\Omega_M$ different strict local minimizers \hat{u} meeting $\|\hat{u}\|_0 = 5 = M$ and $\mathcal{F}_d(\hat{u}) = 500 = \beta M$. This confirms Proposition 3 (p. 14)—obviously d does not belong to the negligible subset Q_M described in (35).

7 Conclusions and perspectives

We provided a detailed analysis of the (local and global) minimizers of a regularized objective \mathcal{F}_d composed of an ℓ_2 data fidelity term and an ℓ_0 penalty weighted by a parameter $\beta > 0$, as given in (1), p. 1. We exhibited easy necessary and sufficient conditions ensuring that a (local) minimizer \hat{u} of \mathcal{F}_d is strict (Theorem 1). The global minimizers of \mathcal{F}_d (whose existence was proven in were shown to be strict as well (Theorem 2). Under mild conditions, \mathcal{F}_d was shown to have a unique global minimizer (Theorem 4). Other interesting results were listed in the Abstract. Below we evoke some perspectives and open questions raised by this work.

- The relationship between the value of the regularization parameter β and the sparsity of the global minimizers of \mathcal{F}_d (Proposition 5, p. 16) can be improved.

- The *generic* linearity in data d of each strict (local) minimizer of \mathcal{F}_d (subsection 3.2, p. 12) should be exploited to better characterize the global minimizers of \mathcal{F}_d .
- Is there a simple way to check if assumption H1 (p. 17) is satisfied by a given matrix $A \in \mathbb{R}^{M \times N}$ when $M < N$ are large? Remark 3 and in particular equation (45), p. 18, could help to discard some non-random matrices. Conversely, one can ask if there is a systematic way to construct matrices A that satisfy H1.

An alternative would be to exhibit families of matrices that satisfy H1 for *generically* large values of $\xi_{\mathcal{K}}(\cdot)$, where the latter quantifiers are defined in equation (49) (p. 21).

- A proper adaptation of the results to matrices A and data d with complex entries should not present inherent difficulties.
- The theory developed here can be extended to MAP energies of the form evoked in (6), p. 2. This is important for the imaging applications mentioned there.
- Based on the comments next to Corollary 1 and the numerical tests in subsection 6.2, one has right of asking for conditions ensuring that the global minimizers of \mathcal{F}_d perform a honorable work. Given the high quality of the numerical results provided in [32, 33], the question deserves attention.

There exist numerous algorithms aimed at approximating a (local) minimizer of \mathcal{F}_d . As a by-product of our research, we got simple rules to verify whether or not an algorithm could find

- a (local) minimizer \hat{u} of \mathcal{F}_d —by checking if \hat{u} satisfies (26) in Corollary 1 (p. 9);
- and if this local minimizer is strict—by testing if the submatrix whose columns are indexed by the support of \hat{u} (i. e. $A_{\hat{\sigma}}$) has full column rank (Theorem 1, p. 10).

Some properties of the minimizers of \mathcal{F}_d given in this work can be inserted in numerical schemes in order to quickly escape from shallow local minimizers.

Many existing numerical methods involve a studious choice of the regularization parameter β and some of them are proven to converge to a local minimizer of \mathcal{F}_d . *We have seen that finding a (strict or nonstrict) local minimizer of \mathcal{F}_d is independent of the value of β* (Corollary 1, p. 9 and Corollary 2, p. 11). There is a *theoretical gap* that needs clarification. Other successful algorithms are not guaranteed to converge to a local minimizer of \mathcal{F}_d . Whenever algorithms do a good job, the choice of β , the assumptions on A and on $\|\hat{u}\|_0$, the iterative scheme and its initialization *obviously* provide a tool for selecting a meaningful solution by minimizing \mathcal{F}_d .

The connection between the existing algorithms and the description of the minimizers exposed in this article deserves a deep exploration.

What conditions ensure that an algorithm minimizing \mathcal{F}_d yields meaningful solutions?

One can expect that such a research gives rise to innovative and more efficient algorithms enabling to compute relevant solutions by minimizing the tricky objective \mathcal{F}_d .

8 Appendix

8.1 Proof of Lemma 1, p. 7

Since $\hat{u} \neq 0$, the definition of $\hat{\sigma}$ shows that $\min_{i \in \hat{\sigma}} |\hat{u}[i]| > 0$. Then ρ in (19) meets $\rho > 0$.

Proof of (i). Since $\sharp \hat{\sigma} \geq 1$, we have

$$\begin{aligned}
i \in \hat{\sigma}, v \in B_\infty(0, \rho) &\Rightarrow \max_{i \in \hat{\sigma}} |v[i]| < \rho \\
&\Rightarrow \max_{i \in \hat{\sigma}} |v[i]| < \min_{i \in \hat{\sigma}} |\hat{u}[i]| \\
&\Rightarrow |\hat{u}[i] + v[i]| \geq |\hat{u}[i]| - |v[i]| \\
&\quad \geq \min_{i \in \hat{\sigma}} |\hat{u}[i]| - \max_{i \in \hat{\sigma}} |v[i]| \geq \rho - \max_{i \in \hat{\sigma}} |v[i]| > 0 \\
&\Rightarrow \hat{u}[i] + v[i] \neq 0 \\
\text{[by (2), p. 2]} &\Rightarrow \phi(\hat{u}[i] + v[i]) = \phi(\hat{u}[i]) = 1. \tag{56}
\end{aligned}$$

