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# INHOMOGENEOUS DIOPHANTINE APPROXIMATION WITH GENERAL ERROR FUNCTIONS

LINGMIN LIAO AND MICHAL RAMS

ABSTRACT. Let  $\alpha$  be an irrational and  $\varphi : \mathbb{N} \rightarrow \mathbb{R}^+$  be a function decreasing to zero. For any  $\alpha$  with a given Diophantine type, we show some sharp estimations for the Hausdorff dimension of the set

$$E_\varphi(\alpha) := \{y \in \mathbb{R} : \|n\alpha - y\| < \varphi(n) \text{ for infinitely many } n\},$$

where  $\|\cdot\|$  denotes the distance to the nearest integer.

## 1. INTRODUCTION

Let  $\alpha$  be an irrational real number. Denote by  $\|\cdot\|$  the distance to the nearest integer. A famous result of Minkowski ([Min57]) in 1907 showed that if  $y \notin \mathbb{Z} + \alpha\mathbb{Z}$ , then for infinitely many  $n \in \mathbb{Z}$ , we have

$$\|n\alpha - y\| < \frac{1}{4|n|}.$$

If  $n$  is restricted to positive integers only, Khintchine ([Khi26]) in 1926 proved that for any real number  $y$ , there exist infinitely many  $n \in \mathbb{N}$  satisfying the Diophantine inequalities:

$$(1.1) \quad \|n\alpha - y\| < \frac{1}{\sqrt{5}n}.$$

We shall always restrict  $n$  to positive integers. Khintchine's result is equivalent to say that the set

$$E(\alpha, c) := \left\{ y \in \mathbb{R} : \|n\alpha - y\| < \frac{c}{n} \text{ for infinitely many } n \right\},$$

is the whole space  $\mathbb{R}$  when the constant  $c$  equals to  $1/\sqrt{5}$ . It is showed by Cassels [Cas50] in 1950 that the set  $E(\alpha, c)$  is of full measure for any constant  $c > 0$ .

However, if the error function (the right-hand side of (1.1)) of the above Diophantine inequalities is replaced by a function decreasing to zero faster than  $c/n$ , the sizes of the sets in question would be of zero Lebesgue measure and then the Hausdorff dimension is involved.

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Define the Diophantine type  $\beta(\alpha)$  of  $\alpha$  by

$$\beta(\alpha) := \sup\{\theta \geq 1 : \liminf_{n \rightarrow \infty} n^\theta \|n\alpha\| = 0\}.$$

In 1999, Bernik and Dodson [BD99] proved that the Hausdorff dimension, denoted by  $\dim_H$ , of the set

$$E_\gamma(\alpha) := \left\{ y \in \mathbb{R} : \|n\alpha - y\| < \frac{1}{n^\gamma} \text{ for infinitely many } n \right\} \quad (\gamma \geq 1),$$

satisfies

$$\frac{1}{\beta(\alpha) \cdot \gamma} \leq \dim_H E_\gamma(\alpha) \leq \frac{1}{\gamma}.$$

In 2003, Bugeaud [Bug03], and independently Schmeling and Troubetzkoy [TS03] improved the above result. They showed that for any irrational  $\alpha$ ,

$$\dim_H E_\gamma(\alpha) = \frac{1}{\gamma}.$$

Now let  $\varphi : \mathbb{N} \rightarrow \mathbb{R}^+$  be a function decreasing to zero. Consider the set

$$E_\varphi(\alpha) := \{y \in \mathbb{R} : \|n\alpha - y\| < \varphi(n) \text{ for infinitely many } n\}.$$

This is the set of well-approximated numbers with a general error function  $\varphi$ . It easily follows from the Borel-Cantelli lemma that the Lebesgue measure of  $E_\varphi(\alpha)$  is zero whenever the series  $\sum_{n=1}^{\infty} \varphi(n)$  converges. But on the other hand, it seems hard to obtain a lower bound of the Lebesgue measure of  $E_\varphi(\alpha) \cap [0, 1]$  when the series  $\sum_{n=1}^{\infty} \varphi(n)$  diverges. For the results on the Lebesgue measure, we refer the readers to [Kur55], [LN12], [Kim12], and the references therein.

