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Abstract

We consider a two period model in which a continuum of agents trade in a context of costly information acquisition and systematic heterogeneous expectations biases. We show that under very weak technical assumptions a market equilibrium exists and the supply and demand functions are strictly monotonic with respect to the price. The equilibrium price is also shown to be the price that maximizes the trading volume. We prove additional properties such as the anti-monotony of the trading volume with respect to the marginal information price.

Keywords:
information acquisition, heterogeneous beliefs, heterogeneous estimations, Grossman-Stiglitz paradox, costly information
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1. Introduction

We consider a continuum of agents that act in a two-period ($t = 0$ and $t = T$) market consisting of a single asset of value V. The value V is constant, deterministic but unknown to the agents. Each agent constructs an

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estimation for V in the form of a normal variable with known mean and variance. The numerical value of the mean, which is not necessarily V and as such can be interpreted as a systematic bias, is given by their estimation method and cannot be changed. However, the variance can be reduced at time $t = 0$ against a cost, which is a known deterministic function of the target variance to be attained. Each agent uses a CARA utility function and constructs the functional mapping each triplet of market price, estimation mean and estimation variance to the optimal number of units to trade. The sum of all such functions from all agents results at time $t = 0$ in aggregate market demand and supply functions; the price of the asset is chosen to clear the market (we prove in particular that except trivial settings such a price exists and is unique). This price can be different from the real value V and in practice it will. The agents close their position at final time $t = T$. This paper investigates the following questions: existence of an equilibrium, continuity of supply and demand functions, and interpretation of the equilibrium price as the price maximizing the liquidity (trading volume).

The paper is organized as follows. The rest of this section presents a literature overview. In Section 2 the model is explained and the fundamental hypothesis 2 is introduced. In Sections 3 and 4 we prove the existence of an equilibrium and important properties of the liquidity (here defined as the transaction volume) among which the fact that the market price also maximizes the trading volume. We apply our results to a Grossmann-Stiglitz framework in Section 4.1. Finally, in Section 5 we show that the liquidity is inversely correlated with the marginal price of information.

1.1. Literature overview

The model has two several ingredients : the existence of heterogeneous beliefs (or expectations) biases among a continuum of agents and the fact that the information is costly (the literature speaks of "information acquisition" cost).

The literature is rich with approaches to model how disagreements between agent estimations' generates investment decisions and trading volume. The importance of the heterogeneity of opinions on the future value of a financial instrument and its use in speculation has been recognized as early as Keynes (see Keynes (1936)) that invokes the "beauty contest" metaphor to explain how speculators would like to predict the future consensus price.

A model of speculative trading in a large economy with a continuum of agents with heterogeneous beliefs was presented in Wu & Guo (2003, 2004)

(see also the references within). They demonstrate the existence of price amplification effects and show that the equilibrium prices can be different from the rational expectation equilibrium price. It is also shown that trading volume is positively related to the directions of price changes and they explain the recurrent presence of diverse beliefs. We also refer to Scheinkman & Xiong (2004) and references within for a survey on how heterogeneous beliefs among agents generate speculation and trading.

The difference-of-opinion approach (see Varian (1985); Harris & Raviv (1993)) does not consider noise agents but on the contrary obtain diverse posterior beliefs from the differences in the way agents interpret common information. The primary focus is on the implications of dispersion in beliefs on the price level or direction. Yet another different method explains diverse posterior beliefs by relaxing the assumption of common prior (see Morris (1996)); the authors also model the learning process which enables a convergence towards a common estimation when more information is available. Such a framework was invoked for modeling asset pricing during initial public offerings, but not for other speculative circumstances. Finally, see also Pagano (1989) that analyses the implications of low liquidity in a market and propose appropriate incentive schemes to shift the market to a equilibrium characterized by a higher number of transactions.

An important advance has been to recognize that the dynamics of the information gathering is important; it was thus established how the presence of private information and noise (liquidity) agents interact with market price and volume (see, for example Grossman & Stiglitz (1980); Long et al. (1990) and Wang (1994) for recent related endeavors). It was thus in particular recognized (the so called "Grossman-Stiglitz paradox") that is not always optimal for the agents to obtain all the information on a particular asset. This remark is important in the following because, as explained in Section 2, our model allows for each agent to choose his level of precision concerning the information to acquire on a given asset. In the classical paper of Verrecchia (1982) and in subsequent related works Jackson (1991); Veldkamp (2006); Ko & Huang (2007); Krebs (2007); Litvinova & Ou-Yang (2003); Peng (2005) a framework is proposed where the information is costly and agents can pay more to lower their uncertainty on the future value of the risky asset. Verrecchia derives a close form solution which requires some particular assumptions, among which the convexity of the cost as function of the precision (inverse of the variance of the estimate). On the contrary our cost function is here only lower semi-continuous. Our approach also differs in a

more fundamental way in that we suppose that heterogeneity of estimations is given but arbitrary, i.e. not centered around the correct price. Moreover, the Verrecchia model relies on the heterogeneity of risk tolerances in the CARA utility function while here the price formation mechanism does not require such an assumption, the heterogeneity in estimations being enough. Also, in this model the endowments of the agents do not play any role and in particular are not required to obtain an equilibrium. The paper also extends a previous work Shen & Turinici (2012) where stronger technical hypothesis were invoked.

2. The model

We consider a two-period model, $t = 0$ and $t = T$ in which a risky security of value V is traded. The value V is unknown to the agents and each participant x in the market constructs at $t = 0$ an estimate $\tilde{A}^x$ for V, $\tilde{A}^x$ being a random variable. For simplicity, we suppose that $\tilde{A}^x$ has a normal distribution, and that $\tilde{A^{x_1}}$ and $\tilde{A^{x_2}}$ are independent if x_1 and x_2 are two distinct agents. Also, we assume that the mean and the variance of $\tilde{A}^x$ are respectively given by A^x and $(\sigma^x)^2$, both mean and variance being known to the agent x. As in Verrecchia (1982) we work with the precision $B^x = 1/(\sigma^x)^2$ instead of the variance $(\sigma^x)^2$.

Note that we do not model here the riskless security but everything works as if the numeraire was the riskless security; from a technical point of view this allows to set the interest rate to zero.

An important remark is that each agent has his own bias attached to his estimate $\tilde{A}^x$ because he has his own procedure to interpret the available information. It may be due to personal optimism or pessimism or be correlated with some exogenous factors, such as overall economic outlooks, commodities evolution, geopolitic factors, that each agent interprets with a specific systematic bias. See also the cited references for additional discussion on how agents interpret the information they obtain. We assume that the bias $A^x - V$ of agent x does not depend on the precision B^x to be attained and only depends on the agent; the value A^x associated to an agent is known only by himself. The agent does not influence A^x in any way during the process of forecasting. Hence, two different agents x_1 and x_2 have generically different biases $A^{x_1} - V$ and $A^{x_2} - V$ and thus different estimation averages A^{x_1} and A^{x_2}. This is not a collateral property of the model. It is instead the mere

reason for which the agents trade. They trade because they have different (heterogeneous) expectations on the final value of the security.

We define $\rho(A)$ to be the distribution of A^x among the agents; neither the law of the distribution $\rho(A)$ nor any moments or statistics are known by the agents. We also introduce the expected value with respect to $\rho(\cdot)$, which is denoted $\mathbb{E}^A$; also see Abarbanell et al. (1995) for related works on how to empirically estimate such a ρ. We do not consider the law of ρ to be normal or have particular properties (except technical hypothesis 8 below).

