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Exact Histogram Specification for Digital Images¹ Using a Variational Approach

Mila Nikolova*, You-Wei Wen[†], Raymond Chan[‡]

Abstract

We consider the problem of exact histogram specification for digital (quantized) images. The goal is to transform the input digital image into an output (also digital) image that follows a prescribed histogram. Classical histogram modification methods are designed for real-valued images where all pixels have different values, so exact histogram specification is straightforward. Digital images typically have numerous pixels sharing the same value. If one imposes the prescribed histogram to a digital image, usually there are numerous ways of assigning the prescribed values to the quantized values of the image. Therefore, exact histogram specification for digital images is an *ill-posed problem*. In order to satisfy the prescribed histogram, all pixels of the input digital image must be rearranged in a strictly ordered way, while preserving the specific features of the input image. Such a task can be realized if we are able to extract additional *representative* information (called *auxiliary attributes*) from the input digital image. This is a real challenge in exact histogram specification for digital images. We propose a new method that efficiently provides a strict ordering for all pixel values. It is based on a well designed variational approach. Noticing that the input digital image contains quantization noise, we minimize a *specialized* objective function whose solution is a real-valued image with slightly reduced quantization noise that remains very close to the input digital image. We show that all the pixels of this real-valued image can be ordered in a strict way with a very high probability. Then transforming the latter image into another digital image satisfying a specified histogram is an easy task. Numerical results show that our method outperforms by far the existing competing methods.

Key words: Exact histogram specification, strict-ordering, variational methods, restoration from quantization noise, smooth nonlinear optimization, convex minimization.

I. INTRODUCTION

The histogram of an image counts the number of pixels at each intensity value. It is a graph that shows the distribution of the intensity values. Image histogram processing alters the value of each individual pixel in order to modify the appearance of the whole image. It is an important image processing task with many real-world applications, such as contrast enhancement [12], [34], [59], [38], segmentation [13], [44], watermarking [16], texture synthesis [53], texture processing [22], [41] and pattern recognition [39], among many others.

In histogram processing, image intensity level is viewed as a random variable characterized by its probability density function. The histogram of an image shows the empirical distribution of the intensity levels of its pixels. One of the basic histogram processing problems is histogram equalization [27], [50]. It aims to find

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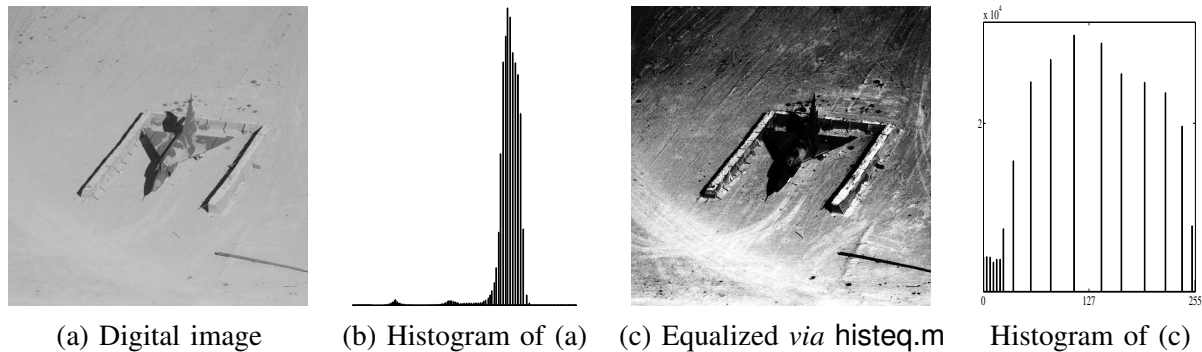


Fig. 1: The original image (a) has gray values in $\{0, \dots, 255\}$ and 11.62% of its pixels equal to 177 (the maximum of the histogram in (b)). Using the built-in Matlab function for histogram equalization (`histeq.m`) yield the image shown in (c) whose histogram, plotted in (d), is far from being uniform.

a transformation so that the output image has a uniform histogram. More generally, given an input image, *histogram specification* or *histogram matching* is the problem to transform it into a similar image that has the pre-specified histogram shape. The pre-specified histogram can be given according to various needs. In the continuous setting the random variable defined by the cumulative distribution function of the intensity levels is uniformly distributed in $[0, 1]$, and hence such a transformation can always be found. However this is usually not the case for digital images.

Numerous methods have been proposed to modify the histogram of an input image. The simplest one is histogram linear stretching [43]. Histogram clipping [52] limits the maximum number of pixels for each intensity level to a given constant and the clipped pixels are then uniformly distributed among the other intensity levels where the numbers of pixels are less than the clip limit. Several other methods were proposed to preserve the mean brightness of the input image [12], [34], [59]. The prescription of a histogram by combining two histograms was considered in [18], [19]. Sapiro and Caselles [55] proposed histogram modification via image evolution equations. Arici *et al.* exhibited a general framework for histogram modification [1]. Based on perceptual consideration, Sen and Sankar [56] establish a rule enabling to prescribe an optimal histogram using the histogram of the input image.

The principle behind histogram specification is straightforward for real-valued (analog) images: the histogram of the input image and the prescribed histogram should be equalized to uniform distribution first, say by T_i and T_t respectively. Then the output image can be obtained from the composite transformation $T_t^{-1} \circ T_i$. Since the images are real-valued, T_i and T_t are one-to-one functions, and hence $T_t^{-1} \circ T_i$ is well-defined. The principle fails, however, for quantized (digital) images, which is the case of all digital imaging systems. The reason is that for quantized images, the intensity levels of all pixels take a limited number of discrete values, see Fig. 1. Therefore their cumulative density functions are staircase functions rather than strictly increasing functions like those for the real-valued images. It is only when one can order the pixels of a quantized image in a strict way that the histogram specification problem can be solved exactly for any prescribed histogram.

Several approaches to get strict ordering for digital (quantized) images were considered in the literature. This problem is the main goal of our paper. Random ordering of the pixels sharing the same intensity level is the simplest way to do the job [51]. One way of avoiding randomness is to separate pixels having the same intensity using the local mean of the four nearest neighbors as was proposed in [29]. The most successful contemporary methods are the Local Mean (LM) approach proposed by Coltuc *et al.* [17] and the wavelet-based approach (WA) elaborated by Wan and Shi in [58]. More details on these methods are given in section

II. Nonetheless, a major drawback of both the LM and the WA methods is that in practice, they can order strictly only a limited portion of the pixels, and this set of pixels is smaller for images whose histogram is not that uniform to begin with.

In this paper, we propose a specialized variational method that enables us to order in a strict way the pixel values of a digital image by *slightly reducing* the quantization noise that they contain. A sketch of our approach was given in a conference paper [11]. Here we provide a theoretical analysis of our variational method and show that the pixels of the images that it yields can be ordered in a strict way with a high probability. We present a lot of experimental results showing that our method outperforms by far its main competitors—the local mean [17] and the wavelet-based [58] methods—both in terms of strict ordering of the pixels and of the quality of the output images.

The outline of the paper is as follows. In Section II, we give comments on sorting algorithm for exact histogram specification. Section III is devoted to the minimizers of the proposed variational functional. In Section IV, we summarize the algorithm for exact histogram specification. In Section V numerical tests are provided where our algorithm is compared with the LM and the WA methods. Concluding remarks are given in Section VI.

II. SORTING ALGORITHMS

Let \mathbf{u} be an M -by- N image obtained by digitizing an analog image \mathbf{u}_o (on a discrete grid) with range in some interval $[0, a)$. We assume the possible quantized values that \mathbf{u} can take are from

$$\mathcal{P} \stackrel{\text{def}}{=} \{p_1, \dots, p_L\} \quad (1)$$

and that p_k are in increasing order. For 8-bit images, $\mathcal{P} = \{0, \dots, 255\}$. In the following we will express \mathbf{u} as an n -vector by concatenating the columns in \mathbf{u} . Here $n = MN$. Denote $\mathbb{I}_q \stackrel{\text{def}}{=} \{1, \dots, q\}$ for any positive integer q and $\Omega_k \stackrel{\text{def}}{=} \{i \in \mathbb{I}_n \mid \mathbf{u}[i] = p_k\}$, $k = 1, 2, \dots, L$. The associated histogram of \mathbf{u} is the L -tuple $\mathbf{h}_{\mathbf{u}} = (|\Omega_1|, |\Omega_2|, \dots, |\Omega_L|)$, where $|\cdot|$ stands for cardinality. The image \mathbf{u} with histogram $\mathbf{h}_{\mathbf{u}}$ is a result of quantization of the original real-valued image \mathbf{u}_o . This amounts to set to the same intensity level p_k all the values of \mathbf{u}_o on the interval $[t_{k-1}, t_k)$, $k = 1, \dots, L$. Then $|\Omega_k| = n \int_{t_{k-1}}^{t_k} p(\mathbf{u}_o) dt$, where $p(\mathbf{u}_o)$ is the empirical probability density function of \mathbf{u}_o , and $t_0 = 0$ and $t_L = a$.

Let the pre-specified histogram read $\mathbf{h} = (h_1, h_2, \dots, h_L)$. The classical way of defining exact histogram specification is designed for real-value images like \mathbf{u}_o . Such images are almost nowhere constant [28] and exact histogram specification is straightforward. However, in practice, we only have the quantized image \mathbf{u} , obtained from an original real-valued (analog) image \mathbf{u}_o . The digital \mathbf{u} can seldom be totally-ordered—its pixels have only L possible magnitudes, see e.g. Fig. 1. Thus, the number of pixels in \mathbf{u} is generally much larger than the number of intensity levels, i.e., $n \gg L$. It is therefore very likely that some Ω_k , $k = 1, \dots, L$, will meet $|\Omega_k| > h_k$. In order to satisfy the prescribed histogram, some pixels in Ω_k will have values mapped to other intensity levels. The number of ways of selecting the pixels to assign to other intensity levels is very large. Consequently, this problem is ill-posed.

The key to a stable solution is to create a total ordering of the pixels in the same Ω_k by learning some auxiliary information from the digital image \mathbf{u} . Suppose that for any pixel $i \in \mathbb{I}_n$, $(K-1)$ auxiliary information $\kappa_1[i], \dots, \kappa_{K-1}[i]$ are created. Then an ascending order “ \prec ” for all pixels in \mathbb{I}_n may be obtained using the K -tuples

$$(\mathbf{u}[i], \kappa_1[i], \dots, \kappa_{K-1}[i]), \quad \forall i \in \mathbb{I}_n.$$

To facilitate the discussions, set $\kappa_0[i] \stackrel{\text{def}}{=} \mathbf{u}[i]$. For any two pixels i and j in \mathbb{I}_n , we say that $i \prec j$ if for some $-1 \leq \ell < (K - 1)$ we have

$$\kappa_s[i] = \kappa_s[j] \text{ for all } 0 \leq s \leq \ell \quad \text{and} \quad \kappa_{\ell+1}[i] < \kappa_{\ell+1}[j].$$

For good choices of auxiliary information and K sufficiently large, one can in principle sort all pixels i in \mathbb{I}_n according to the ordering \prec . That is, we can order the pixels i in \mathbb{I}_n in such a way that $i_1 \prec i_2 \prec \dots \prec i_n$. Once such a strict-ordering is obtained, matching the input histogram to the prescribed one is straightforward. This can be done by dividing the ordered list $\{i_\ell\}_{\ell=1}^n$ from left to right into L groups. Thus the first h_1 pixels i_1, i_2, \dots, i_{h_1} belong to the first group, and are assigned the intensity of p_1 . The next h_2 pixels $i_{h_1+1}, \dots, i_{h_1+h_2}$ belong to the second group and are assigned the intensity of p_2 , and so on until all pixels are assigned to their new intensities.

