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Convex computation of the region of attraction of polynomial control systems

Didier Henrion[1,2,3], Milan Korda[4]

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Abstract

We address the long-standing problem of computing the region of attraction (ROA) of a target set (typically a neighborhood of an equilibrium point) of a controlled nonlinear system with polynomial dynamics and semialgebraic state and input constraints. We show that the ROA can be computed by solving a convex linear programming (LP) problem over the space of measures. In turn, this problem can be solved approximately via a classical converging hierarchy of convex finite-dimensional linear matrix inequalities (LMIs). Our approach is genuinely primal in the sense that convexity of the problem of computing the ROA is an outcome of optimizing directly over system trajectories. The dual LP on nonnegative continuous functions (approximated by polynomial sum-of-squares) allows us to generate a hierarchy of semialgebraic outer approximations of the ROA at the price of solving a sequence of LMI problems with asymptotically vanishing conservatism. This sharply contrasts with the existing literature which follows an exclusively dual Lyapunov approach yielding either nonconvex bilinear matrix inequalities or conservative LMI conditions.

Keywords: Region of attraction, polynomial control systems, occupation measures, linear matrix inequalities (LMIs), convex optimization, viability theory, capture basin.

1 Introduction

Given a nonlinear control system, a state-constraint set and a target set (e.g. a neighborhood of an attracting orbit or an equilibrium point), the constrained controlled region of attraction (ROA) is the set of all initial states that can be steered with an admissible control to the target set without leaving the state-constraint set. The target set

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can be required to be reached at a given time or at any time before a given time[1]. The problem of computing the ROA (and variations thereof) lies at the heart of viability theory (see, e.g., [4]) and goes by many other names, e.g., the reach-avoid or target-hitting problem (see, e.g., [21]); in the language of viability theory the ROA itself is sometimes referred to as the capture basin [4].

We show that, in the case of polynomial dynamics, semialgebraic state-constraint, input-constraint and target sets, the computation of the ROA boils down to solving an infinite-dimensional linear programming (LP) problem in the cone of nonnegative Borel measures. Our approach is genuinely primal in the sense that we optimize over state-space system trajectories modeled with occupation measures [19, 9].

In turn, this LP can be solved approximately by a classical hierarchy of finite-dimensional convex linear matrix inequality (LMI) relaxations. The dual LP on nonnegative continuous functions and its LMI relaxations on polynomial sum-of-squares provide explicitly an asymptotically converging sequence of nested semialgebraic outer approximations of the ROA.

Most of the existing literature on ROA computation follows Zubov's approach [22, 32, 12] and uses a dual Lyapunov certificate; see [29], the survey [10], Section 3.4 in [11], and more recently [27] and [6] and the references therein. These approaches either enforce convexity with conservative LMI conditions (whose conservatism is difficult if not impossible to evaluate systematically) or they rely on nonconvex bilinear matrix inequalities (BMIs), with all their inherent numerical difficulties. In contrast, we show in this paper that the problem of computing the ROA has actually a convex infinite-dimensional LP formulation, and that this LP can be solved with a hierarchy of convex finite-dimensional LMIs with asymptotically vanishing conservatism.

We believe that our approach is closer in spirit to set-oriented approaches [7], level-set and Hamilton-Jacobi approaches [17, 21, 23] or transfer operator approaches [30], even though we do not discretize w.r.t. time and/or space. In our approach, we model a measure with a finite number of its moments, which can be interpreted as a frequency-domain discretization (by analogy with Fourier coefficients which are moments w.r.t. the unit circle).

Another way to evaluate the contribution of our paper is to compare it with the recent works [15, 13] which deal with polynomial approximations of semialgebraic sets. In these references, the sets to be approximated are given a priori (as a polynomial sublevel set, or as a feasibility region of a polynomial matrix inequality). In contrast, in the current paper the set to be approximated (namely the ROA of a nonlinear dynamical system) is not known in advance, and our contribution can be understood as an application and extension of the techniques of references [15, 13] to sets defined implicitly by differential equations.

The benefits of our occupation measure approach are overall the convexity of the problem of finding the ROA, and the availability of publicly available software to implement and solve the hierarchy of LMI relaxations.

[1]The cases of an arbitrarily long but finite time and of asymptotic convergence are not addressed in this paper.

Our primary focus in this paper is the computation of the constrained finite-time controlled region of attraction of a given set. This problem is formally stated in Section 2 and solved using occupation measures in Section 4; the occupation measures themselves are introduced in Section 3. A dual problem on the space of continuous functions is discussed in Section 5. The hierarchy of finite-dimensional LMI relaxations of the infinite dimensional LP is described in Section 6. An extension to the free final time case is sketched in Section 7. Numerical examples are presented in Section 8, and we conclude in Section 9.

2 Problem statement

Consider the control system

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t) \in X, \quad u(t) \in U, \quad t \in [0, T] \tag{1}$$

with a given polynomial vector field f with entries $f_i \in \mathbb{R}[t, x, u]$, $i = 1, \ldots, n$, given final time $T > 0$, and given compact semialgebraic state and input constraints

$$\begin{array}{l} x(t) \in X := \{x \in \mathbb{R}^n \ : \ g_{Xi}(x) \geq 0, i = 1, 2, \ldots, n_X\}, \quad t \in [0, T] \\ u(t) \in U := \{u \in \mathbb{R}^m \ : \ g_{Ui}(u) \geq 0, i = 1, 2, \ldots, n_U\}, \quad t \in [0, T] \end{array} \tag{2}$$

with $g_{Xi} \in \mathbb{R}[x]$, $g_{Ui} \in \mathbb{R}[u]$. Given a compact semialgebraic target set

$$X_T := \{x \in \mathbb{R}^n \ : \ g_{Ti}(x) \geq 0, i = 1, 2, \ldots, n_T\} \subset X,$$

with $g_{Ti} \in \mathbb{R}[x]$, let

$$\begin{array}{rl} \mathcal{X}(x_0) := \big\{x(\cdot) \ : x(t) = x_0 + \displaystyle\int_0^t f(\tau, x(\tau), u(\tau)) d\tau, & \\ u(t) \in U, \, x(t) \in X, \, x(T) \in X_T, \, \forall t \in [0, T]\big\} & \end{array} \tag{3}$$

denote the set of all absolutely continuous admissible controlled trajectories $x(\cdot)$ starting from x_0, generated by an admissible control $u(\cdot) \in L^1([0, T]; \mathbb{R}^m)$.

The constrained controlled region of attraction (ROA) is then defined as

$$X_0 := \{x_0 \in X \ : \ \mathcal{X}(x_0) \neq \emptyset\}. \tag{4}$$

In words, the ROA is the set of all initial conditions for which there exists an admissible controlled trajectory. By construction the set X_0 is bounded and unique.

In the sequel we propose an infinite-dimensional LP approach to computing ROA X_0 and show how this reformulation can be approximated by a sequence of LMI problems converging to the solution to the LP.

3 Occupation measures

In the paper we use the following notations:

- $I_A(\cdot)$ is the indicator function of a set A, i.e., a function equal to 1 on A and 0 elsewhere;
- λ denotes the Lebesgue measure on $X \subset \mathbb{R}^n$ such that

$$\lambda(A) = \int_X I_A(x)\, d\lambda(x) = \int_X I_A(x) dx = \int_A dx$$

  is the standard n-dimensional volume of a set $A \subset X$;
- $\operatorname{spt}\mu$ denotes the support of a measure μ, that is, the closed set of all points x such that $\mu(A) > 0$ for every neighborhood A of x.

3.1 Liouville's equation

Given an initial condition x_0 and an admissible trajectory $x(\cdot \mid x_0) \in \mathcal{X}(x_0)$ with its corresponding control $u(\cdot \,|\, x_0) \in L^1([0,T];\mathbb{R}^m)$ that we assume to be a measurable function of x_0, define the *occupation measure*

$$\mu(A \times B \times C \,|\, x_0) := \int_0^T I_{A\times B\times C}(t, x(t \,|\, x_0), u(t \,|\, x_0))\, dt$$

for all subsets $A \times B \times C$ in the Borel σ-algebra of subsets of $[0,T] \times X \times U$. Next, for a set K let $M(K)$ denote the Banach space of signed Borel measures supported on K, so that a measure $\nu \in M(K)$ can be interpreted as a function that takes any subset of K and returns a number in $\mathbb{R}$. Alternatively, elements of $M(K)$ can be interpreted as linear functionals acting on the Banach space of continuous functions $C(K)$, that is, as elements of the dual space $C(K)'$. The action of a measure $\nu \in M(K)$ on a test function $v \in C(K)$ can be modeled with the duality pairing

$$\langle \nu,\, v \rangle \;:= \int_K v(z)\, d\nu(z).$$

Define further the linear operator $\mathcal{L} : C^1([0,T] \times X) \to C([0,T] \times X \times U)$ by

$$v \mapsto \mathcal{L}v := \frac{\partial v}{\partial t} + \sum_{i=1}^n \frac{\partial v}{\partial x_i} f_i(t,x,u) = \frac{\partial v}{\partial t} + \operatorname{grad} v \cdot f$$

and its adjoint operator $\mathcal{L}' : C([0,T] \times X \times U)' \to C^1([0,T] \times X)'$ by the adjoint relation

$$\langle \mathcal{L}'\nu, v \rangle := \langle \nu, \mathcal{L}v \rangle = \int_{[0,T]\times X\times U} \mathcal{L}v(t,x,u)\, d\nu(t,x,u)$$

for all $\nu \in M([0,T] \times X \times U) = C([0,T] \times X \times U)'$ and $v \in C^1([0,T] \times X)$. This operator is sometimes expressed symbolically as

$$\nu \mapsto \mathcal{L}'\nu = -\frac{\partial \nu}{\partial t} - \sum_{i=1}^n \frac{\partial (f_i \nu)}{\partial x_i} = -\frac{\partial \nu}{\partial t} - \operatorname{div} f\nu$$

where the derivatives of measures are understood in the sense of distributions (i.e., via their action on suitable test functions), and the change of sign comes from the integration by parts formula.

