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# Explicit computation of the electrostatic energy for an elliptical charged disc

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## Abstract

This letter describes a method to obtain an explicit expression for the electrostatic energy of a charged elliptical infinitely thin disc. The charge distribution is assumed to be polynomial. Such explicit values for this energy are fundamental for assessing the accuracy of boundary element codes. The main tools used are an extension of Copson's method and a diagonalization, given by Lepington and Levine, of the single-layer potential operator associated with the electrostatic potential created by a distribution of charges on the elliptical disc.

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## 1. Introduction

In recent years, integral equations became an essential tool for solving both industrial and scientific problems in electromagnetism and acoustics. The assessment of the accuracy delivered by such codes, in particular their handling of the involved singular integrals, is a major issue. Here, we present a method to derive an analytical expression for the electrostatic energy of a charged elliptical infinitely thin plate, providing a mean for the validation of these codes.

Let us denote by  $A = \{(x_1, x_2) \in \mathbb{R}^2 \text{ with } x_1^2/a^2 + x_2^2/b^2 - 1 < 0 \text{ and } a > b\}$  the ellipse with major and minor semi-axis  $a$  and  $b$ . Let  $f$  be the electrostatic potential generated by a density of charges  $\sigma$  distributed over  $A$

$$f(\mathbf{x}) = \frac{1}{4\pi} \int_A \frac{\sigma(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} ds_{\mathbf{y}} \quad \text{for all } \mathbf{x} \text{ in } A, \quad (1)$$

with  $\mathbf{x} = (x_1, x_2)$ . The units have been chosen so that the electric permittivity of air is 1. The electrostatic energy  $I$  can be expressed in either the two following forms:

$$I_{\sigma} = \int_A f(\mathbf{x}) \overline{\sigma(\mathbf{x})} ds_{\mathbf{x}} = \int_A \frac{1}{4\pi} \int_A \frac{\overline{\sigma(\mathbf{x})} \sigma(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} ds_{\mathbf{x}} ds_{\mathbf{y}}. \quad (2)$$

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We aim in this letter in proving and in numerically illustrating the following theorem, where  $\varepsilon$  is the eccentricity of the ellipse  $A$  given by  $\varepsilon = \sqrt{1 - b^2/a^2}$ .

**Theorem 1.1.** *Let  $\sigma(\mathbf{x}) = \alpha_0 + \alpha_1 x_1/a + \alpha_2 x_2/b$ , with  $\alpha \in \mathbb{R}^3$ , be the distribution of charges over  $A$ . The corresponding electrostatic energy is given by*

$$I_\sigma = \frac{8ab^2}{15\pi} \left[ (5\alpha_0^2 + \alpha_2^2) K(\varepsilon) + (\alpha_1^2 - \alpha_2^2) \frac{K(\varepsilon) - E(\varepsilon)}{\varepsilon^2} \right],$$

with  $K(\varepsilon)$  and  $E(\varepsilon)$  the complete elliptic integrals of first and second kind

$$K(\varepsilon) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \varepsilon^2 \sin^2 \phi}} \quad \text{and} \quad E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 \phi} d\phi. \quad (3)$$

## 2. Diagonalization of the electrostatic energy

Following [1], we consider the spheroidal coordinate system  $(\theta, \varphi)$  giving a parametrization of  $A$  in terms of the unit half-sphere

$$x_1 = a \sin \theta \cos \varphi \quad \text{and} \quad x_2 = b \sin \theta \sin \varphi, \quad \text{with } \theta \in [0, \pi/2], \varphi \in [0, 2\pi]. \quad (4)$$

The elemental area associated with the new variables is  $ab \cos \theta \sin \theta d\theta d\varphi$ . In these spheroidal coordinates, the electrostatic potential  $f$  defined in (1) and the electrostatic energy can also be written in terms of  $\theta$  and  $\varphi$  as

$$\begin{cases} f(\theta, \varphi) = \frac{ab}{4\pi} \int_0^{\pi/2} \int_0^{2\pi} \frac{g(\theta', \varphi')}{d(\theta, \varphi, \theta', \varphi')} \sin \theta' d\theta' d\varphi', \\ I_\sigma = ab \int_0^{\pi/2} \int_0^{2\pi} \overline{f(\theta, \varphi)} g(\theta, \varphi) \sin \theta d\theta d\varphi, \end{cases}$$

with  $g(\theta, \varphi) = \sigma(\theta, \varphi) \cos \theta$  and  $d(\theta, \varphi, \theta', \varphi')$  the distance separating  $\mathbf{x}$  from  $\mathbf{y}$   $d = |\mathbf{x} - \mathbf{y}|$  expressed in the spheroidal coordinates (4).

