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# *A polynomial-time log barrier algorithm for linear programming with simplex-like features.*

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This paper presents a polynomial-times log barrier algorithm which solves linear programming which has some simplex-like features.

As simple log barrier, this algorithm is probably less efficient than central path log barrier but still polynomial times. But, it is suitable for hybridization.

**Keywords:** Linear programming, polynomial times log barrier, simplex, hybridization

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## 1 Introduction

Linear programming is the very studied task of optimizing a linear criterion under linear equality and inequality constraints. This problem has been first tackled by exponential algorithms like simplex Dantzig (1955) or perceptron Rosenblatt (1958). The first polynomial time algorithm solving this problem is the ellipsoid method Khachiyan (1979); Gr tschel et al. (1981) which is mostly theoretic. Today, the state of the art of polynomial time algorithm for linear programming is central-path log-barrier algorithm Nesterov and Nemirovskii (1994); Nemirovski (2004) (despite the recent Chubanov method Chubanov (2015) could be a challenger).

Yet, it is known that simplex algorithm solves simple instances much more quickly than interior point algorithm (while being exponential on hard ones). Simple instances are typically instance with very low number of variables and/or constraints, and/or non degenerated instances with very low number of vertices (non degeneracy is important Megiddo (1986)). Indeed, in such instances, simplex like algorithms are good both in theory (they are strongly polynomial for such instances), and, in practice, while log barrier is an overkill. In addition, Allamigeon et al. (2018) recently proves a strong negative result on central path log barrier method: they are not strongly polynomial.

Thus, it could be interesting to run simultaneously different algorithms like 1 simplex/perceptron and 1 polynomial-times one (either ellipsoid or log barrier). Each algorithm will run only twice slower than if it was alone, but, the resulting portfolio algorithm terminates as the faster ones. In particular, such portfolio algorithm is polynomial-times in all cases (thank to ellipsoid/log barrier) while being as fast as simplex on simple instances.

Now, if both algorithms could help each other, it could produce an even better ones. Yet, it seems not trivial to make simplex/perceptron and ellipsoid/log barrier helping each other. For example, it seems impossible to help the ellipsoid algorithm. As the only invariant of the algorithm is that the current

ellipsoid contains a solution, moving the center even in relevant direction could break the invariant (or required to increase the volume of the ellipsoid which is as bad).

The contribution of this paper is to prove that it is possible to design a log barrier algorithm (polynomial even if probably not competitive alone with central path log barrier) which has some simplex like features, and, and thus which seems suitable for hybridation with a simplex/perceptron like algorithm. This statement is clarified in next section which also presents the algorithm. Then, a discussion section presents the difference with classical log barrier.

## Notation

The set of matrices of size  $M \times N$  on real numbers is written  $\mathbb{R}^{M \times N}$  (the same with  $\mathbb{Q}$  for rational numbers or  $\mathbb{Z}$  for integer numbers). If  $A$  is a matrix from  $\mathbb{R}^{M \times N}$ , then, the transposed matrix is written  $A^T \in \mathbb{R}^{N \times M}$ . Also,  $A_i$  is the row  $i$  of  $A$  considered as a  $1 \times N$  matrix. If  $I$  is the set of indexes,  $A_I$  is the submatrix when keeping only row  $i$  from  $I$ . Vectors of dimension  $N$  are considered as matrix with size  $N \times 1$ , but, matrices  $1 \times 1$  are considered as number: so, if  $v$  is a vector  $v_i$  is the value of component  $i$ .  $\mathbf{0}$ ,  $\mathbf{1}$  are the vectors with all components being 0 or 1, and,  $\mathbf{b}_i$  the  $i$  vector of the natural basis i.e. all components are 0 except component  $i$  with is 1.

The set of  $M \times N$  matrix is a vectorial space with an addition written  $+$ , and, a product between a scalar and a matrix written  $\times$ . Independently, the set of all matrices of any size has a product (not scalar) which is represented by juxtaposition of matrices: if  $A \in \mathbb{R}^{I \times J}$ , and,  $B \in \mathbb{R}^{J \times K}$ , then,  $AB$  is the matrix product of  $A$  and  $B$  which is in  $\mathbb{R}^{I \times K}$ . Thus, the scalar product between two vectors  $u$  and  $v$  is written  $u^T v$  (no related to the product of a scalar  $\lambda$  and vector/matrix written with  $\times$ ).

$Det(A)$  is the maximal determinant of any submatrix  $A_I$  with size of  $I$  being the number of columns. And, the kernel of a matrix  $A$  is  $Ker(A) = \{x \in \mathbb{Q}^N / Ax = \mathbf{0}\}$  ( $A$  can be not square).

Finally, if  $f$  is a derivable function of  $z$ ,  $\nabla_z f$  is the gradient of  $f$  in  $z$  and  $\nabla_z^2 f$  is the hessian of  $f$  in  $z$ .

This paper uses Greek letters (except  $\lambda$ ) only to design abstract object (object which could not exist (when proving contradiction), optimal solution which are not decidable, not rational numbers...). Only exception is  $\lambda$  which used consistently with Nemirovski (2004) to design an important quantity in the Newton descent process.

## 2 Decoupled log barrier algorithm

### 2.1 Reduced homogeneous linear feasibility

The offered algorithm requires some preprocessing of the input linear program allowing to cast it in a reduced homogeneous linear feasibility. Importantly, any linear program without any assumption can be casted in such working form (see appendix A). So, this is **not** an assumption on the input, but just a pre processing step.

A reduced homogeneous linear feasibility instance is given by any matrix  $A$  (of arbitrary size  $N$ ,  $M$  i.e.  $A \in \mathbb{Z}^{M \times N}$ ) with  $Ker(A) = \{\mathbf{0}\}$ , and, with the property that there exists  $\psi \in \mathbb{R}^N$  such that  $A\psi \geq \mathbf{1}$ , and, if  $\psi^T \psi$  is written  $\Psi$ , with  $\log(\Psi)$  is bounded by a polynomial in the binary size of  $A$ . The reduced homogeneous linear feasibility task is to find  $x \in \mathbb{Q}^N$  such that  $Ax > \mathbf{0}$  (equivalently  $Ax \geq \mathbf{1}$ ).

## 2.2 Pseudo vertex

For all reduced homogeneous linear feasibility instances  $A \in \mathbb{Z}^{M \times N}$  and vectors  $x \in \mathbb{Q}^N$ , let  $e = \min_{m \in \{1, \dots, M\}} A_m x$  and  $E = \{m \in \{1, \dots, M\} / A_m x = e\} = \arg \min_{m \in \{1, \dots, M\}} A_m x$ .

The set  $E$  is called the **pseudo vertex** of  $x$  regarding  $A$ . This concept of pseudo vertex is somehow a gap between vertex based algorithm (e.g. simplex) and interior point one.

In particular,  $\forall i, j \in E, A_i x = A_j x$ . So,  $x$  is in the kernel of the matrix formed from  $A_E$  after subtracting  $A_j$  (any  $i \in E$  works) to all rows. So, not all subsets of  $\{1, \dots, M\}$  could be pseudo vertex.

Also, let  $K$  be the size of the vertex  $E$ , if there is  $u \in \mathbb{Q}^K$  such that  $A_E A_E^T u = \mathbf{1}$  and  $u \geq \mathbf{0}$ , then,  $u$  is called a **proper greedy improvement**. In particular, if  $A_E A_E^T$  is not singular, there is a proper greedy improvement iff  $(A_E A_E^T)^{-1} \mathbf{1} \geq \mathbf{0}$ .

The interesting point of successive proper greedy improvements (see also section 4) is that it corresponds somehow to the projection of the current point on maximal pseudo vertex. Typically, If  $u$  is a proper greedy improvement of  $x$  regarding  $A$ , then, either  $AA_E^T u > \mathbf{0}$ , or, one could consider  $y = x + l \times A_E^T u$  with  $l = \max_{t \in \mathbb{R}} \min_{m \in \{1, \dots, M\}} A_m(x + t \times A_E^T u)$ . Importantly, let  $E = \arg \min_{m \in \{1, \dots, M\}} A_m x$  and  $F = \arg \min_{m \in \{1, \dots, M\}} A_m y$ , then,  $E$  is a strict subset of  $F$  which is also a pseudo vertex.