If $\hat{\sigma}^c = \emptyset$ the result is proven. Let $\hat{\sigma}^c \neq \emptyset$. Then $\hat{u}[i] = 0 = \phi(\hat{u}[i])$, $\forall i \in \hat{\sigma}^c$. Statement (i) is obtained by inserting the latter and (56) into

$$\sum_{i \in \mathbb{I}_N} \phi(\hat{u}[i] + v[i]) = \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i] + v[i]) + \sum_{i \in \hat{\sigma}^c} \phi(\hat{u}[i] + v[i]).$$

Proof of (ii). Using that $\|A(\hat{u} + v) - d\|^2 = \|A\hat{u} - d\|^2 + \|Av\|^2 + 2\langle Av, A\hat{u} - d \rangle$, one obtains

$$\begin{aligned}
v \in B_\infty(0, \rho) \setminus K_{\hat{\sigma}} &\Rightarrow \mathcal{F}_d(\hat{u} + v) = \|A\hat{u} - d\|^2 + \|Av\|^2 + 2\langle Av, A\hat{u} - d \rangle \\
\text{[by Lemma 1(i)]} &\quad + \beta \sum_{i \in \hat{\sigma}} \phi(\hat{u}[i]) + \beta \sum_{i \in \hat{\sigma}^c} \phi(v[i]) \\
\text{[using (4), p. 2]} &= \mathcal{F}_d(\hat{u}) + \|Av\|^2 + 2\langle Av, A\hat{u} - d \rangle + \beta \sum_{i \in \hat{\sigma}^c} \phi(v[i]) \\
&\geq \mathcal{F}_d(\hat{u}) - |2\langle v, A^T(A\hat{u} - d) \rangle| + \beta \sharp \hat{\sigma}^c \\
\text{[by Hölder's inequality]} &\geq \mathcal{F}_d(\hat{u}) - 2\|v\|_\infty \|A^T(A\hat{u} - d)\|_1 + \beta \sharp \hat{\sigma}^c. \tag{57}
\end{aligned}$$

If $\sharp \hat{\sigma}^c = 0$ then $K_{\hat{\sigma}} = \mathbb{R}^N$, so $v \in \mathbb{R}^0$ and $\|v\|_\infty = 0$, hence the inequality.

Let $\sharp \hat{\sigma}^c \geq 1$. The definition of ρ in (19) shows that

$$\begin{aligned}
v \in B_\infty(0, \rho) \setminus K_{\hat{\sigma}} &\Rightarrow -\|v\|_\infty > -\rho \geq -\frac{\beta}{2(\|A^T(A\hat{u} - d)\|_1 + 1)} \\
&\Rightarrow -2\|v\|_\infty \|A^T(A\hat{u} - d)\|_1 + \beta \sharp \hat{\sigma}^c > -\frac{2\beta \|A^T(A\hat{u} - d)\|_1}{2(\|A^T(A\hat{u} - d)\|_1 + 1)} + \beta \sharp \hat{\sigma}^c > 0.
\end{aligned}$$

Introducing the last inequality into (57) shows that for $\sharp \hat{\sigma}^c \geq 1$, the inequality in (ii) is strict.

8.2 Proof of Proposition 4, p. 15

If $\hat{u} = 0$, the statement is obvious. We focus on $\hat{u} \neq 0$. For an arbitrary $i \in \mathbb{I}_N$, define

$$\hat{u}^{(i)} \stackrel{\text{def}}{=} (\hat{u}[1], \dots, \hat{u}[i-1], 0, \hat{u}[i+1], \dots, \hat{u}[N]) \in \mathbb{R}^N.$$

We shall use the equivalent formulation of \mathcal{F}_d given in (3). Clearly¹⁹,

$$\mathcal{F}_d(\hat{u}) = \mathcal{F}_d(\hat{u}^{(i)} + e_i \hat{u}[i]) = \|A\hat{u}^{(i)} + a_i \hat{u}[i] - d\|^2 + \beta \sum_{j \in \mathbb{I}_N} \phi(\hat{u}^{(i)}[j]) + \phi(\hat{u}[i]).$$

¹⁹Using the definition of $\hat{u}^{(i)}$, we have $\hat{u}^{(i)} = A_{(\mathbb{I}_N \setminus \{i\})} \hat{u}_{(\mathbb{I}_N \setminus \{i\})}$, hence $A\hat{u}^{(i)}$ is independent of $\hat{u}[i]$.

Consider $f_i : \mathbb{R} \rightarrow \mathbb{R}$ as given below

$$f_i(t) \stackrel{\text{def}}{=} \mathcal{F}_d(\hat{u}^{(i)} + e_i t). \quad (58)$$

Since \hat{u} is a global minimizer of \mathcal{F}_d , for any $i \in \mathbb{I}_N$, we have

$$\begin{aligned} f_i(\hat{u}[i]) &= \mathcal{F}_d(\hat{u}^{(i)} + e_i \hat{u}[i]) \\ &\leq \mathcal{F}_d(\hat{u}^{(i)} + e_i t) = f_i(t), \quad \forall t \in \mathbb{R}. \end{aligned}$$

Equivalently, for any $i \in \mathbb{I}_N$, $f_i(\hat{u}[i])$ is the global minimum of $f_i(t)$ on \mathbb{R} . Below we will determine the global minimizer(s) $\hat{t} = \hat{u}[i]$ of f_i as given in (58), i. e.

