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Main result on self concordant Perceptron.

Adrien CHAN-HON-TONG

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Abstract

This paper is a summary of paper hal-02491694 focusing on the main theorem.

1 Claim

The paper hal-02491694 focuses on the problem of

Finding x such that $Ax > \mathbf{0}$ for a given matrix $A \in \mathbb{Z}^{M \times N}$ such that one such x exists.

which is linked with general linear programming (see the complete paper for details).

definition: Let introduce the self concordant function:

$$F(v) = \frac{v^T A A^T v}{2} - \mathbf{1}^T \log(v) = \sum_{i,j=1}^M v_i v_j \times A_i A_j^T - \sum_{m=1}^M \log(v_m) \quad (1)$$

theorem: $A \in \mathbb{Q}^{M \times N}$ a normalized linear feasibility instance with $Ax \geq \mathbf{1}$

- $F(\frac{1}{M}\mathbf{1}) \leq 1 + M \log(M)$
- F has a minimum (let write it F^*) with $-F^* \leq M \log(x^T x)$
- for all v , $F(v) - F^* \leq \frac{1}{2Mx^T x + 2} \Rightarrow A A^T v > \mathbf{0}$

2 Proof of the claim

2.1 Existence of a minimum

Let recall Cauchy inequality: $\forall u, v$ two vectors $u^T v \leq \sqrt{u^T u \times v^T v}$.

As A is normalized, then for all m , $A_m A_m^T = 1$. So from Cauchy, $\forall i, j$, $A_i A_j^T \leq 1$. So, $F(\frac{1}{M}\mathbf{1}) = \frac{1}{M^2} \sum_{i,j} A_i A_j^T + M \log(M) \leq 1 + M \log(M)$.

Then, again from Cauchy $(A^T v)^T x \leq \sqrt{v^T A A^T v \times x^T x}$. But $(A^T v)^T x = v^T (Ax)$. And, by definition $Ax \geq \mathbf{1}$. Injecting this last inequality is interesting when $v \geq \mathbf{0}$: $\forall v \geq \mathbf{0}$, $v^T \mathbf{1} \leq v^T (Ax) = (A^T v)^T x \leq \sqrt{v^T A A^T v \times x^T x}$.

Also when $v \geq \mathbf{0}$, then, both $v^T(Ax) = (A^T v)^T x$ and $\sqrt{v^T AA^T v \times x^T x}$ are positive. So, one can take the square: $\forall v \geq \mathbf{0}, \frac{(v^T \mathbf{1})^2}{x^T x} \leq v^T AA^T v$. And, independently $(v^T \mathbf{1})^2 > v^T v$ because $v \geq \mathbf{0}$. So $\forall v \geq \mathbf{0}, \frac{v^T v}{x^T x} \leq v^T AA^T v$.

Let introduce $f(t) = \frac{t^2}{2x^T x} - \log(t)$, from previous inequality it stands that $F(v) \geq \sum_m f(v_m)$.

Now, f is a single variable function which goes to infinity when t goes to 0 ($t^2 \rightarrow 0$ but $-\log(t) \rightarrow \infty$) or to infinity (t^2 grows faster than $\log(t)$). So, f has a minimum and so F too. Let call them f^* and F^* .

As f and F are smooth, the minimums are characterized by a null derivative or gradient. $f'(t) = \frac{t}{x^T x} - \frac{1}{t}$, so, $f'(\sqrt{x^T x}) = 0$, so $f^* = f(\sqrt{x^T x}) = \frac{1}{2} - \frac{1}{2} \log(x^T x) \geq -\log(x^T x)$. Thus, the minimum of F verifies $F^* \geq M f^* \geq -M \log(x^T x)$.

So the two first assertions of the main theorem are proven.

2.2 Normalization, linearization and lemmas

Independently, let remark that $\theta(t) = F(tv) = \frac{v^T AA^T v}{2} t^2 - \mathbf{1}^T \log(v) - M \log(t)$ is minimal when $v^T AA^T v = M$. So for any w , one could build a $v = \mu w$ such that $v^T AA^T v = M$ and $F(v) \leq F(w)$. In other words, it stands that $F\left(\sqrt{\frac{M}{v^T AA^T v}} v\right) \leq F(v)$.

So, let consider $v \geq \mathbf{0}$ such that $v^T AA^T v = M$. As, $v^T AA^T v \geq \frac{(\mathbf{1}v)^2}{x^T x}$, no v_m could be higher than $\sqrt{Mx^T x}$ i.e. $\mathbf{0} \leq v \leq \sqrt{Mx^T x} \mathbf{1}$.