The next step consists in introducing a well chosen spectral basis of the half sphere involving the even Legendre functions  $Q_n^m$  normalized by

$$\int_0^\pi Q_n^m(\cos \theta) Q_{n'}^m(\cos \theta) \sin \theta d\theta = \delta_{n,n'}. \quad (5)$$

This basis yields a block diagonalization of the convolution operator (see [1])

$$\begin{aligned} \frac{1}{d} &= \frac{1}{\sqrt{ab}} \sum_{n=0}^{\infty} \sum_{\substack{m=-n \\ n-m \text{ even}}}^n \sum_{\substack{m'=-n \\ n-m' \text{ even}}}^n d_{mm'}^n Q_n^m(\cos \theta) Q_n^{m'}(\cos \theta') e^{i(m\varphi - m'\varphi')}, \\ \text{with } d_{mm'}^n &= \frac{Q_n^m(0) Q_n^{m'}(0)}{2n+1} \int_0^{2\pi} \frac{e^{i(m-m')\varphi}}{\sqrt{\frac{b}{a} \cos^2 \varphi + \frac{a}{b} \sin^2 \varphi}} d\varphi. \quad (6) \end{aligned}$$

The functions  $f$  and  $g$  can be expanded in this basis as

$$u(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{\substack{m=-n \\ n-m \text{ even}}}^n u_n^m Q_n^m(\cos \theta) e^{im\varphi} \quad \text{with } u = f \text{ or } g \quad (7)$$

with

$$u_n^m = \frac{1}{\pi} \int_0^{\pi/2} \int_0^{2\pi} u(\theta, \varphi) Q_n^m(\cos \theta) e^{-im\varphi} \sin(\theta) d\theta d\varphi. \quad (8)$$

Due to (6), coefficients  $f_n^m$  are related to  $g_n^m$  by

$$f_n^m = \frac{\sqrt{ab}}{4} \sum_{\substack{m'=-n \\ n-m' \text{ even}}}^n d_{mm'}^n g_{n'}^{m'}. \quad (9)$$

Moreover, the orthogonal properties of the spectral basis yield

$$I_\sigma = \frac{\pi}{4} (ab)^{3/2} \sum_{n,m,m'} d_{mm'}^n g_n^m \overline{g_{n'}^{m'}},$$

where we have shorten the notation by making implicit the range of the summation index. Indices  $n, n'$  are varying from 0 to  $\infty$ , and  $m, m'$  are such that  $|m| \leq n$ ,  $|m'| \leq n'$ ,  $n - m$  and  $n - m'$  even. Substituting expression (6) for  $d_{mm'}^n$  and introducing the eccentricity of the ellipse  $\varepsilon$ , we get

$$I_\sigma = \pi ab^2 \sum_{n,m,m'} g_n^m \overline{g_{n'}^{m'}} \frac{Q_n^m(0) Q_{n'}^{m'}(0)}{2n+1} \int_0^{\pi/2} \frac{\cos(m-m')\varphi}{\sqrt{1-\varepsilon^2 \cos^2 \varphi}} d\varphi. \quad (10)$$

### 3. Electrostatic energy for an affine distribution of charges

This section is dedicated to the calculation of the electrostatic energy generated by an affine density of charges.

**Proof of Theorem 1.1.** Let  $\sigma_0(\mathbf{x}) = 1$ ,  $\sigma_1(\mathbf{x}) = x_1/a$  and  $\sigma_2(\mathbf{x}) = x_2/b$ . In view of the symmetry of  $A$  with respect to  $x_1$  and  $x_2$ , we have

$$\int_A \frac{1}{4\pi} \int_A \frac{\sigma_i(\mathbf{x})\sigma_j(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} ds_{\mathbf{x}} ds_{\mathbf{y}} = 0 \quad \text{for } i \neq j.$$

Consequently, the electrostatic energy  $I_\sigma$  can be expanded as

$$I_\sigma = \alpha_0^2 I_{\sigma_0} + \alpha_1^2 I_{\sigma_1} + \alpha_2^2 I_{\sigma_2}$$

The result will follow from the computation of  $I_{\sigma_0}$ ,  $I_{\sigma_1}$  and  $I_{\sigma_2}$ .