## 2.3 Main statement

Now that the notions of reduced homogeneous linear feasibility, and, proper greedy improvement are defined, the main theorem of this paper can be stated:

*Theorem:*

**There exists an algorithm solving the reduced homogeneous linear feasibility problem in polynomial times, characterized by an inner state  $x$  such that, at any point during runtime,  $x$  can be replaced by  $x + lA_E^T u$  with  $E$  the pseudo vertex and  $u$  a proper greedy improvement, and,  $0 \leq l \leq \max_{t \in \mathbb{R}} \min_{m \in \{1, \dots, M\}} A_m(x + t \times u)$  without increasing the complexity of the algorithm.**

*Algorithm:*

Let  $\log(v)$  and  $\frac{1}{v}$  the vectors such that  $\forall m \in \{1, \dots, M\}$  being  $\log(v_m)$  and  $\frac{1}{v_m}$ .

The offered algorithm is a Newton descent on log barrier function  $F(v)$  from  $]0, \infty[^M$  to  $\mathbb{R}$  linked with the inner state  $x(v)$  from  $]0, \infty[^M$  to  $\mathbb{R}^N$  defined by:

$$x(v) = A^T v = \sum_{m=1}^M v_m A_m^T$$

$$F(v) = x(v)^T x(v) - \mathbf{1}^T \log(v)$$

$$= v^T A A^T v - \mathbf{1}^T \log(v) = \sum_{i,j=1}^M v_i v_j \times A_i A_j^T - \sum_{i=1}^M \log(v_i)$$

## 2.4 Proof

### 2.4.1 Minimum of $F$ is a solution

**Lemma:**  $F$  goes to  $\infty$  when any of the  $v_{m \in \{1, \dots, M\}}$  goes to 0 or to  $\infty$ .

From Cauchy, for all  $x$ ,  $\psi^T x \leq \sqrt{\psi^T \psi} \times \sqrt{x^T x} = \sqrt{\Psi} \times \sqrt{x^T x}$ . So,  $\frac{1}{\Psi}(\psi^T x)^2 \leq x^T x$ . Let stress that  $\Psi$  can be assumed higher than 1 (scaling  $\psi$  with a number higher than 1 is possible). In particular, for  $x(v)$ , it gives  $\frac{1}{\Psi}(\psi^T A^T v)^2 \leq x(v)^T x(v)$  but  $A\psi \geq \mathbf{1}$  so  $\frac{1}{\Psi}(\mathbf{1}^T v)^2 \leq x(v)^T x(v)$ . As  $v > \mathbf{0}$ ,  $(\mathbf{1}v)^2 \geq v^T v$ , so,  $\frac{1}{\Psi}v^T v \leq x(v)^T x(v)$ .

So,  $F(v) \geq \frac{1}{\Psi}v^T v - \mathbf{1} \log(v)$ .

Let  $\Phi(t) = \frac{1}{\Psi}t^2 - \log(t)$  from  $]0, \infty[$  to  $\mathbb{R}$ .  $\Phi(t)$  goes trivially to  $\infty$  when  $t$  goes to 0 or  $\infty$ , so,  $\Phi$  has a minimum  $\Phi^*$ .  $\Phi'(t) = \frac{2}{\Psi}t - \frac{1}{t}$ , and,  $\Phi'(t) = 0$  as an unique solution. It leads to  $\Phi^* = \Phi(\sqrt{\frac{\Psi}{2}}) = \frac{1}{2} - \frac{1}{2} \log(\Psi) + \frac{1}{2} \log(2)$ . In particular,  $\Phi^*$  is polynomial in the binary size of  $A$  (due to assumption on  $\log(\Psi)$ ).

Then,  $F(v) \geq \sum_{m \in \{1, \dots, M\}} \Phi(v_m)$ . A fortiori, for all  $m \in \{1, \dots, M\}$ ,  $F(v) \geq (M-1)\Phi^* + \Phi(v_m)$

with  $\Phi(v_m)$  going to infinity is  $v_m$  goes to 0 or infinity.

Independently,  $\nabla_v F = 2 \times Ax(v) - \frac{1}{v}$ .

**Implications:**  $F$  has a minimum  $F^* = F(\nu)$  with  $\nu \in ]0, \infty[^M$ . So,  $\nabla_\nu F = \mathbf{0}$ . So,  $Ax(\nu) = \frac{1}{2\nu} > \mathbf{0}$ .

So, the minimum of  $F$  is a solution to the homogeneous reduced linear feasibility problem (currently,  $\nu$  is even the only root of  $\nabla_v F = \mathbf{0}$ ).

### 2.4.2 Approximating the minimum of $F$ leads to a solution

Let  $\Gamma = \max_{m \in \{1, \dots, M\}} A_m A_m^T$  (let stress that  $\log(\Gamma)$  is polynomial in the binary size of  $A$ ).

**Starting point:**

Let  $v_0 = \left( \frac{1}{\sqrt{A_1 A_1^T}}, \dots, \frac{1}{\sqrt{A_M A_M^T}} \right)^T$ . From Cauchy,  $A_i A_j^T \leq \sqrt{A_i A_i^T} \sqrt{A_j A_j^T}$ , so,  $x(v_0)^T x(v_0) \leq M^2$ , and, so  $F(v_0) \leq M^2 + \frac{1}{2} \sum_{m \in \{1, \dots, M\}} \log(A_m A_m^T) \leq M^2 + M \log(\Gamma)$ . In particular,  $F(v_0)$  is polynomial in the binary size of  $A$ .

**Precise bound on  $v$ :**

Independently, as  $\Phi''(t) \geq 0$ , it holds that  $\forall t, s$  if  $t \geq s$  then  $\Phi(t+s) \geq \Phi(s) + t\Phi'(s)$ . In particular  $\Phi'(\Psi) = 2 - \frac{1}{\Psi} \geq 1$  (as  $\Psi$  can be assumed higher than 1). So,  $\forall t > \Psi$ ,  $\Phi(t+\Psi) \geq \Phi(\Psi) + t = \Psi - \log(\Psi) + t$ .

So, there is two cases:

- either  $\Phi(\Psi) = \Psi - \log(\Psi) \geq F(v_0) - (M-1)\Phi^*$ , then, for any  $k \in \{1, \dots, M\}$ ,  $v_k \geq \Psi$  implies that  $F(v) \geq (M-1)\Phi^* + \Phi(v_k) \geq (M-1)\Phi^* + \Phi(\Psi) \geq (M-1)\Phi^* + F(v_0) - (M-1)\Phi^* \geq F(v_0)$
- either  $\Phi(\Psi) = \Psi - \log(\Psi) \leq F(v_0) - (M-1)\Phi^*$ , then for any  $k \in \{1, \dots, M\}$ ,  $v_k \geq F(v_0) - (M-1)\Phi^* + \log(\Psi)$  implies that  $F(v) \geq (M-1)\Phi^* + \Phi(v_k) = (M-1)\Phi^* + \Phi(F(v_0) - (M-1)\Phi^* + \log(\Psi) - \Psi) + \Psi \geq (M-1)\Phi^* + \Phi(\Psi) + F(v_0) - (M-1)\Phi^* + \log(\Psi) - \Psi = (M-1)\Phi^* + \Psi - \log(\Psi) + F(v_0) - (M-1)\Phi^* + \log(\Psi) - \Psi = F(v_0)$

In both case, if any  $v_k \geq \max(\Psi, F(v_0) - (M-1)\Phi^* + \log(\Psi))$  ( $k \in \{1, \dots, M\}$ ), then,  $F(v) \geq F(v_0)$ .

**Implications:**

Now, let assume that  $F(v) \leq F(v_0)$  and  $F(v) - F^* \leq \varepsilon$  and there is  $k$  such that  $A_k x(v) \leq 0$ .

There is two cases: either  $v_k \geq \frac{1}{4}$  (hard one) or  $v_k \leq \frac{1}{4}$  (easy one).

If  $v_k \geq \frac{1}{4}$ , let consider  $\delta = \frac{1}{4v_k^2 \times A_k A_k^T}$  ( $\delta \in [0, 1]$ ) and  $\omega$  such that  $\omega_m = v_m$  but  $\omega_k = (1 + \delta)v_k$ .