In this paper, we are concerned with the Hausdorff dimension of the set  $E_\varphi(\alpha)$ . We can find a natural upper bound:

$$\dim_H E_\varphi(\alpha) \leq \limsup_{n \rightarrow \infty} \frac{\log n}{-\log \varphi(n)}.$$

It can also be proved that for almost all real numbers  $\alpha$ , the above inequality becomes an equality. However, in [FW06], Fan and Wu constructed an example which shows that the equality is not always true. In fact, they found a Liouville number  $\alpha$  and constructed an error function  $\varphi$  such that

$$\dim_H E_\varphi(\alpha) = \liminf_{n \rightarrow \infty} \frac{\log n}{-\log \varphi(n)} < \limsup_{n \rightarrow \infty} \frac{\log n}{-\log \varphi(n)}.$$

So in general case, the dimension formula seems mystery.

Recently, Xu [Xu10] made a progress, he proved the following theorem.

**Theorem 1.1** (Xu). *For any  $\alpha$ , we have the following estimation*

$$\limsup_{n \rightarrow \infty} \frac{\log q_n}{-\log \varphi(q_n)} \leq \dim_H(E_\varphi(\alpha)) \leq \limsup_{n \rightarrow \infty} \frac{\log n}{-\log \varphi(n)},$$

where  $q_n$  denotes the denominator of the  $n$ -th convergent of the continued fraction of  $\alpha$ .

As a corollary, Xu proved that for any irrational number  $\alpha$  with Diophantine type 1,

$$\dim_H(E_\varphi(\alpha)) = \limsup_{n \rightarrow \infty} \frac{\log n}{-\log \varphi(n)}.$$

For the simplicity, let us denote

$$u_\varphi := \limsup_{n \rightarrow \infty} \frac{\log n}{-\log \varphi(n)} \quad l_\varphi := \liminf_{n \rightarrow \infty} \frac{\log n}{-\log \varphi(n)}.$$

In this paper, we prove the following results.

**Theorem 1.2.** *For any  $\alpha$  with Diophantine type  $\beta$ , we have*

$$\min \left\{ u_\varphi, \max \left\{ l_\varphi, \frac{1 + u_\varphi}{1 + \beta} \right\} \right\} \leq \dim_H(E_\varphi(\alpha)) \leq u_\varphi.$$

**Corollary 1.3.** *If  $\beta \leq 1/u_\varphi$ , then*

$$\dim_H(E_\varphi(\alpha)) = u_\varphi.$$

**Example 1.4.** Take  $\beta = 2$ ,  $u = 1/2$  and  $l = 1/3$ . We can construct an irrational  $\alpha$  such that for all  $n$ ,  $q_n^2 \leq q_{n+1} \leq 2q_n^2$ . Define

$$\varphi(n) = \max \{ n^{-1/l}, q_k^{-1/l} \} \quad \text{if } q_{k-1}^{u/l} < n \leq q_k^{u/l}.$$

Then by Corollary 1.3, we have

$$\lim_{n \rightarrow \infty} \frac{\log q_n}{-\log \varphi(q_n)} = l < u = \dim_H(E_\varphi(\alpha)).$$

Thus the lower bound of Xu (Theorem 1.1) is not optimal.

The next two theorems show that the estimations in Theorem 1.2 are sharp.

**Theorem 1.5.** *For any irrational  $\alpha$  and for any  $0 \leq l < u \leq 1$ , with  $u > 1/\beta$ , there exists a decreasing function  $\varphi : \mathbb{N} \rightarrow \mathbb{R}^+$ , with  $l_\varphi = l$  and  $u_\varphi = u$ , such that*

$$\dim_H(E_\varphi(\alpha)) = \max \left\{ l, \frac{1 + u}{1 + \beta} \right\} < u.$$

**Theorem 1.6.** *Suppose  $0 \leq l < u \leq 1$ . There exists a decreasing function  $\varphi : \mathbb{N} \rightarrow \mathbb{R}^+$ , with  $l_\varphi = l$  and  $u_\varphi = u$ , such that for any  $\alpha$  with  $\beta < \infty$ ,*

$$\dim_H(E_\varphi(\alpha)) = u.$$

## 2. THREE STEPS DIMENSION

The goal of this section is to prove Proposition 2.2 which will be the base of our dimension estimation (compare [Xu10, Section 3]).

