



HAL
open science

On the fast Khintchine spectrum in continued fractions

Fan Ai-Hua, Lingmin Liao, Bao-Wei Wang, Jun Wu

► **To cite this version:**

Fan Ai-Hua, Lingmin Liao, Bao-Wei Wang, Jun Wu. On the fast Khintchine spectrum in continued fractions. Monatshefte für Mathematik, 2013, 171, pp.329–340. hal-00723315

HAL Id: hal-00723315

<https://hal.science/hal-00723315>

Submitted on 9 Aug 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON THE FAST KHINTCHINE SPECTRUM IN CONTINUED FRACTIONS

AIHUA FAN, LINGMIN LIAO, BAOWEI WANG[†], AND JUN WU

ABSTRACT. For $x \in [0, 1)$, let $x = [a_1(x), a_2(x), \dots]$ be its continued fraction expansion with partial quotients $\{a_n(x), n \geq 1\}$. Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ be a function with $\psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$. In this note, the fast Khintchine spectrum, i.e., the Hausdorff dimension of the set

$$E(\psi) := \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{\psi(n)} \sum_{j=1}^n \log a_j(x) = 1 \right\}$$

is completely determined without any extra condition on ψ .

1. INTRODUCTION

Continued fraction expansions are induced by the Gauss transformation $T : [0, 1) \rightarrow [0, 1)$ given by

$$T(0) := 0, \quad T(x) = \frac{1}{x} \pmod{1}, \quad \text{for } x \in (0, 1).$$

Let $a_1(x) = \lfloor x^{-1} \rfloor$ ($\lfloor \cdot \rfloor$ stands for the integral part) and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$. Each irrational number $x \in [0, 1)$ admits a unique infinite continued fraction expansion of the form

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}}}. \quad (1.1)$$

Sometimes, (1.1) is written as $x = [a_1, a_2, \dots]$. The integers a_n are called the partial quotients of x . The n -th convergent $p_n(x)/q_n(x)$ of x is given by $p_n(x)/q_n(x) = [a_1, \dots, a_n]$.

The continued fraction is tightly connected with the classic Diophantine approximation. For example, for any $\nu \geq 2$, the well-known Jarník set

$$\left\{ x : |x - p/q| < q^{-\nu}, \text{ for infinitely many } (p, q) \in \mathbb{Z}^2 \right\}$$

is equal to $J_{\nu-2}$, where for any $\beta > 0$, the set J_β is defined by continued fractions as

$$J_\beta := \left\{ x : a_{n+1}(x) \geq q_n(x)^\beta, \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

2000 *Mathematics Subject Classification.* 11K50, 28A80.

Key words and phrases. Continued fractions, Fast Khintchine spectrum, Hausdorff dimension.

[†] Corresponding author.

The Gauss transformation is identified with an infinite symbolic dynamical system if we consider the partial quotients as symbols. The appearance of infinite symbols brings us new phenomena in relative to the case of finite symbols. For example, consider the set

$$\left\{ x \in [0, 1) : \mathbb{A} \left\{ \frac{1}{n} \#\{1 \leq j \leq n : a_j(x) = 1\} \right\}_{n \geq 1} = [0, 1] \right\}$$

where $\mathbb{A}(E)$ denotes the set of the accumulation points of a set E . The Hausdorff dimension of this set is $1/2$ (see [9]), while in b -adic expansion a similar set is of Hausdorff dimension 0 (see [12]). Another example is that the multifractal spectrum of the level sets of the Khintchine constant

$$\left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \log a_j(x) = \xi \right\}$$

is neither concave nor convex [3]. Because of the difference from the finite symbolic dynamical systems and of the observed new phenomena, continued fractions attracted much attention. One can find rich properties of the continued fraction dynamical system in [2, 3, 5, 6, 7, 10, 11, 14] and related works therein.

Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$. Define

$$E(\psi) = \left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{\log a_1(x) + \cdots + \log a_n(x)}{\psi(n)} = 1 \right\}.$$

When $\psi(n) = \lambda n$ for some $\lambda > 0$, the set $E(\psi)$ is a level set of the classic Khintchine constant. Besides a detailed spectrum analysis of the classic Khintchine constant in [3], the authors also studied the fast Khintchine spectrum, i.e. the Hausdorff dimension of $E(\psi)$ when $\psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$. But the result for the latter case is incomplete. Only under the conditions that $\lim_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(n)} = b$ and $\lim_{n \rightarrow \infty} (\psi(n) - \psi(n-1)) = \infty$, the dimension of $E(\psi)$ was given [3]. In this note, we show that these extra conditions are unnecessary for determining the dimension of $E(\psi)$ in the case of fast Khintchine spectrum.