From a theoretical point of view it is interesting to explore the situation when $\mathbb{E}^A(A) = V$. This means that the average estimate is V, so that the agents are neither overpricing nor underpricing the security with respect to its (unknown) value. However, we will see that this does not necessarily indicate that the market price is V.

The only parameter the agent can control is the accuracy of the result, i.e. the precision B^x. However, this has a cost: they need to pay $f(b)$ to obtain precision b. The precision cost function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined on positive numbers. By convention, we can assume that $f(b) = \infty$ for any $b < 0$. See also Peng & Xiong (2003) for an example involving a power function and Peng (2005) for a structural model to motivate such a function.

Such a model is relevant in the case of high expense for information sources, for instance news broadcasting fees. The expense also involves the reward of research personnel or the need for more accurate numerical computations.

Based on his estimations the agent x decides at time $t = 0$ to trade a quantity of θ^x security units. When θ^x is positive, the agent is long, so he buys the security, whereas when θ^x is negative, he is short: he sells it.

Hence, each agent is characterized by three parameters: his mean estimate A^x, the precision B^x of the estimate (that comes at a cost $f(B^x)$) and the quantity of traded units, θ^x.

The agents buys of sells the security at time $t = 0$ by formulating demand and supply functions depending on the price. The market price at time $t = 0$ is chosen to clear the aggregate total demand/supply, no other different category of participants in the market exists.

We set the investment horizon of all agents to be the final time $t = T$ which is the time at which each agent sells / buys back the initial position. Each agent supposes that this final transaction takes place at a price in agreement with his initial estimation.

In order to describe the model for the market price, we introduce the basic

notions of respectively total supply and demand at price $p \geq 0$. Namely the total demand and supply at time $t = 0$ are respectively denoted by $D(p)$ and $S(p)$ and are defined as follows:

$$D(p) = \mathbb{E}^A(\theta_+), \quad S(p) = \mathbb{E}^A(\theta_-), \tag{1}$$

where for any real number a we define $a_+ = \max\{a, 0\}$, $a_- = \max\{-a, 0\}$.

A price p^* such that $S(p^*) = D(p^*)$ is said to clear the market. Indeed, from definitions of $D(\cdot)$ and $S(\cdot)$ this is equivalent to say that $\mathbb{E}^A(\theta) = 0$ i.e., at the price p^*, the overall (signed) demand is zero. Note that such a price may not exist or may not be unique. Hence, one of the goals of the paper is to prove existence and uniqueness of p^*.

The transaction volume at some price p is the number of units that can be exchanged at that price and it is defined as follows

$$TV(p) = \min\{S(p), D(p)\}. \tag{2}$$

A price p^* for which $TV(\cdot)$ reaches its maximum is of particular interest because it maximizes the total number of security units being exchanged. Note that such a price may not exist, and may also be non-unique.

Let us recall the following result (see Shen & Turinici (2012) for the proof):

Theorem 1. *Suppose that functions $S(p), D(p)$ are continuous and positive, $S(0) = 0$ and $\lim_{p\to\infty} D(p) = 0$.*

A/ if $S(p)$ is increasing, not identically null, and $D(p)$ is decreasing then there exists at least a $p^* < \infty$ such that $S(p^*) = D(p^*)$; moreover $TV(p^*) \geq TV(p)$ for all $p \geq 0$;

B/ Suppose now that in addition $S(p)$ is strictly increasing and $\lim_{p\to\infty} S(p) > 0$, whereas $D(p)$ is strictly decreasing and such that $D(0) > 0$. Then the following statements are true.

1/ There exists a unique p_1^* such that $S(p_1^*) = D(p_1^*)$;

2/ There exists a unique p_2^* such that $TV(p_2^*) \geq TV(p)$ for all $p \geq 0$;

3/ Moreover $p_1^* = p_2^*$.

Recall that $F : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ is called lower semi-continuous (denoted "l.s.c.") if for any $x \in \mathbb{R}_+$

$$F(x) \leq \liminf_{y\to x} F(y). \tag{3}$$

A function G such that $-G$ is l.s.c. is called upper semi-continuous (denoted "u.s.c.").

For any function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ we define

$$\underline{\zeta(x)} = \liminf_{y \to x} \zeta(y), \quad \underline{\zeta'(x)} = \liminf_{y \to x} \frac{\zeta(y) - \zeta(x)}{y - x} \tag{4}$$

In particular $\underline{f'(0)} = \liminf_{y \to 0} \frac{f(y) - f(0)}{y}$. Denote by $\left(\underline{f'(0)}\right)_+$ its positive part.

Let us introduce the fundamental hypothesis.

Hypothesis 2. *We say that a function $f : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies hypothesis 2 if $\underline{f(0)} < \infty$, f is lower semi-continuous and there exists $\beta > 0$ such that*

$$\liminf_{x \to \infty} \frac{f(x)}{x^{1+\beta}} > 0. \tag{5}$$

Remark 3. *The quantity $\underline{f(0)} < \infty$ represents the residual cost, when precision approaches zero, to enter the market. It is not related to the precision (because there is none in the limit) but to the fixed costs to trade on the market (independent of the quantity). If the fixed costs are infinite then the market is surely particular.*

The assumption $\underline{f(0)} < \infty$ implies, by lower semi-continuity, that $f(0) < \infty$ and is realistic in that it demands that the price of zero precision be finite. In fact one can hardly imagine why someone will pay anything for zero precision (it suffices to do nothing to have zero precision) so in practice one should set $f(0) = 0$.

In order to model the choices of the agents, we consider that the agents maximize a CARA-type expected utility function (see Arrow (1965)) i.e., if the output is the random variable X they maximize $\mathbb{E}(-e^{-\lambda X})$; note that if X is normal with mean $\mathbb{E}(X)$ and variance $var(X)$ then maximizing $\mathbb{E}(-e^{-\lambda X})$ is equivalent to maximizing the mean-variance utility function $\mathbb{E}(X) - \frac{\lambda}{2} var(X)$. We will make more explicit in equation (6) what will be the convention for degenerate normal variables with infinite variance. The parameter $\lambda \in \mathbb{R}_+$ is called the risk aversion coefficient. Note that all agents have here the same utility function (cf. also Grossman (1977, 1978) that argue that differences in preferences are not a major factor in explaining the magnitude of trade in speculative markets).

Of course, the expected wealth of the agent at time $t = T$ is a function of θ^x and B^x. It is computed under the assumption that each agent enters the transaction (buys or sells) at time $t = 0$ at the market price and exits the transaction (sells or buys) at time $t = T$ at a price coherent with his estimation, i.e. we condition on the available information at time $t = 0$. Thus, for a given price p, which is not necessarily the market equilibrium price $\overline{\mathcal{P}}$, the average expected wealth at time $t = T$ of the agent x denoted by u^x is given by: $u^x = \theta^x(A^x - p) - f(B^x)$. The variance of the wealth, denoted by v^x is given by: $v^x = \frac{(\theta^x)^2}{B^x}$.