The Local Mean (LM) and the Wavelet Approach (WA), evoked in the Introduction, extract such auxiliary information from the input quantized image. The LM approach of Coltuc *et al.* [17] uses the average intensities of neighboring pixels: given two pixels with the same intensity value, the means over the neighborhoods centered at each pixel are compared to order these pixels. If the mean values are still the same, then larger neighborhoods are chosen to compute the average intensity. This procedure is repeated until all pixels are ordered. The authors claim that $K = 6$ is ‘‘appropriate for any application’’ [17]. Wan and Shi argue that the LM approach fails to keep sharp the edges in the output image [58]. They proposed to order the pixels according to the absolute values of its wavelet coefficients. This approach tends to amplify the noise since a noise in a smooth region may be mistaken as an edge and hence is sharpened. Post-processing approach or iterative methods can be applied to suppress the amplified noises [3].

III. A SPECIALIZED VARIATIONAL APPROACH

We use a different paradigm to obtain auxiliary information. The available image \mathbf{u} contains quantization noise. Our strategy is to build a real-valued image where this noise is slightly reduced. Even though the original real-valued \mathbf{u}_o is unknown, some general priors such as the presence of edges and fine structures in \mathbf{u}_o can be employed to produce such a restored version of \mathbf{u} . A subtle task like this can be handled by means of a well conceived variational method.

Given \mathbf{u} , its real-valued version $\hat{\mathbf{f}}$ is defined as the minimizer $\arg \min_{\mathbf{f} \in \mathbb{R}^n} \mathcal{J}(\mathbf{f}, \mathbf{u})$ of a smooth, convex regularized cost function $\mathcal{J}(\cdot, \mathbf{u}) : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\mathcal{J}(\mathbf{f}, \mathbf{u}) = \Psi(\mathbf{f}, \mathbf{u}) + \beta \Phi(\mathbf{f}), \quad (2)$$

where both the data fitting term Ψ and the regularization term Φ are asymptotically nearly affine. Here $\beta > 0$ is a weighting parameter. These terms read

$$\begin{aligned} \Psi(\mathbf{f}, \mathbf{u}) &= \sum_{i \in \mathbb{I}_n} \psi(\mathbf{f}[i] - \mathbf{u}[i]), \\ \Phi(\mathbf{f}) &= \sum_{i \in \mathbb{I}_n} \sum_{j \in \mathcal{N}_i} \phi(\mathbf{f}[i] - \mathbf{f}[j]), \end{aligned} \quad (3)$$

where the functions $\psi : \mathbb{R} \rightarrow \mathbb{R}$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ have to fulfill some special requirements given below, and \mathcal{N}_i is the set of the four or the eight adjacent neighbors of pixel i in the image, for every $i \in \mathbb{I}_n$. We consider either Neumann or mirror boundary conditions on the set $\{\mathcal{N}_i \mid i \in \mathbb{I}_n\}$.

The functions ψ and ϕ depend on a parameter $\alpha_1 > 0$ and $\alpha_2 > 0$, respectively. The pivotal condition on (ψ, ϕ) in (3) is that they satisfy H1 and H2 described below.

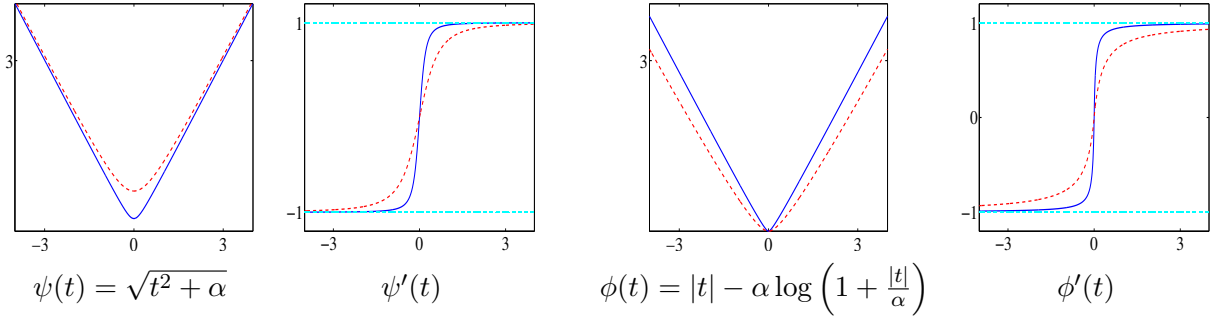


Fig. 2: The first two functions in Table I where the plots for $\alpha = 0.05$ are in blue solid curves and the plots for $\alpha = 0.5$ are in red dashed curves.

H1. The functions $\psi : \mathbb{R} \rightarrow \mathbb{R}$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ are C^s for $s \geq 2$, symmetric and satisfy

$$t \in \mathbb{R} \quad \Rightarrow \quad \psi''(t) > 0 \quad \text{and} \quad \phi''(t) > 0 ,$$

where ψ'' and ϕ'' denote the second derivatives with respect to t .

Moreover, (ψ, ϕ) are nearly affine beyond a small neighborhood of the origin. More precisely,

H2. Their first derivatives ψ' and ϕ' with respect to t are continuous and strictly decreasing in $\alpha_1 > 0$ and α_2 , respectively, where

$$\alpha_1 > 0 \quad \text{and} \quad \alpha_2 > 0 \quad \Rightarrow \quad \lim_{t \rightarrow \infty} \psi'(t, \alpha_1) = 1 \quad \text{and} \quad \lim_{t \rightarrow \infty} \phi'(t, \alpha_2) = 1 ,$$

$$t \in \mathbb{R} \quad \Rightarrow \quad \lim_{\alpha_1 \rightarrow 0} \psi'(t, \alpha_1) = 1, \quad \lim_{\alpha_1 \rightarrow \infty} \psi'(t, \alpha_1) = 0 \quad \text{and} \quad \lim_{\alpha_2 \rightarrow 0} \phi'(t, \alpha_2) = 1, \quad \lim_{\alpha_2 \rightarrow \infty} \phi'(t, \alpha_2) = 0 .$$

Good choices for ψ and ϕ , meeting H1 and H2, are given in Table I. Customarily, such functions are involved only in the regularization term, combined with a quadratic data-fidelity term, see e.g. [8], [2]. In order to achieve our objectives, they are pertinent to define the data fitting term as well.

	ψ, ϕ	ψ', ϕ'	ψ'', ϕ''
f1	$\sqrt{t^2 + \alpha}$	$\frac{t}{\sqrt{t^2 + \alpha}}$	$\frac{t}{(\sqrt{t^2 + \alpha})^3}$
f2	$\alpha \log \left(\cosh \left(\frac{t}{\alpha} \right) \right)$	$\tanh \left(\frac{t}{\alpha} \right)$	$\frac{1}{\alpha} \left(1 - \left(\tanh \left(\frac{t}{\alpha} \right) \right)^2 \right)$
f3	$ t - \alpha \log \left(1 + \frac{ t }{\alpha} \right)$	$\frac{t}{\alpha + t }$	$\frac{\alpha}{(\alpha + t)^2}$

TABLE I: Relevant choices for ψ and ϕ obeying H1 and H2. The size of the neighborhood of zero where these functions are not “nearly affine” is controlled by the parameter $\alpha > 0$.

Remark (a): Motivation to choose a \mathcal{J} satisfying H1 and H2. Before going into the details, we explain the intuition behind the demands H1 and H2 addressed to \mathcal{J} in (2). Our main concern is to obtain a restoration $\widehat{\mathbf{f}}$ of \mathbf{u} whose pixels are all different from each other while being close to \mathbf{u} but “better” than \mathbf{u} . Since ϕ satisfies H1, we will get minimizers $\widehat{\mathbf{f}}$ that are generically nowhere constant [47]. Indeed, natural images have been shown to be almost nowhere constant [28]. Rather, they present large variations, edges and fine structures. The trend of the large variations is available in the input image \mathbf{u} . Requirement H2 on ϕ enables

the recovery of edges and details and in this way some removal of the quantization noise. For instance, $\alpha > 0$ in f1, Table I, should be just small enough. Selecting a small $\beta > 0$ (compared to the range of \mathbf{u}) enhances fitting to data \mathbf{u} . If the neighborhood near the origin where ψ is not quasi-affine goes to zero (e.g. α is almost zero in f1, Table I), ψ tends to the absolute value function. The latter is known to generate minimizers $\hat{\mathbf{f}}$ containing a certain number of entries equal to the relevant entries of \mathbf{u} [46]. These minimizers may still contain numerous equally valued pixels, hence such a scenario must be avoided. If ψ obeys H1 and H2 (e.g., for f1, Table I, we should take $\alpha > 0$ very small), the components of $\hat{\mathbf{f}}$ will certainly be close but different from the relevant entries of \mathbf{u} . So the level sets of the digital image \mathbf{u} are not destroyed, but only refined. This is the reason why we can say that details are preserved in $\hat{\mathbf{f}}$. Undoubtedly, pixels in $\hat{\mathbf{f}}$ must change from those in \mathbf{u} no more than a given value (for example $|0.1|$). The soundness of these arguments is illustrated in Fig. 3. We can thus claim that the objective \mathcal{J} defined using (2) and (3), where ψ and ϕ satisfy H1 and H2 is *detail preserving*. Rigorous details on this point are presented in subsection III-C.

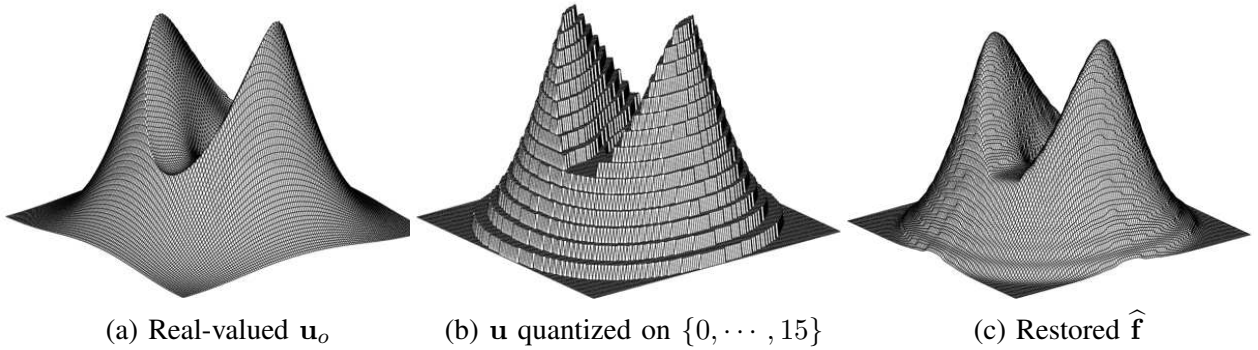


Fig. 3: The restored $\hat{\mathbf{f}}$ in (c) is obtained by minimizing $\mathcal{J}(\cdot, \mathbf{u})$ where $\psi(t) = \sqrt{\alpha_1 + t^2}$ and $\phi(t) = \sqrt{\alpha_2 + t^2}$.