Two functions ψ and $\tilde{\psi}$ defined on \mathbb{N} are said to be *equivalent* if $\frac{\psi(n)}{\tilde{\psi}(n)} \rightarrow 1$ as $n \rightarrow \infty$.

Theorem 1.1. *Let $\psi : \mathbb{N} \rightarrow \mathbb{N}$ with $\psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$. If ψ is equivalent to an increasing function, then $E(\psi) \neq \emptyset$ and*

$$\dim_H E(\psi) = \frac{1}{1+b}, \quad \text{with } b = \limsup_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(n)}.$$

Otherwise, $E(\psi) = \emptyset$.

Remark 1. The method used in [3] does not apply to general ψ . This is explained in Section 3 below.

Remark 2. The upper bound of $\dim_H E(\psi)$ is the difficult part of the proof of Theorem 1.1. As a byproduct of the proof, we get that for any $\beta > 0$, the Hausdorff dimension of the set

$$J_\beta^* := \left\{ x \in J_\beta : \lim_{n \rightarrow \infty} \frac{\log q_n(x)}{n} = \infty \right\}$$

is $1/(2 + \beta)$, i.e. one half of the dimension of the Jarník set J_β . A detailed explanation is given at the end of this paper.

2. PRELIMINARY

This section is devoted to fixing some notation, recalling some elementary properties enjoyed by continued fractions and citing some technical lemmas in dimension estimation.

Throughout this paper, we use $[\cdot]$ to denote the integral part of a real number, $|A|$ the diameter of a set $A \subset \mathbb{R}$, \mathcal{H}^s the s -dimensional Hausdorff measure, and \dim_H the Hausdorff dimension of a subset of $[0, 1)$.

Recall that for any irrational number $x \in [0, 1)$, $p_n(x)$ and $q_n(x)$ are the numerator and denominator of the n -th convergent of x . It is known that $p_n = p_n(x)$ and $q_n = q_n(x)$ can be obtained recursively by the following relations.

$$p_n = a_n(x)p_{n-1} + p_{n-2}, \quad q_n = a_n(x)q_{n-1} + q_{n-2} \quad (2.1)$$

with the conventions $p_0 = q_{-1} = 0$ and $p_{-1} = q_0 = 1$. For each $n \geq 1$,

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n. \quad (2.2)$$

For any $n \geq 1$ and $(a_1, a_2, \dots, a_n) \in \mathbb{N}^n$, define

$$I_n(a_1, a_2, \dots, a_n) = \{x \in [0, 1) : a_1(x) = a_1, \dots, a_n(x) = a_n\},$$

which is the set of points beginning with (a_1, \dots, a_n) in their continued fraction expansions, and is called a *cylinder* of order n .

Note that p_n and q_n are determined by the first n partial quotients of x . So all points in $I_n(a_1, \dots, a_n)$ determine the same p_n and q_n . Hence sometimes, we write $p_n = p_n(a_1, \dots, a_n)$ and $q_n = q_n(a_1, \dots, a_n)$ to denote $p_n(x)$ and $q_n(x)$ for $x \in I_n(a_1, \dots, a_n)$.

Proposition 2.1 ([8]). *For any $n \geq 1$ and $(a_1, \dots, a_n) \in \mathbb{N}^n$, let q_n be given recursively by (2.1). The cylinder $I_n(a_1, \dots, a_n)$ is an interval with the endpoints p_n/q_n and $(p_n + p_{n-1})/(q_n + q_{n-1})$. Then*

$$\frac{1}{2q_n^2} \leq \left| I_n(a_1, \dots, a_n) \right| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2}. \quad (2.3)$$

For each $n \geq 1$, $q_n(a_1, \dots, a_n) \geq 2^{(n-1)/2}$ and

$$\prod_{k=1}^n a_k \leq q_n(a_1, \dots, a_n) \leq 2^n \prod_{k=1}^n a_k. \quad (2.4)$$