Given a test function $v \in C^1([0,T] \times X)$, it follows from the above definition of the occupation measure μ that

$$\begin{aligned} v(T, x(T)) &= v(0, x(0)) + \int_0^T \dot{v}(t, x(t\,|\,x_0))\, dt \\ &= v(0, x(0)) + \int_0^T \mathcal{L}v(t, x(t\,|\,x_0), u(t\,|\,x_0))\, dt \\ &= v(0, x(0)) + \int_{[0,T]\times X\times U} \mathcal{L}v(t,x,u)\, d\mu(t,x,u\,|\,x_0). \end{aligned} \tag{5}$$

Now consider that the initial state is not a single point but that its distribution in space is modeled by an *initial measure* $\mu_0 \in M(X)$, and that to each initial state x_0 an admissible control function $u(\cdot \mid x_0) \in L^1([0,T];\mathbb{R}^m)$ is assigned in such a way that $x(\cdot \mid x_0)$ is admissible[2]. Then we can define the average occupation measure $\mu \in M([0,T] \times X \times U)$ by

$$\mu(A \times B \times C) := \int_X \mu(A \times B \times C\,|\,x_0)\, d\mu_0(x_0), \tag{6}$$

and the *final measure* $\mu_T \in M(X_T)$ by

$$\mu_T(B) := \int_X I_B(x(T\,|\,x_0))\, d\mu_0(x_0). \tag{7}$$

It follows by integrating (5) with respect to μ_0 that

$$\int_{X_T} v(T,x)\, d\mu_T(x) = \int_X v(0,x)\, d\mu_0(x) + \int_{[0,T]\times X\times U} \mathcal{L}v(t,x,u)\, d\mu(t,x,u)$$

or more concisely

$$\langle \mu_T,\, v(T,\cdot)\rangle = \langle \mu_0,\, v(0,\cdot)\rangle + \langle \mu,\, \mathcal{L}v\rangle \quad \forall\, v \in C^1([0,T]\times X), \tag{8}$$

which is a linear equation linking the nonnegative measures μ_T, μ_0 and μ. Denoting δ_t the Dirac measure at a point t and $\otimes$ the product of measures, we can write $\langle \mu_0,\, v(0,\cdot)\rangle = \langle \delta_0 \otimes \mu_0,\, v\rangle$ and $\langle \mu_T,\, v(T,\cdot)\rangle = \langle \delta_T \otimes \mu_T,\, v\rangle$. Then, Eq. (8) can be rewritten equivalently using the adjoint $\mathcal{L}'$ as

$$\langle \mathcal{L}'\mu,\, v\rangle = \langle \delta_T \otimes \mu_T, v\rangle - \langle \delta_0 \otimes \mu_0,\, v\rangle \quad \forall\, v \in C^1([0,T]\times X),$$

and since this equation is required to hold for all test functions v, we obtain a linear operator equation

$$\mathcal{L}'\mu = \delta_T \otimes \mu_T - \delta_0 \otimes \mu_0. \tag{9}$$

[2]The measure μ_0 can be thought of as the probability distribution of x_0 although we do not require that its mass be normalized to one.

This equation is classical in fluid mechanics and statistical physics, where $\mathcal{L}'$ is usually written using distributional derivatives of measures as remarked above; then the equation is referred to as Liouville's partial differential equation (PDE).

Each family of admissible trajectories starting from a given initial distribution $\mu_0 \in M(X)$ satisfies Liouville's equation (9). The converse may not hold in general although for the computation of the ROA the two formulations can be considered equivalent, at least from a practical viewpoint. Let us briefly elaborate more on this subtle point now.

3.2 Relaxed ROA

The control system $\dot{x}(t) = f(t, x(t), u(t))$, $u(t) \in U$, can be viewed as a differential inclusion

$$\dot{x}(t) \in f(t, x(t), U) := \{f(t, x(t), u) \ : \ u \in U\}. \tag{10}$$

We show in Lemma 4 in Appendix A that any triplet of measures satisfying Liouville's equation (9) corresponds to a family of trajectories of the convexified inclusion

$$\dot{x}(t) \in \text{conv } f(t, x(t), U) \tag{11}$$

starting from the initial distribution μ_0, where conv denotes the convex hull. Let us denote the set of absolutely continuous admissible trajectories of (11) by

$$\bar{\mathcal{X}}(x_0) := \{x(\cdot) : \dot{x}(t) \in \text{conv } f(t, x(t), U),\ x(0) = x_0,\ x(t) \in X\ x(T) \in X_T,\ \forall\, t \in [0, T]\}.$$

Given a family[3] of admissible trajectories of the convexified inclusion starting from an initial distribution μ_0, the occupation and final measures can be defined in a complete analogy via (6) and (7), but now there are only the time and space arguments in the occupation measure, not the control argument. In Appendix A, we state and prove the correspondence between the convexified inclusion (11) and the measures satisfying the Liouville equation (9).

Define now the relaxed region of attraction as

$$\bar{X}_0 := \{x_0 \in X \ : \ \bar{\mathcal{X}}(x_0) \neq \emptyset\}.$$

Clearly $X_0 \subset \bar{X}_0$ and the inclusion can be strict; see Appendix C for concrete examples. However, by the Filippov-Ważewski relaxation Theorem [5], the trajectories of the original inclusion (10) are dense (w.r.t. the metric of uniform convergence of absolutely continuous functions of time) in the set of trajectories of the convexified inclusion (11). This implies that the relaxed region of attraction $\bar{X}_0$ corresponds to the region of attraction of the original system but with infinitesimally dilated constraint sets X and X_T; see Appendix B for more details. Therefore, we argue that there is little difference between the two ROAs from a practical point of view. Nevertheless, because of this subtle distinction we make the following standing assumption in the remainder of the paper.

[3]Each such family can be described by a measure on $C([0, T]; \mathbb{R}^n)$ which is supported on the absolutely continuous solutions to (11). Note that there may be more than one trajectory corresponding to a single initial condition x_0 since the inclusion (11) may admit multiple solutions.

Assumption 1 *Control system (1) is such that* $\lambda(X_0) = \lambda(\bar{X}_0)$.

In other words, the volume of the classical ROA X_0 is assumed to be equal to the volume of the relaxed ROA $\bar{X}_0$. Obviously, this is satisfied if $X_0 = \bar{X}_0$, but otherwise these sets may differ by a set of zero Lebesgue measure. Any of the following conditions on control system (1) is *sufficient* for Assumption 1 to hold:

- $\dot{x}(t) \in f(t, x(t), U)$ with $f(t, x, U)$ convex for all t, x,
- $\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t)$, $u(t) \in U$ with U convex,
- uncontrolled dynamics $\dot{x}(t) = f(t, x(t))$,

as well as all controllability assumptions allowing the application of the constrained Filippov-Ważewski Theorem; see, e.g., [8] and the discussion around Assumption I in [9].

3.3 ROA via optimization

The problem of computing ROA X_0 can be reformulated as follows:

$$\begin{array}{rcll} q^* & = & \sup & \lambda(\operatorname{spt}\mu_0) \\ & & \text{s.t.} & \mathcal{L}'\mu = \delta_T \otimes \mu_T - \delta_0 \otimes \mu_0 \\ & & & \mu \geq 0,\ \mu_0 \geq 0,\ \mu_T \geq 0 \\ & & & \operatorname{spt}\mu \subset [0,T] \times X \times U,\ \operatorname{spt}\mu_0 \subset X,\ \operatorname{spt}\mu_T \subset X_T, \end{array} \tag{12}$$

where the given data are f, X, X_T, U, and the supremum is over a vector of nonnegative measures $(\mu, \mu_0, \mu_T) \in M([0,T] \times X \times U) \times M(X) \times M(X_T)$. Problem (12) is an infinite-dimensional optimization problem on the cone of nonnegative measures.

Lemma 1 *The optimal value of problem (12) is equal to the volume of the ROA* X_0*, that is,* $q^* = \lambda(X_0)$.

Proof By definition of the ROA, for any initial condition $x_0 \in X_0$ there is an admissible trajectory in $\mathcal{X}(x_0)$. Therefore for any initial measure μ_0 with $\operatorname{spt}\mu_0 \subset X_0$ there exist an occupation measure μ and a final measure μ_T such that the constraints of problem (12) are satisfied. Thus, $q^* \geq \lambda(X_0) = \lambda(\bar{X}_0)$, where the equality follows from Assumption 1.

Now we show that $q^* \leq \lambda(X_0) = \lambda(\bar{X}_0)$. Suppose that a triplet of measures (μ_0, μ, μ_T) is feasible in (12) and that $\lambda(\operatorname{spt}\mu_0 \setminus \bar{X}_0) > 0$. From Lemma 4 in Appendix A there is a family of admissible trajectories of the inclusion (11) starting from μ_0 generating the (t, x)-marginal of the occupation measure μ and the final measure μ_T. However, this is a contradiction since no trajectory starting from $\operatorname{spt}\mu_0 \setminus \bar{X}_0$ can be admissible. Thus, $\lambda(\operatorname{spt}\mu_0 \setminus \bar{X}_0) = 0$ and so $\lambda(\operatorname{spt}\mu_0) \leq \lambda(\bar{X}_0)$. Consequently, $q^* \leq \lambda(\bar{X}_0) = \lambda(X_0)$. □

4 Primal LP on measures

The key idea behind the presented approach consists in replacing the optimization over the support of the measure μ_0 by the maximization of its mass under the constraint that μ_0 is dominated by the Lebesgue measure. This leads to the following LP:

$$\begin{array}{rcll} p^* & = & \sup & \mu_0(X) \\ & & \text{s.t.} & \mathcal{L}'\mu = \delta_T \otimes \mu_T - \delta_0 \otimes \mu_0 \\ & & & \mu \geq 0,\ \lambda \geq \mu_0 \geq 0,\ \mu_T \geq 0 \\ & & & \operatorname{spt}\mu \subset [0,T] \times X \times U,\ \operatorname{spt}\mu_0 \subset X,\ \operatorname{spt}\mu_T \subset X_T, \end{array} \tag{13}$$

where the supremum is over a vector of nonnegative measures $(\mu, \mu_0, \mu_T) \in M([0,T] \times X \times U) \times M(X) \times M(X_T)$. In problem (13) the constraint $\lambda \geq \mu_0$ means that $\lambda(A) \geq \mu_0(A)$ for all sets $A \subset X$. Note how the objective functions differ in problems (12) and (13).

The following theorem is then almost immediate.

Theorem 1 *The optimal value of LP problem (13) is equal to the volume of the ROA X_0, that is, $p^* = \lambda(X_0)$. Moreover, the supremum is attained by the restriction of the Lebesgue measure to the ROA X_0.*

Proof: Since the constraint set of problem (13) is tighter than that of problem (12), we have by Lemma 1 that $\lambda(\operatorname{spt}\mu_0) \leq \lambda(X_0)$ for any feasible μ_0. From the constraint $\mu_0 \leq \lambda$ we get $\mu_0(X) = \mu_0(\operatorname{spt}\mu_0) \leq \lambda(\operatorname{spt}\mu_0) \leq \lambda(X_0)$ for any feasible μ_0. Therefore $p^* \leq \lambda(X_0)$. But by definition of the ROA X_0, the restriction of the Lebesgue measure to X_0 is feasible in (13), and so $p^* \geq \lambda(X_0)$. Consequently $p^* = \lambda(X_0)$. □

Now we reformulate problem (13) to an equivalent form more convenient for dualization and subsequent theoretical analysis. To this end, let us define the complementary measure (a slack variable) $\hat{\mu}_0 \in M(X)$ such that the inequality $\lambda \geq \mu_0 \geq 0$ in (13) can be written equivalently as the constraints $\mu_0 + \hat{\mu}_0 = \lambda,\ \mu_0 \geq 0,\ \hat{\mu}_0 \geq 0$. Then problem (13) is equivalent to

$$\begin{array}{rcll} p^* & = & \sup & \mu_0(X) \\ & & \text{s.t.} & \mathcal{L}'\mu = \delta_T \otimes \mu_T - \delta_0 \otimes \mu_0 \\ & & & \mu_0 + \hat{\mu}_0 = \lambda \\ & & & \mu \geq 0,\ \mu_0 \geq 0,\ \mu_T \geq 0,\ \hat{\mu}_0 \geq 0 \\ & & & \operatorname{spt}\mu \subset [0,T] \times X \times U,\ \operatorname{spt}\mu_0 \subset X,\ \operatorname{spt}\mu_T \subset X_T,\ \operatorname{spt}\hat{\mu}_0 \subset X. \end{array} \tag{14}$$

5 Dual LP on functions

In this section, we derive a dual formulation of problem (14) (and hence (13)) on the space of continuous functions. A certain super-level set of any feasible solution to the dual problem yields an outer approximation to the ROA X_0.