Below we exhibit other salient properties of the minimizer $\hat{\mathbf{f}}$ of \mathcal{J} under H1 and H2 that enable a reasonable strict ordering of the pixels of natural images. It is worth noticing that they are independent of the particular shapes of ψ and ϕ , provided that they meet H1 and H2. The latter remark presents a challenging topic for future research.

A. Preliminary results

The differences $\{\mathbf{f}[i] - \mathbf{f}[j] \mid j \in \mathcal{N}_i\}$ for all $i \in \mathbb{I}_n$ in (3) can be rewritten using finite difference operators $\mathbf{g}_i \in \mathbb{R}^n$, $1 \leq i \leq r$, where r is the total number of these operators. Then Φ reads

$$\Phi(\mathbf{f}) = \sum_{i \in \mathbb{I}_r} \phi(\mathbf{g}_i^T \mathbf{f}) . \quad (4)$$

Let us denote

$$G = \begin{bmatrix} \mathbf{g}_1^T \\ \vdots \\ \mathbf{g}_r^T \end{bmatrix} \in \mathbb{R}^{r \times n} .$$

According to (3) and the adopted boundary conditions (Neumann or mirror), we have $\ker G = \{c\mathbb{1} \mid c \in \mathbb{R}\}$, where $\mathbb{1}$ is a vector composed of ones.

We shall study how the minimizer $\hat{\mathbf{f}}$ of \mathcal{J} behaves as a function of data \mathbf{u} . This goal motivates the definition below which was originally introduced in [45]. For clarity, it is restated in a way adapted to this work.

Definition 1. A function $\mathcal{F} : \mathcal{O} \rightarrow \mathbb{R}^n$, where \mathcal{O} is an open domain in \mathbb{R}^n , is said to be a minimizer function relevant to the family of functions $\mathcal{J}(\cdot, \mathbf{u})$ if for every $\mathbf{u} \in \mathcal{O}$, the point $\hat{\mathbf{f}} = \mathcal{F}(\mathbf{u})$ is a strict local minimizer of $\mathcal{J}(\cdot, \mathbf{u})$.

The next lemma is a straightforward extension of the Implicit Functions Theorem [4]. Its proof can be found e.g. in [23, Theorem 6, p. 34] or in [33, Lemma 6.1.1, p. 268]. In what follows, D_i^j stands for the j th order differential of a function with respect to the i th variable¹.

Lemma 1. Suppose that $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is any function which is \mathcal{C}^s , with $s \geq 2$. Fix $\mathbf{u} \in \mathbb{R}^n$. Let $\hat{\mathbf{f}} \in \mathbb{R}^n$ be such that $D_1 \mathcal{J}(\hat{\mathbf{f}}, \mathbf{u}) = 0$ and $D_1^2 \mathcal{J}(\hat{\mathbf{f}}, \mathbf{u})$ is positive definite. Then there exists an open neighborhood of \mathbf{u} , say \mathcal{O} , and a unique \mathcal{C}^{s-1} minimizer function $\mathcal{F} : \mathcal{O} \rightarrow \mathbb{R}^n$ such that $\mathcal{F}(\mathbf{u}) = \hat{\mathbf{f}}$.

We will see that in our case, the minimizer function is uniquely defined on \mathbb{R}^n .

Proposition 1. Let $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ in (2) satisfy H1. Then for any $\beta > 0$, \mathcal{J} has a unique minimizer function $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ which is \mathcal{C}^{s-1} continuous.

The proof of the proposition is given in Appendix VII-A. The components of the minimizer function \mathcal{F} read \mathcal{F}_i , $i \in \mathbb{I}_n$. A diagonal matrix A with diagonal entries $a[i]$, $i \in \mathbb{I}_n$, is denoted by $A = \text{diag}(\{a[i]\}_{i=1}^n)$.

Lemma 2. Let $\beta > 0$ be arbitrary and $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ in (2) satisfy H1. Then its Hessian matrix $H(\mathbf{u}) \stackrel{\text{def}}{=} D_1^2 \mathcal{J}(\mathcal{F}(\mathbf{u}), \mathbf{u})$, given by

$$H(\mathbf{u}) = \text{diag}(\{\psi''(\mathcal{F}_i(\mathbf{u}) - \mathbf{u}[i])\}_{i=1}^n) + \beta G^T \text{diag}(\{\phi''(\mathbf{g}_i^T \mathcal{F}(\mathbf{u}))\}_{i=1}^r) G \quad (5)$$

is invertible. Consequently, the differential $D\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of the minimizer function \mathcal{F} of \mathcal{J} reads

$$D\mathcal{F}(\mathbf{u}) = \begin{bmatrix} D\mathcal{F}_1(\mathbf{u}) \\ \dots \\ D\mathcal{F}_p(\mathbf{u}) \end{bmatrix} = (H(\mathbf{u}))^{-1} \text{diag}(\{\psi''(\mathcal{F}_i(\mathbf{u}) - \mathbf{u}[i])\}_{i=1}^n)$$

and satisfies

$$\text{rank}(D\mathcal{F}(\mathbf{u})) = n, \quad \forall \mathbf{u} \in \mathbb{R}^n. \quad (6)$$

This simple lemma, proven in Appendix VII-B, underlies the main theoretical results in this work.

B. Crucial features of the minimizers of $\mathcal{J}(\cdot, \mathbf{u})$ and discussion

First, we show that the minimizer function \mathcal{F} is non-expansive.

Proposition 2. Let for an arbitrary $\beta > 0$, the cost function $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ in (2) satisfy H1. Then its minimizer function $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is nonexpansive:

$$(\mathbf{u}, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n \quad \Rightarrow \quad \|\mathcal{F}(\mathbf{u} + \zeta) - \mathcal{F}(\mathbf{u})\|_2 \leq \|\zeta\|_2. \quad (7)$$

One can note that \mathcal{F} is Lipschitz with constant 1. The proof is given in Appendix VII-E.

Our main questions are whether or not some entries of a minimizer $\hat{\mathbf{f}}$ of $\mathcal{J}(\cdot, \mathbf{u})$ can: (i) take equal values and (ii) be equal to some components of the input image \mathbf{u} . For (i), we consider the set \mathcal{G} which is composed

¹E.g., $D_1^j \mathcal{J}$ is the j th differential of \mathcal{J} in (2) with respect to \mathbf{f} and $D_2^j \mathcal{J}$ —with respect to \mathbf{u} .

of all operators that yield the difference between any two pixels in an image:

$$\mathcal{G} \stackrel{\text{def}}{=} \bigcup_{(i,j) \in \mathbb{I}_n \times \mathbb{I}_n} \{g \in \mathbb{R}^n \mid g[i] = -g[j] = 1, i \neq j, (i, j) \in \mathbb{I}_n, g[k] = 0, k \in \mathbb{I}_n \setminus (i \cup j)\} . \quad (8)$$

Observe that all difference operators in (4) satisfy $\mathbf{g}_i \in \mathcal{G}, \forall i \in \mathbb{I}_r$.

Our main result is stated below. Its proof is presented in Appendix VII-C. As usual, the Lebesgue measure on \mathbb{R}^n is denoted by $\mathbb{L}^n(\cdot)$.

Theorem 1. *Let $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ in (2) satisfy H1. For its minimizer function $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, define the sets $K_{\mathcal{G}}$ and $K_{\mathcal{I}}$ as follows:*

$$K_{\mathcal{G}} = \bigcup_{g \in \mathcal{G}} \{\mathbf{u} \in \mathbb{R}^n \mid g^T \mathcal{F}(\mathbf{u}) = 0\}, \quad (9)$$

$$K_{\mathcal{I}} = \bigcup_{i \in \mathbb{I}_n} \bigcup_{j \in \mathbb{I}_n} \{\mathbf{u} \in \mathbb{R}^n \mid \mathcal{F}_i(\mathbf{u}) = \mathbf{u}[j]\} . \quad (10)$$

Then $K_{\mathcal{G}}$ and $K_{\mathcal{I}}$ are closed in \mathbb{R}^n and obey

$$\mathbb{L}^n(K_{\mathcal{G}}) = 0 \quad \text{and} \quad \mathbb{L}^n(K_{\mathcal{I}}) = 0 .$$

The result holds true for any $\beta > 0$.

The sets $K_{\mathcal{G}}$ in (9) contains all possible $\mathbf{u} \in \mathbb{R}^n$ such that a minimizer $\hat{\mathbf{f}} = \mathcal{F}(\mathbf{u})$ of $\mathcal{J}(\cdot, \mathbf{u})$ might have two equal entries, $\hat{\mathbf{f}}[i] = \hat{\mathbf{f}}[j]$ for some $i \neq j$ where $(i, j) \in \mathbb{I}_n \times \mathbb{I}_n$. The set $K_{\mathcal{I}}$ in (10) contains all possible $\mathbf{u} \in \mathbb{R}^n$ such that the minimizer $\hat{\mathbf{f}} = \mathcal{F}(\mathbf{u})$ might contain some entries equal to data entries, $\mathcal{F}_i(\mathbf{u}) = \mathbf{u}[j]$ for some $(i, j) \in \mathbb{I}_n \times \mathbb{I}_n$.

According to Theorem 1,

- The elements of $K_{\mathcal{G}}$ are highly exceptional in \mathbb{R}^n . Indeed, the subset $\mathbb{R}^n \setminus K_{\mathcal{G}}$ contains an *open and dense* subset of \mathbb{R}^n . The chance that a truly random $\mathbf{u} \in \mathbb{R}^n$ —i.e. a \mathbf{u} following a non-singular probability distribution on \mathbb{R}^n —happens to be in $K_{\mathcal{G}}$ can be ignored in practice. Conversely, $\mathcal{F}_i(\mathbf{u}) \neq \mathcal{F}_j(\mathbf{u})$, for $i \neq j$, is a *generic* property² of the minimizer $\mathcal{F}(\mathbf{u})$ of $\mathcal{J}(\cdot, \mathbf{u})$, as given in (2) and satisfying H1.
- The event $\mathcal{F}_i(\mathbf{u}) = \mathbf{u}[j]$ is highly exceptional, as anticipated in Remark (a) “Motivation”. The minimizer functions $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ relevant to \mathcal{J} *generically* satisfy $\mathcal{F}_i(\mathbf{u}) \neq \mathbf{u}[j]$, for all $(i, j) \in \mathbb{I}_n \times \mathbb{I}_n$.

These are general results that hold for any real data $\mathbf{u} \in \mathbb{R}^n$.