$$\hat{t} = \hat{u}[i] = \arg \min_{t \in \mathbb{R}} f_i(t).$$

Let us focus on an arbitrary $i \in \mathbb{I}_N$. In detail, the function f_i reads

$$\begin{aligned} f_i(t) &= \|A\hat{u}^{(i)} + a_i t - d\|^2 + \beta \sum_{j \in \mathbb{I}_N} \phi(\hat{u}^{(i)}[j]) + \beta \phi(t) \\ &= \|A\hat{u}^{(i)} - d\|^2 + \|a_i\|^2 t^2 + 2t \langle a_i, A\hat{u}^{(i)} - d \rangle + \beta \sum_{j \in \mathbb{I}_N} \phi(\hat{u}^{(i)}[j]) + \beta \phi(t) \\ &= \|a_i\|^2 t^2 + 2t \langle a_i, A\hat{u}^{(i)} - d \rangle + \beta \phi(t) + C_i, \end{aligned} \quad (59)$$

where

$$C_i = \|A\hat{u}^{(i)} - d\|^2 + \beta \sum_{j \in \mathbb{I}_N} \phi(\hat{u}^{(i)}[j]).$$

Note that C_i does not depend on t . The function f_i has two local minimizers denoted by \hat{t}_0 and \hat{t}_1 . The first one is

$$\hat{t}_0 = 0 \quad \Rightarrow \quad f_i(\hat{t}_0) = C_i. \quad (60)$$

The other one, $\hat{t}_1 \neq 0$, corresponds to $\phi(t) = 1$. From (59), \hat{t}_1 solves

$$2\|a_i\|^2 t + 2\langle a_i, A\hat{u}^{(i)} - d \rangle = 0.$$

Hence

$$\hat{t}_1 = -\frac{\langle a_i, A\hat{u}^{(i)} - d \rangle}{\|a_i\|^2} \quad \Rightarrow \quad f_i(\hat{t}_1) = -\frac{\langle a_i, A\hat{u}^{(i)} - d \rangle^2}{\|a_i\|^2} + \beta + C_i. \quad (61)$$

Next we check if \hat{t}_0 or \hat{t}_1 is a global minimizer of f_i . From (60) and (61) we get

$$f_i(\hat{t}_0) - f_i(\hat{t}_1) = \frac{\langle a_i, A\hat{u}^{(i)} - d \rangle^2}{\|a_i\|^2} - \beta.$$

Furthermore,

$$\begin{aligned} f_i(\hat{t}_0) < f_i(\hat{t}_1) &\quad \Rightarrow \quad \hat{u}[i] = \hat{t}_0 = 0, \\ f_i(\hat{t}_1) < f_i(\hat{t}_0) &\quad \Rightarrow \quad \hat{u}[i] = \hat{t}_1 = -\frac{\langle a_i, A\hat{u}^{(i)} - d \rangle}{\|a_i\|^2}, \\ f_i(\hat{t}_0) = f_i(\hat{t}_1) &\quad \Rightarrow \quad \hat{t}_0 \text{ and } \hat{t}_1 \text{ are global minimizers of } f_i. \end{aligned} \quad (62)$$

In particular, we have

$$\begin{aligned}
f_i(\hat{t}_1) \leq f_i(\hat{t}_0) &\Leftrightarrow \langle a_i, A\hat{u}^{(i)} - d \rangle^2 \geq \beta \|a_i\|^2 & (63) \\
\left[\text{by (62)} \right] &\Rightarrow |\hat{u}[i]| = \frac{|\langle a_i, A\hat{u}^{(i)} - d \rangle|}{\|a_i\|^2} \\
\left[\text{by (63)} \right] &\geq \frac{\sqrt{\beta} \|a_i\|}{\|a_i\|^2} = \frac{\sqrt{\beta}}{\|a_i\|}.
\end{aligned}$$

It is clear that the conclusion holds true for any $i \in \mathbb{I}_N$.

8.3 Proof of Theorem 2, p. 15

Proof of (i). Given $v \in \mathbb{R}^N$, the asymptotic function $(\mathcal{F}_d)_\infty(v)$ of \mathcal{F}_d reads [2, p. 49]

$$(\mathcal{F}_d)_\infty(v) = \liminf_{v' \rightarrow v, t \rightarrow \infty} \frac{\mathcal{F}_d(tv')}{t} = \begin{cases} 0 & \text{if } v \in \ker(A), \\ +\infty & \text{if } v \notin \ker(A). \end{cases} \quad (64)$$

Hence $\ker(\mathcal{F}_d)_\infty = \ker(A)$ and

$$(\mathcal{F}_d)_\infty(v) \geq 0, \quad \forall v \in \mathbb{R}^N \setminus \{0\}. \quad (65)$$

Let $\{v_k\}$ be any sequence belonging to the set $S_{\mathcal{F}_d} \subset \mathbb{R}^N$ given below:

$$S_{\mathcal{F}_d} = \left\{ \{v_k\} : \|v_k\| \rightarrow \infty, \mathcal{F}_d(v_k) \text{ bounded above and } \frac{v_k}{\|v_k\|} \rightarrow v \in \ker(A) \right\}.$$

For k large enough, $v_k \in \ker(A)$ since otherwise the quadratic term in $\mathcal{F}_d(v_k)$ cannot be bounded above. For such a k , we have $\mathcal{F}_d(v_k) = \|d\|^2 + \beta \|v_k\|_0$. Let

$$z_k \stackrel{\text{def}}{=} \frac{v_k}{\|v_k\|} \quad \text{and} \quad \rho_k = \|v_k\|.$$

Then $\lim_{k \rightarrow \infty} z_k = v$ and²⁰

$$\mathcal{F}_d(v_k - \rho_k z_k) = \|\rho_k A z_k + d\|^2 + \beta \|v_k - \rho_k z_k\|_0 = \|d\|^2 < \mathcal{F}_d(v_k). \quad (66)$$

All conditions in [2, Theorem 3.4.1] are satisfied²¹ which ensures that \hat{U} is nonempty.