Thus, for a given price p (not necessarily the market equilibrium price $\overline{\mathcal{P}}$) the fact that agent x optimizes his CARA utility function is equivalent to say that he optimizes with respect to θ^x and B^x his mean-variance utility:

$$J(\theta^x, B^x) = \begin{cases} \theta^x(A^x - p) - f(B^x) - \frac{\lambda}{2}\frac{(\theta^x)^2}{B^x} & if \quad B^x, \theta^x > 0 \\ -\infty & if \quad B^x = 0, \theta^x > 0 \\ -f(0) & if \quad B^x = \theta^x = 0 \end{cases} . \tag{6}$$

3. Existence of the transaction volume

Each agent x is characterized by his own bias A^x. The agents consider the market price as being fixed, which means they cannot influence it directly. They do not know any statistics on ρ so the market price is not informative directly, but the acquired information is. Therefore, their strategy depend on two values: the bias A and the market price p.

Under hypothesis 2, the agent chooses the optimal pair of precision $B_{opt}(p, A; f)$ and demand / supply $\theta_{opt}(p, A; f)$, i.e. the value of the pair maximizing the following expression:

$$\mathcal{J}(y, z) = \begin{cases} y(A - p) - f(z) - \frac{\lambda}{2}\frac{y^2}{z} & if \quad y, z > 0 \\ -\infty & if \quad z = 0, y > 0 \\ -f(0) & if \quad y = z = 0 \end{cases} , \tag{7}$$

so that:

$$\mathcal{J}(\theta_{opt}(p, A; f), B_{opt}(p, A; f)) \geq \mathcal{J}(y, z), \quad \forall y, z \geq 0. \tag{8}$$

Let $g_{p,A;f}(X) = \frac{(p-A)^2}{2\lambda}X - f(X)$ and α be the function defined by $\alpha(p, A) = \frac{(p-A)^2}{2\lambda}$. To ease the notations we sometimes write only $g_{p,A}$, g_p or g instead

of $g_{p,A;f}$ and $\theta_{opt}(p,A)/B_{opt}(p,A)$ instead of $\theta_{opt}(p,A;f)/B_{opt}(p,A;f)$; same for α instead of $\alpha(p,A)$.

Lemma 4. *Under hypothesis 2, for any p and A, there exists a pair* $(B_{opt}(p,A), \theta_{opt}(p,A))$ *such that* (8) *is satisfied.*

Proof. Since f satisfies hypothesis 2 then there exists x_1 and some constant C_1 such that $f(x) \geq C_1 x^{1+\beta}$ for all $x \geq x_1$. In particular for $x > \max\left\{\left(\frac{\alpha}{2C_1}\right)^{1/\beta}, \left(\frac{f(0)}{2C_1}\right)^{1/(1+\beta)}\right\}$: $g(x) < -f(0) = g(0)$. Since f is l.s.c. then g is u.s.c.; it follows that g attains its maximum on $\mathbb{R}_+$ in the interval $\left[0, \max\left\{\left(\frac{\alpha}{2C_1}\right)^{1/\beta}, \left(\frac{f(0)}{2C_1}\right)^{1/(1+\beta)}\right\}\right]$. We set $B_{opt}(p,A)$ to be one such maximum (it may not be unique) and set $\theta_{opt}(p,A) = \frac{(A-p)B_{opt}(p,A)}{\lambda}$.

Note that $B_{opt}(p,A) = 0$ implies $\theta_{opt}(p,A) = 0$ thus

$$\forall y > 0 : \ \mathcal{J}(\theta_{opt}(p,A), B_{opt}(p,A)) > -\infty = \mathcal{J}(y,0). \tag{9}$$

When $y = z = 0$ one has:

$$\mathcal{J}(0,0) = g(0) \leq g(B_{opt}(p,A)) = \mathcal{J}(\theta_{opt}(p,A), B_{opt}(p,A)). \tag{10}$$

Let $y, z > 0$. Since $\mathcal{J}$ as function of the first argument is a parabola with negative coefficient it follows that:

$$\mathcal{J}(y,z) \leq \mathcal{J}(\frac{(A-p)z}{\lambda}, z) = g(z) \leq g(B_{opt}(p,A)) = \mathcal{J}(\theta_{opt}(p,A), B_{opt}(p,A)). \tag{11}$$

■

Remark 5. *Note that the formula* $\theta_{opt}(p,A) = \frac{(A-p)B_{opt}(p,A)}{\lambda}$ *is completely compatible with previous works, see Grossman (1976) p575, although here we have no hypothesis on budget constraints and the riskless interest rate is neglected.*

In order to prove the existence of an equilibrium we need the following auxiliary results.

Lemma 6. *Under hypothesis 2, let* (p_1, A_1), (p_2, A_2) *be such that* $\alpha_1 \leq \alpha_2$, *where* $\alpha_k = \alpha(p_k, A_k)$. *Then* $B_{opt}(p_1, A_1) \leq B_{opt}(p_2, A_2)$. *We say that* $B_{opt}(p, A)$ *is increasing with respect to* α. *In particular, for fixed* A, *we have:*

- $B_{opt}(p, A)$ *is increasing with respect to* p *on the interval* $]A, \infty[$;
- $B_{opt}(p, A)$ *is decreasing with respect to* p *on the interval* $]0, A[$.

Proof. Let, for $k = 1, 2$: $B_k = B_{opt}(p_k, A_k)$. Recall that B_k optimizes $\alpha_k B - f(B)$ with respect to B. Then:

$$\begin{aligned} &\alpha_1 B_1 - f(B_1) \geq \alpha_1 B_2 - f(B_2) = \alpha_2 B_2 - f(B_2) + (\alpha_1 - \alpha_2) B_2 \\ &\geq \alpha_2 B_1 - f(B_1) + (\alpha_1 - \alpha_2) B_2. \end{aligned} \tag{12}$$

Thus, $\alpha_1 B_1 \geq \alpha_2 B_1 + (\alpha_1 - \alpha_2) B_2$ and hence $(\alpha_1 - \alpha_2)(B_1 - B_2) \geq 0$, which gives the conclusion. ■

Lemma 7. *Under hypothesis 2, let* $\alpha_n = \alpha(p_n, A_n)$, $n \geq 0$, *be a sequence such that* $\alpha_n \underset{n \to +\infty}{\to} \alpha_0$ *but* $B_{opt}(p_n, A_n)$ *does not converge to* $B_{opt}(p_0, A_0)$. *The set of such* α_0 *is at most countable. In particular, if* p *is fixed, then the set of* A *such that* $B_{opt}(p, A)$ *is discontinuous with respect to* A *is countable. An analogous result holds if* A *is fixed.*

Proof. Let $B_n = B_{opt}(p_n, A_n)$, for $n \geq 0$. Without loss of generality, we only investigate the case when $\alpha_n \underset{n \to +\infty}{\searrow} \alpha_0$. Then, we have $B_n \geq B_0$, $\forall n \geq 0$.

Since B_n does not converge to B_0, let $\eta = \left(\lim_{n \to +\infty} B_n \right) - B_0$. Note that $\eta > 0$ and $B_n \geq B_0 + \eta$, $\forall n \geq 0$. Also recall that:

$$\alpha_n B_n - f(B_n) \geq \alpha_n B - f(B), \ \forall B. \tag{13}$$

Yet, since $-f$ is u.s.c.,

$$\alpha_0(B_0 + \eta) - f(B_0 + \eta) \geq \limsup_{n \to \infty} \alpha_n B_n - f(B_n), \tag{14}$$

and for fixed B, $\alpha_n B - f(B) \underset{n \to +\infty}{\to} \alpha_0 B - f(B)$. In the limit when $n \to \infty$, it holds that

$$\alpha_0(B_0 + \eta) - f(B_0 + \eta) \geq \alpha_0 B - f(B), \ \forall B. \tag{15}$$

This implies that $B_0 + \eta$ is also a maximum for $\alpha_0 B - f(B)$. From this we deduce that g_{α_0} has at least two distinct maximums, B_0 and $B_0 + \eta$.