Remark (b): Discussion. Let us denote by $\mathcal{S}_{\mathcal{P}}^n$ the set of all $M \times N$ images whose $n = MN$ pixels values belong to the quantization set \mathcal{P} in (1):

$$\mathcal{S}_{\mathcal{P}}^n = \{u \in \mathbb{R}^n \mid u[i] \in \mathcal{P}, \forall i \in \mathbb{I}_n\} . \quad (11)$$

Recall that $|\mathcal{P}| = L$, the cardinality of $\mathcal{S}_{\mathcal{P}}^n$ is $|\mathcal{S}_{\mathcal{P}}^n| = L^n$. Even though $|\mathcal{S}_{\mathcal{P}}^n|$ is a *huge* number³, the set $\mathcal{S}_{\mathcal{P}}^n$ is finite, hence

$$\mathbb{L}^n(\mathcal{S}_{\mathcal{P}}^n) = 0 .$$

Hence the question: *what can we say about the possible intersections $K_{\mathcal{G}} \cap \mathcal{S}_{\mathcal{P}}^n$ and $K_{\mathcal{I}} \cap \mathcal{S}_{\mathcal{P}}^n$?*

²Here we use the terminology in [20].

³For 512×512 , 8-bits images, this value is 255^{512^2} —the amount of all 512×512 , 8-bits pictures that people can ever take with their digital cameras.

Assume that \mathcal{P} is composed of L integers. Let $N_{\mathcal{P}}$ be the set of the functions that map \mathcal{P}^n onto \mathcal{P} . We would not like that the minimizer function \mathcal{F} has some components belonging to $N_{\mathcal{P}}$. Number theory gives limited answers to the question of the kind of functions being able to be in $N_{\mathcal{P}}$. For $n = 1$, every function applied to an integer $u \in \mathcal{P}$ and yielding an integer $\hat{f} = f(u)$ is of the form

$$f(u) = \sum_{k \in \mathbb{I}_u} b_k \binom{u}{k} \quad \text{where} \quad \binom{u}{k} = \frac{u(u-1) \cdots (u-k+1)}{k!},$$

where *all* b_k are integers. The question was initially posed in [49]. This result along with some refinements can be found in [9]. However, $f(u) \in N_{\mathcal{P}}$ requires also that $f(u) \in \mathcal{P}$, which *drastically limits*⁴ the functions of this form that fall into $N_{\mathcal{P}}$. In the case of several variables, some polynomial functions may belong to $N_{\mathcal{P}}$, see [10]. More generally, Diophantine equations [30] can also be cast in the class of polynomial functions. Under the (severe) restriction $f(u) \in \mathcal{P}$, some of them also live in $N_{\mathcal{P}}$. To the best of our knowledge, no other families of functions were shown to be able to belong to $N_{\mathcal{P}}$.

However, given the expression for $D\mathcal{F}$ in Lemma 2, it is not difficult to see that no component \mathcal{F}_i of our minimizer function \mathcal{F} can have a polynomial expression.

Let $\mathbf{u} = c\mathbb{1}$ for some $c \in \mathcal{P}$. Then for any $\beta > 0$, the minimizer $\hat{\mathbf{f}}$ of $\mathcal{J}(\cdot, \mathbf{u})$ reads $\hat{\mathbf{f}} = \mathbf{u}$. Indeed, since $c\mathbb{1} \in \ker G$, we have $\Psi(\mathbf{u}, \mathbf{u}) = 0$ and $\Phi(\mathbf{u}) = \beta r \phi(0)$, so \mathcal{J} reaches its lower bound for $\hat{\mathbf{f}} = \mathbf{u}$. Hence all constant digital images meet

$$\{c\mathbb{1} \mid c \in \mathcal{P}\} \in K_G \cap \mathcal{S}_{\mathcal{P}}^n \quad \text{and} \quad \{c\mathbb{1} \mid c \in \mathcal{P}\} \in K_{\mathcal{I}} \cap \mathcal{S}_{\mathcal{P}}^n.$$

Consequently,

$$K_G \cap \mathcal{S}_{\mathcal{P}}^n \neq \emptyset \quad \text{and} \quad K_{\mathcal{I}} \cap \mathcal{S}_{\mathcal{P}}^n \neq \emptyset.$$

One can ask what histogram modification would be needed for a constant image. However, there are other *simple images* that belong to $(K_G \cup K_{\mathcal{I}}) \cap \mathcal{S}_{\mathcal{P}}^n$. We can reasonably conjecture the following:

- $K_G \cap \mathcal{S}_{\mathcal{P}}^n$ is *essentially* composed of simple (synthetic, in practice) images. For most of them, if some histogram modification was needed, it should be defined in a proper way.
- The ratios

$$\frac{|(K_G \cap \mathcal{S}_{\mathcal{P}}^n)|}{|\mathcal{S}_{\mathcal{P}}^n|} \quad \text{and} \quad \frac{|(K_{\mathcal{I}} \cap \mathcal{S}_{\mathcal{P}}^n)|}{|\mathcal{S}_{\mathcal{P}}^n|}$$

should be numbers close to zero.

Being impossible to prove such a conjecture, we cannot *guarantee* that for *any digital image* $\mathbf{u} \in \mathcal{P}^n$, the entries of the minimizer $\hat{\mathbf{f}} = \mathcal{F}(\mathbf{u})$ can be ordered in a strictly increasing way. However, the numerical results on 8-bit images in Table V (in section V) shows that our method enables strict ordering for all tested images.

C. Selection of the parameters

Objectives \mathcal{J} of the form given by (2)–(3) belong to a wider family of objectives satisfying H1 and H2 that were studied by the authors in [5]. Below we sketch some of the main results of [5] as far as they are important to tune our applications. Using H1, ψ' is odd and strictly increasing on \mathbb{R} for any $\alpha_1 > 0$. Therefore, for any $\alpha_1 > 0$, $t \mapsto \psi'(t, \alpha_1)$ is invertible on $(-1, 1)$. So the function

$$b(y, \alpha_1) \stackrel{\text{def}}{=} (\psi')^{-1}(y, \alpha_1) \in \mathbb{R}, \quad \forall y \in (-1, +1), \quad \forall \alpha_1 > 0, \quad (12)$$

⁴These polynomial functions yield also arbitrarily large values that exceed the bounded set \mathcal{P} .

is well defined. Combining our assumptions with [5, Remark 1] shows that for any α_1 fixed, $y \mapsto b(y, \alpha_1)$ is continuous, odd and strictly increasing. Under Neumann boundary conditions, $|\mathcal{N}_i|$ is constant for any $i \in \mathbb{I}_n$. Let us denote $\eta \stackrel{\text{def}}{=} |\mathcal{N}_i|$. If we consider the four nearest neighbors then $\eta = \mathcal{N}_i = 4$ and for the eight nearest neighbors, $\eta = 8$. Given \mathbf{u} , let $\hat{\mathbf{f}}$ be the minimizer of $\mathcal{J}(\cdot, \mathbf{u})$. According to [5, Theorems 1 and 2]

$$\beta < \frac{1}{\eta} \quad \Rightarrow \quad b(c\beta\eta, \alpha_1) \leq \|\hat{\mathbf{f}} - \mathbf{u}\|_\infty < b(\beta\eta, \alpha_1), \quad (13)$$

where

$$c = \phi'(\nu_{\mathbf{u}} - 2b(\beta\eta, \alpha_1), \alpha_2) < 1 \quad (14)$$

and $\nu_{\mathbf{u}}$ is a variable computed from the input image. It is worth emphasizing that the upper bound in (13) is independent of $\phi(t, \alpha_2)$. Using our (stronger) assumptions and [5, Lemma 2] entails that for $\beta\eta < 1$ fixed, $\alpha_1 \mapsto b(\beta\eta, \alpha_1)$ is strictly decreasing on $(0, +\infty)$ and $b(\beta\eta, \alpha_1) \searrow 0$ as $\alpha_1 \searrow 0$. Hence we can choose $\beta \in (0, 1/\eta)$ and $\alpha_1 > 0$ such that the upper bound in (13) equals any fixed number. In our application, we will require that this bound is quite small, around 0.1 in order to guarantee that $\hat{\mathbf{f}}$ is close enough to \mathbf{u} . It is also shown in [5, Theorem 3] that $\nu_{\mathbf{u}} - 2b(\beta\eta, \alpha_1) > 0$ is generically quite a large number for real-world digital images. From (14) and H2, decreasing $\alpha_2 > 0$ towards zero increases c close to one, hence $\|\hat{\mathbf{f}} - \mathbf{u}\|_\infty$ can be arbitrarily close to $b(\beta\eta, \alpha_1)$.

For instance, if $\psi(t) = \sqrt{t^2 + \alpha_1}$, $\phi(t) = \sqrt{t^2 + \alpha_2}$ and $|\mathcal{N}_i| = 4$, we have $b(y, \alpha_1) = y\sqrt{\frac{\alpha_1}{1-y^2}}$, hence

$$\beta < \frac{1}{4} \quad \Rightarrow \quad b(\beta\eta, \alpha_1) = 4\beta\sqrt{\frac{\alpha_1}{1-(4\beta)^2}} \quad \text{and} \quad c = \frac{\nu_{\mathbf{u}} - 2b(\beta\eta, \alpha_1)}{(\nu_{\mathbf{u}} - 2b(\beta\eta, \alpha_1))^2 + \alpha_2}. \quad (15)$$

IV. NUMERICAL SCHEME

A. Algorithm

Given an input image \mathbf{u} , the minimizer $\hat{\mathbf{f}}$ of $\mathcal{J}(\cdot, \mathbf{u})$ provides auxiliary information to sort the pixels of \mathbf{u} . More precisely, 2-tuples $(\mathbf{u}[i], \hat{\mathbf{f}}[i])$ are used as described next

$$\forall i \neq j, \quad \left\{ \begin{array}{l} \left[\mathbf{u}[i] < \mathbf{u}[j] \right] \quad \text{or} \quad \left[\mathbf{u}[i] = \mathbf{u}[j] \quad \text{and} \quad \hat{\mathbf{f}}[i] < \hat{\mathbf{f}}[j] \right] \quad \Rightarrow \quad i \prec j ; \\ \left[\mathbf{u}[i] > \mathbf{u}[j] \right] \quad \text{or} \quad \left[\mathbf{u}[i] = \mathbf{u}[j] \quad \text{and} \quad \hat{\mathbf{f}}[i] > \hat{\mathbf{f}}[j] \right] \quad \Rightarrow \quad i \succ j . \end{array} \right. \quad (16)$$

The resulting algorithm for exact histogram specification using the variational approach presented in section III is summarized in Algorithm 1.

Algorithm 1 Exact Histogram Specification Using Variational Approach

Input: the input image \mathbf{u} , the specified histogram \mathbf{h} and the intensity value set \mathbf{p} .

Output: the specified image \mathbf{v} .