Proof of (ii). Let \hat{u} be a global minimizer of \mathcal{F}_d . Set $\hat{\sigma} = \sigma(\hat{u})$.

If $\hat{u} = 0$, (i) follows from Lemma 2 (p. 7). Let $\hat{u} \neq 0$ be a *nonstrict* minimizer of \mathcal{F}_d . Then Theorem 1(ii) (p. 10), fails to hold, hence

$$\dim \ker(A_{\hat{\sigma}}) \geq 1. \quad (67)$$

Choose $v_{\hat{\sigma}} \in \ker(A_{\hat{\sigma}}) \setminus \{0\}$ and set $v = Z_{\hat{\sigma}}(v_{\hat{\sigma}})$. Select an $i \in \hat{\sigma}$ satisfying $v[i] \neq 0$. Define \tilde{u} by

$$\tilde{u} \stackrel{\text{def}}{=} \hat{u} - \hat{u}[i] \frac{v}{v[i]}. \quad (68)$$

²⁰The inequality in (66) is strict because $\|v_k\| \rightarrow \infty$ implies $\|v_k\|_0 \geq 1$.

²¹More precisely, it is asked that \mathcal{F}_d is lower semi-continuous and proper, that (65) holds true and that for every sequence $\{v_k\} \in S_{\mathcal{F}_d}$, there exist $\{z_k\}$ and $0 < \rho_k \leq \|v_k\|$ such that for k large enough $\mathcal{F}_d(v_k - \rho_k z_k) < \mathcal{F}_d(v_k)$ and $\|\lim_{k \rightarrow \infty} z_k - v\| < 1$, see [2, p. 97].

We have $\tilde{u}[i] = 0$ and $\hat{u}[i] \neq 0$. Set $\tilde{\sigma} \stackrel{\text{def}}{=} \sigma(\tilde{u})$. Then

$$\tilde{\sigma} \subsetneq \hat{\sigma} \quad \text{hence} \quad \#\tilde{\sigma} \leq \#\hat{\sigma} - 1. \quad (69)$$

Using that $v_{\hat{\sigma}} \frac{\hat{u}[i]}{v[i]} \in \ker(A_{\hat{\sigma}})$, jointly with the expression for \tilde{u} in (68), shows that

$$A\hat{u} = A_{\hat{\sigma}}\hat{u}_{\hat{\sigma}} = A_{\hat{\sigma}} \left(\hat{u}_{\hat{\sigma}} - v_{\hat{\sigma}} \frac{\hat{u}[i]}{v[i]} \right) = A_{\hat{\sigma}}\tilde{u}_{\hat{\sigma}} = A_{\tilde{\sigma}}\tilde{u}_{\tilde{\sigma}} = A\tilde{u}.$$

In the latter expression, we also used the relation (12) on p. 5. Then

$$\begin{aligned} \mathcal{F}_d(\tilde{u}) &= \|A\tilde{u} - d\|^2 + \beta \#\tilde{\sigma} \\ &\leq \mathcal{F}_d(\hat{u}) - \beta = \|A\hat{u} - d\|^2 + \beta(\#\hat{\sigma} - 1). \end{aligned}$$

It follows that \hat{u} is not a global minimizer, hence (67) is false. Therefore $\text{rank}(A_{\hat{\sigma}}) = \#\hat{\sigma}$ and \hat{u} is a strict minimizer of \mathcal{F}_d (Theorem 1).

8.4 Proof of Proposition 5, p. 16

Given $\mathbf{K} \in \mathbb{I}_{\mathbf{M}-1}$, set

$$U_{\mathbf{K}+1} \stackrel{\text{def}}{=} \bigcup_{\omega \subset \mathbb{I}_{\mathbf{N}}} \{ \bar{u} : \bar{u} \text{ solves } (\mathcal{P}_{\omega}) \text{ and } \|\bar{u}\|_0 \geq \mathbf{K} + 1 \}. \quad (70)$$

- Let $U_{\mathbf{K}+1} \neq \emptyset$. By Proposition 1 (p. 7), for any $\beta > 0$, \mathcal{F}_d has a (local) minimum at each $\bar{u} \in U_{\mathbf{K}+1}$. Thus

$$\bar{u} \text{ is a (local) minimizer of } \mathcal{F}_d \text{ and } \|\bar{u}\|_0 \geq \mathbf{K} + 1 \quad \Leftrightarrow \quad \bar{u} \in U_{\mathbf{K}+1}. \quad (71)$$

Then for any $\beta > 0$

$$\mathcal{F}_d(\bar{u}) \geq \beta(\mathbf{K} + 1), \quad \forall \bar{u} \in U_{\mathbf{K}+1}. \quad (72)$$

Let \tilde{u} be defined by:

$$\tilde{u} \text{ solves } (\mathcal{P}_{\omega}) \text{ for some } \omega \in \Omega_{\mathbf{K}}.$$