Now we mention some known results concerning the dimension of sets in continued fractions. Let $\{s_n\}_{n \geq 1}$ be a sequence of integers and $\ell \geq 2$ be some fixed integer. Set

$$F(\{s_n\}_{n=1}^{\infty}; \ell) = \{x \in [0, 1) : s_n \leq a_n(x) < \ell s_n, \text{ for all } n \geq 1\}.$$

Lemma 2.2 ([3]). *Under the assumption that $\frac{1}{n} \sum_{k=1}^n s_k \rightarrow \infty$ as $n \rightarrow \infty$, one has*

$$\dim_H F(\{s_n\}_{n=1}^{\infty}; \ell) = \liminf_{n \rightarrow \infty} \frac{\log(s_1 s_2 \cdots s_n)}{2 \log(s_1 s_2 \cdots s_n) + \log s_{n+1}}.$$

Lemma 2.3 ([3]).

$$\dim_H \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{\log q_n(x)}{n} = \infty \right\} = \frac{1}{2}.$$

3. PROOF OF THEOREM 1.1

Notice that $E(\psi) = E(\tilde{\psi})$ if ψ and $\tilde{\psi}$ are equivalent. We can assume that ψ is increasing because of the following simple lemma.

Lemma 3.1. *The set $E(\psi) \neq \emptyset$ if and only if ψ is equivalent to an increasing function.*

Proof. If $E(\psi)$ is nonempty, take an $x_0 \in E(\psi)$. Then put

$$\tilde{\psi}(n) = \lfloor \log a_1(x_0) + \cdots + \log a_n(x_0) \rfloor$$

for all $n \geq 1$. Clearly $\tilde{\psi}$ is increasing. The functions ψ and $\tilde{\psi}$ are equivalent.

On the other hand, if ψ is increasing, we have a point $x \in E(\psi)$ such that for each $n \geq 1$

$$a_n(x) = \lfloor e^{\psi(n) - \psi(n-1) + 1} \rfloor.$$

□

Now we can proceed the proof of Theorem 1.1 with the assumption that ψ is increasing.

• **LOWER BOUND.** Apply Lemma 2.2 to $s_n = \lfloor e^{\psi(n) - \psi(n-1)} \rfloor$ and $\ell = 2$. Let

$$F = \left\{ x \in [0, 1) : \lfloor e^{\psi(n) - \psi(n-1)} \rfloor \leq a_n(x) < 2 \lfloor e^{\psi(n) - \psi(n-1)} \rfloor, \text{ for all } n \geq 1 \right\}$$

which is subset of $E(\psi)$. We get immediately that

$$\dim_H E(\psi) \geq \frac{1}{1+b}.$$

• **UPPER BOUND.** This is the main part of the proof.

Let us first recall the method used in [3] under the extra condition that $\lim_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(n)} = b \geq 1$. Especially when $b > 1$, we constructed a set containing $E(\psi)$ by posing precise restrictions on each partial quotients, namely

$$\left\{ x \in [0, 1) : e^{L_n} \leq a_n(x) \leq e^{M_n}, \text{ when } n \gg 1 \right\}, \quad (3.1)$$

where (with a small $\epsilon > 0$)

$$L_n = \frac{\psi(n)}{1 + \epsilon} - \frac{\psi(n-1)}{1 - \epsilon} \quad \text{and} \quad M_n = \frac{\psi(n)}{1 - \epsilon} - \frac{\psi(n-1)}{1 + \epsilon}.$$

By a standard covering argument, together with $\lim_{n \rightarrow \infty} \frac{\psi(n+1)}{\psi(n)} = b$, we get the exact upper bound of the dimension of $E(\psi)$. But as far as a general function ψ is concerned, the above argument fails. For example, take

$$\psi(n) = (k+2)!, \quad \text{when } k! \leq n < (k+1)!.$$

Then the set in (3.1) reads as

$$\left\{ x \in [0, 1) : \begin{cases} e^{c_1(k+2)!} \leq a_n(x) \leq e^{c_2(k+2)!}, & \text{when } n = k!; \\ 1 \leq a_n(x) \leq e^{c_3(k+2)!}, & \text{when } k! < n < (k+1)!. \end{cases} \right\}$$

for suitably chosen constants c_1, c_2, c_3 . According to Lemma 2.2, this set has Hausdorff dimension $\geq 1/2$. However, the dimension of $E(\psi)$ is equal to zero by Theorem 1.1.