$F(\omega) - F(v) = x(\omega)^T x(\omega) - x(v)^T x(v) - \log(1 + \delta)$  But,  $x(\omega) = x(v) + \delta v_k \times A_k^T$ . So,  $F(\omega) - F(v) = x(v)^T x(v) + 2\delta \times A_k x(v) + \delta^2 v_k^2 \times A_k A_k^T - x(v)^T x(v) - \log(1 + \delta)$ . As,  $A_k x(v) \leq 0$  it means that  $F(\omega) - F(v) \leq \delta^2 v_k^2 \times A_k A_k^T - \log(1 + \delta)$

But  $\forall t \in [0, 1]$ ,  $\frac{t}{2} \leq \log(t + 1)$ , so  $F(\omega) - F(v) \leq \delta^2 v_k^2 \times A_k A_k^T - \frac{\delta}{2} = \frac{1}{16v_k^2 \times A_k A_k^T} - \frac{1}{8v_k^2 \times A_k A_k^T} = -\frac{1}{8v_k^2 \times A_k A_k^T}$

So,  $F(\omega) - F(v) \leq -\frac{1}{8v_k^2 \times A_k A_k^T}$ . But,  $F(\omega) \geq F^*$  because  $F^*$  is the minimum, and  $F(v) - F^* \leq \varepsilon$ , so  $\frac{1}{8v_k^2 \times A_k A_k^T} \leq \varepsilon$ . But,  $v_k \leq \max(\Psi, F(v_0) - (M - 1)\Phi^* + \log(\Psi))$ , so  $\frac{1}{8v_k^2 \times A_k A_k^T} \geq \frac{1}{8 \max(\Psi, F(v_0) - (M - 1)\Phi^* + \log(\Psi))^2 \times A_k A_k^T}$ .

So, it is not possible that  $\varepsilon \leq \frac{1}{8 \max(\Psi, F(v_0) - (M - 1)\Phi^* + \log(\Psi))^2 \times A_k A_k^T}$ .

For the easy case  $v_k \geq \frac{1}{4}$ , it is sufficient to consider  $\delta = \frac{1}{A_k A_k^T}$  (less than 1) because it also leads to  $F(\omega) - F(v) \leq \delta^2 v_k^2 \times A_k A_k^T - \log(1 + \delta)$  i.e.  $F(\omega) - F(v) \leq \frac{1}{16A_k A_k^T} - \log(1 + \frac{1}{A_k A_k^T}) \leq \frac{1}{16A_k A_k^T} - \frac{1}{2A_k A_k^T} \leq -\frac{7}{16A_k A_k^T}$ . This is impossible as soon as  $\varepsilon \leq \frac{7}{16\Gamma}$

Let  $\chi = 8\Gamma(\max(\Psi, F(v_0) - (M - 1)\Phi^* + \log(\Psi), 2))^2$  ( $\log(\chi)$  is polynomial in the binary size of  $A$ ): if  $F(v) \leq F(v_0)$  and  $F(v) - F^* \leq \frac{1}{\chi}$ , then  $Ax(v) > \mathbf{0}$ .

### 2.4.3 Minimizing $F$

In previous subsection, it is proven that if  $F(v) \leq F(v_0)$  and  $F(v) - F^* \leq \frac{1}{\chi}$ , then  $Ax(v) > \mathbf{0}$ . Yet,  $F$  is a self concordant function because a positive quadratic function has positive second derivative and null third derivative ( $\nabla_v^2(x(v)^T x(v)) = 2 \times x(1)^T x(1) > 0$ ) and a sum of log is a self concordant function Nemirovski (2004).

So, it is possible to find  $v$  such that  $F(v) \leq F(v_0)$  and  $F(v) - F^* \leq -\log(1 - \varepsilon) - \varepsilon$  in  $O(F(v_0) - F^* + \log \log(\frac{1}{\varepsilon}))$  steps Nemirovski (2004). In particular, for  $\varepsilon \leq 1$ ,  $-\log(1 - \varepsilon) - \varepsilon \leq \frac{1}{4}\varepsilon^2$ . So, it is possible to find  $v$  such that  $F(v) \leq F(v_0)$  and  $F(v) - F^* \leq \varepsilon$  in  $O(F(v_0) - F^* + \log \log(\frac{1}{\varepsilon}))$  steps (the square goes into the  $O$ ).

So it is possible to find  $F(v) - F^* \leq \frac{1}{\chi}$  and  $F(v) \leq F(v_0)$  in  $O(F(v_0) - F^* + \log \log(\chi))$  steps. Importantly, all the descent can be done with infinite arithmetic precision (plus rounding to keep acceptable size) without evaluating log like for a classical log barrier. In other words, performing computation with infinite precision (using rational representation) by rounding all variables on fixed denominator grid is possible without breaking the convergence (only twice Newton steps are required), and, without evaluating log. Only square root approximation with clear termination criterion is required. This point is presented in appendix B.

So, it is possible to solve  $Ax > \mathbf{0}$  in  $O(F(v_0) - F^* + \log \log(\chi))$  steps, with  $F(v_0)$ ,  $-F^*$ ,  $\log(\chi)$  all polynomial in the binary size of  $A$ .

### 2.4.4 Proper greedy improvement decreases $F$

At this point, applying standard gradient descent on  $F$  allows to solve linear programming in polynomial time. Yet classical log barriers do it too. But, the contribution of this paper is to show that this algorithm

is helped by proper greedy improvement.

Importantly, the task of producing  $x$  such that  $Ax \geq \mathbf{0}$  and  $x \neq \mathbf{0}$  is sufficient to solve homogeneous reduced linear feasibility because it allows iteratively to build  $x$  such that  $Ax > \mathbf{0}$  (see appendix A). Newton descent will directly produce  $Ax > \mathbf{0}$ , but, the coupling with proper greedy improvement may result in  $Ax \geq \mathbf{0}$   $x \neq \mathbf{0}$ .

First,  $-\log$  is an decreasing function, so any proper greedy improvement  $u$  verifies  $\mathbf{1}^T \log(v + u) > \mathbf{1}^T \log(v)$  (as  $u > \mathbf{0}$ ).

Then, let stress that  $A_E^T u = x(u)$  (currently this is an abusive equality because  $u$  has not the same dimension in both member: left  $u$  correspond to right  $u_E$  other component being 0).

Now, for the quadratic part,  $x(v + u)^T x(v + u) = (x(v) + x(u))^T (x(v) + x(u))$  because  $x$  is just a linear function. So,  $x(v + u)^T x(v + u) = x(v)^T x(v) + 2x(u)^T x(v) + x(u)^T x(u) = x(v)^T x(v) + 2x(u)^T (x(v) + x(u)) - x(u)^T x(u)$ . But  $u$  is a proper greedy improvement starting from a not solution point and terminating in a not solution point. So, for all  $k$  such that  $u_k \neq 0$ ,  $A_k x(v) < 0$  and  $A_k x(v + u) \leq 0$ . So  $x(u)^T x(v) < 0$  and  $x(u)^T (x(v) + x(u)) \leq 0$ . In particular,  $x(v)^T x(v) + 2x(u)^T (x(v) + x(u)) - x(u)^T x(u) < x(v)^T x(v)$ .

So, both the  $x$  part and the log part decreases when  $v$  becomes  $v + u$  with  $u$  a proper greedy improvement. In other words, if  $u$  is a greedy proper improvement (which is not a solution), then  $l = \max_{t \in \mathbb{R}} \min_{m \in \{1, \dots, M\}} A_m(x(v) + t \times x(u))$  is defined and either  $F(v + lu) < F(v)$  or  $x(v) + l \times x(u)$  is a solution ( $A(x(v) + l \times x(u)) \geq \mathbf{0}$  and  $v + l \times u > v \neq \mathbf{0}$ ).

### 2.4.5 Decreasing $F$ does not break the convergence

The fact that proper greedy improvements decreases  $F$  does not imply that it could not break the convergence. It is required to go into the mechanisms of the newton descent to check if modifying  $v$  could break the process.

Hopefully, the descent has two phases (see Nemirovski (2004)) characterized by a quantity depending on derivative of  $F$ :  $\lambda(v) = \sqrt{(\nabla_v F)^T (\nabla_v^2 F)^{-1} (\nabla_v F)}$ .

First phase has high  $\lambda(v)$ :  $\lambda(v) \geq \frac{1}{4}$  with each Newton step leading to a decreasing of  $F$  higher than  $\frac{1}{4} - \log(\frac{5}{4}) \geq \frac{1}{50}$  independently from  $\lambda$ . This step is trivially fasten by proper greedy improvement which decreases  $F$  while having no side effect.

The second phase has low  $\lambda(v)$ :  $\lambda(v) \leq \frac{1}{4}$ . In this phase, there is a quadratic convergence of  $\lambda$  to 0:  $\lambda(v - \frac{1}{1+\lambda(v)} (\nabla_v^2 F)^{-1} (\nabla_v F)) \leq 2\lambda(v)^2$ , and very interestingly,  $F(v) - F^* \leq -\log(1 - \lambda(v)) - \lambda(v)$  Nemirovski (2004) (this is not trivial at all because  $v$  could have been very low implying that  $\lambda$  is low but neither  $\nabla_v F$  or  $F$ ).