Consider the LP problem

$$\begin{array}{rcll} d^* & = & \inf & \displaystyle\int_X w(x)\, d\lambda(x) \\ & & \text{s.t.} & \mathcal{L}v(t,x,u) \le 0, & \forall\, (t,x,u) \in [0,T] \times X \times U \\ & & & w(x) \ge v(0,x) + 1, & \forall\, x \in X \\ & & & v(T,x) \ge 0, & \forall\, x \in X_T \\ & & & w(x) \ge 0, & \forall\, x \in X, \end{array} \tag{15}$$

where the infimum is over $(v, w) \in C^1([0,T] \times X) \times C(X)$. The interpretation of the dual is intuitive: the constraint $\mathcal{L}v \le 0$ forces v to decrease along trajectories and hence necessarily $v(0,x) \ge 0$ on X_0 because of the constraint $v(T,x) \ge 0$ on X_T. Consequently, $w(x) \ge 1$ on X_0. This instrumental observation is formalized in the following Lemma.

Lemma 2 *If $\mathcal{L}v \le 0$ on $[0,T] \times X \times U$, $v(T,\cdot) \ge 0$ on X_T and $w \ge v(0,\cdot) + 1$ on X, then $w \ge 1$ on X_0.*

Proof: By definition of X_0, given any $x_0 \in X_0$ there exists $u(t)$ such that $x(t) \in X$, $u(t) \in U$ for all $t \in [0,T]$ and $x(T) \in X_T$. Therefore, since $\mathcal{L}v \le 0$ on $[0,T] \times X \times U$ and $v(T,\cdot) \ge 0$ on X_T,

$$0 \le v(T, x(T)) = v(0,x_0) + \int_0^T \mathcal{L}v(t, x(t), u(t))\, dt \le v(0,x_0) \le w(x_0) - 1. \qquad \square$$

We have the following salient result:

Theorem 2 *There is no duality gap between primal LP problems (13) and (14) on measures and dual LP problem (15) on functions, in the sense that $p^* = d^*$.*

Proof: To streamline the exposition, let

$$\begin{aligned} \mathcal{C} &:= C([0,T] \times X \times U) \times C(X) \times C(X_T) \times C(X), \\ \mathcal{M} &:= M([0,T] \times X \times U) \times M(X) \times M(X_T) \times M(X), \end{aligned}$$

and let $\mathcal{K}$ and $\mathcal{K}'$ denote the positive cones of $\mathcal{C}$ and $\mathcal{M}$ respectively. Note that the cone $\mathcal{K}'$ of nonnegative measures of $\mathcal{M}$ can be identified with the topological dual of the cone $\mathcal{K}$ of nonnegative continuous functions of $\mathcal{C}$. The cone $\mathcal{K}'$ is equipped with the weak-* topology; see [20, Chapter 5].

Then the LP problem (14) can be rewritten as

$$\begin{array}{rcll} p^* & = & \sup & \langle \gamma,\, c \rangle \\ & & \text{s.t.} & \mathcal{A}'\gamma = \beta \\ & & & \gamma \in \mathcal{K}', \end{array} \tag{16}$$

where the infimum is over the vector

$$\gamma := (\mu, \mu_0, \mu_T, \hat{\mu}_0),$$

the linear operator $\mathcal{A}' : \mathcal{K}' \to C^1([0,T] \times X)' \times M(X)$ is defined by

$$\mathcal{A}'\gamma := (\mathcal{L}'\mu + \delta_0 \otimes \mu_0 - \delta_T \otimes \mu_T \ , \ \mu_0 + \hat{\mu}_0)\,,$$

the right hand side of the equality constraint in (16) is the vector of measures

$$\beta := (0\,,\, \lambda) \in M([0,T] \times X) \times M(X),$$

the vector function in the objective is

$$c := (0, 1, 0, 0) \in \mathcal{C},$$

and the objective function itself is

$$\langle \gamma,\, c\rangle = \int_X d\mu_0 = \mu_0(X).$$

The LP problem (16) can be interpreted as a dual to the LP problem

$$\begin{array}{rcll} d^* & = & \inf & \langle \beta,\, (v,w)\rangle \\ & & \text{s.t.} & \mathcal{A}(v,w) - c \in \mathcal{K}, \end{array} \tag{17}$$

where the infimum is over $(v,w) \in C^1([0,T] \times X) \times C(X)$, and the linear operator $\mathcal{A} : C^1([0,T] \times X) \times C(X) \to \mathcal{C}$ is defined by

$$\mathcal{A}v := (-\mathcal{L}v,\ \ w - v(0,\cdot),\ \ v(T,\cdot),\ \ w)$$

and satisfies the adjoint relation $\langle \mathcal{A}'\gamma,\, v\rangle = \langle \gamma,\, \mathcal{A}v\rangle$. The LP problem (17) is exactly the LP problem (15).

To conclude the proof we use an argument similar to that of [18, Section C.4]. From [2, Theorem 3.10] there is no duality gap between LPs (16) and (17) if the supremum p^* is finite and the set $P := \{(\mathcal{A}'\gamma,\, \langle \gamma,\, c\rangle) : \gamma \in \mathcal{K}'\}$ is closed in the weak-* topology of $\mathcal{K}'$. The fact that p^* is finite follows readily from the constraint $\mu_0 + \hat{\mu}_0 = \lambda$, $\hat{\mu}_0 \geq 0$, and from compactness of X. To prove closedness, we first remark that $\mathcal{A}'$ is weakly-* continuous[4] since $\mathcal{A}(v,w) \in \mathcal{C}$ for all $(v,w) \in C^1([0,T] \times X) \times C(X)$. Then we consider a sequence $\gamma_k \in \mathcal{M}$ and we want to show that its accumulation point $\lim_{k\to\infty}(\mathcal{A}'\gamma_k,\, \langle \gamma_k,\, c\rangle)$ belongs to P. Since the supports of the measures are compact and p^* is finite (hence μ_0, μ_T and $\hat{\mu}_0$ are bounded) and since $T < \infty$ (hence μ is bounded), the sequence γ_k is bounded and from the weak-* compactness of the unit ball (Alaoglu's Theorem, see [20, Chapter 5]) there is a subsequence γ_{k_i} that converges weakly-* to an element $\gamma \in \mathcal{M}$ so that $\lim_{i\to\infty}(\mathcal{A}'\gamma_{k_i},\, \langle \gamma_{k_i},\, c\rangle) \in P$. $\square$

In the above proof we observed that the equivalent LPs (13) and (14) are formulated in the dual of a Banach space equipped with the appropriate topology. It follows that the supremum in LPs (13) and (14) is attained (by the restriction of the Lebesgue measure to X_0), a statement already proved by a different means in Theorem 1. In contrast, the

[4]The weak-* topology on $C^1([0,T] \times X)' \times M(X)$ is induced by the standard topologies on C^1 and C – the topology of uniform convergence of the function and its derivative on C^1 and the topology of uniform convergence on C.

infimum in problem (15) is not attained in $C^1([0,T]\times X)\times C(X)$, but the w-component of feasible solutions of (15) converges to the discontinuous indicator I_{X_0} as we show next.

Before we state our convergence results, we recall the following types of convergence of a sequence of functions $w_k : X \to \mathbb{R}$ to a function $w : X \to \mathbb{R}$ on a compact set $X \subset \mathbb{R}^n$. As $k \to \infty$, the functions w_k converge to w:

- in L^1 norm if $\int_X |w_k - w|\, d\lambda \to 0$,
- in Lebesgue measure if $\lambda(\{x \,:\, |w_k(x) - w(x)| \geq \epsilon\}) \to 0 \;\forall\, \epsilon > 0$,
- almost everywhere if $\exists\, B \subset X$, $\lambda(B) = 0$, such that $w_k \to w$ pointwise on $X\backslash B$,
- almost uniformly if $\forall\, \epsilon > 0, \exists\, B \subset X$, $\lambda(B) < \epsilon$, such that $w_k \to w$ uniformly on $X\backslash B$.

We also recall that convergence in L^1 norm implies convergence in Lebesgue measure and that almost uniform convergence implies convergence almost everywhere (see [3, Theorems 2.5.2 and 2.5.3]). Therefore we will state our results in terms of the stronger notions of L^1 norm and almost uniform convergence.

Theorem 3 *There is a sequence of feasible solutions to problem (15) such that its w-component converges from above to I_{X_0} in L^1 norm and almost uniformly.*

Proof: By Theorem 1, the optimal solution to the primal is attained by the restriction of the Lebesgue measure to X_0. Consequently,

$$p^* = \int_X I_{X_0} d\lambda. \tag{18}$$

By Theorem 2, there is no duality gap ($p^* = d^*$), and therefore there exists a sequence $(v_k, w_k) \in C^1([0,T]\times X)\times C(X)$ feasible in (15) such that

$$p^* = d^* = \lim_{k\to\infty} \int_X w_k(x)\, d\lambda(x). \tag{19}$$

From Lemma 2 we have $w_k \geq 1$ on X_0 and since $w_k \geq 0$ on X, we have $w_k \geq I_{X_0}$ on X for all k. Thus, subtracting (18) from (19) gives

$$\lim_{k\to\infty} \int_X (w_k(x) - I_{X_0}(x))\, d\lambda(x) = 0,$$

where the integrand is nonnegative. Hence w_k converges to I_{X_0} in L^1 norm. From [3, Theorems 2.5.2 and 2.5.3] there exists a subsequence converging almost uniformly. □

6 LMI relaxations

In this section we show how the infinite dimensional LP problem (14) can be approximated by a hierarchy of LMI problems with the approximation error vanishing as the relaxation order tends to infinity. The dual LMI problem then yields a converging sequence of outer approximations to the ROA.

The measures in equation (8) (or (9)) are fully determined by their values on a family of functions whose span is dense in $C^1([0,T]\times X)$. Hence, since all sets are assumed compact, the family of test functions in (8) can be restricted to any polynomial basis (since polynomials are dense in the space of continuous functions on compact sets equipped with the supremum norm). The basis of our choice is the set of all monomials.