Now we are going to prove the upper bound of $\dim_H E(\psi)$ for a general function ψ . Since ψ is increasing, we always have $b \geq 1$. We distinguish two cases: $b = 1$ and $b > 1$.

Case 1. $b = 1$. Lemma 2.3 serves for this case. According to the estimation (2.4), since $\psi(n)/n \rightarrow \infty$ as $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \frac{\log q_n(x)}{\psi(n)} = \lim_{n \rightarrow \infty} \frac{\log a_1(x) + \dots + \log a_n(x)}{\psi(n)}.$$

Thus Lemma 2.3 gives us

$$\dim_H E(\psi) \leq \frac{1}{2} = \frac{1}{1+b}.$$

Case 2. $b > 1$. Fix an $\epsilon > 0$. Choose a sequence of integers $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ with n_1 large enough and for each $k \geq 1$ one has

$$\psi(n_k + 1) \geq \psi(n_k)b(1 - \epsilon), \quad n_k \leq \epsilon\psi(n_k). \quad (3.2)$$

For each $N \geq 1$, let

$$E_N(\psi) = \left\{ x \in [0, 1) : (1 - \epsilon) < \frac{1}{\psi(n)} \sum_{j=1}^n \log a_j(x) < (1 + \epsilon), \quad \forall n \geq N \right\}.$$

Then

$$E(\psi) \subset \bigcup_{N \geq 1} E_N(\psi).$$

To estimate the dimension of $E_N(\psi)$ for $N \geq 1$, we proceed in three steps.

Step i. Find a cover of $E_N(\psi)$. For any $n \geq N$, set

$$D_n(\epsilon) = \left\{ (a_1, \dots, a_n) \in \mathbb{N}^n : (1 - \epsilon) < \frac{1}{\psi(n)} \sum_{j=1}^n \log a_j < (1 + \epsilon) \right\}. \quad (3.3)$$

For every $(a_1, \dots, a_n) \in D_n(\epsilon)$, we define

$$D_{n+1}(\epsilon; (a_1, \dots, a_n)) = \left\{ a_{n+1} \in \mathbb{N} : (a_1, \dots, a_n, a_{n+1}) \in D_{n+1}(\epsilon) \right\}.$$

Clearly, by the definition of $D_n(\epsilon)$, we have

$$E_N(\psi) \subset \bigcap_{n=N}^{\infty} \mathfrak{D}_n(\epsilon), \text{ with } \mathfrak{D}_n(\epsilon) = \bigcup_{(a_1, \dots, a_n) \in D_n(\epsilon)} I_n(a_1, \dots, a_n). \quad (3.4)$$

Now instead of considering the intersections in (3.4) from $n = N$ until $n = \infty$, we only consider the intersection of two consecutive terms. Namely, for any $n \geq N$,

$$E_N(\psi) \subset (\mathfrak{D}_n(\epsilon) \cap \mathfrak{D}_{n+1}(\epsilon)) = \bigcup_{(a_1, \dots, a_n) \in D_n(\epsilon)} J_n(a_1, \dots, a_n),$$

where

$$J_n(a_1, \dots, a_n) = \bigcup_{a_{n+1} \in D_{n+1}(\epsilon; (a_1, \dots, a_n))} I_{n+1}(a_1, \dots, a_n, a_{n+1}).$$

Hence, for each $n \geq N$, we get a cover of $E_N(\psi)$:

$$\{J_n(a_1, \dots, a_n) : (a_1, \dots, a_n) \in D_n(\epsilon)\}. \quad (3.5)$$

Thus the s -dimensional Hausdorff measure of $E_N(\psi)$ can be estimated as

$$\mathcal{H}^s(E_N(\psi)) \leq \liminf_{n \rightarrow \infty} \sum_{(a_1, \dots, a_n) \in D_n(\epsilon)} |J_n(a_1, \dots, a_n)|^s. \quad (3.6)$$

As we shall see, $J_n(a_1, \dots, a_n)$ is a union of cylinders of order $(n + 1)$, say $I_{n+1}(a_1, \dots, a_n, a_{n+1})$ with a taking large values (Lemma 3.3). Using this fact, the length of $J_n(a_1, \dots, a_n)$ will be well estimated.