#### 3.1. Computation of $I_{\sigma_0}$

For  $\sigma(\mathbf{x}) = \sigma_0(\mathbf{x}) = 1$ , the function  $g(\theta, \varphi) = \cos \theta$  does not depend on  $\varphi$ . The  $g_n^m$  coefficients are independent from  $a$  and  $b$ , and since  $g_n^m = 0$  for all

$m \neq 0$ , the function  $g$  can be expanded as  $g(\theta) = \sum_{n=0}^{+\infty} g_n^0 Q_n^0(\cos \theta)$ . Due to (10), the electrostatic energy  $I_{\sigma_0}$  depends only on  $a$  and  $b$  and is given by

$$I_{\sigma_0}(a, b) = \pi ab^2 \sum_{n=0}^{+\infty} \frac{|g_n^0|^2 (Q_n^0(0))^2}{2n+1} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \varepsilon^2 \cos^2 \varphi}} = \kappa ab^2 K(\varepsilon), \quad (11)$$

with  $\varepsilon = \sqrt{1 - b^2/a^2}$  and  $\kappa$  a constant depending neither on  $a$  nor on  $b$ . The constant  $\kappa$  is deduced from the classical case of an unit circle which has been detailed for example in [2]

$$I_{\sigma_0}(1, 1) = \int_{\mathcal{C}} \frac{1}{4\pi} \int_{\mathcal{C}} \frac{1}{|\mathbf{x} - \mathbf{y}|} ds_{\mathbf{x}} ds_{\mathbf{y}} = 4/3. \quad (12)$$

Comparing (11) and (12), this yields to  $\kappa = 8/3\pi$  since  $K(0) = \pi/2$ . Therefore,

$$I_{\sigma_0} = \frac{8}{3\pi} ab^2 K(\varepsilon). \quad (13)$$

### 3.2. Computation of $I_{\sigma_1}$

For  $\sigma(\mathbf{x}) = \sigma_1(\mathbf{x}) = x_1/a$ , the function  $g$  is given by  $g(\theta, \varphi) = \sin \theta \cos \theta \cos \varphi$ . In that case, the  $g_n^m$  are zero except for  $|m| = 1$ . By definition of the Legendre functions, we have  $Q_n^{-1} = -Q_n^1$ . As the function  $g$  is even, it comes that  $g_n^{-1} = -g_n^1$ , and thus

$$I_{\sigma_1} = \pi ab^2 \sum_n \frac{(g_n^1 Q_n^1(0))^2}{2n+1} \left[ 2 \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \varepsilon^2 \cos^2 \varphi}} + 2 \int_0^{\pi/2} \frac{\cos 2\varphi d\varphi}{\sqrt{1 - \varepsilon^2 \cos^2 \varphi}} \right].$$

Due to (3), it comes

$$I_{\sigma_1}(a, b) = \kappa ab^2 \frac{K(\varepsilon) - E(\varepsilon)}{\varepsilon^2}. \quad (14)$$

To determine the constant  $\kappa$ , we consider again the case of an unit circle. In this case,  $I_{\sigma_1}$  can be explicitly computed, see Appendix A, and is given by  $I_{\sigma_1}(1, 1) = 2/15$ . Evaluating (14) at  $a = b = 1$  we get

$$I_{\sigma_1}(1, 1) = \kappa \frac{\pi}{4} \quad \text{since} \quad \lim_{\varepsilon \rightarrow 0} \frac{K(\varepsilon) - E(\varepsilon)}{\varepsilon^2} = \frac{\pi}{4}. \quad (15)$$

It follows that  $\kappa = 8/15\pi$  and therefore we have

$$I_{\sigma_1} = \frac{8}{15\pi} ab^2 \frac{K(\varepsilon) - E(\varepsilon)}{\varepsilon^2}. \quad (16)$$

### 3.3. Computation of $I_{\sigma_2}$

For  $I_{\sigma_2}$ , we consider  $\sigma(\mathbf{x}) = \sigma_2(\mathbf{x}) = x_2/b$ , meaning  $g(\theta, \varphi) = \sin \theta \cos \theta \sin \varphi$ . We still have  $g_n^m = 0$  except for  $|m| = 1$ , but in that case,  $g_n^{-1} = g_n^1$ . Thus

$$I_{\sigma_2} = \pi ab^2 \sum_{n=0}^{+\infty} \frac{(g_n^1 Q_n^1(0))^2}{2n+1} \left[ 2 \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\varepsilon^2 \cos^2 \varphi}} - 2 \int_0^{\pi/2} \frac{\cos 2\varphi d\varphi}{\sqrt{1-\varepsilon^2 \cos^2 \varphi}} \right]$$

Moreover, both integrals  $I_{\sigma_1}$  and  $I_{\sigma_2}$  are equal on  $\mathbb{C}$  due to symmetry. We obtain

$$I_{\sigma_2} = \frac{8}{15\pi} ab^2 \left( K(\varepsilon) - \frac{K(\varepsilon) - E(\varepsilon)}{\varepsilon^2} \right). \quad (17)$$

## 4. Numerical tests and conclusion

Tables 1 and 2 yield a comparison of exact values given by an analytical expression with numerical approximate values obtained by the boundary element code CESC of CERFACS with a  $P^1$  continuous elements. It can be observed that the two values coincide at least up to the fourth decimal digit. Table 3 depicts the maximum relative error for each of tables 1 and 2 cases, which is less than 0.35 per mil.