Let also remark that $F(v+w) = \frac{v^T AA^T v}{2} + \frac{w^T AA^T w}{2} + w^T AA^T v - \mathbf{1}^T \log(v) - \mathbf{1}^T \log(1 + \frac{w}{v}) = F(v) + \frac{w^T AA^T w}{2} + w^T AA^T v - \mathbf{1}^T \log(1 + \frac{w}{v})$

Finally, let consider the following lemmas from basic analysis:

1. $\phi(t) = \frac{1}{2} \alpha t^2 - \log(1+t) \leq \frac{1}{2}(\alpha+1)t^2 - t = \psi(t)$ for $t \geq 0$
2. $\psi(\frac{1}{\alpha+1}) \leq -\frac{1}{2\alpha+2}$
3. $\phi(\frac{1}{\alpha+1}) \leq -\frac{1}{2\alpha+2}$ i.e. $\forall \alpha \geq 0, \frac{1}{2} \frac{\alpha}{(\alpha+1)^2} - \log(1 + \frac{1}{\alpha+1}) \leq -\frac{1}{2\alpha+2}$

Lemma1: $\psi'(t) - \phi'(t) = (\alpha+1)t - 1 - \alpha t + \frac{1}{1+t} = t - 1 + \frac{1}{1+t} = \frac{t^2}{1+t} > 0$, so $\psi(t) - \phi(t)$ always increases. But, $\psi(0) = \phi(0) = 0$ so $\psi(t) \geq \phi(t)$ for $t \geq 0$. Lemma2: $\psi(\frac{1}{\alpha+1}) = \frac{1}{2}(\alpha+1) \frac{1}{(\alpha+1)^2} - \frac{1}{\alpha+1} = -\frac{1}{2\alpha+2}$. lemma3 is just lemma1+lemma2.

2.3 Convergence

Now, either $AA^T v > \mathbf{0}$ (problem solved) or there exists k such that $A_k A^T v \leq 0$.

Let consider this case $A_k A^T v \leq 0$ and $v^T AA^T v = M$, and, let introduce $w = v + \frac{vk}{v_k^2+1} \mathbf{1}_k$.

Then $F(w) = F(v + \frac{vk}{v_k^2+1} \mathbf{1}_k) = F(v) + \frac{A_k A_k^T}{2} (\frac{vk}{v_k^2+1})^2 + A_k A^T v \times \frac{vk}{v_k^2+1} - \log(1 + \frac{1}{v_k^2+1})$. But, $A_k A^T v \leq 0$ (by assumption) and $A_k A_k^T = 1$, so $F(w) \leq F(v) + \frac{1}{2} (\frac{vk}{v_k^2+1})^2 - \log(1 + \frac{1}{v_k^2+1})$.

And, from lemmas just above (consider $\alpha = v_k^2$), $F(w) \leq F(v) - \frac{1}{2v_k^2+2}$.

But, $v_k \leq \sqrt{Mx^T x}$, so, $F(w) \leq F(v) - \frac{1}{2Mx^T x+2}$ which is impossible if $F(v) - F^* < \frac{1}{2Mx^T x+2}$. So, $\forall v > \mathbf{0}$ such that $v^T AA^T v = M$, $F(v) - F^* \leq \frac{1}{2Mx^T x+2} \Rightarrow AA^T v > \mathbf{0}$.

Finally, the requirement that $v^T AA^T v = M$ could be remove because normalizing decreases F : $\forall v > \mathbf{0}$, $F(v) - F^* \leq \frac{1}{2Mx^T x+2} \Rightarrow F(\sqrt{\frac{M}{v^T AA^T v}} v) - F^* \leq \frac{1}{2Mx^T x+2}$

$\Rightarrow \sqrt{\frac{M}{v^T AA^T v}} \times AA^T v > \mathbf{0} \Rightarrow AA^T v > \mathbf{0}$.

This proves the main theorem.

3 Corollary

As F is self concordant [2], damped Newton descent, starting from any v_{start} , builds v such that $F(v) - F^* \leq \frac{1}{2Mx^T x+2}$ in less than $\tilde{O}(F(v_{start}) - F^* + \log \log(2Mx^T x))$ steps.

But, as proven, using $v_{start} = F(\frac{1}{M}\mathbf{1})$ gives $F(v_{start}) \leq M \log(M) + 1$ and $F^* \geq -M \log(x^T x)$.

So, damped Newton descent, starting from $\frac{1}{M}\mathbf{1}$, builds a solution of the linear feasibility in less than $\tilde{O}(M \log(x^T x) + M \log(M) + \log \log(2Mx^T x)) = \tilde{O}(M \log(x^T x))$ steps.

Yet, as for the Perceptron [3], $x^T x \leq N2^{2NB \log(N)}$ where B is the maximal binary size of any entry of A before normalization. This finally gives a $\tilde{O}(NMB)$ number of steps, which becomes $\tilde{O}(L)$ by reintroducing the total binary size $L = NMB$, with each step being the resolution of a $M \times M$ linear system (which can be done in M^ω with $\omega \approx 2.38$ with [1]).

This is currently the best complexity for linear feasibility, and, it could even have implication for linear programming (see the complete article for details).

References

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