Step ii. **Lengths of $J_n(a_1, \dots, a_n)$.** We begin with a fact on $D_{n+1}(\epsilon; a_1, \dots, a_n)$.

Lemma 3.2. For each $(a_1, \dots, a_n) \in D_n(\epsilon)$,

$$D_{n+1}(\epsilon; (a_1, \dots, a_n)) \neq \emptyset.$$

Proof. This follows from the following simple constructions.

- (a) If $\sum_{j=1}^n \log a_j > (1 - \epsilon)\psi(n + 1)$, we choose $a_{n+1} = 1$.
- (b) If $\sum_{j=1}^n \log a_j \leq (1 - \epsilon)\psi(n + 1)$, we can choose

$$a_{n+1} = \left\lfloor \frac{e^{\psi(n+1)}}{a_1 \cdots a_n} \right\rfloor.$$

□

Recall that the sequence of integers $\{n_k\}_{k \geq 1}$ is given in (3.2).

Lemma 3.3. For any $(a_1, \dots, a_{n_k}) \in D_{n_k}(\epsilon)$ and $a_{n_k+1} \in D_{n_k+1}(\epsilon, (a_1, \dots, a_{n_k}))$, we have

$$\log a_{n_k+1} \geq (1 - \epsilon) \left(\frac{b(1 - \epsilon)^2}{1 + \epsilon} - 1 \right) \log q_{n_k} =: \beta \log q_{n_k}, \quad (3.7)$$

Proof. By the definitions of $D_n(\epsilon)$ and the first inequality in (3.2), for any $(a_1, \dots, a_{n_k}) \in D_{n_k}(\epsilon)$ and $a_{n_k+1} \in D_{n_k+1}(\epsilon, (a_1, \dots, a_{n_k}))$, one has

$$\begin{aligned} \sum_{j=1}^{n_k+1} \log a_j &\geq \psi(n_k + 1)(1 - \epsilon) \geq \psi(n_k)b(1 - \epsilon)^2 \\ &\geq \frac{b(1 - \epsilon)^2}{1 + \epsilon} \sum_{j=1}^{n_k} \log a_j. \end{aligned} \quad (3.8)$$

On the other hand, by (2.4) and the second inequality in (3.2), we get

$$q_{n_k}(a_1, \dots, a_{n_k}) \leq 2^{n_k} \prod_{j=1}^{n_k} a_j \leq \left(\prod_{j=1}^{n_k} a_j \right)^{\frac{1}{1-\epsilon}}. \quad (3.9)$$

Combining (3.8) and (3.9), we obtain the desired result. \square

Now return back to the cover of $E_N(\psi)$ given in (3.5) especially when $n = n_k$. We estimate the length of $J_{n_k}(a_1, \dots, a_{n_k})$ for every $(a_1, \dots, a_{n_k}) \in D_{n_k}(\epsilon)$. For $n = n_k$, by (3.7) and Proposition 2.1, we have

$$|J_n(a_1, \dots, a_n)| \leq \sum_{a: a \geq q_n^\beta} \left| \frac{a \cdot p_n + p_{n-1}}{a \cdot q_n + q_{n-1}} - \frac{(a+1)p_n + p_{n-1}}{(a+1)q_n + q_{n-1}} \right|.$$

By (2.2), for all $a \in \mathbb{N}$, the differences appearing in the series have the same sign depending only the parity of n . Thus the series is telescopic. Since $\frac{(a+1)p_n + p_{n-1}}{(a+1)q_n + q_{n-1}}$ tends to p_n/q_n as $a \rightarrow \infty$, we get

$$|J_n(a_1, \dots, a_n)| \leq \left| \frac{q_n^\beta p_n + p_{n-1}}{q_n^\beta q_n + q_{n-1}} - \frac{p_n}{q_n} \right| = \frac{1}{(q_n^\beta q_n + q_{n-1})q_n} \leq \frac{1}{q_n^{2+\beta}}.$$

Consider the liminf in (3.6) along the subsequence $\{n_k\}_{k \geq 1}$, then we obtain

$$\mathcal{H}^s(E_N(\psi)) \leq \liminf_{k \rightarrow \infty} \sum_{(a_1, \dots, a_{n_k}) \in D_{n_k}(\epsilon)} \left(\frac{1}{q_{n_k}} \right)^{s(2+\beta)}. \quad (3.10)$$

The last step is devoted to estimating the summation in (3.10) under a suitable choice of s .