$a$	0.5	0.7	0.9	1.1	1.3	1.5
$I_{\sigma_0}^{\text{comp}}$	0.1666	0.2741	0.3939	0.5234	0.6608	0.8048
$I_{\sigma_0}^{\text{exact}}$	0.1666	0.2741	0.3939	0.5234	0.6608	0.8048
$I_{\sigma_1}^{\text{comp}} \times 10^{-1}$	0.0417	0.1455	0.3651	0.7535	1.3715	2.2781
$I_{\sigma_1}^{\text{exact}} \times 10^{-1}$	0.0417	0.1456	0.3656	0.7543	1.3717	2.2800
$I_{\sigma_2}^{\text{comp}} \times 10^{-2}$	0.4167	0.6280	0.8426	1.0585	1.2748	1.4910
$I_{\sigma_2}^{\text{exact}} \times 10^{-2}$	0.4167	0.6280	0.8427	1.0586	1.2748	1.4911

Table 1: Exact and computed values of the electrostatic energy  $I_{\sigma_i}^{\text{exact}}$  and  $I_{\sigma_i}^{\text{comp}}$  for an elliptical disc with a minor axis  $b = 0.5$ , given by (13), (16), (17).

$a$	0.75	0.9	1.05	1.2	1.35	1.5
$I_{\sigma}^{\text{comp}}$	2.7734	3.6157	4.5162	5.4706	6.4761	7.5313
$I_{\sigma}^{\text{exact}}$	2.7736	3.6159	4.5165	5.4708	6.4763	7.5316

Table 2: Affine density of charges  $\sigma = x_1 + 2x_2 + 3$  for an elliptical disc with  $b = 0.5$ .

	$I_{\sigma_0}$	$I_{\sigma_1}$	$I_{\sigma_2}$	$I_{\sigma}$
$\varepsilon_{\text{rel}}$	0.055 <sup>0</sup> / <sub>00</sub>	0.195 <sup>0</sup> / <sub>00</sub>	9.1 × 10 <sup>-4</sup> <sup>0</sup> / <sub>00</sub>	0.334 <sup>0</sup> / <sub>00</sub>

Table 3: Maximum value of the relative error  $\varepsilon_{\text{rel}} = \max |I_{\sigma_i}^{\text{exact}} - I_{\sigma_i}^{\text{comp}}|$  in per mil for an elliptical disc for  $b = 0.5$  and  $a = 0.5 : 0.05 : 1.5$ .

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## Appendix A. The case of an unit circle disc

Let  $C$  be the circle with radius 1. We aim here in computing the integral

$$I_C = \frac{1}{4\pi} \int_C \frac{x_1 y_1}{|\mathbf{x} - \mathbf{y}|} ds_{\mathbf{x}} ds_{\mathbf{y}}$$

This integral is rewritten in polar coordinates ( $r, \phi$  for  $\mathbf{x}$  and  $\rho, \phi'$  for  $\mathbf{y}$ )

$$I_C = \frac{1}{4\pi} \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} \frac{r \cos \phi \rho \cos \phi'}{\sqrt{r^2 + \rho^2 - 2r\rho \cos(\phi - \phi')}} r dr d\phi \rho d\rho d\phi'$$

and evaluated thanks to the formula (3.4.5) of [3, p70]

$$\int_0^{2\pi} \frac{\cos \phi d\phi}{\sqrt{\rho^2 + r^2 - 2r\rho \cos(\phi - \phi')}} = \frac{4 \cos \phi'}{\rho r} \int_0^{\min(\rho, r)} \frac{t^2 dt}{\sqrt{\rho^2 - t^2} \sqrt{r^2 - t^2}}.$$

This leads to

$$I_C = \frac{1}{\pi} \int_0^1 \int_0^1 \int_0^{\min(\rho, r)} \frac{r \rho t^2}{\sqrt{\rho^2 - t^2} \sqrt{r^2 - t^2}} dr d\rho dt \int_0^{2\pi} \cos^2 \phi' d\phi'.$$

This integral is symmetric in  $\rho$  and  $r$ , and since  $\int_0^{2\pi} \cos^2 \phi' d\phi' = \pi$ , we have

$$I_C = 2 \int_{t=0}^1 t^2 \int_{\rho=t}^1 \frac{\rho}{\sqrt{\rho^2 - t^2}} \int_{r=\rho}^1 \frac{r}{\sqrt{r^2 - t^2}} dr d\rho dt = \frac{2}{15}.$$

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