Let $\mathbb{R}_k[x]$ denote the vector space of real multivariate polynomials of total degree less than or equal to k. Each polynomial $p(x)\in\mathbb{R}_k[x]$ can be expressed in the monomial basis as

$$p(x)=\sum_{|\alpha|\le k} p_\alpha x^\alpha,$$

where α runs over the multi-indices (vectors of nonnegative integers) such that $|\alpha| = \sum_i \alpha_i \le k$, and $x^\alpha = \prod_i x^{\alpha_i}$. A polynomial $p(x)$ is identified with its vector of coefficients $p := (p_\alpha)$ whose entries are indexed by α. Given a vector of real numbers $y := (y_\alpha)$ indexed by α, we define the linear functional $L_y : \mathbb{R}_k[x]\to\mathbb{R}$ such that

$$L_y(p) := p'y := \sum_\alpha p_\alpha y_\alpha$$

where the prime here denotes transposition. When entries of y are moments of a measure ν, i.e.,

$$y_\alpha = \int x^\alpha d\nu(x),$$

the linear functional models integration of a polynomial w.r.t. ν, i.e.,

$$L_y(p) \;=\; \langle \nu,\, p\rangle \;=\; \int p(x)\,d\nu(x) \;=\; \int \sum_\alpha p_\alpha x^\alpha d\nu(x) = \sum_\alpha p_\alpha \int x^\alpha d\nu(x) = p'y.$$

When this linear functional acts on the square of a polynomial p of degree k, it becomes a quadratic form in the polynomial coefficients space, and we denote by $M_k(y)$ and call the *moment matrix* of order k the matrix of this quadratic form, which is symmetric and linear in y:

$$L_y(p^2) = p'M_k(y)p.$$

Finally, given a polynomial $g(x)\in\mathbb{R}[x]$ we define the *localizing matrix* $M_k(g,\,y)$ by the equality

$$L_y(gp^2) = p'M_k(g,\,y)p.$$

The matrix $M_k(g,\,y)$ is also symmetric and linear in y.

Let y, y_0, y_T and $\hat{y}_0$ respectively denote the sequences of moments of measures μ, μ_0, μ_T and $\hat{\mu}_0$ such that the constraints in problem (14) are satisfied. Then it follows that these

sequences satisfy an infinite-dimensional linear system of equations corresponding to the equality constraints of problem (14) written explicitly as

$$\int_{X_T} v(T,x)\,d\mu_T(x) - \int_X v(0,x)\,d\mu_0(x) - \int_{[0,T]\times X\times U} \mathcal{L}v(t,x,u)\,d\mu(t,x,u) = 0,$$

$$\int_X w(x)\,d\mu_0(x) + \int_X w(x)\,d\hat{\mu}_0(x) = \int_X w(x)\,d\lambda(x)$$

for the particular choice of test functions $v(t,x) = t^\alpha x^\beta$ and $w(x) = x^\beta$ for all $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}^n$. Let us denote by

$$A_k(y, y_0, y_T, \hat{y}_0) = b_k$$

the finite-dimensional truncation of this system obtained by considering only the test functions of total degree less than or equal to $2k$. Let further

$$d_{Xi} := \left\lceil \frac{\deg g_{Xi}}{2} \right\rceil, \quad d_{Ui} := \left\lceil \frac{\deg g_{Ui}}{2} \right\rceil, \quad d_{Ti} := \left\lceil \frac{\deg g_{Ti}}{2} \right\rceil,$$

and consider the primal linear semidefinite programming problem

$$\begin{array}{rcll} p_k^* & = & \max & (y_0)_0 \\ & & \text{s.t.} & A_k(y, y_0, y_T, \hat{y}_0) = b_k \\ & & & M_k(y) \succeq 0, & M_{k-d_{Xi}}(g_{Xi}, y) \succeq 0, & i = 1,2,\ldots,n_X \\ & & & M_{k-1}(t(T-t), y) \succeq 0, & M_{k-d_{Ui}}(g_{Ui}, y) \succeq 0, & i = 1,2,\ldots,n_U \\ & & & M_k(y_0) \succeq 0, & M_{k-d_{Xi}}(g_{Xi}, y_0) \succeq 0, & i = 1,2,\ldots,n_X \\ & & & M_k(y_T) \succeq 0, & M_{k-d_{Ti}}(g_{Ti}, y_T) \succeq 0, & i = 1,2,\ldots,n_T \\ & & & M_k(\hat{y}_0) \succeq 0, & M_{k-d_{Xi}}(g_{Xi}, \hat{y}_0) \succeq 0, & i = 1,2,\ldots,n_X \end{array} \tag{20}$$

where the notation $\succeq 0$ stands for positive semidefinite and the minimum is over sequences $(y, y_0, y_T, \hat{y}_0)$ truncated to degree $2k$. Problem (20) is a semidefinite program (SDP), where a linear function is minimized subject to convex linear matrix inequality (LMI) constraints, or equivalently a finite-dimensional LP in the cone of positive semidefinite matrices.

For the remainder of the section we make the following standard assumption.

Assumption 2 *One of the polynomials modeling the sets X, U resp. X_T, is equal to $g_{Xi}(x) = R_X^2 - \|x\|_2^2$, $g_{Ui}(u) = R_U^2 - \|u\|_2^2$ resp. $g_{Ti}(x) = R_T^2 - \|x\|_2^2$, with R_X, R_U resp. R_T sufficiently large constants.*

Assumption 2 is made without loss of generality since X, U and X_T are bounded, and polynomials modeling ball constraints can be added to the semialgebraic descriptions of these sets.

The dual to the SDP problem (20) is given by

$$\begin{array}{rcll} d_k^* & = & \inf & w'l \\ & & \text{s.t.} & -\mathcal{L}v(t,x,u) = p(t,x,u) + q_0(t,x,u)t(T-t) \\ & & & \quad + \sum_{i=1}^{n_X} q_i(t,x,u) g_{Xi}(x) + \sum_{i=1}^{n_U} r_i(t,x,u) g_{Ui}(x) \\ & & & w(x) - v(0,x) - 1 = p_0(x) + \sum_{i=1}^{n_X} q_{0i}(x) g_{Xi}(x) \\ & & & v(T,x) = p_T(x) + \sum_{i=1}^{n_T} q_{Ti}(x) g_{Ti}(x) \\ & & & w(x) = s_0(x) + \sum_{i=1}^{n_X} s_{0i}(x) g_{Xi}(x), \end{array} \tag{21}$$

where l is the vector of Lebesgue moments over X indexed in the same basis in which the polynomial $w(x)$ with coefficients w is expressed. The minimum is over polynomials $v(t,x) \in \mathbb{R}_{2k}[t,x]$ and $w \in \mathbb{R}_{2k}[x]$, and polynomial sum-of-squares $p(t,x,u)$, $q_i(t,x,u)$, $i = 0,1,\ldots,n_X$, $r_i(t,x,u)$, $i = 1,\ldots,n_U$, $p_0(x)$, $p_T(x)$, $q_{0i}(x)$, $q_{Ti}(x)$, $s_0(x)$, $s_{0i}(x)$, $i = 1,\ldots,n_X$ of appropriate degrees. The constraints that polynomials are sum-of-squares can be written explicitly as LMI constraints (see, e.g., [18]), and the objective is linear in the coefficients of the polynomial $w(x)$; therefore problem (21) can be formulated as an SDP.

Theorem 4 *There is no duality gap between primal LMI problem (20) and dual LMI problem (21), i.e.* $p_k^* = d_k^*$.

In order to prove Theorem 4 we rewrite primal LMI problem (20) in a vectorized form as follows

$$\begin{array}{rcll} p^* & = & \min & \mathbf{c}'\mathbf{y} \\ & & \text{s.t.} & \mathbf{A}\mathbf{y} = \mathbf{b} \\ & & & \mathbf{e} + \mathbf{D}\mathbf{y} \in \mathbf{K}, \end{array} \tag{22}$$

where $\mathbf{y} := [y',\, y_0',\, y_T', \hat{y}_0']'$ and $\mathbf{K}$ is a direct product of cones of positive semidefinite matrices of appropriate dimensions, here corresponding to the moment matrix and localizing matrix constraints. The notation $\mathbf{e} + \mathbf{D}\mathbf{y} \in \mathbf{K}$ means that vector $\mathbf{e} + \mathbf{D}\mathbf{y}$ contains entries of positive semidefinite moment and localizing matrices, and by construction matrix $\mathbf{D}$ has full column rank (since a moment matrix is zero if and only if the corresponding moment vector is zero). Dual LMI problem (21) then becomes

$$\begin{array}{rcll} d^* & = & \max & \mathbf{b}'\mathbf{x} - \mathbf{e}'\mathbf{z} \\ & & \text{s.t.} & \mathbf{A}'\mathbf{x} + \mathbf{D}'\mathbf{z} = \mathbf{c} \\ & & & \mathbf{z} \in \mathbf{K}, \end{array} \tag{23}$$

and we want to prove that $p^* = d^*$. The following instrumental result is a minor extension of a classical lemma of the alternatives for primal LMI (22) and dual LMI (23). The notation $\operatorname{int} \mathbf{K}$ stands for the interior of $\mathbf{K}$.

Lemma 3 *If matrix* $\mathbf{D}$ *has full column rank, exactly one of these statements is true:*

- *there exists* $\mathbf{x}$ *and* $\mathbf{z} \in \operatorname{int} \mathbf{K}$ *such that* $\mathbf{A}'\mathbf{x} + \mathbf{D}'\mathbf{z} = \mathbf{c}$
- *there exists* $\mathbf{y} \neq 0$ *such that* $\mathbf{A}\mathbf{y} = 0$, $\mathbf{D}\mathbf{y} \in \mathbf{K}$ *and* $\mathbf{c}'\mathbf{y} \leq 0$.

Proof of Lemma 3: A classical lemma of alternatives states that if matrix $\bar{\mathbf{D}}$ has full column rank, then either there exists $\mathbf{z} \in \operatorname{int} \mathbf{K}$ such that $\bar{\mathbf{D}}'\mathbf{z} = \bar{\mathbf{c}}$ or there exists $\bar{\mathbf{y}}$ such that $\bar{\mathbf{D}}\bar{\mathbf{y}} \in \mathbf{K}$ and $\bar{\mathbf{c}}'\bar{\mathbf{y}} \leq 0$, but not both, see e.g. [28, Lemma 2] for a standard proof based on the geometric form of the Hahn-Banach separation theorem. Our proof then follows from restricting this lemma of alternatives to the null-space of matrix $\mathbf{A}$. More explicitly, there exists $\mathbf{x}$ and $\mathbf{z}$ such that $\mathbf{A}'\mathbf{x} + \mathbf{D}'\mathbf{z} = \mathbf{c}$ if and only if $\mathbf{z}$ is such that $\bar{\mathbf{D}}'\mathbf{z} = \bar{\mathbf{c}}$ with $\bar{\mathbf{D}} = \mathbf{D}\mathbf{F}$, $\bar{\mathbf{c}} = \mathbf{F}'\mathbf{c}$ for $\mathbf{F}$ a full-rank matrix such that $\mathbf{A}\mathbf{F} = 0$. Matrix $\bar{\mathbf{D}}$ has full column rank since it is the restriction of the full column rank matrix $\mathbf{D}$ to the null-space of $\mathbf{A}$. $\square$.