During this second phase modifying  $F$  may have side effect of increasing  $\lambda$  (this could break the convergence). But, this second phase is very *fast*, and, one could have just forbid proper greedy improvement during this phase. Now, it is even not required because:  $F(v) - F^* \leq -\log(1 - \lambda(v)) - \lambda(v)$  so small  $\lambda$  implies small  $F(v) - F^*$ , but,  $F(v - \frac{1}{1+\lambda(v)} (\nabla_v^2 F)^{-1} (\nabla_v F)) \leq F(v) - \lambda(v) + \log(1 + \lambda(v))$ ,  $F(v) - F^*$  small also implies small  $\lambda$ . Precisely,  $-\log(1 - \lambda(v)) - \lambda(v) \leq \frac{\lambda(v)^2}{2(1-\lambda(v))}$  So, for  $\lambda(v) \leq \frac{1}{4}$ ,  $-\log(1 - \lambda(v)) - \lambda(v) \leq \lambda(v)^2$ . And, for  $\lambda(v) \leq \frac{1}{4}$ ,  $-\lambda(v) + \log(1 + \lambda(v)) \leq \frac{1}{2}\lambda(v)^2$ . So,  $\frac{1}{2}\lambda(v)^2 \leq F(v) - F^* \leq \lambda(v)^2$ . Or, in other words,  $F(v) - F^* \leq \lambda(v)^2 \leq 2(F(v) - F^*)$ . So, the second phase can in reality be expressed using  $F - F^*$  only.

So decreasing  $F$  can not break the convergence neither in the first nor the second phase.

### 3 Discussion

#### 3.1 Are proper greedy improvements really interesting ?

If  $A_i x(v) < 0$ , it is possible to increase  $v_i$  a little ( $v_{i+} = -\frac{A_i x(v)}{A_i A_i^T}$ ) to both decrease  $x(v)^T x(v)$  and increase  $\log(v)$ . So, it will decrease  $F$ . Yet, such minor improvement are not interesting (except if  $-\frac{A_i x(v)}{A_i A_i^T} > \mu v_i$  with large  $\mu$  at the early beginning of the algorithm), because, the yield/efficiency of such improvement (per times) is generally lower than the Newton descent alone. Indeed, algorithm of first order based on the idea to do  $v_{i+} = -\frac{A_i x(v)}{A_i A_i^T}$  when  $A_i x(v) < 0$  is somehow the perceptron, and, it is exponential.

Now, improvements which maintains  $E$  are different because they consider the set of most violated constraints, and, makes them less violated, leading  $E$  to increase as set (or leading to a solution). So, there can not have more than  $M$  successive proper greedy improvements before reaching a point which can not be improved this way.

For example, let consider  $\min_{Ax \geq b} c^T x$ . Given an admissible point  $x$  saturating the set of constraints  $E$ , if there exists  $y$  such that  $A_E y = \mathbf{0}$ , and,  $c^T y < 0$ , then,  $x = x + l \times y$  either produces an unbounded point, or, leads  $E$  to increase. By doing so (at most  $M$  times), one reaches a new  $x$  saturating a new  $\mathcal{E}$  which is maximal as there is no  $y$  such that  $A_E y = \mathbf{0}$ , and,  $c^T y < 0$ . If, one find  $z$  such that  $Az \geq b$  and  $c^T z < c^T y$ , then, this process could be restarted from  $z$ , and, importantly will never saturating  $\mathcal{E}$  again (otherwise, there was some  $y$ ). Such process may converge efficiently if number of maximal set  $\mathcal{E}$  is low, and, if it is easy to produce a  $z$  allowing to escape the vertex  $\mathcal{E}$ : this is somehow the simplex algorithm.

Now, using proper greedy improvement while reaching some maximal vertex, and, then, Newton step, one could *hope* to have a polynomial times algorithm which is, in addition, strongly polynomial when the number of maximal sets  $\mathcal{E}$  is low (even under degeneracy). This is the motivation of this paper. Obviously, this paper does much less: first it only tackles reduced homogeneous linear feasibility. This way, there is no direct equivalence of the previous process (moving toward  $\mathbf{0}$  is always a way to improve the satisfaction of most violated constraints), and, the algorithm only considers proper greedy improvement. In addition, performing Newton step has in reality no reason to produce a  $z$  allowing to escape the vertex  $\mathcal{E}$ :  $F$  would decrease, but there is no reason that it improves the satisfaction of most violated constraints.

Yet, this is still a bit of hybridization: let consider the algorithms described in figure 1. The first algorithm never performs line 14 twice on the same set  $E$ , and, never performs more than  $M$  line 4 successively without reaching a line 14. As line 14 is exponential, and, as there can exists an exponential number of sets  $E$ , this algorithm is exponential, but, strongly polynomial if each line 14 are strongly polynomial, and, if the number of set  $E$  reaching line 14 is low. Now, this paper offers to replace line 14 by Newton step using the offered function: the global algorithm is polynomial (no more than  $M$  greedy steps between each Newton step - greedy steps being harmless). But, potentially, the algorithm can be strongly polynomial when the number of set  $E$  reaching Newton step is low (but not always due to the limitation to proper greedy improvement, and, the fact that there may have looping from  $E$  point of view). Alternatively, someone may just run each simultaneously and swap  $v$  for one to the other when one is dominated. Importantly, considering  $F(l \times v)$ , one could see that  $F$  always decreases when  $v$  is normalized such that  $x(v)^T x(v) = \frac{M}{2}$ , so comparison between  $v$  of log barrier vs  $v$  of perceptron can be done with normalized  $x(v)^T x(v)$ . Indeed,  $F(l \times v) = l^2 \times x(v)^T x(v) - \mathbf{1}^T \log(v) - M \log(l)$  (because  $x(l \times v) = l \times x(v)$  and  $\log(ab) = \log(a) + \log(b)$ ). But, one could consider  $h(l) = F(l \times v)$ :  $h$



**Algorithm 1** recurrent perceptron vs greedy plus log barrier

---

//  $A \in \mathbb{Z}^{M \times N}$  with  $\text{Ker}(A) = \{\mathbf{0}\}$ , and, there exists  $\psi$  such that  $A\psi > \mathbf{0}$   
 // both return solution of the reduced homogeneous linear feasibility problem  
 recurrent\_perceptron( $A$ ) =

```

1:  $v = \mathbf{1}$ 
2:  $H = []$ 
3: while  $\neg AA^T v \geq \mathbf{0}$  do
4:    $e = \min_{m \in \{1, \dots, M\}} A_m x(v)$ 
5:    $E = \{m \in \{1, \dots, M\} / A_m x(v) = e\}$ 
6:   if  $A_E A_E^T$  not singular and  $(A_E A_E^T)^{-1} \mathbf{1} \geq \mathbf{0}$  then
7:      $w = (A_E A_E^T)^{-1} \mathbf{1}$ 
8:   else
9:     if  $E \in H$  then
10:       $w = v - H[E]$ 
11:    else
12:       $H[E] = v$ 
13:       $w = \text{recurrent\_perceptron}(A_E)$ 
14:     $v = v + l \times w$  with  $l = \max_q \min_m A_m(A^T(v + q \times w))$  (or exit)

```

greedy\_and\_barrier( $A$ ) =

```

1:  $v = v_0$ 
2:  $H = []$ 
3: while  $\neg AA^T v \geq \mathbf{0}$  do
4:    $e = \min_{m \in \{1, \dots, M\}} A_m x(v)$ 
5:    $E = \{m \in \{1, \dots, M\} / A_m x(v) = e\}$ 
6:   if  $A_E A_E^T$  not singular and  $(A_E A_E^T)^{-1} \mathbf{1} \geq \mathbf{0}$  then
7:      $w = (A_E A_E^T)^{-1} \mathbf{1}$ 
8:      $v = v + l \times w$ 
9:     continue
10:  else
11:    if  $E \in H$  then
12:       $w = v - H[E]$ 
13:    if  $x(v + l \times w)^T x(v + l \times w) < x(v)^T x(v)$  and  $w \geq \mathbf{0}$  then
14:       $v = v + l \times w$ 
15:    continue
16:   $H[E] = v$ 
17:  perform a Newton step on  $F$ 

```

---

goes to  $\infty$  when  $l$  goes to 0 or  $\infty$ , so,  $h$  has a minimum but  $h'(l) = 2l \times x(v)^T x(v) - M \frac{1}{l}$  has a single root, so  $h$  is minimum when:  $2l^2 \times x(v)^T x(v) = M$  i.e.  $l^2 = \frac{M}{2x(v)^T x(v)}$  i.e. when  $x(v)^T x(v) = \frac{M}{2}$ . This normalization is not crucial for the convergence even if potentially interesting (currently exact normalization is not possible as root is not necessarily rational). But could be very interesting to improve hybridization between log barrier and perceptron.