- 1: Initialize α_1, α_2 and β according to (13) for b small enough;
 - 2: $\hat{\mathbf{f}} = \arg \min_{\mathbf{f}} \mathcal{J}(\mathbf{f}, \mathbf{u})$;
 - 3: Set the tuple $\kappa[i] \leftarrow (\mathbf{u}[i], \hat{\mathbf{f}}[i])$, $\forall i \in \mathbb{I}_n$;
 - 4: Start from the first pixels on the ordering list, assign the first $\mathbf{h}[1]$ pixels with intensity value $\mathbf{p}[1]$, the next $\mathbf{h}[2]$ pixels with intensity value $\mathbf{p}[2]$, and so on until all pixels are assigned to their new intensities; the resulting image is the specified image \mathbf{v} .
 - 5: **return** \mathbf{v} .
-

Step 2 in the algorithm is aimed to reach a solution $\hat{\mathbf{f}}$ such that (16) can be applied to all pairs (i, j) , $i \neq j$ ¹¹. This algorithm is quite general and can be applied using any functions (ψ, ϕ) obeying H1 and H2—for some examples see Table I.

A clear advantage of our method compared with the LM and the WA methods is that in our case ordering is obtained by comparing only two images—the input \mathbf{u} and the minimizer $\hat{\mathbf{f}}$. The LM method [17] needs to compare (and keep in memory) at least 6 images. For WA method, this number is 9 because the Haar wavelet basis with 2×2 filters and 4×4 filters is used.

B. Implementation of Algorithm 1

There are many algorithms in literature to compute the minimizer $\hat{\mathbf{f}}$ of $\mathcal{J}(\cdot, \mathbf{u})$ in step 2 of the Algorithm. These include partial differential equation based methods such as explicit method [54], semi-implicit method [36], operator splitting [42], lagged diffusivity fixed point iterations [57], Polak-Ribière CG method [6], majorization-minimization algorithms [7], [24], [31], Nesterov algorithms [40] and so on. We used a pseudo-conjugated Polak-Ribière minimization method [6].

For our pixel sorting method, we minimize \mathcal{J} as defined by (3) where \mathcal{N}_i is the set of the four adjacent neighbors and

$$\psi(t) = \sqrt{t^2 + \alpha_1} \quad \text{and} \quad \phi(t) = \sqrt{t^2 + \alpha_2} . \quad (17)$$

We have $\eta = |\mathcal{N}_i| = 4$ hence we must use $\beta < 0.25$. We select

$$\alpha_1 = \alpha_2 = 0.05 \quad \text{and} \quad \beta = 0.1 . \quad (18)$$

According to (15) and (17) we are guaranteed that

$$\|\hat{\mathbf{f}} - \mathbf{u}\|_\infty < 0.0976,$$

which is a reasonable bound to obtain a real-valued image close enough to \mathbf{u} . For the images shown in Fig. 4, the constant $\nu_{\mathbf{u}}$ in (14) ranges between 52 and 205. For our choice of α_2 , the constant c in (13) satisfies $c \in (1 - 10^{-5}, 1)$. Therefore, for all these images we have the good property that

$$\|\hat{\mathbf{f}} - \mathbf{u}\|_\infty \lesssim 0.0976 .$$

C. Some practical limitations

In spite of the theory presented in section III, it can occur that the minimizer $\hat{\mathbf{f}}$ of $\mathcal{J}(\cdot, \mathbf{u})$ contains some equally valued pixels. Several important reasons are mentioned next.

- The real numbers that a computer treats are in fact just a large but finite set of numbers. E.g. Matlab cannot distinguish numbers smaller than 2.2×10^{-16} .
- The theory supposes that we deal with *exact* minimizers of \mathcal{J} , yet in practice we cannot get such minimizers. Note that a \mathcal{J} satisfying H1 and H2 contains large nearly flat regions, so its minimization is not easy. The algorithm being initialized with the digital input image \mathbf{u} , an *inexact* minimizer $\hat{\mathbf{f}}$ of \mathcal{J} might contain pixels with equal values. Indeed, we have observed that in such cases, increasing the precision of the minimization method usually enables to strictly sort all pixels of $\hat{\mathbf{f}}$. However, in some cases this could require long computations and for practical reasons, a compromise is done by fixing the maximum number of iterations.

- If the digital image contains large constant regions or if it involves several equal patterns having the same background (this typically can arise for images having a low compression rate), one should expect that the resultant $\hat{\mathbf{f}}$ can have pixels sharing the same value. There might be some strategies to improve sorting in case of failure. It is easy to see that the minimizer function $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is differentiable with respect to all parameters $\alpha_1, \alpha_2, \beta$. These parameters can slightly be changed so that $\hat{\mathbf{f}}$ remains close enough to \mathbf{u} —rigorous tools for such modifications are provided in subsection III-C. Also, instead of the four adjacent neighbors, one may consider the 8 nearest neighbors. Any such change modifies the possible intersections of $K_{\mathcal{G}}$ and $K_{\mathcal{I}}$ in Theorem 1 with the set $S_{\mathcal{P}}^n$ in (11). These questions are not considered here as they would lead to a trickier algorithm.
- Last, we remind that the set $K_{\mathcal{G}} \cup S_{\mathcal{P}}^n$ is not empty.

V. EXPERIMENTAL RESULTS

Two practical problems are considered next: contrast compression restoration and equalization inversion using exact histogram specification. Our method to order the pixels was applied as described in (17) and (18). We ran the Polak-Ribière CG minimization with stopping rule given by $\|D_1\mathcal{J}(\hat{\mathbf{f}}^k, \mathbf{u})\|_{\infty} \leq n \times 10^{-6}$ and limiting the iteration number to 35. Our method was compared with the local mean (LM) algorithm [17] for $K = 6$ and with the wavelet-based algorithm (WA) [58] for Haar wavelet for $K = 9$, as recommended by the authors. The experiments were performed using a Mac OS X 10.7.2 and MATLAB v7.12 on a MacBook Air Laptop with an Intel Core i5 1.7 GHZ processor and 4GB of RAM.

Here we present numerical results on 15 digital images of different sizes with gray values in $\{0, \dots, 255\}$. They are available at <http://sipi.usc.edu/database/>. We selected images with various sizes, quality and content (presence or quasi-absence of edges, textures, nearly flat regions). The images and their histograms are shown in Fig. 4. We observe that most of these images have quite a singular histogram.

In order to measure the results quantitatively, we start out with a given true quantized image \mathbf{w} with histogram $\mathbf{h}_{\mathbf{w}}$, we degrade it to obtain an input quantized image \mathbf{u} . By applying the three methods on \mathbf{u} with prescribed histogram $\mathbf{h}_{\mathbf{w}}$, we obtain an output image \mathbf{v} which is in fact a restored version of \mathbf{w} . We use peak-signal-to-noise-ratio to measure the quality of the output image \mathbf{v} with respect to \mathbf{w} , defined as $\text{PSNR} = 20 \log_{10}(255NM/\|\mathbf{v} - \mathbf{w}\|_2)$.

One should keep in mind that

- the performance of LM and WA decrease when the number of gray-values in the input image decreases;
- LM is less efficient for larger image sizes.

A. Restoration of Contrast Compression

The input image \mathbf{u} is obtained from \mathbf{w} by the degradation: $\mathbf{u} = \text{fix}(\mathbf{w}/2^{8-k})$, where $k = 1, 2, \dots, 7$. We show the process of contrast compression in Figure 5. This situation arises when a picture is taken with insufficient exposure time, or when we want to compress the image by reducing the number of intensity levels. For example, a 7-bit image can be obtained from an 8-bit image by using $k = 7$. In the tests, we used LM, WA and our method to obtain the output images \mathbf{v} having a prescribed histogram $\mathbf{h}_{\mathbf{w}}$.

We compare LM, WA and our algorithm. The comparison results are shown in Table II–IV for 3, 4, 5 bits per pixel, respectively, where the best results are printed in boldface. We see from the PSNR values that our method outperforms LM and WA in quality in all cases; this can be explained by the constraint that $\|\hat{\mathbf{f}} - \mathbf{u}\|_{\infty} < 0.1$ and that our algorithm enables a strongly better strict ordering.

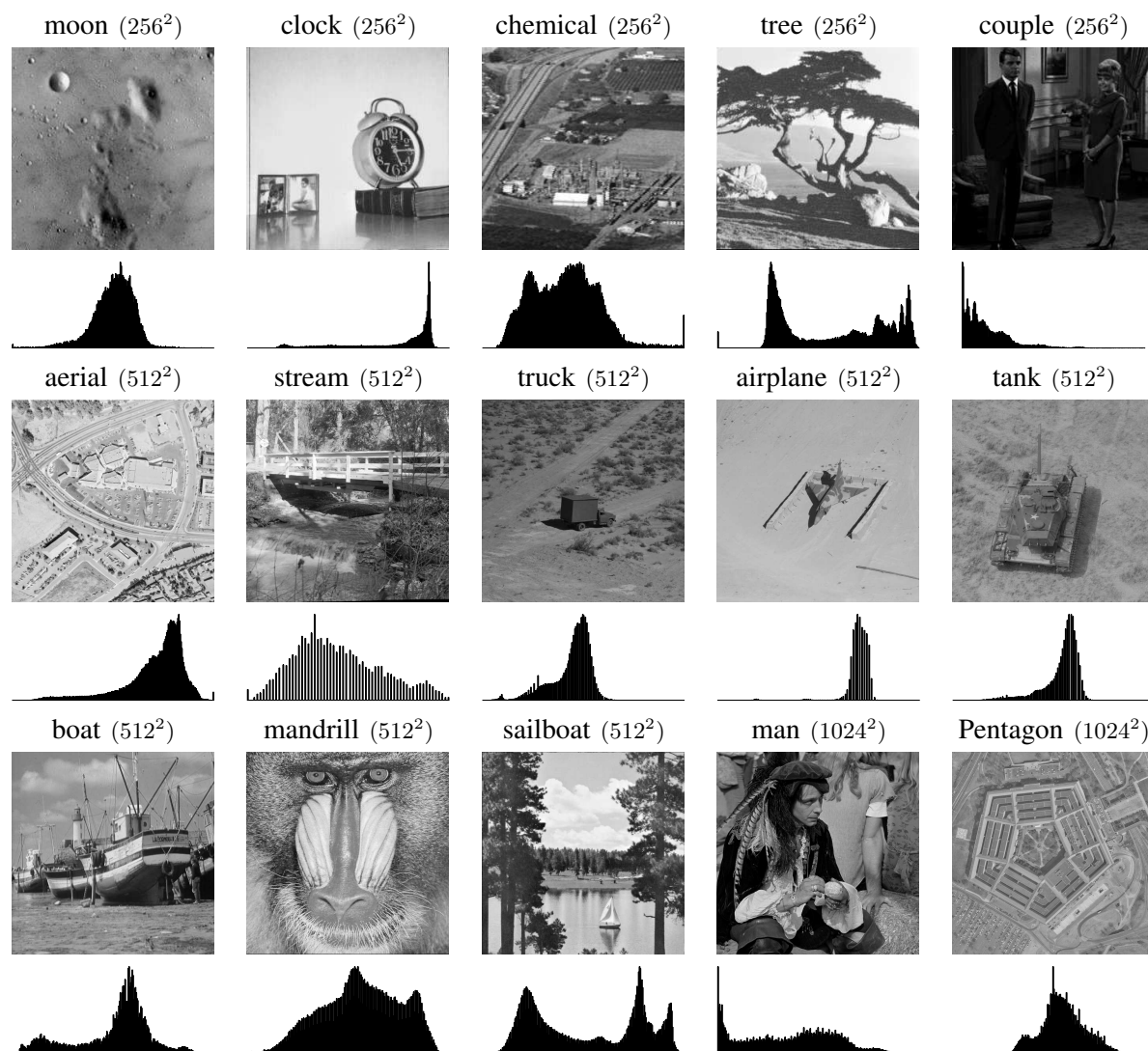


Fig. 4: All 15 digital images used in the tests in this section and their histograms. The gray values of these images belong to $\{0, \dots, 255\}$.