As a direct corollary of Proposition 2.2, we will also give a new proof of Xu's theorem (Theorem 1.1) at the end of this section.

Let us start with a technical lemma.

**Lemma 2.1.** *Let  $1 > a > b$  and  $1 > c > d$ . Then for any  $\delta \in [0, 1]$  we have*

$$\frac{\log(\delta a + (1 - \delta)c)}{\log(\delta b + (1 - \delta)d)} \geq \min\left(\frac{\log a}{\log b}, \frac{\log c}{\log d}\right).$$

*Proof.* Denote

$$s := \min\left(\frac{\log a}{\log b}, \frac{\log c}{\log d}\right).$$

Then

$$\frac{\log(\delta a + (1 - \delta)c)}{\log(\delta b + (1 - \delta)d)} \geq \frac{\log(\delta b^s + (1 - \delta)d^s)}{\log(\delta b + (1 - \delta)d)}.$$

By concavity of the function  $x \rightarrow x^s$ , we have

$$\delta b^s + (1 - \delta)d^s \leq (\delta b + (1 - \delta)d)^s$$

and the assertion follows.  $\square$

Let  $\alpha$  be an irrational number with Diophantine type  $\beta(\alpha) > 1$ . Recall that  $q_n$  is the denominator of the  $n$ -th convergent of the continued fraction of  $\alpha$ . Let  $B \geq 1$  and suppose there exists a sequence of natural numbers  $\{n_i\}$  such that

$$(2.1) \quad \frac{\log q_{n_i+1}}{\log q_{n_i}} \rightarrow B.$$

Let  $\{m_i\}$  be a sequence of natural numbers such that  $q_{n_i} < m_i \leq q_{n_i+1}$ . By passing to subsequences, we suppose the limit

$$N := \lim_{i \rightarrow \infty} \frac{\log m_i}{\log q_{n_i}}$$

exists. Then obviously,  $1 \leq N \leq B$ .

Let  $K > 1$ . Denote

$$E_i := \left\{ y \in \mathbb{R} : \|n\alpha - y\| < \frac{1}{2}q_{n_i}^{-K} \text{ for some } n \in (m_{i-1}, m_i] \right\}.$$

Let

$$E := \bigcap_{i=1}^{\infty} E_i \quad \text{and} \quad F := \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_i.$$

**Proposition 2.2.** *If  $\{n_i\}$  is increasing sufficiently fast then*

$$\dim_H E = \dim_H F = S,$$

where

$$S = S(N, B, K) := \min \left( \frac{N}{K}, \max \left( \frac{1}{K}, \frac{1}{1+B-N} \right) \right).$$

*Proof.* As  $F \supset E$ , we only need to get the lower bound for  $\dim_H E$  and the upper bound for  $\dim_H F$ . For the former, we will use the Frostman Lemma, and for the latter, we will use a natural cover.

We will distinguish two cases:  $B \geq K$  and  $B < K$ . Notice the following fact.

**Fact:** If  $B \geq K$  then

$$\frac{N}{K} > \frac{1}{1+B-N}, \quad \text{and} \quad S = \max \left( \frac{1}{K}, \frac{1}{1+B-N} \right).$$

If  $B < K$ , then

$$\frac{1}{K} < \frac{1}{1+B-N}, \quad \text{and} \quad S = \min \left( \frac{N}{K}, \frac{1}{1+B-N} \right).$$