Let α be such that g_α has at least two distinct minimums x_α^1 and x_α^2 with $x_\alpha^1 < x_\alpha^2$; we associate to α a rational number q_α such that $q_\alpha \in]x_\alpha^1, x_\alpha^2[$. Take α and $\tilde{\alpha}$ such that $\alpha \neq \tilde{\alpha}$, to fix notations suppose $\alpha < \tilde{\alpha}$. Then by the previous result $x_\alpha^2 \leq x_{\tilde{\alpha}}^1$; moreover $q_\alpha < x_\alpha^2 \leq x_{\tilde{\alpha}}^1 < q_{\tilde{\alpha}}$ i.e. $q_\alpha \neq q_{\tilde{\alpha}}$. Thus the set of α such that g_α has at least two distinct minimums is of cardinality less than the cardinality of $\mathbb{Q}$, i.e., at most countable. Since continuity can only fail when g_α has non-unique maximum the conclusion follows ■

Hypothesis 8. *We say that $\rho(A)$ satisfies hypothesis 8 if ρ is absolutely continuous with respect to the Lebesgue measure and :*

$$\int_0^\infty A^{1+2/\beta} \rho(A) dA < \infty. \tag{16}$$

Lemma 9. *Let $S(f,p)$ and $D(f,p)$ (or in short notation $S(p)$ and $D(p)$ when function f is implicit) be defined by:*

$$S(f,p) = \frac{1}{2\lambda} \int_0^\infty (A-p)_- B_{opt}(p, A; f) \rho(A) dA, \tag{17}$$

$$D(f,p) = \frac{1}{2\lambda} \int_0^\infty (A-p)_+ B_{opt}(p, A; f) \rho(A) dA. \tag{18}$$

Then under hypothesis 2 and 8 $S(p)$ and $D(p)$ are finite, continuous and monotonic. Moreover $S(0) = 0 = \lim_{p\to\infty} D(p)$.

Proof. To prove that $S(p)$ and $D(p)$ are finite we recall that maximum of $g_{p,A}$ is attained on $\left[0, \max\left\{\left(\frac{\alpha}{2C_1}\right)^{1/\beta}, \left(\frac{f(0)}{2C_1}\right)^{1/(1+\beta)}\right\}\right]$, i.e., $B_{opt}(p,A) \leq \max\left\{\left(\frac{\alpha}{2C_1}\right)^{1/\beta}, \left(\frac{f(0)}{2C_1}\right)^{1/(1+\beta)}\right\}$. Recalling that $\alpha = \frac{(A-p)^2}{2\lambda}$ it follows that both integrals are bounded (modulo some constant) by $\int_0^\infty A^{1+2/\beta}\rho(A)dA$ i.e., $S(p)$ and $D(p)$ are finite for all $p \geq 0$.

Let $p_n \underset{n\to+\infty}{\nearrow} p$. For any X, the set of A such that $B_{opt}(X,A)$ is discontinuous is at most countable. Denote it by $\mathcal{B}_X$. Let $\mathcal{B} = \mathcal{B}_p \cup \left(\bigcup_{n=1}^{+\infty} \mathcal{B}_{p_n}\right)$. $\mathcal{B}$ is also clearly countable and thus $\rho(\mathcal{B}) = 0$.

Let $\zeta_n(A) = (A - p_n)_- B_{opt}(p_n, A)$ and $\zeta(A) = (A - p)_- B_{opt}(p, A)$. Then $\lim_{n\to+\infty} \zeta_n(A) = \zeta(A)$, for all A except at the most for A in the null set $\mathcal{B}$. Also, the sequence ζ_n is increasing.

Then from the Beppo-Levi theorem, it holds:

$$\lim_{n\to+\infty} S(p_n) = \lim_{n\to+\infty} \frac{1}{2\lambda} \int_0^{+\infty} (A - p_n)_- B_{opt}(p_n, A)\rho(A)dA$$
$$= \frac{1}{2\lambda} \int_0^{+\infty} (A - p)_- B_{opt}(p, A)\rho(A)dA = S(p). \tag{19}$$

This proves sequential continuity of $S(p)$ and thus its continuity. The monotonicity is a consequence of the monotonicity of $B_{opt}(p, A)$. This result also holds for the demand $D(p)$, noting that $-D(p)$ is increasing and lower-bounded.

The property $S(0) = 0$ is trivial; to prove $\lim_{p\to\infty} D(p) = 0$ it suffices to use the above upper bound for $B_{opt}(p, A)$ and $\lim_{p\to\infty} \int_p^{\infty} A^{1+2/\beta}\rho(A)dA = 0$

■

Recall that $S(p)$ is increasing on $[0, +\infty[$ but to use Theorem 1 we need to prove its strict monotonicity.

Lemma 10. *Under hypothesis 2 and 8 and supposing* $\left(\underline{f'(0)}\right)_+ < \infty$ *the following hold:*

1. $S(p)$ *is strictly increasing on* $]\sqrt{2\lambda \left(\underline{f'(0)}\right)_+} + \inf(supp(\rho)), +\infty[$;
2. $S(0) = 0$;
3. $\lim_{p\to+\infty} S(p) > 0$.
4. $D(p)$ *is strictly decreasing on* $[0, \sup(supp(\rho)) - \sqrt{2\lambda \left(\underline{f'(0)}\right)_+}]$;
5. *if* $\sup(supp(\rho)) > \sqrt{2\lambda \left(\underline{f'(0)}\right)_+}$ *then* $D(0) > 0$;
6. $\lim_{p\to+\infty} D(p) = 0$.

Remark 11. *The hypothesis* $\left(\underline{f'(0)}\right)_+ < \infty$ *will be relaxed in Section 4, cf. Theorem 17.*

Proof. Note that $\left(\underline{f'(0)}\right)_+ < \infty$ implies in particular continuity of $f(B)$ at $B = 0$. Let p and p' be such that $p > p' > A \geq 0$:

$$S(p) - S(p') = \frac{1}{2\lambda}\int_0^\infty \left[(A-p)_- B_{opt}(p,A) - (A-p')_- B_{opt}(p',A)\right]\rho(A)dA$$
$$= \frac{1}{2\lambda}\int_0^\infty \left[(A-p)_- B_{opt}(p,A) - (A-p')_- B_{opt}(p,A)\right]\rho(A)dA$$
$$+\frac{1}{2\lambda}\int_0^\infty \left[(A-p')_- B_{opt}(p,A) - (A-p')_- B_{opt}(p',A)\right]\rho(A)dA. \tag{20}$$

Since B_{opt} is increasing if $p > A$,

$$\frac{1}{2\lambda}\int_0^\infty (A-p')_-(B_{opt}(p,A) - B_{opt}(p',A))\rho(A)dA \geq 0. \tag{21}$$

Hence,

$$S(p) - S(p') \geq \frac{1}{2\lambda}\int_0^\infty ((A-p)_- - (A-p')_-)B_{opt}(p,A)\rho(A)dA \tag{22}$$

Note that $A < p' < p$ implies that $((A-p)_- - (A-p')_-) > 0$. So, for strict inequality it is sufficient to prove that $B_{opt}(p,A) > 0$ with A in the support of ρ. Yet

$$B_{opt}(p,A) = \arg\max_B g_p(B) = \arg\max_B(\alpha B - f(B)). \tag{23}$$

Therefore we only need to prove that there exists B such that $\alpha B - f(B) > 0$ with A in the support of ρ. A sufficient condition is that the upper limit of derivative of $\alpha B - f(B)$ at $B = 0$ be strictly positive. This means $\alpha - \left(\underline{f'(0)}\right)_+ > 0$ which is equivalent to: $\frac{(p-A)^2}{2\lambda} > \left(\underline{f'(0)}\right)_+$. Recalling that $p > A$, the latter condition can be rewritten as $p - A > \sqrt{2\lambda\left(\underline{f'(0)}\right)_+}$ or else $p > A + \sqrt{2\lambda\left(\underline{f'(0)}\right)_+}$, for at least one A in the support of ρ. Therefore $S(p) - S(p') > 0$ as soon as p is in $]\sqrt{2\lambda\left(\underline{f'(0)}\right)_+} + \inf(supp(\rho)), +\infty[$. This

implies strict monotony for $S(p)$ on $]\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}+\inf(supp(\rho)),+\infty[$, and hence also on the interval $[\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}+\inf(supp(\rho)),+\infty[$.