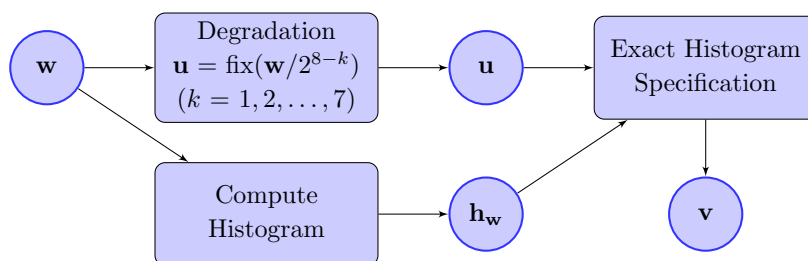


Fig. 5: Contrast Compression.

One important indicator for a good exact histogram specification algorithm is to see if it can establish a strict ordering for all the pixels. If a sorting method yields two or more pixels sharing the same value we call the pixels failure pixels. Table II–IV shows the percentages of failure pixels produced by the three methods. We find that LM and WA have a high number of equal-valued pixels.

Image	PSNR			Fail %			CPU		
	LM	WA	Ours	LM	WA	Ours	LM	WA	Ours
moon	31.15	30.70	31.68	52.29	75.75	0.04	0.44	0.96	1.08
clock	31.32	30.68	32.60	67.93	76.58	30.12	0.37	0.74	0.74
chemical	30.20	29.24	30.47	34.28	55.97	0.04	0.40	0.77	1.08
tree	29.86	29.20	30.38	46.45	59.63	6.15	0.43	0.82	1.02
couple	29.30	28.77	29.74	75.90	82.72	12.83	0.34	0.77	1.03
aerial	29.39	28.62	29.74	57.83	67.72	3.19	1.68	4.49	4.19
stream	29.22	28.60	29.46	34.68	57.85	0.06	1.78	4.32	3.68
truck	32.03	31.34	32.36	69.49	89.60	2.33	1.70	4.55	3.90
airplane	31.16	30.76	32.86	88.27	94.37	41.20	1.36	4.37	3.88
tank	31.27	30.74	31.67	75.13	89.74	0.60	1.79	4.71	3.97
boat	30.59	29.75	31.18	63.46	78.81	7.76	1.67	4.33	3.65
mandrill	28.65	28.21	28.95	27.13	46.73	0.10	1.75	4.60	3.97
sailboat	30.72	29.74	31.34	56.14	71.63	6.40	1.63	4.10	3.48
man	31.27	30.42	31.62	75.60	87.92	6.89	8.27	20.81	12.70
Pentagon	30.33	29.84	30.66	68.82	87.35	0.58	9.56	23.52	13.66
means	30.43	29.77	30.98	59.56	74.83	7.89	2.21	5.59	4.14

TABLE II: Data are 3-bits images. The PSNR (dB) between the true image w and the output image v .

Image	PSNR			Fail %			CPU		
	LM	WA	Ours	LM	WA	Ours	LM	WA	Ours
moon	35.25	34.74	35.51	16.75	40.14	0.00	0.47	1.20	0.95
clock	36.14	35.66	37.52	52.54	62.66	18.17	0.32	0.77	1.15
chemical	34.71	33.83	35.05	8.70	20.01	0.01	0.41	0.81	1.00
tree	34.76	34.34	34.95	22.15	36.02	4.05	0.34	0.72	1.13
couple	35.11	34.43	35.63	49.56	60.04	6.96	0.44	0.73	0.89
aerial	34.46	33.76	34.95	28.56	42.21	0.36	1.79	4.38	3.87
stream	34.18	33.73	34.49	9.61	20.25	0.00	1.76	5.62	4.12
truck	35.93	35.23	36.19	39.81	60.85	0.50	1.92	5.92	3.50
airplane	37.72	37.28	38.70	75.92	88.18	7.88	1.99	4.64	3.99
tank	35.49	34.90	35.69	44.58	64.84	0.01	2.04	5.63	4.03
boat	35.67	34.77	36.13	32.86	52.41	0.29	2.06	5.18	4.22
mandrill	33.40	33.12	33.75	8.15	17.16	0.03	1.88	4.80	4.17
sailboat	35.29	34.65	35.52	27.94	46.96	0.51	1.76	4.70	4.33
man	35.78	34.94	36.13	46.59	64.64	3.43	8.84	24.76	11.91
Pentagon	34.71	34.32	35.16	37.84	61.45	0.05	10.04	26.14	12.33
means	34.73	34.07	35.20	32.80	48.80	2.53	2.33	6.53	4.07

TABLE III: Data are 4-bits images. The PSNR (dB) between the true image w and the output image v .

B. Histogram Equalization Inversion

We consider the application of the histogram equalization inversion, which is to recover the true quantized image w with histogram \mathbf{h}_w from its specified version \mathbf{u} . This was the beacon application used in [17] to demonstrate the interest of the LM method. Let $\mathbf{u} = T(\mathbf{w}, \mathbf{h})$ be the process to specify the image w with histogram \mathbf{h}_w to an image \mathbf{u} such that the histogram of \mathbf{u} is \mathbf{h} . The quantized image w can be exactly

Image	PSNR			Fail %			CPU		
	LM	WA	Ours	LM	WA	Ours	LM	WA	Ours
moon	39.69	39.37	39.81	0.62	3.44	0.00	0.47	0.98	1.16
clock	41.24	40.79	42.29	39.10	48.76	8.25	0.40	0.75	0.83
chemical	39.52	38.95	39.78	0.72	2.21	0.00	0.49	0.76	1.14
tree	39.78	39.51	40.17	13.40	18.34	1.16	0.33	0.77	0.93
couple	40.56	39.85	41.17	24.82	33.59	0.80	0.41	1.22	1.00
aerial	39.70	39.19	40.20	12.38	19.05	0.08	1.85	4.35	4.67
stream	39.99	39.72	40.14	1.82	4.16	0.00	1.71	4.30	4.05
truck	40.31	39.77	40.52	8.03	22.80	0.00	2.20	4.75	3.87
airplane	41.97	41.75	42.16	60.54	77.30	0.26	1.83	4.41	3.75
tank	39.89	39.45	40.03	8.41	16.95	0.00	1.96	4.55	3.94
boat	39.86	39.34	40.11	7.11	19.02	0.05	1.62	4.32	3.96
mandrill	38.71	38.52	38.93	0.36	1.91	0.00	1.60	3.75	8.69
sailboat	39.74	39.45	39.88	9.94	19.91	0.03	1.64	4.30	8.76
man	40.24	39.68	40.44	15.38	26.30	0.18	8.25	20.98	12.60
Pentagon	39.30	39.07	39.56	5.47	20.11	0.01	8.51	21.64	11.84
means	39.40	39.07	39.93	13.40	21.80	0.60	2.20	5.47	4.87

TABLE IV: Data are 5-bits images. The PSNR (dB) between the true image w and the output image v .

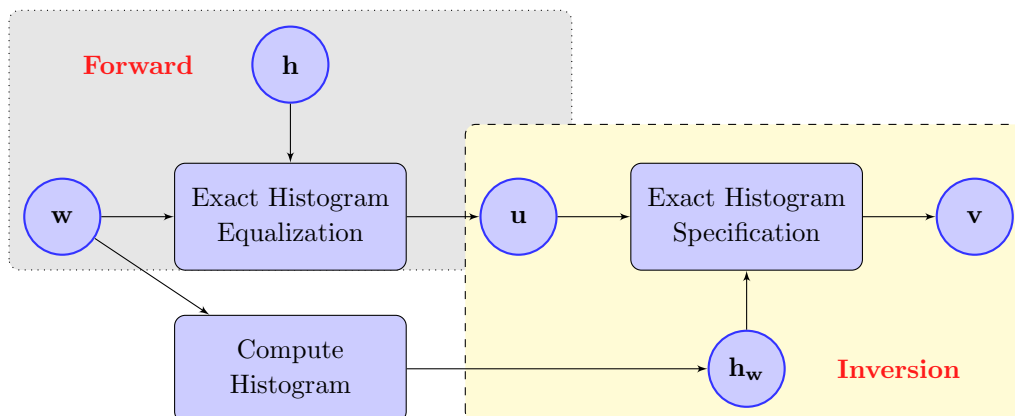


Fig. 6: Histogram Equalization Inversion. A true quantized image w with histogram h_w is equalized to the input image u using exact histogram specification method. The task is to restore w from the input image u and the prescribed histogram h_w using the same exact histogram specification method. The specified image v is a restored version of w .

recovered by $T(u, h_w)$ under the hypothesis of order preservation by the method. Since the ordering among the pixels of w is not identical with that among pixels of u , the recovered image $v = T(u, h_w)$ is just an approximation of w . We show the process of histogram equalization inversion in Figure 6.

Table V shows the PSNR, the failure percentage, the CPU running time of the results by the three methods. We notice from Table V that WA method yields better PSNR than LM method in all images. In all these cases, our results are much better. Whereas our method can sort all the pixels in all the 8-bit images tested, the only 8-bit image where the other two methods successfully sort is the "mandrill" image. All these results show that the ordering enabled by our method better fits natural digital images.

Figures 8 give the enlarged portions of the difference images on "Aerial", "Couple", "Man" images, the corresponding enlarged portions of the input image are shown in Figure 7. We can discern more features in

the difference images by both LM method and WA method than by our method. It demonstrates that our algorithm yields the best restoration.

Image	PSNR			Fail %			CPU		
	LM	WA	Ours	LM	WA	Ours	LM	WA	Ours
moon	47.39	46.50	47.83	0.07	0.11	0.00	0.67	1.10	3.58
clock	51.69	51.56	51.79	0.53	2.41	0.00	0.65	1.20	2.01
chemical	49.34	48.90	49.67	0.03	0.10	0.00	0.67	1.06	3.05
tree	51.94	51.84	52.02	0.03	0.18	0.00	0.54	1.12	2.81
couple	44.23	43.71	44.66	1.38	1.97	0.00	0.60	1.13	2.35
aerial	48.36	48.06	50.06	0.00	0.02	0.00	2.86	6.08	9.35
stream	44.76	45.00	45.08	0.37	0.71	0.00	2.65	6.13	7.98
truck	49.67	49.51	49.96	0.12	0.24	0.00	2.82	5.90	12.43
airplane	46.74	46.26	47.24	1.73	8.96	0.00	3.01	7.40	8.40
tank	48.11	48.15	48.34	0.02	0.06	0.00	2.95	6.70	12.53
boat	49.51	49.58	49.89	0.06	0.09	0.00	2.88	5.97	13.08
mandrill	48.27	49.47	49.76	0.00	0.00	0.00	2.56	5.33	12.68
sailboat	50.85	50.72	51.30	0.01	0.05	0.00	2.58	5.29	17.16
man	49.23	49.18	49.44	0.17	0.40	0.00	13.03	28.01	39.96
Pentagon	50.69	50.62	51.36	0.01	0.01	0.00	13.74	29.32	39.48
means	48.72	48.60	49.23	0.30	1.02	0.00	3.48	7.45	12.46

TABLE V: The PSNR (dB) between the true image w and output images v .