Proof of Theorem 4: First notice that the feasibility set of LMI problem (22) is nonempty and bounded. Indeed, a triplet of zero measures is a trivial feasible point for (13) and hence $(0, 0, 0, \lambda)$ is feasible in (14); consequently a concatenation of truncated moment sequences corresponding to the quadruplet of measures $(0, 0, 0, \lambda)$ is feasible in (22) for each relaxation order k. Boundedness of the even components of each moment vector follows from the structure of the localizing matrices corresponding to the functions from Assumption 2 and from the fact that the masses (zero-th moments) of the measures are bounded because of the constraint $\mu_0 + \hat{\mu}_0 = \lambda$ and because $T < \infty$. Boundedness of the whole moment vectors then follows since the even moments appear on the diagonal of the positive semidefinite moment matrices.

To complete the proof, we follow [28, Theorem 4] and show that boundedness of the feasibility set of LMI problem (22) implies existence of an interior point for LMI problem (23), and then from standard SDP duality it follows readily that $p^* = d^*$ since D has a full column rank; see, e.g., [28, Theorem 5] and references therein.

Let $\bar{\mathbf{y}}$ denote a point in the feasibility set of LMI problem (22), i.e. a vector satisfying $\mathbf{A}\bar{\mathbf{y}} = \mathbf{b}$ and $\mathbf{e} + \mathbf{D}\bar{\mathbf{y}} \in \mathbf{K}$. Suppose that there is no interior point for LMI problem (23), i.e. there is no $\mathbf{x}$ and $\mathbf{z} \in \operatorname{int} \mathbf{K}$ such that $\mathbf{A}'\mathbf{x} + \mathbf{D}'\mathbf{z} = \mathbf{c}$. Then from Lemma 3 there exists $\mathbf{y} \neq 0$ such that $\mathbf{A}\mathbf{y} = 0$, $\mathbf{D}\mathbf{y} \in \mathbf{K}$ and $\mathbf{c}'\mathbf{y} \leq 0$. It follows that there exists a ray $\bar{\mathbf{y}} + t\mathbf{y}$, $t \geq 0$ of feasible points for LMI problem (22), hence implying that the feasibility set is not bounded. □

Now we prove convergence results analogous to those of Theorem 3 as well as a set-wise convergence to the ROA X_0 of certain super-level sets of the polynomial solutions to (21).

Theorem 5 *Let $w_k \in \mathbb{R}_{2k}[x]$ denote the w-component of a solution to the dual LMI problem (21) and let $\bar{w}_k(x) = \min_{i \leq k} w_i(x)$. Then w_k converges from above to I_{X_0} in L^1 norm and $\bar{w}_k$ converges from above to I_{X_0} in L^1 norm and almost uniformly.*

Proof: From Lemma 2 and Theorem 3, for every $\epsilon > 0$ there exists a $(v, w) \in C^1([0, T] \times X) \times C(X)$ feasible in (15) such that $w \geq I_{X_0}$ and $\int_X (w - I_{X_0})\, d\lambda < \epsilon$. Set

$$\tilde{v}(t, x) := v(t, x) - \epsilon t + (T + 1)\epsilon,$$
$$\tilde{w}(x) := w(x) + (T + 3)\epsilon.$$

Since v is feasible in (15), we have $\mathcal{L}\tilde{v} = \mathcal{L}v - \epsilon$, and $\tilde{v}(T, x) = v(T, x) + \epsilon$. Since also $\tilde{w}(x) - \tilde{v}(0, x) \geq 1 + 2\epsilon$, it follows that $(\tilde{v}, \tilde{w})$ is *strictly* feasible in (15) with a margin at least ϵ. Since $[0, T] \times X$ and X are compact, there exist[5] polynomials $\hat{v}$ and $\hat{w}$ of a sufficiently high degree such that $||\tilde{v} - \hat{v}||_\infty < \epsilon$, $||\mathcal{L}\tilde{v} - \mathcal{L}\hat{v}||_\infty < \epsilon$ and $||\tilde{w} - \hat{w}||_\infty < \epsilon$. The pair of polynomials $(\hat{v}, \hat{w})$ is therefore *strictly* feasible in (15) and as a result, under Assumption 2, feasible in (21) for a sufficiently large relaxation order k (this follows from the classical Positivstellensatz by Putinar; see, e.g., [18] or [24]), and moreover $\hat{w} \geq w$. Consequently, $\int_X |\tilde{w} - \hat{w}|\, d\lambda \leq \epsilon\lambda(X)$, and so $\int_X (\hat{w} - w)\, d\lambda \leq \epsilon\lambda(X)(T + 4)$. Therefore

$$\int_X (\hat{w} - I_{X_0})\, d\lambda < \epsilon K, \quad \hat{w} \geq I_{X_0},$$

[5]This follows from an extension of the Stone-Weierstrass theorem that allows for a simultaneous uniform approximation of a function and its derivatives by a polynomial on a compact set; see, e.g., [16].

where $K := [1 + (T + 4)\lambda(X)] < \infty$ is a constant. This proves the first statement since ϵ was arbitrary.

The second statement immediately follows since given a sequence $w_k \to I_{X_0}$ in L_1 norm, there exists a subsequence w_{k_i} that converges almost uniformly to I_{X_0} by [3, Theorems 2.5.2 and 2.5.3] and clearly $\bar{w}_k(x) \leq \min\{w_{k_i}(x) \,:\, k_i \leq k\}$. $\square$

The following Corollary follows immediately from Theorem 5.

Corollary 1 *The sequence of infima of LMI problems (21) converges monotonically from above to the supremum of the LP problem (15), i.e., $d^* \leq d^*_{k+1} \leq d^*_k$ and $\lim_{k\to\infty} d^*_k = d^*$. Similarly, the sequence of maxima of LMI problems (20) converges monotonically from above to the maximum of the LP problem (13), i.e., $p^* \leq p^*_{k+1} \leq p^*_k$ and $\lim_{k\to\infty} p^*_k = p^*$.*

Proof: Monotone convergence of the dual optima d^*_k follows immediately from Theorem 5 and from the fact that the higher the relaxation order k, the looser the constraint set of the minimization problem (21). To prove convergence of the primal maxima observe that from weak SDP duality we have $d^*_k \geq p^*_k$ and from Theorems 5 and 2 it follows that $d^*_k \to d^* = p^*$. In addition, clearly $p^*_k \geq p^*$ and $p^*_{k+1} \leq p^*_k$ since the higher the relaxation order k, the tighter the constraint set of the maximization problem (20). Therefore $p^*_k \to p^*$ monotonically from above. $\square$

Theorem 5 establishes a functional convergence of w_k to I_{X_0} and Corollary 1 a convergence of the primal and dual optima p^*_k and d^*_k to the volume of the ROA $\lambda(X_0) = p^* = d^*$. Finally, the following theorem establishes a set-wise convergence of the unit super-level sets of w_k to X_0.

Theorem 6 *Let $w_k \in \mathbb{R}_{2k}[x]$ denote the w-component of a solution to the dual LMI problem (21) and let $X_{0k} := \{x \in \mathbb{R}^n \,:\, w_k(x) \geq 1\}$. Then $X_0 \subset X_{0k}$,*

$$\lim_{k\to\infty} \lambda(X_{0k} \setminus X_0) = 0, \tag{24}$$

and

$$\lambda(\cap_{k=1}^{\infty} X_{0k} \setminus X_0) = 0. \tag{25}$$

Proof: From Lemma 2, we have $w_k \geq I_{X_0}$ and therefore $w_k \geq I_{X_{0k}} \geq I_{X_0}$. From Theorem 5, we have $w_k \to I_{X_0}$ in L^1 norm on X. Consequently,

$$\begin{aligned}\lambda(X_0) = \int_X I_{X_0}\, d\lambda \;&=\; \lim_{k\to\infty} \int_X w_k\, d\lambda \;\geq\; \lim_{k\to\infty} \int_X I_{X_{0k}}\, d\lambda \;=\; \lim_{k\to\infty} \lambda(X_{0k}) \\ &\geq\; \lim_{k\to\infty} \lambda(\cap_{i=1}^{k} X_{0i}) \;=\; \lambda(\cap_{k=1}^{\infty} X_{0k}).\end{aligned}$$

But since $X_0 \subset X_{0k}$ for all k, we must have

$$\lim_{k\to\infty} \lambda(X_{0k}) = \lambda(X_0) \quad \text{and} \quad \lambda(\cap_{k=1}^{\infty} X_{0k}) = \lambda(X_0).$$

This proves the theorem since $X_{0k} \supset X_0$ and $\cap_{k=1}^{\infty} X_{0k} \supset X_0$. $\square$

7 Free final time

In this section we outline a straightforward extension of our approach to the problem of reaching the target set X_T at *any* time before $T < \infty$ (and not necessarily staying in X_T afterwards).

It turns out that the set of all initial states x_0 from which it is possible to reach X_T at a time $t \leq T$ can be obtained as the support of an optimal solution μ_0^* to the problem

$$\begin{array}{ll} \sup & \mu_0(X) \\ \text{s.t.} & \mathcal{L}'\mu = \delta_T \otimes \mu_T - \mu_0 \\ & \mu \geq 0,\ \lambda \geq \mu_0 \geq 0,\ \mu_T \geq 0 \\ & \operatorname{spt}\mu \subset [0,T] \times X \times U,\ \operatorname{spt}\mu_0 \subset X,\ \operatorname{spt}\mu_T \subset [0,T] \times X_T, \end{array} \tag{26}$$

where the supremum is over a vector of nonnegative measures $(\mu, \mu_0, \mu_T) \in M([0,T] \times X \times U) \times M(X) \times M([0,T] \times X_T)$. Note that the only difference to problem (13) is in the support constraints of the final measure μ_T.

The dual to this problem reads as

$$\begin{array}{lll} \inf & \displaystyle\int_X w(x)\, d\lambda(x) & \\ \text{s.t.} & \mathcal{L}v(t,x,u) \leq 0, & \forall\, (t,x,u) \in [0,T] \times X \times U \\ & w(x) \geq v(0,x) + 1, & \forall\, x \in X \\ & v(t,x) \geq 0, & \forall\, (t,x) \in [0,T] \times X_T \\ & w(x) \geq 0, & \forall\, x \in X. \end{array} \tag{27}$$

The only difference to problem (15) is in the third constraint which now requires that $v(t,x)$ is nonnegative on X_T for *all* $t \in [0,T]$.

All results from the previous sections hold with proofs being almost verbatim copies.

8 Numerical examples

In this section we present three examples to illustrate our approach: a univariate uncontrolled cubic system, an uncontrolled Van der Pol oscillator and a minimum-time control of a double integrator. For numerical implementation, one can either use Gloptipoly 3 [14] to formulate the primal problem on measures and then extract the dual solution provided by SeDuMi [25] or formulate directly the dual SOS problem using, e.g., YALMIP [31] or SOSTOOLS [26].