We already seen that $S(0)=0$. Moreover since the supply is strictly increasing on $[\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}+\inf(supp(\rho)),+\infty[$ and increasing on $[0,+\infty[$, it holds that $\lim_{p\to+\infty} S(p)>0$.

For the monotony of the demand, let p and p' be such that $A>p>p'$. Then:

$$\begin{aligned}
&D(p)-D(p')=\frac{1}{2\lambda}\int_0^\infty\left[(A-p)_+B_{opt}(p,A)-(A-p')_+B_{opt}(p',A)\right]\rho(A)dA\\
&=\frac{1}{2\lambda}\int_0^\infty\left[(A-p)_+B_{opt}(p,A)-(A-p')_+B_{opt}(p,A)\right]\rho(A)dA\\
&+\frac{1}{2\lambda}\int_0^\infty\left[(A-p')_+B_{opt}(p,A)-(A-p')_+B_{opt}(p',A)\right]\rho(A)dA. \qquad (24)
\end{aligned}$$

Since B_{opt} is decreasing for $A>p>p'$, we have:

$$\frac{1}{2\lambda}\int_0^\infty(A-p')_+(B_{opt}(p,A)-B_{opt}(p',A))\rho(A)dA\leq 0. \qquad (25)$$

Hence,

$$D(p)-D(p')\leq\frac{1}{2\lambda}\int_0^\infty((A-p)_+-(A-p')_+)B_{opt}(p,A)\rho(A)dA. \qquad (26)$$

Note that $A>p>p'$ implies that $(A-p)_+-(A-p')_+<0$. For strict inequality it is sufficient to prove that $B_{opt}(p,A)>0$. Using the same arguments as in Lemma 10, we have strict monotony as soon as $\frac{(p-A)^2}{2\lambda}>\left(\underline{f'(0)}\right)_+$.

Recalling that $p<A$, the latter condition can be written as $A-p>\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}$ or else $p<A-\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}$ for at least one A in the support of ρ. Therefore, $D(p)-D(p')<0$ as soon as p is in $]0,\sup(supp(\rho))-\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}[$. This yields strict monotony of $D(p)$ on $]0,\sup(supp(\rho))-\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}[$. Monotony also holds on $[0,\sup(supp(\rho))-\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}]$ by continuity.

Since $\sup(supp(\rho)) - \sqrt{2\lambda\left(\underline{f'(0)}\right)_+} > 0$, we have $B_{opt}(0, A) > 0$ so $D(0) > 0$.

Hence, demand is strictly decreasing. We also saw before that $\lim_{p\to+\infty} D(p) = 0$. ■

The previous results can be summarized as:

Theorem 12. *Under hypothesis 2 and 8 and supposing* $\left(\underline{f'(0)}\right)_+ < \infty$ *the following hold:*

A/ there exists at least a $p^* \geq 0$ *such that* $TV(p^*) \geq TV(p)$, $\forall p \geq 0$, *moreover* $D(p^*) = S(p^*)$.

B/ suppose that $diam(supp(\rho)) > 2\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}$ *then:*

1. *The functions* B_{opt} *and* θ_{opt} *are well defined.*
2. *There exists a unique* $p^* > 0$ *such that* $TV(p^*) \geq TV(p), \quad \forall p \geq 0$. *Moreover* p^* *is the unique solution of the equation* $D(p^*) = S(p^*)$.

Note that the results of Shen & Turinici (2012) are a particular case of this Theorem (any convex C^2 function is in particular l.s.c.).

Remark 13. *If* $diam(supp(\rho)) \leq 2\sqrt{2\lambda\left(\underline{f'(0)}\right)_+}$ *then* $TV \equiv 0$ *and* $S(p) = D(p) = 0$, $\forall p$ *(see Figure 1).*

Remark 14. *Since we assume the distribution* ρ *to be absolutely continuous with respect to the Lebesgue measure, it holds that* $diam(supp(\rho)) > 0$. *Thus one can always find a critical value* λ^* *defined as*

$$\lambda^* = \begin{cases} \frac{diam(supp(\rho))^2}{8\left(\underline{f'(0)}\right)_+} & if \quad \left(\underline{f'(0)}\right)_+ > 0 \\ 0 \quad if \quad \left(\underline{f'(0)}\right)_+ = 0 \end{cases} \tag{27}$$

such that for any $\lambda < \lambda^*$, *the hypothesis of Theorem 12 are satisfied, i.e. there exists a market price maximizing the volume and clearing the market. On the contrary there exists no such market price for* $\lambda \geq \lambda^*$. *The results of Shen & Turinici (2012) are a particular case of this remark. In fact, under*

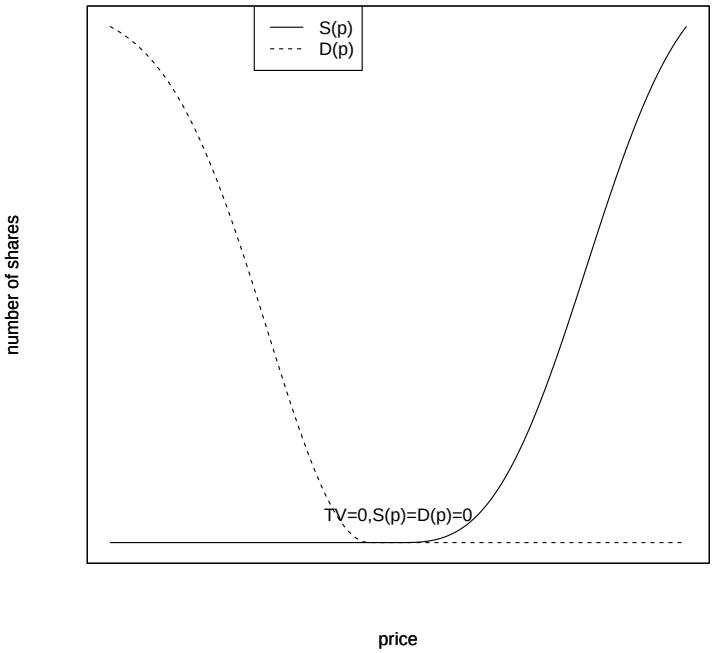

Figure 1: Illustration of Remark 13.

the hypothesis given in Shen & Turinici (2012), $\left(\underline{f'(0)}\right)_+ = f'(0) = 0$ *and thus* $\lambda^* = 0$.

