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MULTIFRACTAL ANALYSIS OF SOME MULTIPLE ERGODIC AVERAGES IN LINEAR COOKIE-CUTTER DYNAMICAL SYSTEMS

AI-HUA FAN, LINGMIN LIAO AND MENG WU

ABSTRACT. In this Note, we study the multiple ergodic averages of a locally constant real-valued function in linear Cookie-Cutter dynamical systems. The multifractal spectrum of these multiple ergodic averages is completely determined.

§1. INTRODUCTION AND STATEMENT OF RESULTS

Consider a piecewise linear map T on the unit interval with m branches ($m \geq 2$). Let $I_0, \dots, I_{m-1} \subset [0, 1]$ be m closed intervals with disjoint interiors. Suppose that on each interval I_i , the restriction $T : I_i \rightarrow [0, 1]$ is bijective and linear with slope e^{λ_i} ($\lambda_i > 0$). Let $J_T := \bigcap_{n=1}^{\infty} T^{-n}[0, 1]$ be the corresponding Cookie-Cutter set. We will study the dynamical system (J_T, T) . Let $\ell \geq 2$ be an integer and let φ be a real-valued function defined on $[0, 1]^\ell$. Let $q \geq 2$ be an integer. We would like to calculate the Hausdorff dimensions of the following level sets

$$L_\varphi(\alpha) := \left\{ x \in J_T : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi(T^k x, T^{qk} x, \dots, T^{q^{\ell-1}k} x) = \alpha \right\} \quad (\alpha \in \mathbb{R}). \quad (1)$$

If $\varphi = g_1 \otimes \dots \otimes g_\ell$ is a tensor product of ℓ real-valued functions, the averages in (1) is nothing but the *multiple ergodic averages* which have been widely studied in ergodic theory. The research for the dimensions of the level sets (1), which was initiated by Fan, Liao and Ma [1], is concerned with multifractal analysis of multiple ergodic averages and has attracted much attention (see e.g. [6, 7, 3, 4, 9, 5, 10, 8]). Most of the results concentrated on the symbolic space where the Lyapunov exponents for the shift transformation are constant. Liao and Rams [8] gave the multifractal analysis of a special multiple ergodic averages for systems with non-constant Lyapunov exponents. They considered the level set in (1) with $m = \ell = q = 2$ and $\varphi = 1_{I_1} \otimes 1_{I_1}$. Liao-Rams' argument was based on computations of Peres and Solomyak [9], which seems inconvenient to be adapted to general case. However, in this note we show how to adapt the arguments in [4] to arbitrary $m, \ell, q \geq 2$ and a class of functions φ .

We assume that φ is locally constant: for any $(a_1, a_2, \dots, a_\ell), (b_1, b_2, \dots, b_\ell) \in I_{i_1} \times I_{i_2} \times \dots \times I_{i_\ell}$ ($0 \leq i_1, i_2, \dots, i_\ell \leq m-1$), we have

$$\varphi(a_1, a_2, \dots, a_\ell) = \varphi(b_1, b_2, \dots, b_\ell).$$

With an abuse of notation, we write $\varphi(a_1, a_2, \dots, a_\ell) = \varphi(i_1, i_2, \dots, i_\ell)$ if $(a_1, a_2, \dots, a_\ell) \in I_{i_1} \times I_{i_2} \times \dots \times I_{i_\ell}$.

For simplicity of notation, we restrict ourselves to the case $\ell = 2$ (the same arguments works for arbitrary $\ell \geq 2$ without any problem). For any $s, r \in \mathbb{R}$,

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consider the non-linear transfer system

$$t_i(s, r)^q = \sum_{j=0}^{m-1} e^{s\varphi(i,j)+r\lambda_i} t_j(s, r), \quad (i = 0, \dots, m-1). \quad (2)$$

The difference between (2) and a similar system in [4] is that the Lyapunov exponents λ_i 's are introduced in (2). It is easy to prove that the system admits a unique strictly positive solution $(t_0(s, r), \dots, t_{m-1}(s, r))$ (see [4]). Define the pressure function by

$$P(s, r) = \log \sum_{j=0}^{m-1} t_j(s, r).$$

It will be shown (Lemma 1) that P is analytic and convex, and even strictly convex if φ is not constant and the λ_i 's are not all the same.