Then

$$\|\tilde{u}\|_0 \leq \mathbf{K}. \quad (73)$$

Set β and $\beta_{\mathbf{K}}$ according to

$$\beta > \beta_{\mathbf{K}} \stackrel{\text{def}}{=} \|A\tilde{u} - d\|^2. \quad (74)$$

For such a β we have

$$\begin{aligned} \mathcal{F}_d(\tilde{u}) &= \|A\tilde{u} - d\|^2 + \beta\|\tilde{u}\|_0 \\ \left[\text{by (73) and (74)} \right] &< \beta + \beta\mathbf{K} = \beta(\mathbf{K} + 1) \\ \left[\text{by (72)} \right] &\leq \mathcal{F}_d(\bar{u}), \quad \forall \bar{u} \in U_{\mathbf{K}+1}. \end{aligned}$$

Let \hat{u} be a global minimizer of \mathcal{F}_d . Then

$$\mathcal{F}_d(\hat{u}) \leq \mathcal{F}_d(\tilde{u}) < \mathcal{F}_d(\bar{u}), \quad \forall \bar{u} \in U_{\mathbf{K}+1}.$$

Using (70)-(71), we find

$$\|\hat{u}\|_0 \leq \mathbf{K}.$$

- $U_{K+1} = \emptyset$ entails that²²

$$\bar{u} \text{ solves } (\mathcal{P}_\omega) \text{ for } \omega \subset \mathbb{I}_N, \ \sharp\omega \geq K+1 \quad \Rightarrow \quad \|\bar{u}\|_0 \leq K. \quad (75)$$

Let \hat{u} be a global minimizer of \mathcal{F}_d . By (75) we have

$$\|\hat{u}\|_0 \leq K.$$

According to Theorem 2(ii) (p. 15), any global minimizer of \mathcal{F}_d is strict, hence $\sigma(\hat{u}) \in \overline{\Omega}_K$.

8.5 Proof of Lemma 8, p. 19

The set T_κ is closed in \mathbb{R}^M in all cases. Since Π_ω and Π_ϖ are orthogonal projectors, we have²³

$$(\Pi_\omega - \Pi_\varpi)^T(\Pi_\omega - \Pi_\varpi) = (\Pi_\omega - \Pi_\varpi)(\Pi_\omega - \Pi_\varpi)^T.$$

Hence $(\Pi_\omega - \Pi_\varpi)$ is a *normal* (unitarily diagonalizable) matrix, i.e. there exist a unitary matrix $V \in \mathbb{R}^{M \times M}$ and an $M \times M$ diagonal matrix such that [27, 20]

$$\Pi_\omega - \Pi_\varpi = V \text{diag}(\lambda_1, \dots, \lambda_M) V^T, \quad (76)$$

where $\lambda_i, \forall i \in \mathbb{I}_M$ are the eigenvalues of $(\Pi_\omega - \Pi_\varpi)$. From H1, $\text{diag}(\lambda_1, \dots, \lambda_M) \neq 0$. Set

$$p \stackrel{\text{def}}{=} \text{rank}(\Pi_\omega - \Pi_\varpi). \quad (77)$$

We remind that the eigenvalues of a projection matrix Π_ω for $\omega \in \Omega$ belong to $\{0, 1\}$. Clearly

$$\varpi = \emptyset \quad \Rightarrow \quad T_\kappa = \left\{ g \in \mathbb{R}^M : g = Vx \text{ and } \sum_{i \in \mathbb{I}_p} x[i]^2 = \kappa \right\}.$$

For $\kappa > 0$, T_κ is isomorphic to a spherical surface and for $\kappa = 0$ to a vector subspace, hence $\dim(T_\kappa) = M - p \leq M - 1$. For $k < 0$, $T_\kappa = \emptyset$. If $\omega = \emptyset$, T_κ is negligible in \mathbb{R}^M as well.

In what follows, consider that $\sharp\omega \geq 1$ and $\sharp\varpi \geq 1$. Observe that

$$\begin{aligned} \lambda = 1 \quad \text{and} \quad (\Pi_\omega - \Pi_\varpi)v = \lambda v \neq 0 &\quad \Leftrightarrow \quad v \in \ker \Pi_\varpi \text{ and } \Pi_\omega v = v \neq 0, \\ \lambda = -1 \quad \text{and} \quad (\Pi_\omega - \Pi_\varpi)v = \lambda v \neq 0 &\quad \Leftrightarrow \quad v \in \ker \Pi_\omega \text{ and } \Pi_\varpi v = v \neq 0. \end{aligned} \quad (78)$$

Indeed, we have

$$(\Pi_\omega - \Pi_\varpi)v = v \quad \Rightarrow \quad v^T \Pi_\omega v - v^T \Pi_\varpi v = \|v\|^2 > 0. \quad (79)$$

Since Π_ω and Π_ϖ are also semipositive definite matrices, we have

$$0 \leq v^T \Pi_\omega v \leq \|v\|^2 \quad \text{and} \quad v^T \Pi_\varpi v \geq 0.$$

The equality on the right side of (79) cannot be reached unless $v^T \Pi_\varpi v = \|\Pi_\varpi v\| = 0$, hence $v \in \ker(\Pi_\varpi)$ and $\Pi_\omega v = v$. The reciprocal is direct. The second equivalence is proven likewise.