8.1 Univariate cubic dynamics

Consider the system given by

$$\dot{x} = x(x - 0.5)(x + 0.5),$$

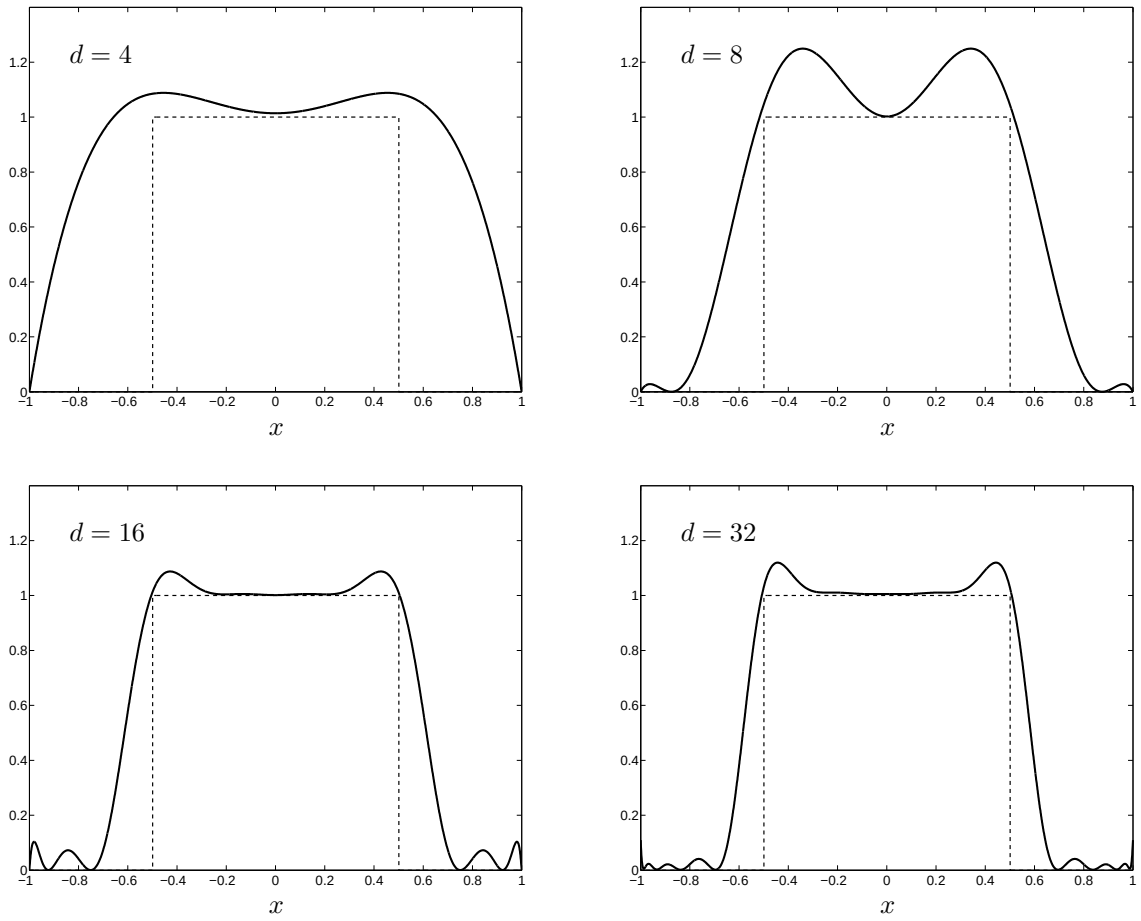

Figure 1: Univariate cubic dynamics – polynomial approximations (solid line) to the ROA indicator function $I_{X_0} = I_{[-0.5,0.5]}$ (dashed line) for degrees $d \in \{4, 8, 16, 32\}$.

the constraint set $X = [-1, 1]$, the final time $T = 100$ and the target set $X_T = [-0.01, 0.01]$. The ROA can in this case be determined analytically as $X_0 = [-0.5, 0.5]$. Polynomial approximations to the ROA for degrees $d \in \{4, 8, 16, 32\}$ are shown in Figure 1. As expected the functional convergence of the polynomials to the discontinuous indicator function is rather slow; however, the set-wise convergence of the unit super-level set is very fast as shown in Table 1. Note that the volume error is not monotonically decreasing – indeed what is guaranteed to decrease is the integral of the approximating polynomial, not the volume of its unit super-level set. Numerically, a better behavior is expected when using alternative polynomial bases (e.g., Chebyshev polynomials) instead of the monomials; see the conclusion for a discussion.

Table 1: Univariate cubic dynamics – relative error of the outer approximation to the ROA $X_0 = [-0.5, 0.5]$ as a function of the approximating polynomial degree.

degree	4	8	16	32
error	31.60 %	3.32 %	0.96 %	1.56 %

8.2 Van der Pol oscillator

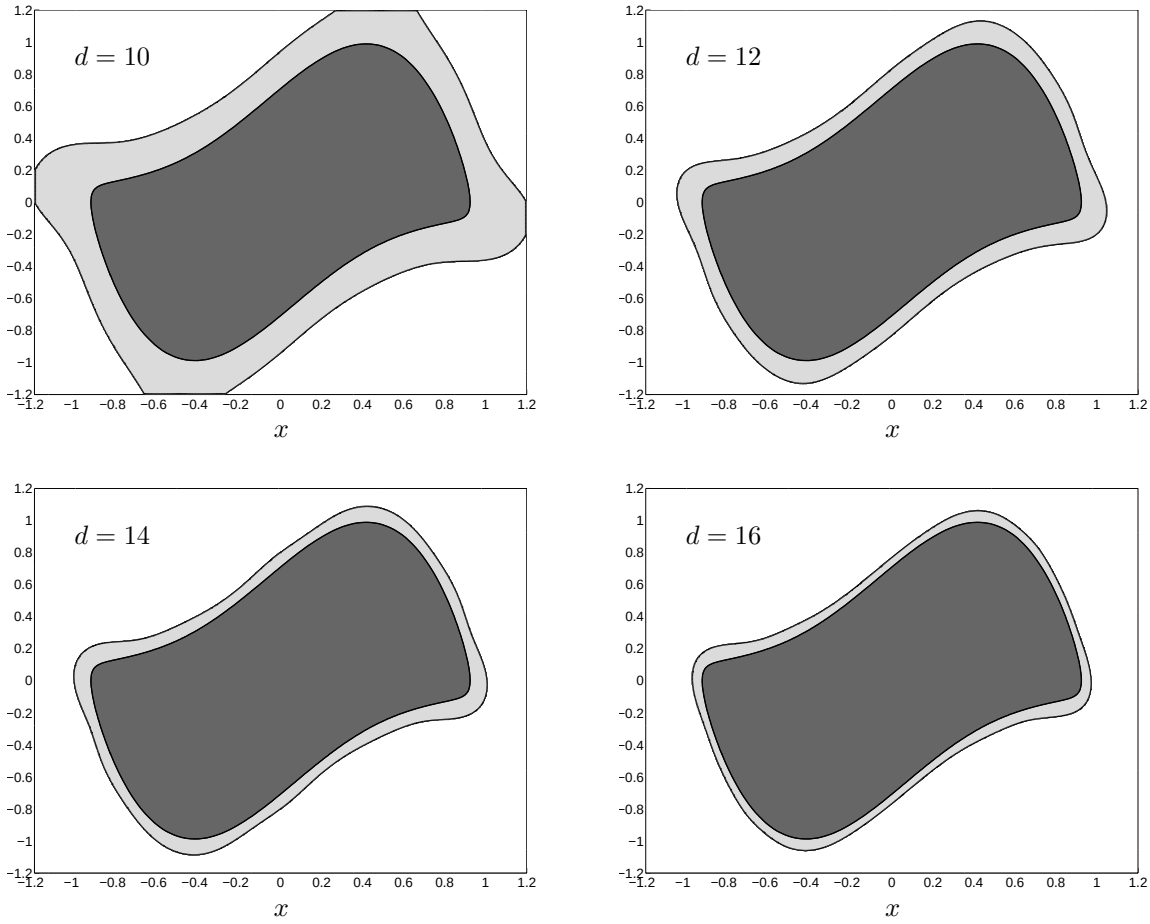

Figure 2: Van der Pol oscillator – polynomial outer approximations (light gray) to the ROA (dark gray) for degrees $d \in \{10, 12, 14, 16\}$.

As a second example consider a scaled version of the uncontrolled reversed-time Van der Pol oscillator given by

$$\begin{aligned}\dot{x}_1 &= -2x_2,\\ \dot{x}_2 &= 0.8x_1 + 10(x_1^2 - 0.21)x_2.\end{aligned}$$

The system has one stable equilibrium at the origin with a bounded region of attraction

$$X_0 \subset X := [-1.2,\, 1.2]^2.$$

In order to compute an outer approximation to this region we take $T = 100$ and $X_T = \{x \,:\, \|x\|_2 \leq 0.01\}$. Plots of polynomial super-level set approximations of degree $d \in \{10, 12, 14, 16\}$ are shown in Figure 2. We observe a relatively fast convergence of the super-level sets to the ROA – this is confirmed by the relative volume error[6] summarized

[6]The relative volume error was computed approximately by Monte Carlo integration.

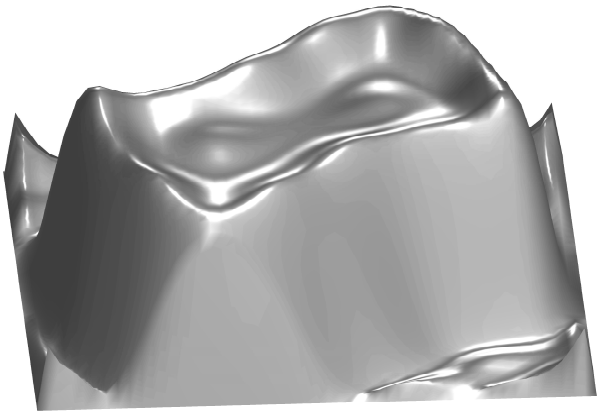

Figure 3: Van der Pol oscillator – a polynomial approximation of degree 18 of the ROA indicator function I_{X_0}.

in Table 2. Figure 3 then shows the approximating polynomial itself for degree $d = 18$. Here too, a better convergence is expected if instead of monomials, a more appropriate polynomial basis is used.

Table 2: Van der Pol oscillator – relative error of the outer approximation to the ROA X_0 as a function of the approximating polynomial degree.

degree	10	12	14	16
error	62.7 %	29.6 %	20.8 %	14.2 %

8.3 Double integrator

In our last example we consider a minimum time control of a double integrator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u.$$

The goal is to find an approximation of the set of initial states X_0 that can be steered to the origin in $T = 1$. Therefore we set $X_T = \{0\}$ and the constraint set such that $X_0 \subset X$, e.g., $X = [-0.7, 0.7] \times [-1.2, 1.2]$. The solution to this problem can be computed analytically as

$$X_0 = \{x \,:\, V(x) \leq 1\},$$

where

$$V(x) = \begin{cases} x_2 + 2\sqrt{x_1 + \frac{1}{2}x_2^2} & \text{if} \quad x_1 + \frac{1}{2}x_2|x_2| > 0, \\ -x_2 + 2\sqrt{-x_1 + \frac{1}{2}x_2^2} & \text{otherwise.} \end{cases}$$

The computation results are depicted in Figure 4; again we observe a relatively fast convergence of the super-level set approximations, which is confirmed by the relative volume errors in Table 3.