Step iii. Bernoulli measures. A family of measures μ_t defined on cylinders is constructed firstly. For each $t > 1$ and for any $(a_1, \dots, a_n) \in \mathbb{N}^n$, set

$$\mu_t(I_n(a_1, \dots, a_n)) = e^{-nP(t)-t \sum_{j=1}^n \log a_j}, \quad (3.11)$$

where $e^{P(t)} = \zeta(t) = \sum_{k=1}^{\infty} k^{-t}$. By Kolmogorov's consistency theorem, μ_t can be extended into a probability measure on $[0, 1)$.

Fix $\epsilon > 0$. By the assumption that $\lim_{n \rightarrow \infty} \psi(n)/n = \infty$, one can choose some integer $N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N(\epsilon)$,

$$nP\left(1 + \frac{\epsilon}{2}\right) \leq \frac{\epsilon}{2}(1 - \epsilon)\psi(n). \quad (3.12)$$

We claim that for each $n \geq N(\epsilon)$ and $(a_1, \dots, a_n) \in D_n$,

$$q_n^{-(1+\epsilon)} \leq \mu_{(1+\epsilon/2)}(I_n(a_1, \dots, a_n)). \quad (3.13)$$

More precisely, for any $(a_1, \dots, a_n) \in D_n$, by (3.3) and (3.12), we have

$$\frac{\epsilon}{2} \sum_{j=1}^n \log a_j \geq nP(1 + \frac{\epsilon}{2}). \quad (3.14)$$

Thus by (2.4) and then (3.14), we get

$$q_n^{-(1+\epsilon)} \leq e^{-(1+\epsilon) \sum_{j=1}^n \log a_j} \leq e^{-nP(1+\frac{\epsilon}{2}) - (1+\frac{\epsilon}{2}) \sum_{j=1}^n \log a_j}.$$

Choose $s = \frac{1+\epsilon}{2+\beta}$ in (3.10). By (3.13), we have

$$\mathcal{H}^{\frac{1+\epsilon}{2+\beta}}(E_N(\psi)) \leq \liminf_{k \rightarrow \infty} \sum_{(a_1, \dots, a_{n_k}) \in D_{n_k}(\epsilon)} \mu_{(1+\epsilon/2)}(I_{n_k}(a_1, \dots, a_{n_k})) \leq 1.$$

Hence

$$\dim_H E(\psi) \leq \sup_{N \geq 1} \left\{ \dim_H E_N(\psi) \right\} \leq \frac{1+\epsilon}{2+\beta}.$$

Then the desired result follows by letting $\epsilon \rightarrow 0$. \square

Final remark: Now we give a remark on the dimension of J_β^* and that of J_β . Recall that J_β^* and J_β are defined in Section 1. For any $(a_1, \dots, a_n) \in \mathbb{N}^n$, we define

$$\tilde{J}_n(a_1, \dots, a_n) = \bigcup_{a_{n+1} \geq q_n^\beta} I_{n+1}(a_1, \dots, a_n, a_{n+1}).$$

Then it is clear that

$$J_\beta = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \tilde{J}_n(a_2, \dots, a_n),$$

where the last union is taken over all $(a_1, \dots, a_n) \in \mathbb{N}^n$. While

$$J_\beta^* \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \tilde{J}_n(a_2, \dots, a_n),$$

where the last union is taken over all $(a_1, \dots, a_n) \in \mathbb{N}^n$ with $\frac{\log q_n}{n}$ being sufficiently large. As a result,

$$\mathcal{H}^s(J_\beta) \leq \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \sum_{(a_1, \dots, a_n) \in \mathbb{N}^n} \left(\frac{1}{q_n} \right)^{s(2+\beta)},$$

while

$$\mathcal{H}^s(J_\beta^*) \leq \liminf_{N \rightarrow \infty} \sum_{n=N}^{\infty} \sum_{(a_1, \dots, a_n) \in \mathbb{N}^n, (\log q_n)/n \text{ large}} \left(\frac{1}{q_n} \right)^{s(2+\beta)}.$$

By (2.3), we know that

$$1 \leq \sum_{a_1, \dots, a_n \in \mathbb{N}^n} q_n^{-2} \leq 2. \quad (3.15)$$

While, by (3.13), we get

$$\sum_{a_1, \dots, a_n: \log q_n/n \text{ large}} q_n^{-(1+\epsilon)} \leq 1. \quad (3.16)$$

Comparing of (3.15) and (3.16) reveals that

$$\dim_H J_\beta \leq \frac{2}{2+\beta}, \quad \dim_H J_\beta^* \leq \frac{1}{2+\beta}.$$

Actually we have proven that $\dim_H J_\beta^* = \frac{1}{2+\beta}$ since $E(\psi)$ can serve as a subset of J_β^* .