²²Let $A \in \mathbb{R}^{4 \times 5}$ read $A = (e_1, e_2, e_3, e_4, e_1)$ and $d = e_1 \in \mathbb{R}^4$. For $K = M - 1 = 3$ one can check that $U_{K+1} = \emptyset$.

²³Here one uses the basic facts that $\Pi_\omega = \Pi_\omega^T = \Pi_\omega \Pi_\omega, \forall \omega \in \Omega$.

From (78) it follows that

$$\lambda \text{ is an eigenvalue of } (\Pi_\omega - \Pi_\varpi) \Rightarrow |\lambda| \leq 1. \quad (80)$$

It is worthwhile to note also that (78) and (80) entail

$$\lambda \neq 0, |\lambda| < 1 \text{ is an eigenvalue of } (\Pi_\omega - \Pi_\varpi) \Leftrightarrow v_{(\lambda)} \notin (\ker(\Pi_\varpi) \cup \ker(\Pi_\omega)), \quad (81)$$

where $v_{(\lambda)}$ denotes the eigenvector of $(\Pi_\omega - \Pi_\varpi)$ relevant to λ .

We shall use a claim on the difference of projection matrices. An alike claim was given in [31, Lemma 5] in the case when $\sharp\omega = \sharp\varpi = k$ and any $2k$ columns of A are linearly independent. In the proof in [31, p. 4676, 2nd column], the cases $|\lambda| = 1$ are omitted. We extend the statement to our more general setting and complete the proof.

Lemma 9 *Let $(\omega, \varpi) \in \bar{\Omega}_{M-1} \times \bar{\Omega}_{M-1}$, $\emptyset \neq \omega \neq \varpi \neq \emptyset$ and H1 holds for $M - 1$. Then there exists an integer $0 \leq r \leq p/2$ such that $(\Pi_\omega - \Pi_\varpi)$ has $2r$ eigenvalues satisfying*

$$\lambda \neq 0 \text{ is an eigenvalue of } (\Pi_\omega - \Pi_\varpi) \Leftrightarrow -\lambda \text{ is an eigenvalue of } (\Pi_\omega - \Pi_\varpi), \quad (82)$$

and $p - 2r$ eigenvalues equal either to $\lambda = 1$ or to $\lambda = -1$. Remind that p is given in (77).

(i) *If $\sharp\omega = \sharp\varpi$ then $p = 2r$ and all nonzero eigenvalues of $(\Pi_\omega - \Pi_\varpi)$ satisfy (82).*

Proof. The proof consists of three parts.

(1). Let $\lambda \neq 0$, $|\lambda| < 1$ be an eigenvalue of $(\Pi_\omega - \Pi_\varpi)$, i. e.

$$(\Pi_\omega - \Pi_\varpi)v_{(\lambda)} = \lambda v_{(\lambda)}, \quad (83)$$

where $v_{(\lambda)} \notin (\ker(\Pi_\varpi) \cup \ker(\Pi_\omega))$ according to (81). Consider the vector²⁴

$$v_{(-\lambda)} = v_{(\lambda)} - (\Pi_\omega + \Pi_\varpi)v_{(\lambda)}. \quad (84)$$

Note²⁵ that $v_{(-\lambda)} \neq 0$. Using that (83) is equivalent to

$$\Pi_\omega v_{(\lambda)} = (\Pi_\varpi + \lambda I_M)v_{(\lambda)} \Leftrightarrow \Pi_\varpi v_{(\lambda)} = (\Pi_\omega - \lambda I_M)v_{(\lambda)},$$

one derives

$$\begin{aligned} (\Pi_\omega - \Pi_\varpi)v_{(-\lambda)} &= (\Pi_\omega - \Pi_\varpi)v_{(\lambda)} - (\Pi_\omega + \Pi_\omega\Pi_\varpi - \Pi_\varpi\Pi_\omega - \Pi_\varpi)v_{(\lambda)} \\ &= -\Pi_\omega\Pi_\varpi v_{(\lambda)} + \Pi_\varpi\Pi_\omega v_{(\lambda)} \\ &= -\Pi_\omega(\Pi_\omega - \lambda I_M)v_{(\lambda)} + \Pi_\varpi(\Pi_\varpi + \lambda I_M)v_{(\lambda)} \\ &= -(\Pi_\omega - \Pi_\varpi)v_{(\lambda)} + \lambda(\Pi_\omega + \Pi_\varpi)v_{(\lambda)} \\ &= -\lambda(v_{(\lambda)} - (\Pi_\omega + \Pi_\varpi)v_{(\lambda)}) \\ &= -\lambda v_{(-\lambda)}. \end{aligned} \quad (85)$$

Hence $-\lambda$ is an eigenvalue of $(\Pi_\omega - \Pi_\varpi)$. From (81), $v_{(-\lambda)} \notin (\ker(\Pi_\varpi) \cup \ker(\Pi_\omega))$.

²⁴It is easy to see that if we had $v_{(\lambda)} \in (\ker(\Pi_\varpi) \cup \ker(\Pi_\omega))$, then (84) would yield $v_{(-\lambda)} = 0$.

²⁵If $v_{(-\lambda)} = 0$, (83) and (84) lead to $\Pi_\omega v_{(\lambda)} = \frac{1+\lambda}{2}v_{(\lambda)}$ and $\Pi_\varpi v_{(\lambda)} = \frac{1-\lambda}{2}v_{(\lambda)}$. This system makes sense only if $\lambda \in \{-1, +1\}$ which contradicts the assumption that $|\lambda| < 1$ or equivalently that $v_{(\lambda)} \notin (\ker(\Pi_\varpi) \cup \ker(\Pi_\omega))$.