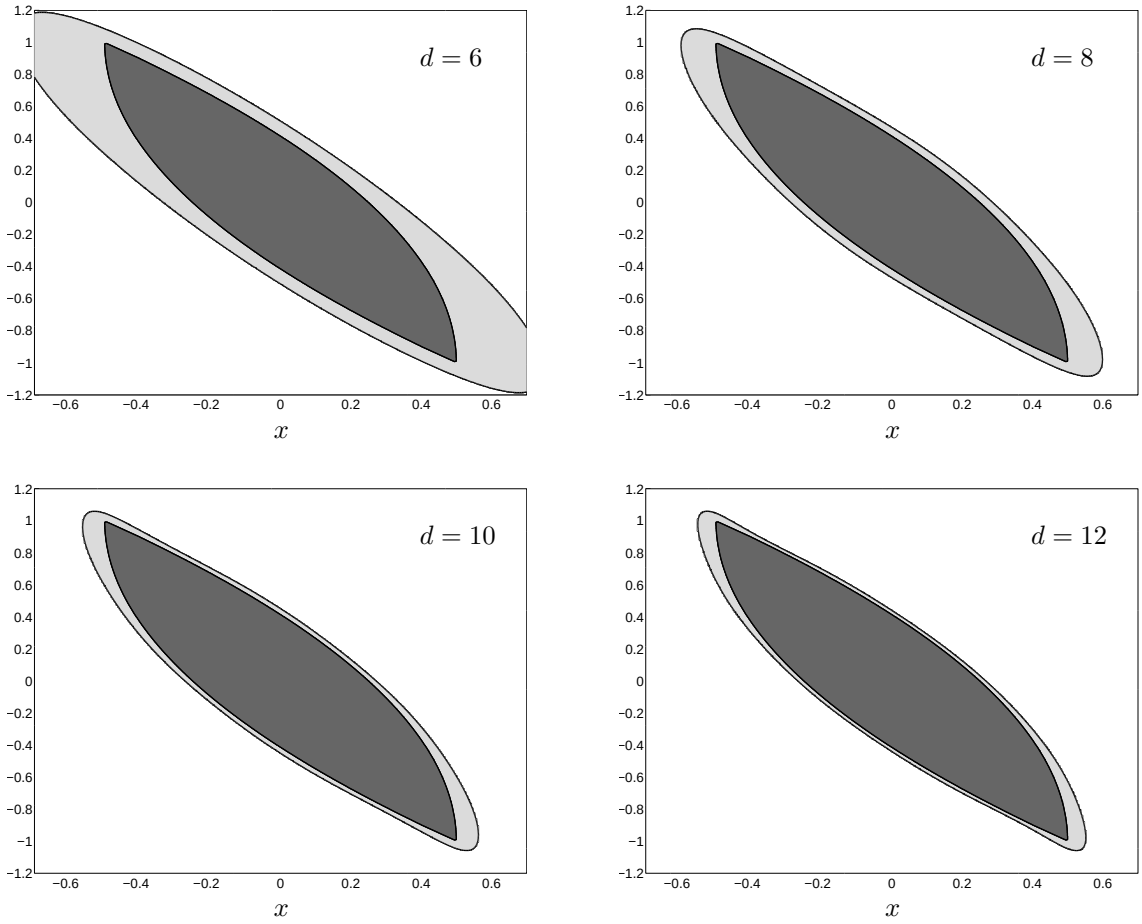

Figure 4: Double integrator – polynomial outer approximations (light gray) to the ROA (dark gray) for degrees $d \in \{6, 8, 10, 12\}$.

9 Conclusion

The main contributions of this paper can be summarized as follows:

- contrary to most of the existing systems control literature, we propose a convex formulation for the problem of computing the controlled region of attraction;
- our approach is constructive in the sense that we rely on standard hierarchies of finite-dimensional LMI relaxations whose convergence can be guaranteed theoretically and for which public-domain interfaces and solvers are available;
- we deal with polynomial dynamics and semialgebraic input and state constraints, therefore covering a broad class of nonlinear control systems;
- the outer approximation obtained is relatively simple in the sense that it is given by a super-level set of a single polynomial of degree given a priori;
- additional properties (e.g., convexity) of the approximations can be enforced by additional constraints on the polynomial (e.g., Hessian being negative definite).

Table 3: Double integrator – relative error of the outer approximation to the ROA X_0 as a function of the approximating polynomial degree.

degree	6	8	10	12
error	78.9 %	34.6 %	21.1 %	16.6 %

The problem of computing the reachability set, i.e. the set of all states that can be reached from a given set of initial conditions under input and state constraints, can be addressed with the same techniques. Basically, the initial and final measures exchange roles. Computation of maximum (controlled) invariant sets should also be amenable to our approach. Furthermore, there is a straightforward extension to piecewise polynomial dynamics defined over a semialgebraic partition of the state and input spaces – one measure is then defined for each region of the partition. Our approach should also allow for extensions to discrete-time controlled systems, stochastic systems (either discrete-time controlled Markov processes or controlled SDEs) and/or uncertain systems.

The hierarchy of LMI relaxations described in this paper generates a sequence of nested outer approximations of the ROA, but it should also be possible, using a similar approach, to compute valid inner approximations.

Numerical examples indicate that the choice of monomials as a dense basis for the set of continuous functions on compact sets, while mathematically appropriate (and notationally convenient), is not always satisfactory regarding convergence and quality of the approximations. However, this is not peculiar to ROA computation problems – a similar behavior was already observed when computing the volume (and moments) of semialgebraic sets in [15]. To achieve better performance, we recommend the use of alternative polynomial bases such as (appropriately scaled tensor products of) Chebyshev polynomials.

Appendix A

In this Appendix we state and prove the correspondence between the Liouville PDE on measures (9) and the convexified differential inclusion (11). Let $\bar{\mu}(t,x)$ denote the (t,x)-marginal of the occupation measure μ defined through (6), that is,

$$\bar{\mu}(A \times B) := \mu(A \times B \times U) \quad \forall\, A \subset [0,T],\ B \subset X.$$

Lemma 4 *Let (μ_0, μ, μ_T) be a triplet of measures satisfying the Liouville equation (9) such that* $\operatorname{spt}\mu_0 \subset X$, $\operatorname{spt}\mu \subset [0,T] \times X \times U$ *and* $\operatorname{spt}\mu_T \subset X_T$. *Then there exists a family of absolutely continuous admissible trajectories of (11) starting from μ_0 (i.e., trajectories in $\bar{\mathcal{X}}(x_0)$) such that the occupation measure and the terminal measure generated by this family of trajectories are equal to $\bar{\mu}$ and μ_T, respectively.*

Proof: First, disintegrate the occupation measure μ as $d\mu(t,x,u) = d\nu(u \mid t,x) d\bar{\mu}(t,x)$, where $d\nu(u \mid t,x)$ is a stochastic kernel, i.e. a probability measure on U for each (t,x) which is measurable in (t,x) for a fixed first argument. Then we can rewrite equation (8)

as

$$\begin{aligned}\int_{X_T} v(T,\cdot)\,d\mu_T &= \int_X v(0,\cdot)\,d\mu_0 + \int_{[0,T]\times X}\int_U \frac{\partial v}{\partial t} + \operatorname{grad} v\cdot f(t,x,u)\,d\nu(u\,|\,t,x)\,d\bar{\mu}(t,x)\\ &= \int_X v(0,\cdot)\,d\mu_0 + \int_{[0,T]\times X} \frac{\partial v}{\partial t} + \operatorname{grad} v\cdot\Big[\int_U f(t,x,u)\,d\nu(u\,|\,t,x)\Big]\,d\bar{\mu}(t,x)\\ &= \int_X v(0,\cdot)\,d\mu_0 + \int_{[0,T]\times X} \frac{\partial v}{\partial t} + \operatorname{grad} v\cdot \bar{f}(t,x)\,d\bar{\mu}(t,x), \end{aligned} \tag{28}$$

where

$$\bar{f}(t,x) := \int_U f(t,x,u)\,d\nu(u\,|\,t,x) \in \operatorname{conv} f(t,x,U).$$

Therefore we will study the trajectories of the differential equation

$$\dot{x}(t) = \bar{f}(t,x(t)). \tag{29}$$

In the remainder of the proof we show that the measures μ_T and $\bar{\mu}$ are generated by a family of absolutely continuous trajectories of this differential equation (which is clearly a subset of trajectories of the convexified inclusion (11)) starting from μ_0. Note that the vector field $\bar{f}$ is only known to be measurable, so this equation may not admit a unique solution.

Observe that the t-marginal of μ (and hence of $\bar{\mu}$) is equal to the Lebesgue measure restricted to $[0,T]$ scaled by $\rho := \mu_0(X)$ $(=\mu_T(X))$ – this follows by plugging in the family of test functions $v(t,x) = t^k$, $k \in \mathbb{N}$, into equation (8). Therefore we can disintegrate $\bar{\mu}$ as

$$d\bar{\mu}(t,x) = d\mu_t(x)dt, \tag{30}$$

where $d\mu_t(x)$ is a stochastic kernel on X given t scaled by ρ and dt is the standard Lebesgue measure. The kernel μ_t can be thought of as the distribution[7] of the state at time t. The kernel μ_t is defined uniquely dt-almost everywhere, and we will show that there is a version such that the function $t \mapsto \int_X w(x)\,d\mu_t(x)$ is absolutely continuous for all $w \in C^1(X)$ and such that the continuity equation

$$\frac{d}{dt}\int_X w(x)\,d\mu_t(x) = \int_X \operatorname{grad} w(x)\cdot \bar{f}(t,x)\,d\mu_t(x) \quad \forall\, w \in C^1(X) \tag{31}$$

is satisfied almost everywhere on $[0,T]$ with the initial condition μ_0.