Indeed, the second statement follows by noting  $1/K < 1/B$ . For the first statement, if  $N \geq K$  then it is obviously true because the right hand side is smaller than 1. Otherwise, we have

$$\frac{K-N}{N} < K-N,$$

hence

$$\frac{K}{N} < 1 + K - N.$$

Since  $B \geq K$ , we have

$$1 + B - N \geq 1 + K - N > K/N.$$

*Distribution of the points.*

Now, let us study the distribution of the points  $\{n\alpha \pmod{1}\}$ . Let  $\{n_i\}$  be a fast increasing sequence satisfying (2.1). By passing to a subsequence, we can always assume that  $\{n_i\}$  grows as fast as we wish; the exact conditions on the rate of growth will be clear later. Denote

$$N_i := m_i - m_{i-1}.$$

By passing to a subsequence, we can suppose that  $N_i \geq q_{n_i}$ .

The three steps theorem tells us how the points  $\{n\alpha \pmod{1}\}_{n=m_{i-1}+1}^{m_i}$  are distributed on the unit circle: there are  $q_{n_i}$  groups of points, each consisting of  $\lfloor N_i/q_{n_i} \rfloor$  ( $\lfloor \cdot \rfloor$  denotes the integer part) points, the distances between points inside each group are equal to  $\xi_i := \|q_{n_i}\alpha\|$  and the distances between groups are  $\zeta_i := \|q_{n_{i-1}}\alpha\| - (\lfloor N_i/q_{n_i} \rfloor - 1)\|q_{n_i}\alpha\|$ .

In the first case, i.e.,  $B \geq K$ , we have  $\xi_i \leq q_{n_i}^{-K}$  for all  $i$  big enough, hence the intervals  $[n\alpha - q_{n_i}^{-K}/2, n\alpha + q_{n_i}^{-K}/2]$  intersect each other (inside each group). So  $E_i$  consists of  $M_i := q_{n_i}$  intervals of length  $y_i := (\lfloor N_i/q_{n_i} \rfloor - 1)\xi_i + q_{n_i}^{-K}$ . By noting that  $\|q_n\alpha\|$  is comparable with  $q_{n+1}^{-1}$ , we have

$$y_i = (\lfloor N_i/q_{n_i} \rfloor - 1)\xi_i + q_{n_i}^{-K} = q_{n_i}^{-\min(K, 1+B-N)+O(\varepsilon)}.$$

In the second case, i.e.,  $B < K$ , for big  $i$ ,  $E_i$  consists of  $N_i$  intervals of length  $z_i := q_{n_i}^{-K}$ .

As  $q_{n_{i+1}} \gg q_{n_i+1}$ , we can freely assume that for any  $\varepsilon > 0$ , each component of  $E_i$  contains at least  $M_{i+1}^{1-\varepsilon}$  (in the first case) or  $N_{i+1}^{1-\varepsilon}$  (in the second case) components of  $E_{i+1}$ .

*Calculations.*

We will distribute a probability measure  $\mu$  in the most natural way: the measure attributed to each component of  $F_i = E_1 \cap \dots \cap E_i$  is the same. Here we distribute the measure only on those components of  $F_i$  that are components of  $E_i$ , i.e., at all stages we count only components completely contained in previous generation sets.

**Case 1:**  $B \geq K$ . At level  $i$  we have at least  $M_i^{1-\varepsilon}$  components of  $F_i$ , each of size  $y_i$  and inside each component of  $F_{i-1}$ , the components of  $F_i$  are in equal distance  $c_i := \zeta_i - q_{n_i}^{-K}$ .

Let  $x \in E$ . For  $y_i \leq r < y_{i-1}$ , consider

$$(2.2) \quad f(r) = \frac{\log \mu(B_r(x))}{\log r}.$$

Notice that the convex hull of components of  $F_i$  intersecting  $B_r(x)$  has measure at most  $3\mu(B_r(x))$  and length at most  $6r$ . For simplicity, we can assume that the interval  $B_r(x)$  is a convex hull of some components of  $F_i$  contained in one component of  $F_{i-1}$ . Hence,

$$(2.3) \quad f(ny_i + (n-1)c_i) \geq \frac{\log(nM_i^{-(1-\varepsilon)})}{\log(ny_i + (n-1)c_i)}.$$

As the right hand side of equation (2.3) is the ratio of logarithms of two functions, both linear in  $n$  and smaller than 1, by Lemma 2.1 the minimum of  $f(r)$  in range  $(y_i, y_{i-1})$  is achieved at one of endpoints. We have

$$(2.4) \quad f(y_i) \geq (1-\varepsilon) \frac{-\log M_i}{\log y_i} = \max\left(\frac{1}{K}, \frac{1}{1+B-N}\right) + O(\varepsilon)$$

and the same holds for  $f(y_{i-1})$ . Recalling the fact at the beginning of the proof, we get the lower bound by Frostman Lemma.