By (81), the eigenvectors (83) and (84) yield *all* eigenvalues of $(\Pi_\omega - \Pi_\varpi)$ meeting $0 < |\lambda| < 1$. Let $2r' \geq 0$ denote their cardinality²⁶. If $2r' = p$, the lemma is proven.

(2). If $p > 2r'$, then by (80), there are $p - 2r'$ eigenvalues living in $\{-1, +1\}$. Therefore $(\Pi_\omega - \Pi_\varpi)$ has $2r$, for $r \geq r' \geq 0$, eigenvalues²⁷ satisfying (82) and $p - 2r \geq 0$ eigenvalues equal either to $+1$ or to -1 .

(3). Consider that $p > 2r'$ and that $\sharp\omega = \sharp\varpi$. More details are needed to complete claim (i).

According to (80) and (81), $(\Pi_\omega - \Pi_\varpi)$ has $p - 2r'$ eigenvalues with $|\lambda| = 1$ corresponding to eigenvectors in $(\ker(\Pi_\varpi) \cup \ker(\Pi_\omega))$ meeting the equivalences in (78). From the latter,

$$\begin{aligned} \lambda = 1 \quad \text{and} \quad (\Pi_\omega - \Pi_\varpi)v = \lambda v \neq 0 &\Leftrightarrow v \in \ker(\Pi_\varpi) \cap \text{range}(\Pi_\omega) , \\ \lambda = -1 \quad \text{and} \quad (\Pi_\omega - \Pi_\varpi)v = \lambda v \neq 0 &\Leftrightarrow v \in \ker(\Pi_\omega) \cap \text{range}(\Pi_\varpi) . \end{aligned} \tag{86}$$

where we notice that $v \in \text{range}(\Pi_\omega)$ is equivalent to $\Pi_\omega v = v$, that

$$(\ker(\Pi_\varpi) \cap \text{range}(\Pi_\omega)) \subset \text{range}(\Pi_\omega - \Pi_\varpi) ,$$

as well as that $(\ker(\Pi_\omega) \cap \text{range}(\Pi_\varpi)) \subset \text{range}(\Pi_\omega - \Pi_\varpi)$. Define

$$H_\varpi = \ker(\Pi_\varpi) \cap \text{range}(\Pi_\omega) , \tag{87}$$

$$H_\omega = \ker(\Pi_\omega) \cap \text{range}(\Pi_\varpi) . \tag{88}$$

The first claim in [5, Lemma 4.1.] tells us that

$$\ker(\Pi_\varpi) = P_\varpi \oplus Q_\varpi ,$$

where

$$\begin{aligned} P_\varpi &= \ker(\Pi_\varpi) \cap \left(\ker(\Pi_\varpi) \cap (\ker(\Pi_\omega))^\perp \right)^\perp , \\ Q_\varpi &= \ker(\Pi_\varpi) \cap (\ker(\Pi_\omega))^\perp . \end{aligned}$$

Using De Morgan's law and the fact that $\text{range}(\Pi_\omega) = (\ker(\Pi_\omega))^\perp$, $\forall \omega \in \Omega$, yield

$$\begin{aligned} P_\varpi &= \ker(\Pi_\varpi) \cap (\text{range}(\Pi_\varpi) + \ker(\Pi_\omega)) = \ker(\Pi_\varpi) \cap \ker(\Pi_\omega) , \\ Q_\varpi &= \ker(\Pi_\varpi) \cap \text{range}(\Pi_\omega) = H_\varpi . \end{aligned}$$

Therefore

$$\ker(\Pi_\varpi) = (\ker(\Pi_\varpi) \cap \ker(\Pi_\omega)) \oplus H_\varpi . \tag{89}$$

²⁶We can have $r' = 0$. For instance, let $A_\omega = (e_3, e_4) \in \mathbb{R}^{4 \times 2}$ and $A_\varpi = (e_1, e_2) \in \mathbb{R}^{4 \times 2}$. Then $\Pi_\omega = (0, 0, e_3, e_4) \in \mathbb{R}^{4 \times 4}$ and $\Pi_\varpi = (e_1, e_2, 0, 0) \in \mathbb{R}^{4 \times 4}$, so

$$\Pi_\omega - \Pi_\varpi = (-e_1, -e_2, e_3, e_4) .$$

The eigenvectors of $\Pi_\omega - \Pi_\varpi$ are $e_i \in \mathbb{R}^4$, $1 \leq i \leq 4$ and the eigenvalues $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = \lambda_4 = 1$. We have

$$\Pi_\omega v_1 = \Pi_\omega v_2 = 0 \quad \text{and} \quad \Pi_\varpi v_3 = \Pi_\varpi v_4 = 0 .$$

²⁷We can have $r = 0$. Let $a_1 \in \mathbb{R}^M \setminus \{0\}$ and $a_2 \in \mathbb{R}^M \setminus \{0\}$ meet $a_1 \neq c a_2$, $\forall c \in \mathbb{R}$. Let $A_\omega = (a_1, a_2)$ and $A_\varpi = (a_1)$. Then $\text{rank}(\Pi_\omega - \Pi_\varpi) = 1$ and $\ker(\Pi_\omega) \subsetneq \ker(\Pi_\varpi)$. For a $v \in \ker(\Pi_\varpi) \setminus \ker(\Pi_\omega)$ the unique eigenvalue of $(\Pi_\omega - \Pi_\varpi)$ is $+1$, see (78).