ACKNOWLEDGEMENT: This work was partially supported by PICS program No. 5727, RFDP20090141120007, NSFC 10901066 and NSFC 11171124. The authors thank the Morningside Center of Mathematics, Beijing for its hospitality.

REFERENCES

- [1] Billingsley, P., Henningsen, I.: Hausdorff dimension of some continued-fraction sets. *Z. Wahrscheinlichkeitstheorie verw. Geb.* 31, 163-173 (1975)
- [2] Fan, A.-H., Liao, L.-M., Ma, J.-H.: On the frequency of partial quotients of regular continued fractions. *Math. Proc. Camb. Phil. Soc.* 148, 179-192 (2010)
- [3] Fan, A.-H., Liao, L.-M., Wang, B. W., Wu, J.: On Kintchine exponents and Lyapunov exponents of continued fractions. *Ergod. Th. Dynam. Sys.* 29, 73-109 (2009)
- [4] Jarník, I.: Zur metrischen Theorie der diophantischen Approximationen. *Proc. Mat. Fyz.* 36, 91-106 (1928)
- [5] Jaerisch, J., Kesseböhmer, M.: The arithmetic-geometric scaling spectrum for continued fractions. *Arkiv för Matematik* 48 (2), 335-360 (2010)
- [6] Kesseböhmer, M., Stratmann, S.: A multifractal analysis for Stern-Brocot intervals, continued fractions and Diophantine growth rates. *J. Reine Angew. Math.* 605, 133-163 (2007)
- [7] Kifer, Y., Peres, Y., Weiss, B.: A dimension gap for continued fractions with independent digits. *Israel J. Math.* 124(1), 61-76 (2001)
- [8] Khintchine, A. Ya.: *Continued Fractions*. P. Noordhoff, Groningen, The Netherlands (1963)
- [9] Liao, L.-M., Ma, J.-H., Wang, B.-W.: Dimension of some non-normal continued fraction sets. *Math. Proc. Cambridge Philos. Soc.* 145 (1), 215-225 (2008)
- [10] Mauldin, R. D., Urbański, M.: Conformal iterated function systems with applications to the geometry of continued fractions. *Trans. Amer. Math. Soc.* 351 (12), 4995-5025 (1999)
- [11] Mayer, D.: On the thermodynamics formalism for the Gauss map. *Comm. Math. Phys.* 130, 311-333 (1990)
- [12] Olsen, L.: Extremely non-normal numbers. *Math. Proc. Cambridge Philos. Soc.* 137 (1), 43-53 (2004)
- [13] Pesin, Y.: *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*. Chicago Lectures in Mathematics, The University of Chicago Press, Chicago (1998)
- [14] Pollicott, M., Weiss, H.: Multifractal analysis of Lyapunov exponent for continued fraction and Manneville-Pomeau transformations and applications to Diophantine approximation. *Comm. Math. Phys.* 207 (1), 145-171 (1999)

LAMFA, UMR 7352 (Ex 6140), CNRS, UNIVERSITÉ DE PICARDIE JULES VERNE, 33, RUE SAINT LEU, 80039 AMIENS CEDEX 1, FRANCE

E-mail address: ai-hua.fan@u-picardie.fr

LAMA, UMR 8050, CNRS, UNIVERSITÉ PARIS-EST CRÉTEIL VAL DE MARNE, 61, AVENUE DU GÉNÉRAL DE GAULLE 94010 CRÉTEIL CEDEX FRANCE

E-mail address: lingmin.liao@u-pec.fr

SCHOOL OF MATHEMATICS AND STATISTICS, HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, 430074 WUHAN, CHINA

E-mail address: bwei_wang@yahoo.com.cn

SCHOOL OF MATHEMATICS AND STATISTICS, HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, 430074 WUHAN, CHINA

E-mail address: wujunyu@public.wh.hb.cn