Let A and B be respectively the infimum and the supremum of the set

$$\left\{ a \in \mathbb{R} : \exists (s, r) \in \mathbb{R}^2 \text{ such that } \frac{\partial P}{\partial s}(s, r) = a \right\}.$$

It will be proved (Lemma 3) that for any $\alpha \in (A, B)$, there exists a unique solution $(s(\alpha), r(\alpha)) \in \mathbb{R}^2$ to the system

$$\begin{cases} P(s, r) &= \alpha s \\ \frac{\partial P}{\partial s}(s, r) &= \alpha. \end{cases} \quad (3)$$

and that $r(\alpha)$ is analytic on (A, B) , increasing on (A, φ_0) and decreasing on (φ_0, B) where $\varphi_0 = \sum_{i,j} \varphi(i, j)/m^2$. Hence the limits $r(A) := \lim_{\alpha \rightarrow A} r(\alpha)$ and $r(B) := \lim_{\alpha \rightarrow B} r(\alpha) := r(B)$ exist.

Let $D_\varphi = \{\alpha \in \mathbb{R} : L_\varphi(\alpha) \neq \emptyset\}$. Our main result is the following.

Theorem 1. *We have $D_\varphi = [A, B]$. For any $\alpha \in [A, B]$, we have $\dim_H L_\varphi(\alpha) = \frac{r(\alpha)}{q \log m}$.*

§2. PROOF OF THEOREM 1

We sketch the proof. Let $f_i : [0, 1] \rightarrow I_i$ be the branches of T^{-1} . Define $\pi : \Sigma_m \rightarrow [0, 1]$ by

$$\pi((\omega_k)_{k=1}^\infty) = \lim_{n \rightarrow \infty} f_{\omega_1} \circ f_{\omega_2} \cdots f_{\omega_n}(0).$$

Then we have $\pi(\Sigma_m) = J_T$. Define the subset $E_\varphi(\alpha)$ of Σ_m which was studied in [3, 4]:

$$E_\varphi(\alpha) := \left\{ (\omega_k)_{k=1}^\infty \in \Sigma_m : \lim_{n \rightarrow \infty} \frac{1}{n} \sum \varphi(\omega_k, \omega_{qk}) = \alpha \right\}.$$

Then with a difference of a countable set, we have $L_\varphi(\alpha) = \pi(E_\varphi(\alpha))$.

In [3, 4], a family of Gibbs-type measures called *telescopic product measures* were used to compute the Hausdorff dimension of $E_\varphi(\alpha)$. Here we construct a similar class of measures in order to determine the Hausdorff dimension of $L_\varphi(\alpha)$. In the following, we suppose that φ is not constant (otherwise the problem is trivial) and the λ_i 's are not the same (otherwise the problem is reduced to the case considered in [3]).

Lemma 1. *For any $s, r \in \mathbb{R}$, the system (2) admits a unique solution $(t_0(s, r), \dots, t_{m-1}(s, r))$ with strictly positive components, which are analytic functions of (s, r) . The pressure function P is strictly convex in \mathbb{R}^2 .*

Proof It is essentially the same proof as in [4], with some small modifications. \square

The solution $(t_0(s, r), t_1(s, r), \dots, t_{m-1}(s, r))$ of the system (2) allows us to define a Markov measure $\mu_{s,r}$ with initial law $(\pi(i))_{i \in S}$ and probability transition matrix $(p_{i,j})_{S \times S}$ defined by

$$\pi(i) = \frac{t_i(s, r)}{t_0(s, r) + \dots + t_{m-1}(s, r)}, \quad p_{i,j} = e^{s\varphi(i,j) + r\lambda_i} \frac{t_j(s, r)}{t_i(s, r)^2}.$$

Then we decompose the set of positive integers \mathbb{N}^* into Λ_i ($q \nmid i$) with $\Lambda_i = \{iq^k\}_{k \geq 0}$ so that $\Sigma_m = \prod_{i: q \nmid i} S^{\Lambda_i}$. Take a copy $\mu_{s,r}$ on each S^{Λ_i} and define the product measure $\mathbb{P}_{s,r}$ of these copies. Then $\mathbb{P}_{s,r}$ is a probability measure on Σ_m .

Let $\nu_{s,r} = \pi_* \mathbb{P}_{s,r} = \mathbb{P}_{s,r} \circ \pi^{-1}$ and let $\underline{D}(\nu_{s,r}, x)$ be the lower local dimension of $\nu_{s,r}$ at x .