In an symmetric way

$$\begin{aligned} \ker(\Pi_\omega) &= P_\omega \oplus Q_\omega , \\ P_\omega &= \ker(\Pi_\omega) \cap \left(\ker(\Pi_\omega) \cap (\ker(\Pi_\varpi))^\perp \right)^\perp = \ker(\Pi_\omega) \cap \ker(\Pi_\varpi) , \\ Q_\omega &= \ker(\Pi_\omega) \cap (\ker(\Pi_\varpi))^\perp = H_\omega , \end{aligned}$$

so

$$\ker(\Pi_\omega) = (\ker(\Pi_\omega) \cap \ker(\Pi_\varpi)) \oplus H_\omega . \quad (90)$$

Since $\#\omega = \#\varpi$ and $\omega \neq \varpi$ are elements of Ω it is obvious that

$$\dim(\ker(\Pi_\omega)) = \dim(\ker(\Pi_\varpi)) . \quad (91)$$

Comparing (89) and (90), combined with (91) and the fact that $2r' < p$ shows that

$$\dim(H_\varpi) = \dim(H_\omega) \stackrel{\text{def}}{=} k \geq 1. \quad (92)$$

Using (86) we have

$$v \in H_\varpi \quad \Leftrightarrow \quad (\Pi_\omega - \Pi_\varpi)v = \lambda v \neq 0 \quad \text{for } \lambda = +1 , \quad (93)$$

$$v \in H_\omega \quad \Leftrightarrow \quad (\Pi_\omega - \Pi_\varpi)v = \lambda v \neq 0 \quad \text{for } \lambda = -1 . \quad (94)$$

Combining (93) and (94), along with (92) shows that $(\Pi_\omega - \Pi_\varpi)$ has simultaneously

- k linearly independent eigenvectors in H_ω corresponding to an eigenvalue equal to $+1$,
- k linearly independent eigenvectors in H_ϖ corresponding to an eigenvalue equal to -1 .

Therefore²⁸ $\text{rank}(\Pi_\omega - \Pi_\varpi) = 2(r' + k) = 2r$ and all eigenvalues of $(\Pi_\omega - \Pi_\varpi)$ meet (82). \square

Using Lemma 9, without loss of generality, assume that

$$\mathbb{I}_r = \{i \in \mathbb{I}_{M-1} : \lambda_i > 0 \text{ and } \lambda_{i+r} = -\lambda_i, \forall i \in \mathbb{I}_r\} . \quad (95)$$

Using (76) and (95), the subset \mathbb{T}_κ in (46) on p. 19 equivalently reads

$$\begin{aligned} \mathbb{T}_\kappa &= \{g \in \mathbb{R}^M : g = Vx \text{ and } x^T \text{diag}(\lambda_1, \dots, \lambda_M)x = \kappa\} \\ &= \{g \in \mathbb{R}^M : g = Vx \text{ and } \sum_{i \in \mathbb{I}_r} \lambda_i (x[i]^2 - x[i+r]^2) \pm \sum_{i=2r+1}^p x[i]^2 = \kappa\} , \end{aligned} \quad (96)$$

where $\lambda_i > 0$ for all $i \in \mathbb{I}_r$. Consider the subset

$$\Theta_\kappa \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^M : \sum_{i \in \mathbb{I}_r} \lambda_i (x[i]^2 - x[i+r]^2) \pm \sum_{i=2r+1}^p x[i]^2 = \kappa \right\} . \quad (97)$$

²⁸A direct examination of $\text{range}(\Pi_\omega - \Pi_\varpi)$ shows also that that $\text{rank}(\Pi_\omega - \Pi_\varpi)$ is an even number.

For any $\kappa \neq 0$, $\Theta_\kappa = \left(x \mapsto \sum_{i \in \mathbb{I}_r} \lambda_i (x[i]^2 - x[i+r]^2) \pm \sum_{i=2r+1}^p x[i]^2 \right)^{-1}(\kappa)$ is a non-degenerate elliptical hyperboloid surface of dimension $(p-1)$. For²⁹ $\kappa = 0$, we have

$$\begin{aligned} \Theta_0 &= \left\{ x \in \mathbb{R}^M : \sum_{i \in \mathbb{I}_r} \lambda_i (x[i]^2 - x[i+r]^2) = 0 \right\} \\ &= \bigcup_{c \in \mathbb{R}} \left\{ x \in \mathbb{R}^M : \sum_{i \in \mathbb{I}_r} \lambda_i x[i]^2 = c \quad \text{and} \quad \sum_{i \in \mathbb{I}_r} \lambda_i x[i+r]^2 = c \right\}. \end{aligned}$$

Clearly, Θ_0 is a double conical surface (symmetric about zero with elliptic sections) whose dimension is $(2r-1)$. For for any $\kappa \in \mathbb{R}$, Θ_κ is a simple surface of dimension $\leq M-1$ meeting $\mathbb{L}^M(\Theta_\kappa) = 0$. Since V in (96) is unitary, T_κ meets the claim.

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²⁹For instance, if $r = 1$ and $M = 2$, the set Θ_0 is composed of two one dimensional vector subspaces in \mathbb{R}^M :

$$\Theta_0 = \{x \in \mathbb{R}^2 : x[1] = x[2]\} \cup \{x \in \mathbb{R}^2 : x[1] = -x[2]\}.$$

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