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# Main result on self concordant Perceptron.

Adrien CHAN-HON-TONG

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## Abstract

This paper is a summary of paper hal-02491694 focusing on the main theorem.

## 1 Claim

The paper hal-02491694 focuses on the problem of:

*Finding  $x$  such that  $Ax > \mathbf{0}$  for a given matrix  $A \in \mathbb{Z}^{M \times N}$  with the prior that one solution to this set of strict inequality exists.*

which is linked with general linear programming (see the complete paper for details).

**definition:** Let introduce the self concordant function:

$$F(v) = \frac{v^T A A^T v}{2} - \mathbf{1}^T \log(v) = \sum_{i,j=1}^M v_i v_j \times A_i A_j^T - \sum_{m=1}^M \log(v_m) \quad (1)$$

**theorem:**  $A \in \mathbb{Q}^{M \times N}$  a normalized linear feasibility instance with  $Ax \geq \mathbf{1}$

- $F(\frac{1}{M}\mathbf{1}) \leq 1 + M \log(M)$
- $F$  has a minimum (let write it  $F^*$ ) with  $-F^* \leq M \log(x^T x)$
- for all  $v$ ,  $F(v) - F^* \leq \frac{1}{2Mx^T x + 2} \Rightarrow A A^T v > \mathbf{0}$

**trivial corollary:** As  $F$  is self concordant [2], damped Newton descent, starting from any  $v_{start}$ , builds  $v$  such that  $F(v) - F^* \leq \frac{1}{2Mx^T x + 2}$  in less than  $\tilde{O}(F(v_{start}) - F^* + \log \log(2Mx^T x))$  steps (if  $F$  has a minimum).

Yet, the main theorem gives exactly the required information:  $F$  has a minimum, and, thus, damped Newton descent, starting from  $\frac{1}{M}\mathbf{1}$ , builds a solution of the linear feasibility in less than  $\tilde{O}(M \log(x^T x) + \log \log(2Mx^T x)) = \tilde{O}(M \log(x^T x))$  steps.

Yet,  $\log(x^T x) = \tilde{O}(L)$  where  $L$  is the binary size of the input matrix (see complete paper). This finally gives a  $\tilde{O}(ML)$  number of steps, with each step being the resolution of a  $M \times M$  linear system (which can be done in  $M^\omega$  arithmetic operation with  $\omega = 3$  with simple algorithm or with  $\omega \approx 2.38$  with [1]).

**comparison with state of the art:** Arithmetic time complexity (operations on  $\mathbb{Q}$  are counted as 1) of this algorithm is  $\tilde{O}(M^\omega ML)$  which is higher than  $\tilde{O}(M^\omega \sqrt{ML})$  for central path log barrier [2] or path following [3].

Yet, this algorithm has some interesting binary complexity features detailed in hal-02491694: [2] has the drawback that some variable becomes very high ( $\tilde{O}(2^{\sqrt{ML}})$ ) while both self concordant Perceptron and [3] avoid this issue. But, [3] requires a careful rounding process based on several implicit constant while self concordant Perceptron has a simple explicit one (but with the drawback of having an higher complexity).

## 2 Proof of the claim

### 2.1 Existence of a minimum

Let recall Cauchy inequality:  $\forall u, v$  two vectors  $u^T v \leq \sqrt{u^T u \times v^T v}$ .

As  $A$  is normalized, then for all  $m$ ,  $A_m A_m^T = 1$ . So from Cauchy,  $\forall i, j$ ,  $A_i A_j^T \leq 1$ . So,  $F(\frac{1}{M} \mathbf{1}) = \frac{1}{M^2} \sum_{i,j} A_i A_j^T + M \log(M) \leq 1 + M \log(M)$ .

Then, again from Cauchy  $(A^T v)^T x \leq \sqrt{v^T A A^T v \times x^T x}$ . But  $(A^T v)^T x = v^T (Ax)$ . And, by definition  $Ax \geq \mathbf{1}$ . Injecting this last inequality is interesting when  $v \geq \mathbf{0}$ :  $\forall v \geq \mathbf{0}$ ,  $v^T \mathbf{1} \leq v^T (Ax) = (A^T v)^T x \leq \sqrt{v^T A A^T v \times x^T x}$ .

Also when  $v \geq \mathbf{0}$ , then, both  $v^T (Ax) = (A^T v)^T x$  and  $\sqrt{v^T A A^T v \times x^T x}$  are positive. So, one can take the square:  $\forall v \geq \mathbf{0}$ ,  $\frac{(v^T \mathbf{1})^2}{x^T x} \leq v^T A A^T v$ . And, independently  $(v^T \mathbf{1})^2 > v^T v$  because  $v \geq \mathbf{0}$ . So  $\forall v \geq \mathbf{0}$ ,  $\frac{v^T v}{x^T x} \leq v^T A A^T v$ .

Let introduce  $f(t) = \frac{t^2}{2x^T x} - \log(t)$ , from previous inequality it stands that  $F(v) \geq \sum_m f(v_m)$ .

Now,  $f$  is a single variable function which goes to infinity when  $t$  goes to 0 ( $t^2 \rightarrow 0$  but  $-\log(t) \rightarrow \infty$ ) or to infinity ( $t^2$  grows faster than  $\log(t)$ ). So,  $f$  has a minimum and so  $F$  too. Let call them  $f^*$  and  $F^*$ .