Fig. 7: Enlarged portions of “Aerial” image, “Couple” image, and “Man” image, respectively.

VI. CONCLUSIONS AND PERSPECTIVES

In this paper, we propose a variational approach for exact histogram specification. Since the energy we minimize is smooth, its minimizers are real-valued which enables us to strictly order the pixels of generically all 8-bit images and to obtain much better ordering results than the competing methods in all cases. Our parameter choice guarantees that $\|\hat{\mathbf{f}} - \mathbf{u}\|_\infty < 0.1$ and the choice of \mathcal{J} is such that this small residual contains some quantization noise. This, combined with the much better ordering rate, explains why the obtained results outperform the existing methods. In all tested cases, our method gave not only a much better ordering, but also yielded the best PSNR. Hence the ordering provided by our method fits better all these various real-world digital images.

Our method is simple and obviously very efficient though we used a standard minimization algorithm. We should envisage the conception of a specialized minimization scheme in order to improve the computation speed. Extension to color images will also be considered in the future.

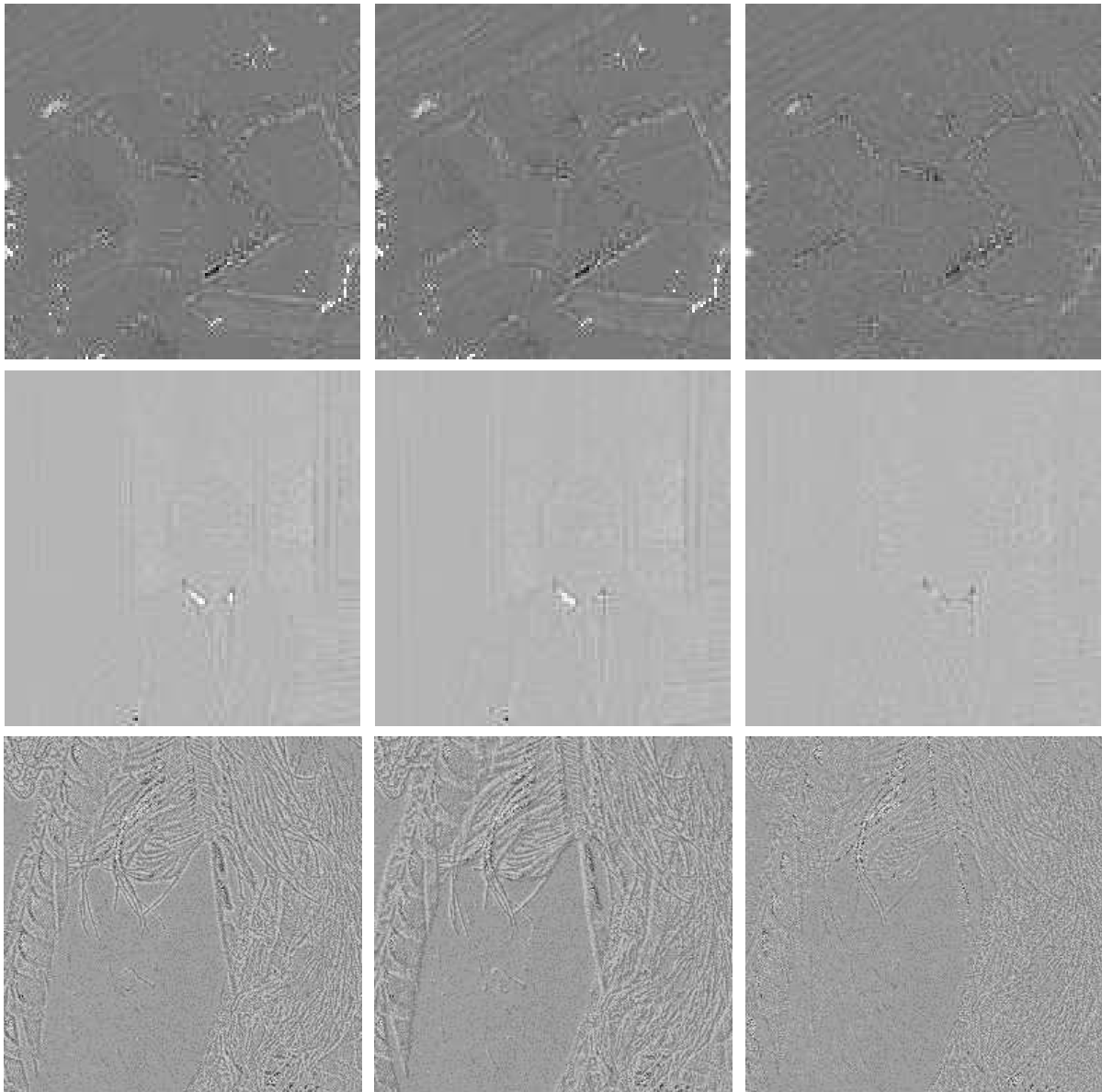


Fig. 8: The enlarged portions of the different images between \mathbf{w} and \mathbf{v} by LM (Left), WA (Middle) and our method (Right) for “Aerial” image (first row), “Couple” image (second row), and “Man” image (third row). Our method yields fewest features in the difference images.

VII. APPENDIX

Given a square matrix A , the expression $A \succ 0$ means that A is positive definite and $A \succeq 0$ that A is positive semi-definite.

A. Proof of Proposition 1

By H1, for any $\mathbf{u} \in \mathbb{R}^n$, the function $\mathcal{J}(\cdot, \mathbf{u})$ in (2) is strictly convex and coercive, hence for any \mathbf{u} and $\beta > 0$, it has a unique minimizer. Each minimizer point $\hat{\mathbf{f}}$ of $\mathcal{J}(\hat{\mathbf{f}}, \mathbf{u})$ is determined by $D_1\mathcal{J}(\hat{\mathbf{f}}, \mathbf{u}) = 0$. We have

$$0 = D_1\mathcal{J}(\hat{\mathbf{f}}, \mathbf{u}) = D_1\Psi(\hat{\mathbf{f}}, \mathbf{u}) + \beta D_1\Phi(\hat{\mathbf{f}}), \quad (19)$$

where

$$D_1\Psi(\widehat{\mathbf{f}}, \mathbf{u}) = \begin{bmatrix} \psi'(\widehat{\mathbf{f}}[1] - \mathbf{u}[1]) \\ \dots \\ \psi'(\widehat{\mathbf{f}}[p] - \mathbf{u}[p]) \end{bmatrix}^T \quad \text{and} \quad D_1\Phi(\widehat{\mathbf{f}}) = \begin{bmatrix} \phi'(\mathbf{g}_1^T\widehat{\mathbf{f}}) \\ \dots \\ \phi'(\mathbf{g}_r^T\widehat{\mathbf{f}}) \end{bmatrix}^T G. \quad (20)$$

Differentiation with respect to $\widehat{\mathbf{f}}$ yet again yields

$$D_1^2\mathcal{J}(\widehat{\mathbf{f}}, \mathbf{u}) = D_1^2\Psi(\widehat{\mathbf{f}}, \mathbf{u}) + \beta D_1^2\Phi(\widehat{\mathbf{f}}) \in \mathbb{R}^{n \times n}. \quad (21)$$

Here, $D_1^2\Psi(\widehat{\mathbf{f}}, \mathbf{u})$ is an $n \times n$ diagonal matrix with strictly positive entries according to H1:

$$D_1^2\Psi(\widehat{\mathbf{f}}, \mathbf{u})[i, i] = \psi''(\widehat{\mathbf{f}}[i] - \mathbf{u}[i]), \quad \forall i \in \mathbb{I}_n. \quad (22)$$

Hence $D_1^2\Psi(\mathcal{F}(\mathbf{u}), \mathbf{u}) \succ 0$. Furthermore,

$$D_1^2\Phi(\widehat{\mathbf{f}}) = G^T \text{diag}(\phi''(\mathbf{g}_1^T\widehat{\mathbf{f}}), \dots, \phi''(\mathbf{g}_r^T\widehat{\mathbf{f}}))G, \quad (23)$$

so $D_1^2\Phi(\widehat{\mathbf{f}}) \succeq 0$. It follows that $D_1^2\mathcal{J}(\widehat{\mathbf{f}}, \mathbf{u}) \succ 0$, for any $\mathbf{u} \in \mathbb{R}^n$.

Consequently, Lemma 1 holds true for any $\mathbf{u} \in \mathbb{R}^n$ and any $\beta > 0$. The same lemma shows that the statement of Proposition 1 holds true for $\mathcal{O} = \mathbb{R}^n$.

B. Proof of Lemma 2

Since \mathcal{F} is a local minimizer function,

$$D_1\mathcal{J}(\mathcal{F}(\mathbf{u}), \mathbf{u}) = 0, \quad \forall \mathbf{u} \in \mathbb{R}^n. \quad (24)$$

We can thus differentiate with respect to \mathbf{u} on both sides of (24) and yields

$$D_1^2\mathcal{J}(\mathcal{F}(\mathbf{u}), \mathbf{u})D\mathcal{F}(\mathbf{u}) + D_2D_1\mathcal{J}(\mathcal{F}(\mathbf{u}), \mathbf{u}) = 0, \quad \forall \mathbf{u} \in \mathbb{R}^n. \quad (25)$$

Note that $D\mathcal{F}(\mathbf{u})$ and $D_2D_1\mathcal{J}(\mathcal{F}(\mathbf{u}), \mathbf{u})$ are $n \times n$ real matrices. The Hessian matrix $H(\mathbf{u}) = D_1^2\mathcal{J}(\mathcal{F}(\mathbf{u}), \mathbf{u})$ can be expanded using (21):

$$H(\mathbf{u}) = D_1^2\Psi(\mathcal{F}(\mathbf{u}), \mathbf{u}) + \beta D_1^2\Phi(\mathcal{F}(\mathbf{u})) \in \mathbb{R}^{n \times n}.$$

Replacing $\widehat{\mathbf{f}}$ by $\mathcal{F}(\mathbf{u})$ in (22) and (23) yields

$$H(\mathbf{u}) = \text{diag}(\{\psi''(\mathcal{F}_i(\mathbf{u}) - \mathbf{u}[i])\}_{i=1}^n) + \beta G^T \text{diag}(\{\phi''(\mathbf{g}_i^T\mathcal{F}(\mathbf{u}))\}_{i=1}^n)G,$$

as stated in (5). Using H1, it is readily seen that

$$\text{diag}(\{\psi''(\mathcal{F}_i(\mathbf{u}) - \mathbf{u}[i])\}_{i=1}^n) \succ 0 \quad \text{and} \quad G^T \text{diag}(\{\phi''(\mathbf{g}_i^T\mathcal{F}(\mathbf{u}))\}_{i=1}^n)G \succeq 0.$$

Then $H(\mathbf{u}) \succ 0$, hence $H(\mathbf{u})$ is invertible.