The critical value λ^* *can be interpreted as the maximum risk aversion allowing the market to function. If the risk aversion becomes larger than the critical value, the market stops and a liquidity crisis occurs. In the latter case, several actions can be proposed to stop the liquidity crisis:*

- lower the perception of risk, i.e. lower the λ *of the agents;*

- make λ^* *higher by lowering* $\left(\underline{f'(0)}\right)_+$*, i.e. lower the marginal cost of information around zero precision. In other words eliminate any entry barriers for new agents on that market by largely spreading information about the real situation of the asset* V*;*

- make λ^* *higher by increasing* $diam(supp(\rho))$*. This means inviting to the market agents with new, different evaluation procedures. This can be carried out for instance by eliminating any entry barrier for newcomers when they*

have a different background and different evaluation procedures.

4. Necessary and sufficient results for general functions

We relax in this section the hypothesis $\left(\underline{f'(0)}\right)_+ < \infty$. For any function f we denote by h^* the Legendre-Fenchel transform (cf. Rockafellar (1970)) of h, by h^{**} the Legendre-Fenchel transform applied twice and so on. We show in this section that the twice Legendre-Fenchel transform f^{**} of the cost function f has remarkable properties i.e., we can replace f by f^{**} for any practical means. In particular this means that from a technical point of view one can suppose f is convex even if the actual function is not.

Theorem 15. *Let f be a function satisfying hypothesis 2. Then*

1. *f^{**} also satisfies hypothesis 2;*
2. *except for a countable set of values $\alpha(p, A)$ we have*

$$B_{opt}(p, A; f) = B_{opt}(p, A; f^{**}),\ \theta_{opt}(p, A; f) = \theta_{opt}(p, A; f^{**}). \tag{28}$$

3. *as a consequence*

$$S(f, p) = S(f^{**}, p),\ D(f, p) = D(f^{**}, p),\ \forall p \geq 0. \tag{29}$$

Proof. To prove point 1 we recall that f^{**} is a convex function and $\forall b \geq 0$: $f^{**}(b) \leq f(b)$. In particular f^{**} is l.s.c. and continuous in 0. Let us now check the growth condition and take β that satisfies hypothesis 2 for f. Take also C_1 as the constant in Lemma 4, i.e., $f(x) \geq C_1 x^{1+\beta}$ for all $x \geq x_1$. Consider now the function

$$f_1(x) = \left\{ \begin{array}{rll} 0 & if & x \leq x_1 \\ C_1 x^{1+\beta} & if & x > x_1 \end{array} \right. . \tag{30}$$

Then it is straightforward to see that

$$f_1^{**}(x) = \left\{ \begin{array}{rll} 0 & if & x \leq x_1 \\ C_1(1+\beta)x_2^{\beta}(x - x_1) & if & x_1 \leq x \leq x_2 \\ C_1 x^{1+\beta} & if & x \geq x_2 \end{array} \right. . \tag{31}$$

where $x_2 = \frac{1+\beta}{\beta} x_1$; of course $f_1 \leq f$ and is l.s.c. Then also $f_1^{**} \leq f^{**}$. But obviously $\liminf_{x\to\infty} \frac{f_1^{**}(x)}{x^{1+\beta}} = C_1 > 0$ hence also $\liminf_{x\to\infty} \frac{f^{**}(x)}{x^{1+\beta}} > 0$.

To prove point 2 we recall that the cost function f is used only as a part of the function g_α. Let us take a point α_0 and x_0 a minimum of g_{α_0}. This implies

$$\alpha_0 x_0 - f(x_0) \geq \alpha_0 x - f(x) \ \forall x \tag{32}$$

which can also be written

$$f(x) \geq f(x_0) + \alpha_0 (x - x_0), \tag{33}$$

i.e., in terms of Rockafellar (1970), the function f has a supporting hyperplane at x_0. Since f has a supporting hyperplane at x_0 this implies that $f(x_0) = f^{**}(x_0)$; recall that f^{**} is the convex hull of f i.e., the largest convex function such that $f^{**} \leq f$. Hence, recalling that for any function $f^{***} = f^*$:

$$\alpha_0 x_0 - f^{**}(x_0) = \alpha_0 x_0 - f(x_0) = f^*(\alpha_0) = f^{***}(\alpha_0) = \max_x \alpha_0 x - f^{**}(x). \tag{34}$$

We thus obtained that x_0 is a maximum of $\alpha_0 x - f^{**}(x)$.

Therefore, if one replaces f by f^{**} the minimization problem involving g_α gives the same solution, except possibly a countable set of values α where the maximum is attained (either for f or f^{**}) in more than one point.

Point 3 is a mere consequence of point 2. ■

For all purposes of calculating aggregate supply and demand we can thus replace f by f^{**} i.e. replace f by its convex hull. Therefore one can work as if f was convex.

Remark 16. *This result is particularly useful when $f(0) \neq \underline{f(0)}$ because in this situation $\left(\underline{f'(0)}\right)_+ = \infty$ thus one cannot use the previous results that guarantee the uniqueness of the market price. When one replaces f by f^{**} it can be shown that $\left(\underline{f'(0)}\right)_+$ becomes finite and the results apply for f^{**}; but the Theorem 15 allows to come back to the function f and obtain the full information on the supply and demand functions and on the market price.*

We obtain the following:

Theorem 17. *Suppose hypothesis 2 and 8 are satisfied. Then at least a price $\overline{\mathcal{P}} \geq 0$ exists such that*

$$TV(\overline{\mathcal{P}}) \geq TV(p), \forall p \geq 0. \tag{35}$$

For this value we also have

$$D(\overline{\mathcal{P}}) = S(\overline{\mathcal{P}}). \tag{36}$$

Furthermore

I If there exists $B > 0$ such that $f(B) < f(0)$ then $D(p; f)$ and $S(p; f)$ are always strictly positive and strictly monotonic, $S(0) = 0 = \lim_{p\to\infty} D(p)$. Moreover $\overline{\mathcal{P}}$ that satisfies (35) is unique.

II Suppose now that $f(B) \geq f(0)$, $\forall B \geq 0$; then the following hold:

a (alternative 1) suppose that $diam(supp(\rho)) > 2\sqrt{2\lambda\left(\underline{(f^{**})'(0)}\right)_+}$ then:

i The functions B_{opt} and θ_{opt} are well defined.

ii The value $\overline{\mathcal{P}}$ that satisfies (35) is unique and $TV(\overline{\mathcal{P}}) > 0$; $\overline{\mathcal{P}}$ is also the unique solution of (36).

b (alternative 2) if on the contrary we suppose that

$$diam(supp(\rho)) \leq 2\sqrt{2\lambda\left(\underline{(f^{**})'(0)}\right)_+}, \tag{37}$$

then $TV(p) = 0$, $\forall p \geq 0$.

Proof. We prove point I. If $f(B^*) < f(0)$ then for all $\alpha \geq 0 : \alpha B^* - f(B^*) > \alpha \cdot 0 - f(0)$ thus $B_{opt}(p, A) > 0$ for all p, A. As a first consequence we obtain $D(p; f) > 0$ for all p and the same for $S(p; f)$. For strict monotonicity it suffices to use same arguments as in the proof of Lemma 10. Of course, $S(0) = 0 = \lim_{p\to\infty} D(p)$ due to Lemma 9.

We continue to proving point II. The point IIa follows from the discussion above.