As  $f$  and  $F$  are smooth, the minimums are characterized by a null derivative or gradient.  $f'(t) = \frac{t}{x^T x} - \frac{1}{t}$ , so,  $f'(\sqrt{x^T x}) = 0$ , so  $f^* = f(\sqrt{x^T x}) = \frac{1}{2} - \frac{1}{2} \log(x^T x) \geq -\log(x^T x)$ . Thus, the minimum of  $F$  verifies  $F^* \geq M f^* \geq -M \log(x^T x)$ .

So the two first assertions of the main theorem are proven.

### 2.2 Normalization, linearization and lemmas

Independently, let remark that  $\theta(t) = F(tv) = \frac{v^T A A^T v}{2} t^2 - \mathbf{1}^T \log(v) - M \log(t)$  is minimal when  $v^T A A^T v = M$ . So for any  $w$ , one could build a  $v = \mu w$  such that  $v^T A A^T v = M$  and  $F(v) \leq F(w)$ . In other words, it stands that  $F\left(\sqrt{\frac{M}{v^T A A^T v}} v\right) \leq F(v)$ .

So, let consider  $v \geq \mathbf{0}$  such that  $v^T A A^T v = M$ . As,  $v^T A A^T v \geq \frac{(\mathbf{1}v)^2}{x^T x}$ , no  $v_m$  could be higher than  $\sqrt{M x^T x}$  i.e.  $\mathbf{0} \leq v \leq \sqrt{M x^T x} \mathbf{1}$ .

Let also remark that  $F(v+w) = \frac{v^T A A^T v}{2} + \frac{w^T A A^T w}{2} + w^T A A^T v - \mathbf{1}^T \log(v) - \mathbf{1}^T \log(1 + \frac{w}{v}) = F(v) + \frac{w^T A A^T w}{2} + w^T A A^T v - \mathbf{1}^T \log(1 + \frac{w}{v})$

Finally, let consider the following lemmas from basic analysis:

1.  $\phi(t) = \frac{1}{2}\alpha t^2 - \log(1+t) \leq \frac{1}{2}(\alpha+1)t^2 - t = \psi(t)$  for  $t \geq 0$
2.  $\psi(\frac{1}{\alpha+1}) \leq -\frac{1}{2\alpha+2}$
3.  $\phi(\frac{1}{\alpha+1}) \leq -\frac{1}{2\alpha+2}$  i.e.  $\forall \alpha \geq 0, \frac{1}{2} \frac{\alpha}{(\alpha+1)^2} - \log(1 + \frac{1}{\alpha+1}) \leq -\frac{1}{2\alpha+2}$

**Lemma1:**  $\psi'(t) - \phi'(t) = (\alpha+1)t - 1 - \alpha t + \frac{1}{1+t} = t - 1 + \frac{1}{1+t} = \frac{t^2}{1+t} > 0$ , so  $\psi(t) - \phi(t)$  always increases. But,  $\psi(0) = \phi(0) = 0$  so  $\psi(t) \geq \phi(t)$  for  $t \geq 0$ . **Lemma2:**  $\psi(\frac{1}{\alpha+1}) = \frac{1}{2}(\alpha+1)\frac{1}{(\alpha+1)^2} - \frac{1}{\alpha+1} = -\frac{1}{2\alpha+2}$ . lemma3 is just lemma1+lemma2.

### 2.3 Convergence

Now, either  $AA^T v > \mathbf{0}$  (problem solved) or there exists  $k$  such that  $A_k A^T v \leq 0$ .

Let consider this case  $A_k A^T v \leq 0$  and  $v^T AA^T v = M$ , and, let introduce  $w = v + \frac{vk}{v_k^2+1} \mathbf{1}_k$ .

Then  $F(w) = F(v + \frac{vk}{v_k^2+1} \mathbf{1}_k) = F(v) + \frac{A_k A_k^T}{2} (\frac{vk}{v_k^2+1})^2 + A_k A^T v \times \frac{vk}{v_k^2+1} - \log(1 + \frac{1}{v_k^2+1})$ . But,  $A_k A^T v \leq 0$  (by assumption) and  $A_k A_k^T = 1$ , so  $F(w) \leq F(v) + \frac{1}{2} (\frac{vk}{v_k^2+1})^2 - \log(1 + \frac{1}{v_k^2+1})$ .

And, from lemmas just above (consider  $\alpha = v_k^2$ ),  $F(w) \leq F(v) - \frac{1}{2v_k^2+2}$ .

But,  $v_k \leq \sqrt{Mx^T x}$ , so,  $F(w) \leq F(v) - \frac{1}{2Mx^T x+2}$  which is impossible if  $F(v) - F^* < \frac{1}{2Mx^T x+2}$ . So,  $\forall v > \mathbf{0}$  such that  $v^T AA^T v = M$ ,  $F(v) - F^* \leq \frac{1}{2Mx^T x+2} \Rightarrow AA^T v > \mathbf{0}$ .

Finally, the requirement that  $v^T AA^T v = M$  could be remove because normalizing decreases  $F$ :  $\forall v > \mathbf{0}, F(v) - F^* \leq \frac{1}{2Mx^T x+2} \Rightarrow F(\sqrt{\frac{M}{v^T AA^T v}} v) - F^* \leq \frac{1}{2Mx^T x+2} \Rightarrow \sqrt{\frac{M}{v^T AA^T v}} \times AA^T v > \mathbf{0} \Rightarrow AA^T v > \mathbf{0}$ .

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