So clearly, this algorithm is only a first step toward merging vertex based algorithm with polynomial times ones. But, still, it is a polynomial times log barrier algorithm with vertex-based features.

### 3.2 Comparison with classical log barrier

Given, a reduced homogeneous linear feasibility  $A$  instance, the classical log barrier algorithm to find a solution is to minimize  $F(x, t, Q) = \min_{x, t} Qt + \sum_m \log(A_m x + t)$ . Indeed, if  $t < \varepsilon$ , but,  $A_i x + t > \varepsilon$ , than, it means that  $A_i x > 0$ .

Precisely, if  $F(v)$  is very low it could means that there is  $i$  with  $A_i x$  very high, and, other  $A_j x^2$  low (positive or negative), or,  $Ax > 0$ . But this allows to find a solution in both cases because, if there is  $i$  with  $A_i x$  very high, and, other  $A_j x^2$  low (positive or negative), then, such  $x$  can be projected into a vertex leading to a point  $y$  such that  $Ay \geq \mathbf{0}$  and  $y \neq \mathbf{0}$ .

Thus, minimizing  $F$  using Newton descent allows to build a solution. Currently, as the convergence depends on the value of  $x_0$ ,  $Q$  can not be too large at first step, but, should not be too low at final step otherwise the final  $x$  is not a solution. So,  $Q$  needs to be updated during runtime. Hopefully, when algorithm is in the second phase, multiplying  $Q$  by some constant allows to make the whole algorithm to converge.

Such, log barrier algorithm allows to solve linear program in polynomial time, and, also allows to tackles raw form (not just homogeneous forms provided here). Today very efficient version of this log barrier (e.g. central path) have interesting practical and theoretic complexity. Central path log barrier may be better than the offered one alone.

Yet, this approach has the drawback of not being suitable for hybridization. First there is the issue of the fact that  $Q$  changes during runtime, so, even if  $F(x', t', Q) < F(x, t, Q)$  it is not sure that this property is still true for  $Q'$ . But, even if  $Q$  is frozen, checking if  $x', t'$  is better than  $x, t$  require either to do it on  $\lambda$  (second phase only) or to approximate log. Indeed, as it is not possible to improve simultaneously all log term (which are correlated), it is not possible to simply improve  $F(x)$ . In the offered algorithm, as each log only contains independent variables, doing so is trivial. Let stress than using stack variables (see for example Allamigeon et al. (2018)) in log is absolutely not equivalent, because, stack variables are linked by equality constraints  $Ax = b$ .

So, there is a real difference with classical log barrier (currently  $F(v)$  is probably less efficient than  $F(x)$  alone, but both are polynomial times, and,  $F(v)$  is more suitable for hybridization than  $F(x)$ ).

## 4 Conclusion

This paper presents a log barrier algorithm for linear programming. This algorithm has polynomial complexity (despite probably not competitive compared to central path log barrier). Yet, the interest of this algorithm is that it has some simplex like features and could have synergy with vertex based algorithm like derived version of perceptron. Such mixing may allow to have an algorithm better than the simple simultaneous run of the two.

It could be interesting to check the behavior of the offered algorithm (coupled with perceptron like algorithm) on central path log barrier worse cases Allamigeon et al. (2018); Deza et al. (2006, 2008)

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## Appendix A: Equivalence on linear programming

The native form of linear program is the task of solving  $\min_{x \in \mathbb{Q}^N / Ax \geq b} cx$  for given  $A \in \mathbb{Z}^{M \times N}$  a matrix,  $b \in \mathbb{Z}^M$  and  $c \in \mathbb{Z}^N$  some vectors.

But, it is well known that solving  $\min_{x \in \mathbb{Q}^N / Ax \geq b} cx$  is equivalent to find a solution  $x \in \mathbb{Q}^{M+N}$  (or prove there is not) to a system of inequality  $Hx \geq h$  with  $(H \in \mathbb{Z}^{(M+N+2) \times (N+M)}$  and  $h \in \mathbb{Z}^{N+M+2})$  thank to primal dual theory. The primal dual has a solution if and only if the original problem is both feasible and bounded. As finding (or prove there is not)  $x$  such that  $Hx \geq h$  is equivalent to solve  $\min_{x, t \in \mathbb{Q}^N \times \mathbb{Q} / Hx + t \geq h, t \geq 0} t$  which is a feasible and bounded problem, it is possible to consider primal dual again (on this last one). Thus, solver can assume that some solution exists, and, that the task is to find it.

Then, as maximal determinant of a sub matrix is bounded by polynomial in the binary size of the matrix, any solution of  $Hx + t \geq h, t \geq 0, -t \geq -\varepsilon$  for a decidable  $\varepsilon$  can be converted into a solution by greedy projection. Indeed, any solution of this last problem can be projected on a vertex. The resulting point is defined by cramer rule for a sub matrix, thus,  $t = 0$  when  $\varepsilon$  is lower than some sub determinant of  $A$ . So, finding  $x$  (which exists) such that  $Hx \geq h$  is equivalent to solve a system of strict inequality  $Gx > g$ .

Finally,  $Gx > g$  is equivalent to find  $x$  such that  $Ax > \mathbf{0}$  because it is always possible to write it  $\begin{pmatrix} G & -g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} >$

**0.** This is also trivially equivalent to produce  $x$  such that  $Ax \geq \mathbf{1}$ .

This way, any linear programming solver can assume without restricting the generality to expect an input  $A \in \mathbb{Z}^{M \times N}$ , with the task of producing  $x \in \mathbb{Q}^N$  such that  $Ax > \mathbf{0}$ , with the prior that such  $x$  exists.

Yet, it is classical that one could remove extra variables until kernel of  $A$  ( $\text{Ker}(A) = \{x \in \mathbb{Q}^N / Ax = \mathbf{0}\}$ ) is reduce to  $\mathbf{0}$ : if there exists  $x, \chi$  such that  $A\chi = \mathbf{0}, \chi \neq \mathbf{0}$  and  $Ax > \mathbf{0}$ , then,  $A(x - \frac{x\chi}{\chi\chi}) = Ax > \mathbf{0}$ , and,  $\chi(x - \frac{x\chi}{\chi\chi}) = 0$ . So, one could just find  $y$  such that  $Ay > \mathbf{0}$  after injecting  $y\chi = 0$  (i.e.  $x_1 = -\frac{1}{x_1} \sum_{n=\{2, \dots, N\}} \chi_n x_n$ ) into  $Ay > \mathbf{0}$  - this does not change the form of the problem).

Finally, if  $I, J$  is a partition of all indexes of rows of  $A$ , then, finding  $A_J x > \mathbf{0}, A_I x = \mathbf{0}$ , and,  $A_I y > \mathbf{0}$  is sufficient to find  $z$  such that  $Az > \mathbf{0}$ . Indeed, even if  $A_J y$  may not be acceptable positive, it will be possible to deal with it by adding  $\theta x$  which does not modify  $A_I y$ . So if one find  $x$  such that  $Ax \geq \mathbf{0}$  and  $Ax \neq \mathbf{0}$ , then it is only required to look for  $y$  verifying the same on null components (this strictly reduces the number of rows, and, possibly the number of variable as extra variable are removed). So, with all this reduction, one could consider reduced homogeneous linear feasibility only.