The upper bound is simpler: for any  $i$ ,  $F$  is contained in  $\bigcup_{n>i} E_n$ . Hence, we can use the components of all  $E_n, n > i$  as a cover for  $F$ . For any  $s$  the

sum of  $s$ -th powers of diameters of components of  $E_n$  is bounded by  $M_n y_n^s$ , and for  $s > \max(\frac{1}{K}, \frac{1}{1+B-N}) + O(\varepsilon)$  it is exponentially decreasing with  $n$ . The upper bound then follows by the definition of Hausdorff dimension.

**Case 2:**  $B < K$ . Once again to obtain the lower bound we will consider the function  $f(r)$  given by (2.2). However, in this case the components of  $F_i$  are not uniformly distributed inside a component of  $F_{i-1}$  but they are in groups. There are at least  $s_i$  groups in distance  $c_i$  from each other, each group is of size  $y_i$  and contains at least  $N_i^{1-\varepsilon}$  components. Inside each group the components of size  $z_i$  are in distance  $d_i := \xi_i - q_{n_i}^{-K}$  from each other.

We need to consider  $z_i \leq r < z_{i-1}$ . This range can be divided into two subranges. The equation (2.3) works for  $y_i \leq r < z_{i-1}$ , while for  $z_i \leq r < y_i$  the same reasoning gives

$$(2.5) \quad f(nz_i + (n-1)d_i) \geq \frac{\log(nN_i^{-(1-\varepsilon)})}{\log(nz_i + (n-1)d_i)}.$$

Like in the first case, Lemma 2.1 implies that the minimum of  $f(r)$  in each subrange is achieved at one of endpoints. We have

$$f(z_i) \geq (1-\varepsilon) \frac{-\log N_i}{\log z_i} = \frac{N}{K} + O(\varepsilon)$$

and the same for  $f(z_{i-1})$ , while  $f(y_i)$  is still given by (2.4). Together with the fact at the beginning of the proof, this gives the lower bound.

To get the upper bound for the dimension of  $F$  we can use two covers. One is given by using the convex hulls of groups of components of  $F_n$  with  $n > i$ . As in the first case (taking into account the fact that  $1/K < 1/(1+B-N)$ ), this cover gives

$$\dim_H F \leq \frac{1}{1+B-N} + O(\varepsilon).$$

The other cover consists of components of  $E_n$  with  $n > i$ . For any  $s$  the sum of  $s$ -th powers of diameters of components of  $E_n$  is bounded by  $N_n z_n^s$ , and for  $s > \frac{N}{K} + O(\varepsilon)$  it is exponentially decreasing with  $n$ . We will choose one of the two covers that gives us the smaller Hausdorff dimension.  $\square$

The statement of Proposition 2.2 could be also written in the following way, fixing  $B$  and  $N$  and varying  $K$ :

$$S(N, B, K) = \begin{cases} 1/K & K < 1+B-N \\ 1/(1+B-N) & 1+B-N \leq K \leq N(1+B-N) \\ N/K & K > N(1+B-N). \end{cases}$$

By Proposition 2.2, we can directly deduce Theorem 1.1.