To prove IIb we need to analyze more in detail the values of $D(p)$ and $S(p)$. Let us now inquire when $B_{opt}(p, A; f^{**}) > 0$: when this is the case then $\alpha B_{opt}(p, A; f^{**}) - f^{**}(B_{opt}(p, A; f^{**})) > \alpha \cdot 0 - f^{**}(0)$ (we exclude the null measure set of α where more than one maximum can exists i.e., we can suppose the inequality to be strict); hence

$$f^{**}(B_{opt}(p, A; f^{**})) < f^{**}(0) + \alpha B_{opt}(p, A; f^{**}), \tag{38}$$

or again for some $\alpha_1 < \alpha$

$$f^{**}(B_{opt}(p, A; f^{**})) \leq f^{**}(0) + \alpha_1 B_{opt}(p, A; f^{**}). \tag{39}$$

Since f^{**} is convex we have for arbitrary $B \in [0, B_{opt}(p, A; f^{**})]$: $f^{**}(B) \leq f^{**}(0) + \alpha_1 B$. But this means $\left(\underline{(f^{**})'(0)}\right)_+ \leq \alpha_1 < \alpha$ i.e., $|A - p| > \sqrt{2\lambda \left(\underline{(f^{**})'(0)}\right)_+}$.

If $D(p)$ is always null the conclusion is reached. Suppose now p exists such that $D(p) > 0$; then at least some A in the support of ρ exists such that $B_{opt}(p, A; f^{**}) > 0$ and $(A - p)_+ > 0$; the three conditions imply

$$\sup(supp(\rho)) - \sqrt{2\lambda \left(\underline{(f^{**})'(0)}\right)_+} > 0. \tag{40}$$

Moreover we have $D(p) = 0$ for $p \geq \sup(supp(\rho)) - \sqrt{2\lambda \left(\underline{(f^{**})'(0)}\right)_+}$.

From (40) and (37) we conclude that

$$0 < \sup(supp(\rho)) - \sqrt{2\lambda \left(\underline{(f^{**})'(0)}\right)_+} \leq \sqrt{2\lambda \left(\underline{(f^{**})'(0)}\right)_+} + \inf(supp(\rho)). \tag{41}$$

A similar reasoning as above shows that $S(p) = 0$ for $p \leq \sqrt{2\lambda \left(\underline{(f^{**})'(0)}\right)_+} + \inf(supp(\rho))$. Therefore for any p either $D(p) = 0$ or $S(p) = 0$ and the conclusion follows. ■

In general, the price $\overline{\mathcal{P}}$ has an implicit dependence on the cost function $f(\cdot)$ with no particular properties. But when the distribution ρ is completely symmetric around some particular value p^1 we obtain the following result:

Theorem 18. *Suppose hypothesis 2 and 8 are satisfied and there exists $p^1 > 0$ such that*

$$\forall y \in \mathbb{R} : \ \rho(p^1 - y) = \rho(p^1 + y), \tag{42}$$

(with the convention that ρ is null on $\mathbb{R}_-$); then we can take in Thm. 17 $\overline{\mathcal{P}} = p^1$.

Proof. The proof results from the remark that, except possibly for a null measure set of values $\alpha(p, A)$, the function $B_{opt}(p, A; f)$ is symmetric around p, i.e., $B_{opt}(p, A; f) = B_{opt}(p, 2p - A; f)$; thus $\theta_{opt}(p, A; f)$ is anti-symmetric. Since the distribution ρ is symmetric then $D(p^1) = S(p^1)$. ■

4.1. An application: the Grossman-Stiglitz framework

We follow Grossman & Stiglitz (1980) to analyze a classical situation where costly information can be used to lower the uncertainty of the estimation. Please however note that in the cited work the equilibrium is realized without modeling the variations in supply and in the absence of the distribution $\rho(A)$.

In the Grossman-Stiglitz model agents can either pay nothing and have a precision B_1 or pay a fixed cost c_b to gain precision up to level $B_2 > B_1$. Thus we know that $f(B) = 0$ for any $B \leq B_1$ and $f(B_2) = c_b$. Taking into account the result of Theorem 17 we can thus propose the following convex function

$$f_{GS}(B) = \begin{cases} 0 & if \quad B \leq B_1 \\ c_b \frac{B-B_1}{B_2-B_1} & if \quad B_1 \leq B \leq B_2 \\ +\infty & if \quad B > B_2 \end{cases} . \tag{43}$$

Since f_{GS} fulfills the hypothesis 2 (with arbitrary $\beta \geq 0$) the results above apply provided that the distribution $\rho(A)$ also fulfills requirements in hypothesis 8: absolute continuity with respect to Lebesgue measure and a moment of order $1 + \epsilon$ (with arbitrary small ϵ) has to exist. Then a (equilibrium) market price exists and is unique. Note that $\underline{f'_{GS}(0)} = 0$ thus $\lambda^*_{GS} = 0$.

The unsigned demand is

$$\theta_{opt}(p, A) = \begin{cases} \frac{(A-p)B_1}{\lambda} & if \quad |A-p| < \frac{2\lambda c_b}{(B_2-B_1)} \\ \frac{(A-p)B_2}{\lambda} & if \quad |A-p| \geq \frac{2\lambda c_b}{(B_2-B_1)} \end{cases} . \tag{44}$$

The optimal precision is either B_1 in the first alternative of equation (44) or B_2 for the second alternative.

5. Transaction volume and marginal costs

We describe in the following the relationship between the cost function f and the trading volume.

Theorem 19. *Suppose that f_1 and f_2 both satisfy hypothesis 2 and that ρ satisfies hypothesis 8.*

A/ Assume that

$$\frac{f_2(y) - f_2(x)}{y - x} \geq \frac{f_1(y) - f_1(x)}{y - x}, \ \forall x, y \geq 0, \ x \neq y. \tag{45}$$

Then $TV_{f_1} \geq TV_{f_2}$.
B/ In particular if f_1 *and* f_2 *are such that*

$$f_1'(X^+) \leq f_2'(X^+),\ f_1'(X^-) \leq f_2'(X^-),\ \forall X \geq 0, \tag{46}$$

(all are lateral derivatives) then $TV_{f_1} \geq TV_{f_2}$.

Remark 20. *Note that if* f_1 *and* f_2 *are convex, both lateral derivatives are defined at each point and A/ implies B/; thus for practical purposes (cf. also section 4) the point B/ is not weaker than point A/.*

Remark 21. *If* $f_1'(X)$ *and* $f_2'(X)$ *exist at some point* X*, then* (46) *implies that* $f_1'(X) \leq f_2'(X)$*. Thus, the above result is a generalization of the analogous theorem in Shen & Turinici (2012).*

Proof. A/ We first show that, except for a countable set of values $\alpha(p, A)$ we have $B_{opt}(p, A; f_1) \geq B_{opt}(p, A; f_2)$. Fix p, A and denote $B_k = B_{opt}(p, A; f_k)$ for $k = 1, 2$. Suppose, by contradiction, that $B_1 < B_2$; recall that, by the optimality of B_1:

$$\alpha B_1 - f_1(B_1) > \alpha B_2 - f_1(B_2), \tag{47}$$

thus

$$\frac{f_1(B_2) - f_1(B_1)}{B_2 - B_1} > \alpha. \tag{48}$$

Note that we wrote strict inequality in (47) because we exclude the countable set of values $\alpha(p, A)$ where the maximum of $g_{p,A}(B) = \alpha B - f_1(B)$ is not unique. We do the same for B_2:

$$\alpha B_2 - f_2(B_2) > \alpha B_1 - f_2(B_1),$$

thus

$$\alpha > \frac{f_2(B_2) - f_2(B_1)}{B_2 - B_1}. \tag{49}$$

Combining equations (48) and (49) we obtain that

$$\frac{f_1(B_2) - f_1(B_1)}{B_2 - B_1} > \frac{f_2(B_2) - f_2(B_1)}{B_2 - B_1}. \tag{50}$$

But this contradicts (45) for $y = B_2$ and $x = B_1$. Thus, with the possible exception of a countable set of values $\alpha(p, A)$ we have $B_{opt}(p, A; f_1) \geq B_{opt}(p, A; f_2)$.