Finally, linear programming is equivalent to the task of finding  $x \in \mathbb{Q}^N$  given  $A \in \mathbb{Z}^{M \times N}$  with  $\text{Ker}(A) = \{\mathbf{0}\}$  such that  $Ax \geq \mathbf{0}, x \neq \mathbf{0}$ , and, with the prior that there exists  $\psi \in \mathbb{R}^N$  such that  $A\psi \geq \mathbf{1}$ . As there is such  $\psi$  which is the solution of an equality system linked to  $A\psi = \mathbf{1}$  is bounded by a polynomial in  $\text{Det}(A)$ , and so,  $\log(\Psi)$  is a polynomial in the binary size of  $A$

## Appendix B: Implementation issues

Newton descent has two implementations issues:

- computing exactly  $\lambda(v)$  is not possible as the root of rational number are not always rational
- taking care of the binary size of  $v$  is critical to bound the number of binary operation

### Computing $\lambda(v)$

Selecting a correct step for the newton descent is in reality not a big issue because it only requires an not-too-high upper bound of  $\lambda(v)$ :

During first phase,  $\lambda(v) > \frac{1}{4}$  and  $F(v - \frac{1}{1+\lambda(v)}(\nabla_v^2 F)^{-1}(\nabla_v F)) \leq F(v) - \frac{1}{50}$

So, if  $\theta \leq \lambda(v) \leq 2\theta$ , then,  $\frac{1}{1+2\theta} \leq \frac{1}{1+\lambda(v)}$  (so it is admissible), and,  $\frac{1}{1+2\theta} \geq \frac{1}{2} \frac{1}{1+\lambda(v)}$ , so  $F(v - \frac{1}{1+2\theta}(\nabla_v^2 F)^{-1}(\nabla_v F)) \leq F(v) - \frac{1}{100}$  (by convexity).

So, the first phase of the newton descent terminates with only twice the number of steps when done with  $\theta \leq \lambda(v) \leq 2\theta$ .

For the second phase, it is simpler:  $\lambda(v - \frac{1}{1+\lambda(v)}(\nabla_v^2 F)^{-1}(\nabla_v F)) \leq 2\lambda(v)^2$ , so, it is sufficient to approximate  $\lambda(v)$  by  $\theta \geq \lambda(v)$  and to check if  $\lambda(v - \frac{1}{1+\theta}(\nabla_v^2 F)^{-1}(\nabla_v F)) \leq \frac{5}{2}\lambda(v)^2$ . One just has to continue to approximate  $\lambda(v)$  until reaching such inequality. It is always possible to find such  $\theta$  as  $\frac{5}{2}\lambda(v)^2 > 2\lambda(v)^2$ . Using  $\frac{5}{2}$  does not break convergence because if  $\lambda(v) \leq \frac{1}{4}$ , then,  $\frac{5}{2}\lambda(v)^2 \leq \frac{5}{32} < \frac{1}{4} = \frac{8}{32}$ . And,  $\lambda(v - \frac{1}{1+\theta}(\nabla_v^2 F)^{-1}(\nabla_v F)) \leq \frac{5}{2}\lambda(v)^2$  is sufficient to ensure a quadratic convergence of  $\lambda$  to 0.

So, even with approximation of  $\lambda(v)$ , one could produce  $v$  such that  $F(v) - F^* \leq \frac{1}{\chi}$  in  $O(F(v_0) - F^* + \log \log(\chi))$

## Rounding numbers

It is critical to select a denominator allowing to round into fixed grid the infinite size arithmetic operations (as  $v$  is bounded, all operations are fixed size if denominator is fixed).

For the first phase, the idea is to ensure the possibility to project each  $v_k$  in a grid while not increasing  $F$  by  $\frac{1}{200}$ : infinite arithmetic precision with  $\theta$  instead of  $\lambda$  leads  $F$  to decrease from  $\frac{1}{100}$ , by forcing  $F$  not to increase by more than  $\frac{1}{200}$  after each rounding, one could be sure that  $F$  decreases by at least  $\frac{1}{200}$  after each step with both rounding and  $\theta$  instead of  $\lambda$ . Importantly, if all  $v_k$  increases during rounding, there is no need to deal with the log.

This (positive) projection on a grid can be seen as  $u$  with  $\mathbf{0} \leq u \leq \rho \times \mathbf{1}$  where  $\rho$  is the size of the grid.

As,  $(v+u)^T AA^T(v+u) = v^T AA^T v + u^T AA^T v + u^T AA^T u$ , the problem is to find  $\rho$  such that  $\mathbf{0} \leq u \leq \rho \times \mathbf{1}$  leads to  $u^T AA^T v + u^T AA^T u \leq \frac{1}{200}$ . This way, all  $v_k$  can be expressed as multiple of  $\rho$ .

Hopefully,  $v_k$  is upper bounded, so  $u^T AA^T v + u^T AA^T u \leq \rho M^2 \Gamma \max(\Psi, F(v_0) - (M-1)\Phi^* + \log(\Psi)) + \rho^2 \Gamma^2 M^2$ . So, safe rounding can be done by selecting  $\rho$  such that  $\rho M^2 \Gamma \max(\Psi, F(v_0) - (M-1)\Phi^* + \log(\Psi)) + \rho^2 \Gamma^2 M^2 \leq \frac{1}{200}$ . For example,  $\rho = \frac{1}{2M^2 \Gamma \max(\Psi, F(v_0) - (M-1)\Phi^* + \log(\Psi))}$ .

Importantly,  $\log(\rho)$  is polynomial in the binary size of  $A$ , as,  $\log(\Gamma)$ ,  $\log(\Psi)$ ,  $\Phi^*$ ,  $F(v_0)$  are also polynomial in the binary size of  $A$ .

For the second phase, there is no issue as the second phase last less than the log of binary size of  $A$ .

## Appendix C: Code sample

This article is almost theoretic: algorithm is proven polynomial formally. And the synergy with greedy improvement is also proven formally.

However, it could be interesting to offer anyway a code sample proving that the algorithm *works* (it does not prove complexity or consistency but it proves something). Typically, the following code works on very small instances.

```
##### COMMON FUNCTION #####

def add(u,v):
    return [u[n]+v[n] for n in range(len(u))]

def mul(l,v):
    return [e*l for e in v]

def dot(u,v):
    return sum([u[n]*v[n] for n in range(len(u))])

def zeros(I,J):
    return [[0]*J] for i in range(I)]

def transpose(A):
    return [[a[j] for a in A] for j in range(len(A[0]))]

def prod(A,B):
    BT = transpose(B)
    return [[dot(a,b) for b in BT] for a in A]
```

```

from fractions import Fraction

# WARNING assume a,b are int
# BUT do not check it (performance issue)
# recall that euclide algorithm is polynomial
# so I assume native pytorch pgcd is too
def lazypgcd(a,b):
    if a==0:
        return abs(b)
    else:
        if b==0:
            return abs(a)
        else:
            return abs(a)//(Fraction(abs(a),abs(b))).numerator

def pgcd(a):
    q = a[0]
    for e in a:
        q = lazypgcd(q,e)
    return q

def convertInZv(v):
    alldenom = lprod([Fraction(e).denominator for e in v])
    tmp = [int(e*alldenom) for e in v]
    q = pgcd(tmp)
    if q==0:
        return v
    else:
        return [e//q for e in tmp]

def convertInZ(A_,B_):
    A,B = [],[]
    for i in range(len(A_)):
        tmp = convertInZv(A_[i]+B_[i])
        tmpbis = [Fraction(e) for e in tmp[len(A_[0]):len(tmp)]]
        A.append(tmp[0:len(A_[0])])
        B.append(tmpbis)
    return A,B

def lprod(l):
    out = 1
    for e in l:
        out*=e
    return out

##### LINEAR ALGEBRA FUNCTION #####

def notnull(A,i,j,k,l):
    for q in range(k,l):
        for p in range(i,j):
            if A[p][q]!=0:
                return (p,q)
    return None

```

```

# gaussian elimination
#
# my current understanding of std gaussian elimination is that
# at step t A[t][t] is the product of all minor determinants
# thus std gaussian elimination is not polynomial
#
# yet, by dividing by the PGCD, one is able to have
# A[t][t] being the minor determinant t
# making the algorithm polynomial
#
# so I believe the following algorithm is polynomial
# anyway, it is known that polynomial algorithm exists for this task
# Bareiss algorithm could be implemented instead if I am wrong !
def GaussianElimination(A_,B_):
    VAR = len(A_[0])
    EQS = len(A_)
    MAXRANK = min(VAR,EQS)

    mapVAR = dict([(i,i) for i in range(VAR)])
    mapEQS = dict([(i,i) for i in range(EQS)])

    A,B = convertInZ(A_,B_)

    #triangulation
    current = 0
    while True:
        # check termination
        pq = notnull(A,current,EQS,current,VAR)
        if pq is None:
            break
        p,q = pq

        #all vars from current to p are useless
        #all eqs from current to q are potentially impossible
        #
        #move them away for simplicity
        tmp = A[current].copy()
        A[current] = A[p].copy()
        A[p] = tmp
        tmp = mapEQS[current]
        mapEQS[current] = mapEQS[p]
        mapEQS[p] = tmp

        for i in range(EQS):
            tmp = A[i][current]
            A[i][current] = A[i][q]
            A[i][q] = tmp
        tmp = mapVAR[current]
        mapVAR[current] = mapVAR[q]
        mapVAR[q] = tmp

    #we have a until some point triangular matrix:
    assert(A[current][current]!=0)
    for i in range(current+1,EQS):