*A new proof of Theorem 1.1:*

The upper bound is easy, and we only show the lower bound. We will apply



Proposition 2.2. Let  $q_{n_i}$  be a sparse subsequence such that

$$\lim_{i \rightarrow \infty} \frac{\log q_{n_{i+1}}}{-\log \varphi(q_{n_{i+1}})} = \limsup_{n \rightarrow \infty} \frac{\log q_n}{-\log \varphi(q_n)} =: L.$$

By passing to a subsequence, suppose the limit

$$\lim_{i \rightarrow \infty} \frac{\log q_{n_{i+1}}}{\log q_{n_i}} =: B$$

exists. Take  $m_i = q_{n_{i+1}}$ . Then  $1 \leq N = B$ . Take  $K = NL^{-1}$  and construct the sets  $E_i$  and  $E$  as in Proposition 2.2. We can easily check that  $E$  is a subset of  $E_\varphi(\alpha)$ . By Proposition 2.2, we have

$$\dim_H E_\varphi(\alpha) \geq \min \{L, 1\}.$$

Then the result follows.

### 3. PROOF OF THEOREM 1.2

The upper bound of Theorem 1.2 is trivial by using the natural covering, hence we will only concern ourselves with the lower bound.

Note that the lower bound in Theorem 1.2 can be written as

$$\max \left\{ l_\varphi, \min \left\{ u_\varphi, \frac{1 + u_\varphi}{1 + \beta} \right\} \right\}.$$

By the result of Bugeaud [Bug03] and Schmeling and Troubetzkoy [TS03], the Hausdorff dimension of  $E_\varphi$  is at least  $l_\varphi$ . So we just need to show it is not smaller than  $\min(u_\varphi, (1 + u_\varphi)/(1 + \beta))$ .

We shall suppose that  $l_\varphi > 0$ , the case  $l_\varphi = 0$  can be done by a limit argument. Since the result is known if  $l_\varphi = u_\varphi$ , we also suppose that  $l_\varphi < u_\varphi$ .

The Diophantine type of the irrational number  $\alpha$  can be defined alternatively by

$$\beta = \limsup_{n \rightarrow \infty} \frac{\log q_{n+1}}{\log q_n}.$$

Choose a sequence  $m_i$  of natural numbers such that

$$\lim_{i \rightarrow \infty} \frac{\log m_i}{-\log \varphi(m_i)} = u_\varphi.$$

Let  $n_i$  be such that  $q_{n_i} < m_i \leq q_{n_i+1}$ . By passing to a subsequence we can assume that

- the sequence  $\log q_{n_{i+1}}/\log q_{n_i}$  has some limit  $B \in [1, \beta]$ ,
- the sequence  $\log m_i/\log q_{n_i}$  has some limit  $N \in [1, B]$ ,
- the sequence  $\{n_i\}$  grows fast enough for Proposition 2.2.

Moreover, we can freely assume that  $N > 1$ : otherwise, by the monotonicity of  $\varphi$ , we would have

$$\lim_{i \rightarrow \infty} \frac{\log q_{n_i}}{-\log \varphi(q_{n_i})} = u_\varphi$$

and the assertion would follow from Theorem 1.1.

Take  $K = N/u_\varphi$ . By the definition of  $m_i$ , for any small  $\delta > 0$ , we have for all large  $i$

$$\varphi(m_i) \geq (m_i)^{-1/u_\varphi - \delta} \geq q_{n_i}^{-K}.$$

Thus by monotonicity of  $\varphi$ ,

$$(3.1) \quad \varphi(n) \geq q_{n_i}^{-K} \quad \forall n \leq m_i.$$

The assumptions of Proposition 2.2 are satisfied, so we can calculate the Hausdorff dimension of the set  $E$  defined in the previous section. By (3.1),  $E \subset E_\varphi$ , so this gives the lower bound for the Hausdorff dimension of  $E_\varphi$ :

$$\dim_H E_\varphi \geq M(N, B) := \min \left( u_\varphi, \max \left( \frac{u_\varphi}{N}, \frac{1}{1 + B - N} \right) \right)$$

and we want to estimate the minimal value of  $M$  for  $B \in [1, \beta]$ ,  $N \in [1, B]$ .

First thing to note is that increasing  $B$  not only decreases  $M(B, N)$  for a fixed  $N$  but also increases the range of possible  $N$ 's. Hence, the minimum of  $M(N, B)$  is achieved for  $B = \beta$ . Denote  $M(N) = M(N, \beta)$ .