The demand and supply of the agents are monotonic and given for $k = 1, 2$ by the formulas:

$$D(f_k, p) = \frac{1}{2\lambda} \int_0^\infty (A - p)_+ B_{opt}(p, A; f_k) \rho(A) dA \tag{51}$$

$$S(f_k, p) = \frac{1}{2\lambda} \int_0^\infty (A - p)_- B_{opt}(p, A; f_k) \rho(A) dA. \tag{52}$$

Denote by $P^A_{f_k}$ the market price for which supply equals demand for the cost function f_k i.e., $D(f_k, P^A_{f_k}) = S(f_k, P^A_{f_k})$. We further take $P^A_{f_2} = \min\{P : D(f_2, P) = S(f_2, P)\}$ and $P^A_{f_1} = \min\{P : D(f_1, P) = S(f_1, P)\}$

It has been proved that $B_{opt}(p, A; f_1) \geq B_{opt}(p, A; f_2)$. Thus, $D(f_1, p) \geq D(f_2, p)$ and $S(f_1, p) \geq S(f_2, p)$, $\forall p$. In particular, $D(f_2, P^A_{f_2}) \leq D(f_1, P^A_{f_2})$.

Let P_1 be the solution of $D(f_1, P_1) = S(f_2, P_1)$. Let us prove that $P_1 \geq P^A_{f_2}$; in fact suppose on the contrary that $P_1 < P^A_{f_2}$. Then

$$D(f_1, P^A_{f_2}) \geq D(f_2, P^A_{f_2}) = S(f_2, P^A_{f_2}) \geq S(f_2, P_1) = D(f_1, P_1) \geq D(f_1, P^A_{f_2}), \tag{53}$$

which means that all inequalities in (53) are in fact equalities, in particular $S(f_2, P^A_{f_2}) = S(f_2, P_1)$ and $D(f_1, P_1) = D(f_2, P^A_{f_2})$. But we also have

$$D(f_1, P_1) \geq D(f_2, P_1) \geq D(f_2, P^A_{f_2}) = D(f_1, P_1) \tag{54}$$

which means again that all are equalities, in particular $D(f_2, P_1) = D(f_2, P^A_{f_2})$. Thus

$$D(f_2, P_1) = D(f_2, P^A_{f_2}) = S(f_2, P^A_{f_2}) = S(f_2, P_1), \tag{55}$$

which means that P_1 is a member of $\{P : D(f_2, P) = S(f_2, P)\}$. But $P^A_{f_2}$ is the minimum of such elements hence we arrive at a contradiction. It follows that $P_1 \geq P^A_{f_2}$.

Similarly we prove that $P_1 \geq P^A_{f_1}$ (see Figure 2). Hence it holds that

$$TV_{f_2} = S(f_2, P^A_{f_2}) \leq \ S(f_2, P_1) = D(f_1, P_1) \leq \ D(f_1, P^A_{f_1}) = TV_{f_1},$$

which concludes the proof.

B/ We prove that (46) implies (45). Of course, it is enough to consider $x < y$. Denote

$$G(y, x) = \frac{f_2(y) - f_2(x)}{y - x} - \frac{f_1(y) - f_1(x)}{y - x}, \ \forall x, y \geq 0, \ x \neq y. \tag{56}$$

Suppose that x_0 and $y_0 > x_0$ exist such that $\xi := G(y_0, x_0) < 0$. Note that

$$G(y, x) = \frac{1}{2}G(y, \frac{x+y}{2}) + \frac{1}{2}G(\frac{x+y}{2}, x). \tag{57}$$

Then $G(y_0, \frac{x_0+y_0}{2}) \leq \xi < 0$ or $G(\frac{x_0+y_0}{2}, x_0) \leq \xi < 0$. Iterating the argument we obtain two convergent sequences x_n and y_n with $\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} x_n = x_\infty$, $x_n < y_n$ and $G(y_n, x_n) \leq \xi < 0$. Up to extracting sub-sequences only three alternatives exist:

1/ $x_\infty \leq x_n < y_n$ for all n

2/ $x_n < y_n \leq x_\infty$ for all n

3/ $x_n \leq x_\infty \leq y_n$ for all n

Alternative 3/ can be reduced to 1/ or 2/ by noting that since $G(y_n, x_n) = \frac{y_n - x_\infty}{y_n - x_n} G(y_n, x_\infty) + \frac{x_\infty - x}{y_n - x_n} G(x_\infty, x)$ then either $G(y_n, x_\infty) \leq \xi$ or $G(x_\infty, x_n) \leq \xi < 0$.

We only prove 1/, the proof of 2/ being completely similar. When $x_\infty \leq x_n < y_n$ we obtain

$$0 > \xi \geq \lim_{n\to+\infty} G(y_n, x_n) = f_2'(x_\infty^+) - f_1'(x_\infty^+) \geq 0, \tag{58}$$

which is a contradiction. Thus (46) implies (45). ■

6. Concluding remarks

The main focus of this work is to establish the existence of an equilibrium and its optimality in terms of trading volumes for the model in the Section 2. The results are proved under minimalistic hypothesis on the cost function and a relationship with the convex hull of the cost function is proved. The model can be used to investigate the determinants of the trading volume and may give hints on how to exit a situation when the volume is abnormally low.

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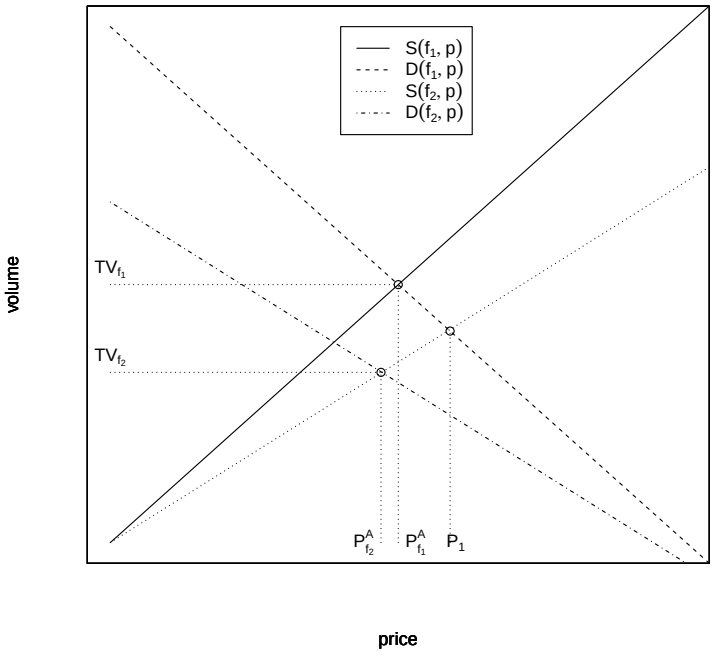

Figure 2: Illustration of the proof of Theorem 19.

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