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Is linear program equivalent to a quadratic size homogeneous linear feasibility ?

Adrien CHAN-HON-TONG

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Abstract

This paper claim to prove that if one is able to solve homogeneous linear feasibility $\exists v / \mathcal{A}v > \mathbf{0}$, then, it can solve linear program $\exists x / Ax > b$ by transforming the linear program into a quadratic size homogeneous linear feasibility problem.

This statement is probably false, as, it would be a very important result. It would be an even more important result, as, Chubanov recently found a very interesting algorithm to solve homogeneous linear feasibility. Hence, it would be interesting for the community to check this result.

1 Introduction

Linear programming is the very studied task of solving $\min_x c^T x$ for given $A \in \mathbb{Q}^{M \times N}$ a matrix, $b \in \mathbb{Q}^M$ and $c \in \mathbb{Q}^N$ some vectors.

Recently, [1] introduces an algorithm with good theoretical properties for homogeneous linear feasibility problems: for deciding if $\exists x \in \mathbb{Q}^N / Hx = \mathbf{0}, x > \mathbf{0}$ given $H \in \mathbb{Q}^{M \times N}$ a full rank matrix. And, [2] should that the corresponding dual is just $\exists v / \mathcal{A}v > \mathbf{0}$ with a specific matrix \mathcal{A} .

But, it is not known if one could take advantage of [1] for generic linear programming. This paper claims (while being aware that this is probably wrong as it is a very strong statement) that if one is able to solve homogeneous linear feasibility $\exists v / \mathcal{A}v > \mathbf{0}$, then, it can solve linear program $\exists x / Ax > b$ by transforming the linear program into a quadratic size homogeneous linear feasibility problem.

Notations

\mathbb{N}, \mathbb{Q} are the sets of integer and rational numbers. \setminus is the ensemble subtraction. For all integers i, j , $[i, j]$ will symbolize the **integer** range i.e. $\{i, i + 1, \dots, j\}$ which is empty if $i > j$ (there will be no ambiguity with the interval in \mathbb{R} as there is no real range in this paper).

For all integers i, j, I, J , \mathbb{Q}^I is the set of I dimensional vectors on \mathbb{Q} , and, $\mathbb{Q}^{I \times J}$ is the set of matrix with I rows and J columns, with values in \mathbb{Q} , and, \cdot_i

designs the i component: a row for a matrix and a rational for vector or a row. \mathbb{Q}^I would be matched with $\mathbb{Q}^{I \times 1}$ i.e. vectors are seen as columns, and, row of a matrix are matched with $\mathbb{Q}^{1, J}$. For all sets $S \subset \mathbb{N}$, A_S, b_S is the submatrix or subvector obtained when keeping only components indexed by $s \in S$. T is the transposition operation i.e. $A_{j,i}^T = A_{i,j}$. $\mathbf{0}$ and $\mathbf{1}$ are the 0 and 1 vector i.e. vector contains only 0 or only 1, and \mathbf{I} is the identity matrix.

2 Massive homogeneous linear feasibility versus linear programming

2.1 Pre processing

Let recall the classical primal dual trick to go from optimization problem into a decision one.

The original goal of linear programming is to solve $\max_{A_{raw}x \leq b_{raw}, x \geq \mathbf{0}} c_{raw}^T x$. Then, it is well known that the dual problem is $\min_{A_{raw}^T y \geq c_{raw}, y \geq \mathbf{0}} b_{raw}^T y$. Now, the primal dual is formed by combining all constraints: $A_{raw}x \leq b_{raw}$, and, $x \geq \mathbf{0}$, and, $A_{raw}^T y \geq c_{raw}$, and $c_{raw}x = b_{raw}y$, and finally, $y \geq \mathbf{0}$. So, the problem $\max_{A_{raw}x \leq b_{raw}, x \geq \mathbf{0}} c_{raw}x$ can be folded into $\exists?x / A_{big}x_{big} \geq b_{big}$ with

$$A_{big} = \begin{pmatrix} -A_{raw} & 0 \\ I & 0 \\ 0 & A_{raw}^T \\ 0 & I \\ c_{raw} & -b_{raw} \\ -c_{raw} & b_{raw} \end{pmatrix} \text{ and } b_{big} = \begin{pmatrix} -b_{raw} \\ 0 \\ c_{raw} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This primal dual problem has a solution i.f.f. the original polytope is neither empty or unbounded.