We are then left with a simple optimization problem of a function of one variable. We can write

$$M(N) = \min \left( u_\varphi, \max \left( \frac{u_\varphi}{N}, \frac{1}{1 + \beta - N} \right) \right).$$

If  $\beta u_\varphi \leq 1$  then  $u_\varphi \leq 1/(1 + \beta - N)$  for all  $N$ , hence

$$\min_N M(N) = u_\varphi \leq \frac{1 + u_\varphi}{1 + \beta}.$$

Otherwise, as  $u_\varphi/N$  is a decreasing and  $1/(1 + \beta - N)$  an increasing function of  $N$ , the global minimum over  $N$  of the maximum of the two is achieved at the point  $N_0$  where they are equal:  $u_\varphi/N_0 = 1/(1 + \beta - N_0)$ , that is for

$$N_0 = \frac{u_\varphi(1 + \beta)}{1 + u_\varphi}.$$

As  $\beta u_\varphi > 1$  implies  $1 < N_0 < \beta u_\varphi \leq \beta$ ,  $N_0$  is inside the interval  $[1, \beta]$ , hence this global minimum is the local minimum we are looking for. Thus, in this case

$$\min_N M(N) = M(N_0) = \frac{1 + u_\varphi}{1 + \beta} < u_\varphi.$$

We are done.

## 4. PROOF OF THEOREMS 1.5 AND 1.6

**Proof of Theorem 1.5:** Let  $\alpha$  be of Diophantine type  $\beta > 1/u$ . Let  $q_{n_i}$  be a sparse subsequence of denominators of convergents such that

$$\beta = \lim_{i \rightarrow \infty} \frac{\log q_{n_i+1}}{\log q_{n_i}}.$$

For any  $0 \leq l < u \leq 1$ , define

$$z = \max\left(l, \frac{1+u}{1+\beta}\right).$$

Note that  $z \leq u$ .

Define also a function  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$  as follows:

$$\varphi(n) := \max\{n^{-1/l}, k_{n_i}^{-1/u}\}, \quad \text{if } k_{n_{i-1}} < n \leq k_{n_i},$$

where

$$k_{n_i} = q_{n_i}^{u/z}.$$

Let  $D_1$  be the set

$$\{y \in \mathbb{R} : \text{for infinitely many } i, \|n\alpha - y\| < k_{n_i}^{-\frac{1}{u}} \text{ for some } n \in (k_{n_{i-1}}, k_{n_i}]\}$$

and  $D_2$  be the set

$$\{y \in \mathbb{R} : \|n\alpha - y\| < n^{-\frac{1}{l}} \text{ for infinitely many } n\}.$$

Clearly,  $E_\varphi(\alpha) = D_1 \cup D_2$ . The Hausdorff dimension of  $D_1$  is given by Proposition 2.2 (with  $B = \beta, K = 1/z, N = u/z$ ):

$$\dim_H D_1 = \min\left(u, \max\left(z, \frac{z}{(1+\beta)z - u}\right)\right) = z$$

(the equality is valid both when  $z = l$  and  $z = (1+u)/(1+\beta)$ ).

By [Bug03] and [TS03] we have

$$\dim_H(D_2) = l.$$

Then the proof is completed.

**Proof of Theorem 1.6:** Construct a sequence  $\{n_i\}_{i \geq 1}$  by recurrence:

$$n_1 = 2, \quad n_{i+1} = 2^{n_i} \quad (i \geq 1).$$

Define a function  $\varphi : \mathbb{N} \rightarrow \mathbb{R}$  as  $\varphi(n) = n_i^{-1/l}$  for  $n \in (n_i, n_i^{u/l})$  and  $\varphi(n) = n^{-1/u}$  elsewhere.

Suppose that  $\dim_H(E_\varphi(\alpha)) < u$ . By Theorem 1.1, no  $q_m$  could be between  $n_i$  and  $n_{i+1}^{1/u}$ . Since  $n_i$  go to infinity very fast,  $\alpha$  cannot be of finite type.

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