```

```

        assert(all([A[i][j]==0 for j in range(current)]))

#first rescale to have minor determinant in diagonal:
#and not the produce of all minor determinant
q = pgcd(A[current])
if q>1:
    A[current] = [e//q for e in A[current]]
    B[current] = mul(Fraction(1,q),B[current])

#then eliminate !
for i in range(current+1,EQS):
    tmp = mul(A[current][current],A[i])
    tmpB = mul(A[current][current],B[i])
    tmpbis = mul(-A[i][current],A[current])
    tmpbisB = mul(-A[i][current],B[current])
    A[i] = add(tmp,tmpbis)
    B[i] = add(tmpB,tmpbisB)

current+=1

#matrix is now triangular
for eq in range(EQS):
    assert(all([A[eq][j]==0 for j in range(min(eq,VAR))]))

#make it diagonal
## first set to 0 extra variable:
for i in range(EQS):
    for j in range(MAXRANK,VAR):
        A[i][j]=0
## THIS IS NOT AN OPERATION ON A
## THIS IS CONSEQUENCE OF VARIABLE SET TO 0

## then normalize
for var in range(MAXRANK):
    if A[var][var]==0:
        break
    B[var] = mul(Fraction(1,A[var][var]),B[var])
    A[var] = mul(Fraction(1,A[var][var]),A[var])

## and diagonalise
reverse = list(range(MAXRANK))
reverse = reverse[::-1]
for var in reverse:
    for i in range(0,var):
        B[i] = add(B[i],mul(-A[i][var],B[var]))
        A[i] = add(A[i],mul(-A[i][var],A[var]))

for eq in range(EQS):
    assert([var for var in range(VAR) if (A[eq][var]!=0 and var!=eq)]==[])

X = zeros(VAR,len(B[0]))
for i in range(MAXRANK):
    for j in range(len(B[0])):
        X[mapVAR[i]][j] = B[mapEQS[i]][j]

```



```

AX = prod(A_,X)
feasible = []
for j in range(len(B_[0])):
    feasible.append(all([AX[i][j]==B_[i][j] for i in range(EQS)]))
return feasible ,X

# returns argmax l such that A(x+lv)>=b
def maximalmoves(A,b,x,v):
    S = [m for m in range(len(A)) if dot(v,A[m])<0]
    assert(S!=[])
    all1 = [-(dot(A[m],x)-b[m])/dot(v,A[m]) for m in S]
    l = min(all1)
    assert(l!=0)
    return l

# takes x as input with Ax>=b
# returns y such that cy<=cx, Ay>=b
# + let D the set of saturated constraint of y
# then A_Dz = 0 => cz=cx
def project_on_vertex(A,b,c,x):
    while True:
        AD = [a for m,a in enumerate(A) if dot(x,a)==b[m]]
        b = [[0].copy() for i in range(len(AD))]
        AD.append(c.copy())
        b.append([-1])

        feasible ,v = GaussianElimination(AD,b)
        if not feasible[0]:
            return x
        else:
            v = [e[0] for e in v]
            l = maximalmoves(A,b,x,v)
            x = add(x,mul(l,v))

##### OPTIMISATION ROUTINE FUNCTIONS #####

### input is a number x>=0
### output is y such that y/2 <= 1+sqrt(x) <= y
### if x<=1 then 2 works
### otherwise, if one find a<=1+sqrt(x)<=b with b-a<=1
### then it means b<=a+1<=2a because a>=1 and so b works
###
### b-a is reduced by a factor 2 each step
### and binary size of the denominator only increases by 1 each step
### so this algorithm only require L^2 operations
### where L is the binary size of x
def approximation2root(x):
    if x<=1:
        return 2

```

```

a = 1
b = x+1
while b-a>1:
    c = (b+a)/Fraction(2)
    tmp = (c-1)**2
    if tmp<x:
        a = c
    else:
        b = c
return c

### return Hadamard bound on maximal determinant of A
### https://en.wikipedia.org/wiki/Hadamard%27s_inequality
def hadamardinequality(A):
    AT = transpose(A)
    maximaldeterminant = 1
    for a in AT:
        maximaldeterminant*=dot(a, a)
    return maximaldeterminant

##### LP FUNCTIONS #####

# primal: max {rawc rawx / rawA rawx<= rawb}
# dual: min {rawb rawy / transpose(rawA) rawy= rawc , rawy>=0}
# primal dual: {rawx / rawA rawx<=rawb,
#               transpose(rawA) rawy =rawc , rawy>=0, rawc rawx=rawb rawy}
# unfolded into A x >= b
# see https://en.wikipedia.org/wiki/Dual_linear_program
# "asymmetric" dual problem
def primaldual(rawA, rawb, rawc):
    A, b = [], []

    #rawA x <= rawb
    for i in len(rawA):
        A.append(mul(-1, rawA[i]) + [0]*len(rawA))
        b.append(-rawb[i])

    #rawA^T y = rawc
    rawAT = transpose(rawA)
    for i in len(rawc):
        A.append([0]*len(rawc) + rawAT[i].copy())
        b.append(rawc[i])
        A.append([0]*len(rawc) + mul(-1, rawAT[i]))
        b.append(-rawc[i])

    #y>=0
    for i in len(rawA):
        tmp = [0]*(len(rawb)+len(rawc))
        tmp[len(rawc)+i]=1
        A.append(tmp)
        b.append(0)

```

```

#rawc x = rawb y
A.append(rawc.copy()+mul(-1,rowb))
b.append(0)
A.append(mul(-1,A[-1]))
b.append(0)

return A,b

### input is a matrix A and a vector b
### with assumption that there exists x with Ax>b
### return such x
def findSolutionStrict(A,b):
    Ab = [a.copy()+[-b[i]] for i,a in enumerate(A)]
    Ab.append([0]*len(A[0]) + [1])
    xt = homogenousfeasibility(Ab)
    return xt[:-1]/xt[-1]

### input is a matrix A and a vector b
### with assumption that there exists x with Ax>=b
### return such x
def findSolutionAb(A_,b_):
    A,b = convertInZ(A_,[[e] for e in b_])
    b = [e[0] for e in b]

    bound = hadamardinequality(A)

    #Ax+t1>b
    #t>0
    #-t>-1/2bound
    A = [a.copy() + [1] for a in A]
    b.append(0)
    A.append([0]*len(A_[0])+[1])
    b.append(-Fraction(1,bound*2+1))
    A.append([0]*len(A_[0])+[-1])

    xt = findSolutionStrict(A,b)

    #by projecting xt on Ax+t1>=b,t>=0 minimizing t, one reaches z,l
    #Az+t1>=b
    #l>=0,l<=t
    #but now l can be expressed by cramer rule - in a vertex
    #so l is either 0 or >=1/bound
    #but l<=t<=1/2bound, so l=0
    #so Az>=b
    z1 = project_on_vertex(A,b,A[-2].copy(),xt)
    assert(z1[-1]==0)

    x = z1[:-1]
    assert(all([dot(A_[i],x)>=b_[i] for i in range(len(A_))]))
    return x

### input is a matrix A and 2 vectors b,c

```

```

### there is x such that Ax>=b
### return y such that Ay>=b and cy=0
### or Ay>=b and there is not z with cz<xy
def findminimum0(A,b,c):
    Amoins = [mul(-1,a) for a in A]
    bmoins = mul(-1,b)
    cmoins = mul(-1,c)
    Amoins.append(mul(-1,c))
    bmoins.append(0)

    A,b = primaldual(Amoins,bmoins,cmoins)
    xy = findSolutionAb(A,b)
    return xy[len(c)]

### input is a matrix A and a vector b
### return x such that Ax>=b or None if there is none
### return such x
def solve(A,b):
    A1 = [a.copy()+[1] for a in A]
    c = [0]*len(A[0])+[-1]

    xt = findminimum0(A1,b,c)
    if xt[-1]==0:
        return xt[:-1]
    else:
        return None

### input is a matrix A and 2 vectors b,c
### there is x such that Ax<=b and if max cx under Ax<=b is bounded
### return an argmax
### either return None
def solveAbc(A,b,c):
    Aprimaldual,bprimaldual = primaldual(A,b,c)
    xy= solve(Aprimaldual,bprimaldual)
    if xy is not None:
        return x[0:len(c)]
    else:
        return None

##### MAIN FUNCTION: strict homogeneous linear feasibility #####
### input is a matrix A
### with assumption that there exists x with Ax>0
### return such x

DEBUG = True

def bilinear(A,v):
    Av = [dot(a,v) for a in A]
    return dot(v,Av)

def findSolution_newton_computation(AA,v):
    nabraA = [dot(AA[i],v) - Fraction(1)/v[i] for i in range(len(v))]
    nabraA2 = [a.copy() for a in AA]
    for i in range(len(v)):