Using (19) and (20), where we consider $\mathcal{F}(\mathbf{u})$ in place of $\widehat{\mathbf{f}}$, shows that

$$D_2D_1\mathcal{J}(\mathcal{F}(\mathbf{u}), \mathbf{u}) = D_2D_1\Psi(\mathcal{F}(\mathbf{u}), \mathbf{u}) = -\text{diag}(\{\psi''(\mathcal{F}_i(\mathbf{u}) - \mathbf{u}[i])\}_{i=1}^n). \quad (26)$$

Then (25) is equivalent to

$$D\mathcal{F}(\mathbf{u}) = -\left(H(\mathbf{u})\right)^{-1} D_2D_1\mathcal{J}(\mathcal{F}(\mathbf{u}), \mathbf{u}) = \left(H(\mathbf{u})\right)^{-1} \text{diag}(\{\psi''(\mathcal{F}_i(\mathbf{u}) - \mathbf{u}[i])\}_{i=1}^n), \quad \forall \mathbf{u} \in \mathbb{R}^n. \quad (27)$$

Obviously, $\text{rank}(D\mathcal{F}(\mathbf{u})) = n$. This result is independent of the value of $\beta > 0$.

C. Proof of Theorem 1

The proof consists of two parts.

(I) Given $g \in \mathcal{G}$, where \mathcal{G} is given in (8), consider the function $f_g : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$f_g(\mathbf{u}) = g^T \mathcal{F}(\mathbf{u}), \quad \forall \mathbf{u} \in \mathbb{R}^n .$$

By Lemma 2, $D\mathcal{F}(\mathbf{u})$ is invertible. Hence⁵ $g^T D\mathcal{F}(\mathbf{u}) \neq 0$ and thus

$$\text{rank}(f_g(\mathbf{u})) = \text{rank}(g^T D\mathcal{F}(\mathbf{u})) = 1, \quad \forall \mathbf{u} \in \mathbb{R}^n .$$

In particular, f_g does not have critical points. Its inverse of the origin K_g reads

$$K_g \stackrel{\text{def}}{=} f_g^{-1}(0) = \{\mathbf{u} \in \mathbb{R}^n \mid g^T \mathcal{F}(\mathbf{u}) = 0\} . \quad (28)$$

By an extension⁶ of the constant rank theorem [4], the subset K_g in (28), supposed nonempty, is a \mathcal{C}^{s-1} manifold of \mathbb{R}^n of dimension $n - 1$. Hence $\mathbb{L}^n(K_g) = 0$ (see e.g. [21], [35]). Using Proposition 1, f_g is \mathcal{C}^{s-1} -continuous, so K_g is closed.

The set $K_{\mathcal{G}}$ in (9) also reads

$$K_{\mathcal{G}} = \bigcup_{g \in \mathcal{G}} K_g .$$

Using that \mathcal{G} is of finite cardinality, it follows that $K_{\mathcal{G}}$ is closed in \mathbb{R}^n and that

$$\mathbb{L}^n(K_{\mathcal{G}}) = 0 .$$

The conclusion is clearly independent of the value of $\beta > 0$.

(II) Given $(i, j) \in \mathbb{I}_n \times \mathbb{I}_n$ (including $i = j$), define the subset $K_{i,j} \subset \mathbb{R}^n$ as

$$K_{i,j} = \mathcal{F}_i^{-1}(\mathbf{u}_j) = \{\mathbf{u} \in \mathbb{R}^n \mid \mathcal{F}_i(\mathbf{u}) = \mathbf{u}_j\} . \quad (29)$$

From Proposition 1, $\mathcal{F}_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is \mathcal{C}^{s-1} continuous, so $K_{i,j}$ is closed in \mathbb{R}^n . By Lemma 2, all rows of $D\mathcal{F}(\mathbf{u}) \in \mathbb{R}^{n \times n}$ are linearly independent, $\forall \mathbf{u} \in \mathbb{R}^n$. Consequently, for any $i \in \mathbb{I}_n$,

$$\text{rank}(D\mathcal{F}_i(\mathbf{u})) = 1, \quad \forall \mathbf{u} \in \mathbb{R}^n .$$

Using the same arguments than in (I), $K_{i,j}$ is a \mathcal{C}^{s-1} submanifold of \mathbb{R}^n of dimension $n - 1$, hence

$$\mathbb{L}^n(K_{i,j}) = 0 .$$

Noticing that $K_{\mathcal{I}}$ in (10) is a finite union of $(n - 1)$ -dimensional submanifolds in \mathbb{R}^n like $K_{i,j}$ entails the result. The independence of these results from $\beta > 0$ is obvious.

⁵If we had $g^T D\mathcal{F}(\mathbf{u}) = 0$ then $g^T = (D\mathcal{F}(\mathbf{u}))^{-1} 0 = 0$ which contradicts the fact that $g \in \mathcal{G}$.

⁶We use the following extension of the constant rank theorem, restated in our context (for details one can check [4, p. 96]). Let f be a \mathcal{C}^s application from an open set $\mathcal{O} \subset \mathbb{R}^n$ to \mathbb{R} . Assume that $Df(\mathbf{u})$ has constant rank r for all $\mathbf{u} \in \mathcal{O}$. Given a $c \in \mathbb{R}$, the inverse image $f^{-1}(c)$ (supposed nonempty) is a \mathcal{C}^s -manifold of \mathbb{R}^n of dimension $n - r$.

D. A Lemma needed to prove Proposition 2

Lemma 3. Let $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ satisfy

$$\begin{aligned} A &= \text{diag}(\{a_i\}_{i=1}^n) \text{ where } a_i > 0, \quad \forall i \in \mathbb{I}_n, \\ B &= B^T, \quad B \succeq 0. \end{aligned}$$

Consider the $n \times n$ matrix $M \stackrel{\text{def}}{=} (A + B)^{-1}A$. Then all eigenvalues of $M^T M$ belong to $(0, 1]$.

Proof: Let λ be an eigenvalue of $(A + B)^{-1}A$ and $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$ a right eigenvector corresponding to λ . Then

$$\begin{aligned} \lambda \mathbf{v} &= M \mathbf{v} \\ \Leftrightarrow \lambda \mathbf{v} &= (A + B)^{-1} A \mathbf{v} \end{aligned} \tag{30}$$

$$\begin{aligned} \Leftrightarrow \lambda(A + B)\mathbf{v} &= A \mathbf{v} \\ \Leftrightarrow \lambda B \mathbf{v} &= (1 - \lambda)A \mathbf{v}. \end{aligned} \tag{31}$$

If $\lambda = 0$ then (31) yields $A \mathbf{v} = 0$ which is impossible because $A \succ 0$, $A^T = A$ and $\mathbf{v} \neq 0$. Hence

$$\lambda \neq 0. \tag{32}$$

Furthermore, (31) yields

$$\lambda \mathbf{v}^T B \mathbf{v} = (1 - \lambda) \mathbf{v}^T A \mathbf{v}.$$

Using that $A \succ 0$ and $B \succeq 0$, the last equation shows that

$$\frac{1}{\lambda} - 1 = \frac{\mathbf{v}^T B \mathbf{v}}{\mathbf{v}^T A \mathbf{v}} \geq 0. \tag{33}$$

Combining the latter inequality with (32) entails that

$$0 < \lambda \leq 1. \tag{34}$$

Hence all eigenvalues of M live in $(0, 1]$.

Using that A is diagonal with positive diagonal entries, we can write down

$$A + B = A^{\frac{1}{2}} \left(I + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}. \tag{35}$$

Then the definition of M shows that⁷

$$\begin{aligned} M^T M &= A(A + B)^{-2} A \\ &= \left(I + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{-2}. \end{aligned} \tag{36}$$

Using (35), the expression in (30) is *equivalent* to

$$\begin{aligned} \lambda \mathbf{v} &= A^{-\frac{1}{2}} \left(I + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{-1} A^{-\frac{1}{2}} A \mathbf{v} \\ \Leftrightarrow \lambda \left(A^{\frac{1}{2}} \mathbf{v} \right) &= \left(I + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{-1} \left(A^{\frac{1}{2}} \mathbf{v} \right). \end{aligned}$$

Combining the last result with (36) yields

$$\begin{aligned} M^T M \left(A^{\frac{1}{2}} \mathbf{v} \right) &= \left(I + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{-1} \left(I + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{-1} \left(A^{\frac{1}{2}} \mathbf{v} \right) \\ &= \lambda \left(I + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{-1} \left(A^{\frac{1}{2}} \mathbf{v} \right) \\ &= \lambda^2 \left(A^{\frac{1}{2}} \mathbf{v} \right). \end{aligned}$$

⁷Remind that $A = A^T$.

Consequently, λ^2 is an eigenvalue of $M^T M$ corresponding to an eigenvector given by $(A^{\frac{1}{2}} \mathbf{v})$. Thus all eigenvalues of $M^T M$ belong to $(0, 1]$ as well.

E. Proof of Proposition 2

Let us denote

$$\begin{aligned} A(\mathbf{u}) &= \text{diag}(\{\psi''(\mathcal{F}_i(\mathbf{u}) - \mathbf{u}[i])\}_{i=1}^n), \\ B(\mathbf{u}) &= G^T \text{diag}(\{\phi''(\mathbf{g}_i^T \mathcal{F}(\mathbf{u}))\}_{i=1}^n) G, \end{aligned}$$

as well as

$$M(\mathbf{u}) = (A(\mathbf{u}) + \beta B(\mathbf{u}))^{-1} A(\mathbf{u}).$$

Then $D\mathcal{F}(\mathbf{u}) = M(\mathbf{u})$. We have

$$\mathcal{F}(\mathbf{u} + \boldsymbol{\zeta}) - \mathcal{F}(\mathbf{u}) = \int_0^1 D\mathcal{F}(\mathbf{u} + t\boldsymbol{\zeta}) \boldsymbol{\zeta} dt = \int_0^1 M(\mathbf{u} + t\boldsymbol{\zeta}) \boldsymbol{\zeta} dt$$

Using Lemma 3 and H1, for any $\mathbf{v} \in \mathbb{R}^n$, all eigenvalues of $(M(\mathbf{v}))^T M(\mathbf{v})$ are in $(0, 1]$. By the definition of the matrix 2-norm [37] one obtains

$$t \in [0, 1] \text{ and } \boldsymbol{\zeta} \in \mathbb{R}^n \quad \Rightarrow \quad \|M(\mathbf{u} + t\boldsymbol{\zeta})\|_2 \leq 1.$$

Noticing that $M(\cdot)$ is a continuous mapping, the mean value theorem (see e.g. [4], [14]) shows that

$$\|\mathcal{F}(\mathbf{u} + \boldsymbol{\zeta}) - \mathcal{F}(\mathbf{u})\|_2 \leq \max_{0 \leq t \leq 1} \|M(\mathbf{u} + t\boldsymbol{\zeta})\|_2 \|\boldsymbol{\zeta}\|_2 \leq \|\boldsymbol{\zeta}\|_2.$$

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