Lemma 2. *Let $\alpha \in (A, B)$. For any $(s, r) \in \mathbb{R}^2$ such that $P(s, r) = \alpha$, for any $x \in L_\varphi(\alpha)$ we have*

$$\underline{D}(\nu_{s,r}, x) \leq \frac{r}{q \log m}.$$

Proof It is the same proof as in [4]. \square

It follows that $\dim_H L_\varphi(\alpha) \leq \frac{r}{q \log m}$. Minimizing the right hand side gives an upper bound:

$$\dim_H E(\alpha) \leq \frac{r(\alpha)}{q \log m},$$

where $r(\alpha)$ together with an $s \in \mathbb{R}$ constitute a (unique) solution of the system (3). The following lemma says that such $r(\alpha)$ is really unique.

Lemma 3. *For any $\alpha \in (A, B)$, there exists a unique solution $(s(\alpha), r(\alpha)) \in \mathbb{R}^2$ to the equation (3). Moreover the function $r(\alpha)$ is analytic on (A, B) , increasing on (A, φ_0) and decreasing on (φ_0, B) where $\varphi_0 = \sum_{i,j} \varphi(i, j)/m^2$ is the average of φ with respect to the Lebesgue measure on Σ_m .*

Proof Similar proof as in Proposition 4.13 of [2]. \square

In order to get the lower bound, we only have to show that $\nu_{s(\alpha), r(\alpha)}$ is supported on $L_\varphi(\alpha)$ and the local dimension of $\nu_{s(\alpha), r(\alpha)}$ is almost everywhere equal to $r(\alpha)/q \log m$. To this end, we need the following law of large numbers (LLN) (see Theorem 6 in [4]).

Lemma 4 ([4]). *Let (F_n) be a sequence of functions defined on $\{0, 1, \dots, m-1\}^2$ such that $\sup_n \sup_{x,y} |F_n(x, y)| < \infty$. For $\mathbb{P}_{s,r}$ -a.e. $\omega \in \Sigma_m$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (F_k(\omega_k, \omega_{qk}) - \mathbb{E}_{\mathbb{P}_s} F_k(\omega_k, \omega_{qk})) = 0.$$

Applying the above lemma to $F_n(\omega_n, \omega_{qn}) = \varphi(\omega_n, \omega_{qn})$ for all n and computing $\mathbb{E}_{\mathbb{P}_{s,r}} \varphi(\omega_n, \omega_{qn})$, we get

Lemma 5. *For $\mathbb{P}_{s,r}$ -a.e. $\omega \in \Sigma_m$, we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \varphi(\omega_k, \omega_{qk}) = \frac{\partial P}{\partial s}(s, r).$$

Lemma 5 implies $\mathbb{P}_{s(\alpha),r(\alpha)}(E_\varphi(\alpha)) = 1$. Hence

$$\nu_{s(\alpha),r(\alpha)}(L_\varphi(\alpha)) = \mathbb{P}_{s(\alpha),r(\alpha)}(\pi^{-1}(L_\varphi(\alpha))) = \mathbb{P}_{s(\alpha),r(\alpha)}(E_\varphi(\alpha)) = 1.$$

Finally, it remains to show the measure $\nu_{s,r}$ is exact in the sense that its local dimension is almost everywhere constant.

Lemma 6. *For $\nu_{s,r}$ -a.e. x we have*

$$D(\nu_{s,r}, x) = \frac{r}{q \log m} + \frac{P(s, r) - s \frac{\partial P}{\partial s}(s, r)}{\lambda(s, r)},$$

where $\lambda(s, r)$ is the expected limit with respect to $\mathbb{P}_{s,r}$ of the Lyapunov exponent $\frac{1}{n} \sum_{k=1}^n \lambda_{\omega_k}$ with $\omega \in \Sigma_m$.

Proof It suffices to apply the LLN to get the expected limits of $\log \mathbb{P}_{s,r}(C_n(\omega))/n$ and of $\log |\pi(C_n(\omega))|/n$ where $C_n(\omega) = \{u \in \Sigma_m : u_k = \omega_k, \forall k \leq n\}$. \square

Thus we have finished the proof for $\alpha \in (A, B)$. If $\alpha = A$ (resp. B), as in the standard multifractal analysis, we use the measure $\nu_{s(\alpha),r(\alpha)}$ and let α tend to A (resp. B).

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