```

```

        nablaA2[i][i]+=Fraction(1)/(v[i]*v[i])

flag,w = GaussianElimination(nablaA2,[[a] for a in nablaA])
assert(flag[0])
w = [e[0] for e in w]

lambda2 = dot(nablaA,w)

return nablaA ,nablaA2 ,w,lambda2

import math
def approximateF(AA,v):
    f = bilinear(AA,v)
    sumlog = 0
    for e in v:
        sumlog-=math.log(float(e))
    return float(f)*0.5+sumlog

def findSolution(A_):
    A,_ = convertInZ(A_,A_)
    N = len(A)

    if DEBUG:
        ### use perceptron algorithm - exponential
        x = A[0].copy()
        while True:
            ax,a = min([(dot(a,x),a) for a in A])
            if ax>=0:
                assert(all([dot(a,x)>0 for a in A_]))
                return x
            else:
                x = add(x,a)

        ### use Newton descent - polynomial
        TA = transpose(A)
        AA = prod(A,TA)

        # F(v) = sum 1/2.v^TAA^Tv - 1^T log(v)
        # Nabla F_v = AA^Tv - 1^T 1/v
        #Nabla2 F_v = AA^T + 1^T 1/v^2
        # lambda(v) = sqrt( Nabla F_v ^T (Nabla2 F_v)^{-1} Nabla F_v )

        ### INIT
        v = [Fraction(1)/dot(a,a) for a in A]
        Q = 2**15

        #####
        ### PHASE 2:
        ### if lambda(v) >= 1/4, then
        ### F(v- 1/(1+lambda(v)) (Nabla2 F_v)^{-1} Nabla F_v ) < F(v)-1/50
        print("PHASE 1")
        while True:
            AA_v = [dot(a,v) for a in AA]

```

```

print(approximateF(AA,v)#,AAv)
if all([aav>0 for aav in AA]):
    x = [dot(a,v) for a in TA]
    assert(all([dot(a,x)>0 for a in A_]))
    return x

nablaA , nablaA2 , w , lambda2 = findSolution_newton_computation(AA,v)

if lambda2 < 1/16:
    break

approxPLUSlambda = approximation2root(lambda2)
w = mul(-Fraction(1)/approxPLUSlambda,w)

#F(v+w)<F(v)-1/100
vtmp = add(v,w)

#BUT upper rounding number is important to be polynomial
while True:
    vround = [Fraction(int(e*Q)+1,Q) for e in vtmp]
    if bilinear(AA,vround)<bilinear(AA,vtmp)+1/200:
        v = vround
        break
    else:
        print("rescale Q")
        Q = Q*Q

#asserting F(v) has decrease of at least 1/200
# is hard due to "log" in F
# yet it should be in theory

#####
### PHASE 2:
### if lambda(v) < 1/4, then
### lambda(v- 1/(1+lambda(v)) (Nabla2 F_v)^{-1} Nabla F_v )
### <= 2lambda(v)^2
### so
### lambda(v- 1/(1+lambda(v)) (Nabla2 F_v)^{-1} Nabla F_v )^2
### <= 4lambda(v)^4
### so we can approximate 1+lambda(v) by theta such that
### lambda(v- 1/theta (Nabla2 F_v)^{-1} Nabla F_v )^2
### <= 5lambda(v)^4
###
### this is enough because 5l^4 < l^2 for l<1/16
###
### theoretically , polynomial complexity requires
### to round number in this second phase
### otherwise complexity could be L^{loglog(L)} (not polynomial)
### yet, it will be for a second version XD

print("phase 2")
while True:
    AA = [dot(a,v)>0 for a in AA]
    if all([aav>0 for aav in AA]):
        x = [dot(a,v) for a in TA]

```

```

    assert(all([dot(a,x)>0 for a in A_]))
    return x

nablaA , nablaA2 , w , lambda2 = findSolution_newton_computation(AA, v)
target = lambda2 * lambda2 * 5

print(lambda2, target) # , AA v)
### find approximation of lambda2 allowing decreasing
lbound = 1
ubound = lambda2 + 1
while True:
    for i in range(30):
        center = (ubound + lbound) / Fraction(2)
        tmp = (center - 1) ** 2
        if center < lambda2:
            lbound = center
        else:
            ubound = center

    vafter = add(v, mul(Fraction(1) / ubound, w))
    _, _, _, l2after = findSolution_newton_computation(AA, vafter)
    print("check current", l2after)

    if l2after < target:
        v = vafter ### rounding should be added
        break

##### MAIN #####
import random

def randomVector(N):
    while True:
        x = [random.randint(-pow(10,5), pow(10,5)) for n in range(N)]
        if dot(x, x) != 0:
            return x

DEBUG = False

def randomMatrix(M, x):
    A = []
    while len(A) != M:
        a = randomVector(len(x))
        if dot(a, x) != 0:
            if dot(a, x) > 0:
                A.append(a)
            else:
                A.append(mul(-1, a))
    return A

A = randomMatrix(60, randomVector(30))
print(findSolution(A))

```

```
DEBUG = True
```

```
print(findSolution(A))
```

Let summarize the proof of the algorithm in the light of this code:

- $F(v) = v^T A A^T v - \log(\prod_n v_i)$
- $F(\frac{1}{\Gamma} \times \mathbf{1}) \leq M^2 - M \log(\Gamma)$  (polynomial in  $L$ )
- $\psi x(v) \leq \sqrt{\Psi x(v)^T x(v)}$  so  $\frac{1}{\Psi}(\psi x(v))^2 \leq x(v)^T x(v)$  so  $\frac{1}{\Psi} \times v^T v \leq x(v)^T x(v)$  so  $\sum_m \Phi(v_m) \leq F(v)$
- so  $F(v) \geq F^* \geq M\Phi^*$  (polynomial in  $L$ )
- and  $v_m \leq \omega = \Phi^{-1}(F(v_0) - (M-1)\Phi^*)$  (polynomial in  $L$  -  $\Phi^{-1}$  is an abusive notation but...)
- if  $A_k A^T v < 0$ , then  $v_k = (1 + \frac{1}{4v_k^2 A_k A_k^T})v_k$  makes  $F$  to decrease by  $\frac{1}{8v_k^2 A_k A_k^T}$
- as  $v$  is bounded,  $A_k A^T v < 0$  is not possible if  $F(v) - F^* \leq \frac{1}{\chi}$  with  $\chi \leq 8\omega^2 \Gamma$  (whose log is polynomial in  $L$  - currently log log is even not required - this way approximating  $\sqrt{\lambda(v)}$  with bisection instead of Newton seems ok)
- so Newton descent allows to solve the problem in  $O(F(v_0) - F^* + \log \log(\chi))$  i.e. in polynomial time
- now, one could remark that each Newton step (with approximate  $\lambda$ ) decreases  $F$  (at least) by  $\delta$  the min of  $\frac{1}{100}$  and  $\frac{1}{4\chi}$
- let note  $ND(v)$  the  $v$  after Newton descent, if instead of  $v_{t+1} = ND(v_t)$ , one consider  $v_{t+1} = \text{round}(ND(ND(v_t)))$  with  $F(\text{round}(v)) \leq F(v) + \delta$ , then it solves the problem by keeping number with fixed number of digits (ceil should be used to avoid log issue)
- as  $v$  is bounded, a denominator of  $M^2 \omega \Gamma^2 \frac{1}{\delta}$  is sufficient for performing such rounding as  $F(\text{round}(v)) = F(v) + 2x(\text{round}(v) - v)^T x(v) + x(\text{round}(v) - v)^T x(\text{round}(v) - v)$  with  $v$  bounded i.e. with a binary size polynomial in  $L$
- global complexity seems high (something like  $M^7 L^5$  when considering the number of iteration  $M^2 L^2$ , plus cost of Newton step  $M^3$  with numbers with binary size  $M^3 L^3$ ), yet it is polynomial !