So, any linear program solver can assume without restricting the generality (as this pre processing is always possible) to want to solve $\exists?x / Ax \geq b$.

Currently, the offered algorithm requires in addition that either there is no x such that $Ax \geq b$ or that there exists x^* such that $Ax^* > b$.

This assumption clearly restrict the generality but it quite classical. Anyway, seeing the paper claim, this restriction is not important at this point.

2.2 Algorithm

Input: $A \in \mathbb{Q}^{M \times N}$, $b \in \mathbb{Q}^M$ such that either $Ax \geq b$ is impossible, either there is $x^* \in \mathbb{Q}^N$ such that $Ax^* > b$

Output: return x^* or a certificate

Algorithm:

$$F = \{m \in \{1, \dots, M\} / b_m > 0\}$$

$$E = \{m \in \{1, \dots, M\} / b_m < 0\}$$

$$G = \{m \in \{1, \dots, M\} / b_m = 0\}$$

Call Chubanov algorithm to find if exists v such that:
 $A_F v > \mathbf{0}$, $A_G v > \mathbf{0}$, and, $\forall (f, e) \in F \times E$, $(b_f A_e - b_e A_f)v > \mathbf{0}$
 If, no such v exists, return the certificate

Else, returns $\left(\max_{f \in F} \frac{b_f}{A_f v} \right) v$

2.3 Proof

If v exists, let $\lambda = \left(\max_{f \in F} \frac{b_f}{A_f v} \right)$ with k such that $\frac{b_k}{A_k v} = \lambda$ and $y = \lambda v$. Then,

- $\forall g \in G$, $A_g y = \lambda A_g v > \mathbf{0}$ as $A_g v > \mathbf{0}$ (definition of v) and $\lambda > 0$ (if $f \in F$, $b_f > 0$ and $A_f v > \mathbf{0}$ so $\frac{b_f}{A_f v} > 0$ and so the max)
- $\forall f \in F$, $A_f y = \left(\max_{m \in F} \frac{b_m}{A_m v} \right) A_f v \geq \frac{b_f}{A_f v} A_f v = b_f$ (because max is greater than individual elements and $A_f v > \mathbf{0}$ by definition of v)
- $\forall e \in E$, $A_e y = \frac{b_k}{A_k v} A_e v$ (by definition of k), but, $(b_f A_e - b_e A_f)v > \mathbf{0}$ by definition of k . So $b_k A_e v - b_e A_k v > \mathbf{0}$, and so, $b_e < \frac{b_k}{A_k v} A_e v$ i.e. $A_e y > b_e$

So, if v exists, then v is the solution of the linear program.

Now, if there is $x^* \in \mathbb{Q}^N$ such that $Ax^* > b$, then,

- $A_F x^* > b_F > \mathbf{0}$
- $A_G x^* > b_G = \mathbf{0}$
- $\forall f \in F$, $A_f(0 \times x^*) = 0 < b_f$ but $A_f(1 \times x^*) > b_f$ (definition of x^*), so there is $0 < l_f < 1$ such that $A_f(l_f \times x^*) = b_f$. Inversely, $\forall e \in E$, $A_e(0 \times x^*) = 0 > b_e$ and $A_e(1 \times x^*) > b_e$ (definition of x^*). So, $\forall 0 < l < 1$, $A_e(l \times x^*) > b_e$. A fortiori $l_f = \frac{b_f}{A_f x^*}$, and, $A_e(l_f \times x^*) > b_e$. By expanding the last, it stands that $A_e\left(\frac{b_f}{A_f x^*} \times x^*\right) > b_e$, $A_e x^* \frac{b_f}{A_f x^*} > b_e$ i.e. $b_f A_e x^* > b_e A_f x^*$. Finally, $(b_f A_e - b_e A_f) x^* > \mathbf{0}$.

So, if x^* exists, then Chubanov algorithm is expected to return v leading to solve the linear program.

Discussion

This paper invites to conclude that linear programming is polynomially equivalent to homogeneous linear feasibility. This may imply that linear programming can be solved in strongly polynomial time thank to Chubanov algorithm !

Seeing the importance of this conclusion, it seems that there is probably a wrong statement in this paper. It could be interesting for the community to check it.

References

- [1] Sergei Chubanov. A polynomial projection algorithm for linear feasibility problems. *Mathematical Programming*, 153(2):687–713, 2015.
- [2] Kees Roos. An improved version of chubanov’s method for solving a homogeneous feasibility problem. *Optimization Methods and Software*, 33(1):26–44, 2018.