Fix $w \in C^1(X)$ and define the test function $v(t,x) := \psi(t)w(x)$, where $\psi \in C^1([0,T])$. Then from equation (28)

$$\begin{aligned}\int_{X_T} \psi(T)w\,d\mu_T &= \int_X \psi(0)w\,d\mu_0 + \int_{[0,T]\times X} \frac{\partial(\psi w)}{\partial t} + \operatorname{grad}(\psi w)\cdot\bar{f}(t,x)\,d\bar{\mu}(t,x)\\ &= \psi(0)\int_X w\,d\mu_0 + \int_0^T\int_X \dot{\psi}(t)w(x) + \psi(t)\operatorname{grad} w(x)\cdot\bar{f}(t,x)\,d\mu_t(x)dt\\ &= \psi(0)\int_X w\,d\mu_0 + \int_0^T \Big[\dot{\psi}\int_X w\,d\mu_t + \psi\int_X \operatorname{grad} w\cdot\bar{f}\,d\mu_t\Big]dt,\end{aligned}$$

[7]It will become clear from the following discussion that for $t = 0$ and $t = T$ this kernel (or a version thereof) coincides with μ_0 and μ_T, respectively; hence there is no ambiguity in notation.

which can be seen as an equation of the form

$$\psi(T)d = \psi(0)c + \int_0^T \dot{\psi}(t)a(t) + \psi(t)b(t)\,dt \quad \forall \psi \in C^1([0,T]), \tag{32}$$

where $c := \int_X w(x)\,d\mu_0(x)$, $d := \int_{X_T} w(x)\,d\mu_T$ and $b(t) := \int_X \operatorname{grad} w \cdot \bar{f}(t,x)\,d\mu_t(x)$ are constants and $a(t)$ is an unknown function. One solution is clearly $a(t) = \int_X w\,d\mu_t$. Now we show that

$$\tilde{a}(t) := c + \int_0^t b(\tau)\,d\tau = \int_X w\,d\mu_0 + \int_0^t \int_X \operatorname{grad} w \cdot \bar{f}\,d\mu_\tau d\tau$$

also solves the equation. Indeed, since from (28) with v replaced by w we have $\tilde{a}(T) = \int_X w\,d\mu_T = d$, integration by parts gives

$$\int_0^T \dot{\psi}(t)\tilde{a}(t)\,dt = \psi(T)d - \psi(0)c - \int_0^T \psi(t)b(t)\,dt,$$

so $\tilde{a}(t)$ indeed solves equation (32). Now we prove that this solution is unique. Since $\tilde{a}$ is a solution we have

$$\psi(T)d = \psi(0)c + \int_0^T \dot{\psi}(t)\tilde{a}(t) + \psi(t)b(t)\,dt,$$

and subtracting this from (32) we get

$$0 = \int_0^T \dot{\psi}(t)[a(t) - \tilde{a}(t)]\,dt \quad \forall\, \psi \in C^1([0,T]),$$

or equivalently

$$0 = \int_0^T \phi(t)[a(t) - \tilde{a}(t)]\,dt \quad \forall\, \phi \in C([0,T]).$$

Since $C([0,T])$ is dense in $L^1([0,T])$, this implies $a(t) = \tilde{a}(t)$ dt-almost everywhere. Consequently, since $C^1(X)$ is separable,

$$\int_X w\,d\mu_t = \int_X w(x)d\mu_0 + \int_0^t \int_X \operatorname{grad} w \cdot \bar{f}\,d\mu_\tau d\tau \quad \forall\, w \in C^1(X) \tag{33}$$

dt-almost everywhere. The right-hand side of this equality is an absolutely continuous function of time for each $w \in C^1(X)$ and the left-hand side is a bounded positive linear functional on $C(X)$ for all $t \in [0,T]$. By continuity, the right-hand side is a bounded positive linear functional on $C^1(X)$ for all $t \in [0,T]$ which can be uniquely extended to a bounded positive linear function on $C(X)$ (since C^1 is dense in C). Therefore, for all $t \in [0,T]$ the right-hand side has a representing measure and hence there is a version of μ_t such that the equality (33) holds for *all* $t \in [0,T]$. With this version of μ_t the function $t \mapsto \int_X w(x)\,d\mu_t(x)$ is absolutely continuous and μ_t solves the continuity equation (31).

To finish the proof, we use [1, Theorem 3.2] which asserts the existence of a nonnegative measure σ on $C([0,T];\mathbb{R}^n)$ which corresponds to a family of absolutely continuous

solutions to ODE (29) whose projection at each time $t \in [0,T]$ coincides with μ_t. More precisely, there is a nonnegative measure $\sigma \in M(C([0,T];\mathbb{R}^n))$ supported on a family of absolutely continuous solutions to ODE (29) such that for all measurable $w : \mathbb{R}^n \to \mathbb{R}$

$$\int_X w(x)\mu_t(x) = \int_{C([0,T];\mathbb{R}^n)} w(x(t))\, d\sigma(x(\cdot)) \quad \forall\, t \in [0,T]. \tag{34}$$

It follows from (30) that

$$\bar{\mu}(A \times B) = \int_{[0,T]\times X} I_A(t) I_B(x)\, d\bar{\mu}(t,x) = \int_0^T I_A(t) \int_X I_B(x)\, d\mu_t(x)\, dt.$$

Therefore, using (34) with $w(x) = I_B(x)$ and Fubini's theorem, we get

$$\bar{\mu}(A \times B) = \int_{C([0,T];\mathbb{R}^n)} \int_0^T I_{A\times B}(t, x(t))\, dt\, d\sigma(x(\cdot)),$$

and so the occupation measure of the family of trajectories coincides with $\bar{\mu}$. Clearly, the initial and the final measures of this family coincide with μ_0 and μ_T as well. As a result σ-almost all trajectories of this family are admissible. The proof is completed by discarding the null-set of trajectories that are not admissible, which does not change the measure σ and the generated measures $\bar{\mu}$, μ_0, μ_T. $\square$

Appendix B

In this Appendix we elaborate further on the discussion from Section 3.2 on the connection between the classical ROA and the relaxed ROA. Let us recall the definition of the classical ROA

$$X_0 := \{x_0 \in X \,:\, \mathcal{X}(x_0) \neq \emptyset\},$$

where

$$\mathcal{X}(x_0) := \{x(\cdot) : \dot{x}(t) \in f(t, x(t), U),\ x(0) = x_0,\ x(t) \in X,\ x(T) \in X_T\},$$

and $x(\cdot)$ is required to be absolutely continuous. Similarly, recall the definition of the relaxed ROA

$$\bar{X}_0 := \{x_0 \in X \,:\, \bar{\mathcal{X}}(x_0) \neq \emptyset\},$$

where

$$\bar{\mathcal{X}}(x_0) := \{x(\cdot) : \dot{x}(t) \in \operatorname{conv} f(t, x(t), U),\ x(0) = x_0,\ x(t) \in X,\ x(T) \in X_T\}$$

with $x(\cdot)$ absolutely continuous. Obviously, it holds

$$X_0 \subset \bar{X}_0, \tag{35}$$

and the question is whether this inclusion is strict or not.

Denote $B_\epsilon(a) := \{x \in \mathbb{R}^n \,:\, ||x-a||_2 < \epsilon\}$ and define the dilated constraint sets

$$X^\epsilon := X \oplus B_\epsilon(0) \quad \text{and} \quad X_T^\epsilon := X_T \oplus B_\epsilon(0),$$

where $\oplus$ denotes the Minkowski sum of two sets. Accordingly, the dilated ROA and the dilated relaxed ROA are

$$\begin{array}{rcl} X_0^\epsilon & := & \{x_0 \in X \,:\, \mathcal{X}^\epsilon(x_0) \neq \emptyset\}, \\ \bar{X}_0^\epsilon & := & \{x_0 \in X \,:\, \bar{\mathcal{X}}^\epsilon(x_0) \neq \emptyset\}, \end{array}$$

where

$$\begin{array}{rcl} \mathcal{X}^\epsilon(x_0) & := & \{x(\cdot) : \dot{x}(t) \in f(t,x(t),U),\ x(0)=x_0,\ x(t) \in X^\epsilon,\ x(T) \in X_T^\epsilon\}, \\ \bar{\mathcal{X}}^\epsilon(x_0) & := & \{x(\cdot) : \dot{x}(t) \in \operatorname{conv} f(t,x(t),U),\ x(0)=x_0,\ x(t) \in X^\epsilon,\ x(T) \in X_T^\epsilon\}. \end{array}$$

Since the constraint sets are compact and the vector field f Lipschitz, it follows from the equivalence between the trajectories of the convexified inclusion (11) and solutions to the Liouville equation (9), stated in Lemma 4 of Appendix A, and from Filippov-Ważewski's relaxation Theorem (see, e.g., [5]) that

$$\bar{X}_0 = \operatorname{spt} \mu_0 \ \subset \bigcap_{\epsilon>0} X_0^\epsilon.$$

In contrast, for all $\epsilon > 0$ it holds

$$X_0^\epsilon = \bar{X}_0^\epsilon.$$

In general inclusion (35) is strict. However, we argue that for most practical purposes the relaxed ROA $\bar{X}_0$ and the true ROA X_0 are the same. Indeed, for any $x_0 \in \bar{X}_0$ there exists a sequence of admissible control functions $u_k(\cdot)$ such that

$$\sup_{t\in[0,T]} \operatorname{dist}_X(x_k(t)) \to 0 \quad \text{and} \quad \operatorname{dist}_{X_T}(x_k(T)) \to 0 \quad \text{as} \quad k \to \infty,$$

where $x_k(\cdot)$ denotes the solution to the ODE (1) corresponding to the control function $u_k(\cdot)$, and $\operatorname{dist}_A(x) := \inf\{\|z-x\|_2 \,:\, z \in A\}$ denotes the distance to a set A.

Appendix C

In this Appendix we describe two contrived examples of control systems (1) for which the relaxed ROA $\bar{X}_0$ is strictly larger than the classical ROA X_0; see Appendix B for definitions.

Let $f(t,x,u) = u$, $U = \{-1,+1\}$, $X = X_T = \{0\}$ for, e.g., $T = 1$. Obviously there is no admissible trajectory in $\mathcal{X}(0)$, whereas there is a feasible triplet of measures satisfying (9) given by $\mu_0 = \delta_0$, $\mu_T = \delta_0$ and $\mu = \lambda_{[0,1]} \otimes \delta_0 \otimes \frac{1}{2}(\delta_{-1} + \delta_{+1})$, where $\lambda_{[0,1]}$ denotes the restriction of the Lebesgue measure to $[0,1]$. Therefore in this case $X_0 = \emptyset \neq \bar{X}_0 = \{0\}$, but $\lambda(X_0) = \lambda(\bar{X}_0)$. Assumption 1 is therefore satisfied. Note that the relaxed solution corresponds to an infinitely fast chattering of the control input between -1 and

+1 which can be arbitrarily closely approximated by chattering solutions of finite speed; the singleton constraint set X, however, renders such solutions infeasible.

Another example for which the gap (e.g., in volume) between $\bar{X}_0$ and X_0 can be as large as desired is the following. Consider $\dot{x} = u \in \mathbb{R}^2$ with

$$x \in X := B_R((-1-R,0)) \cup [-1,1] \times \{0\} \cup B_1((+2,0)) \subset \mathbb{R}^2$$

for a given $R > 0$ and with $u \in U := \{-1,1\}^2$ and $X_T := B_1((+2,0))$. Then $X_0 = X_T$ is strictly smaller than $\bar{X}_0 = X$, and $\lambda(X_0) = \pi$, whereas $\lambda(\bar{X}_0) = (1+R^2)\pi$. Assumption 1 is therefore not satisfied for $R > 0$. In this example, regular solutions starting in the left ball cannot transverse the line to the right ball; this is, by contrast, possible for the relaxed solutions using an infinitely fast chattering.

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