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MODEL CHECKING VECTOR ADDITION SYSTEMS WITH ONE ZERO-TEST *

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Abstract. We design a variation of the Karp-Miller algorithm to compute, in a forward manner, a finite representation of the cover (*i.e.*, the downward closure of the reachability set) of a vector addition system with one zero-test. This algorithm yields decision procedures for several problems for these systems, open until now, such as place-boundedness or LTL model-checking. The proof techniques to handle the zero-test are based on two new notions of cover: the *refined* and the *filtered* cover. The refined cover is a hybrid between the reachability set and the classical cover. It inherits properties of the reachability set: equality of two refined covers is undecidable, even for usual Vector Addition Systems (with no zero-test), but the refined cover of a Vector Addition System is a recursive set. The second notion of cover, called the filtered cover, is the central tool of our algorithms. It inherits properties of the classical cover, and in particular, one can effectively compute a finite representation of this set, even for Vector Addition Systems with one zero-test.

1. Introduction

Context: verifying properties of Vector Addition Systems. Petri Nets, Vector Addition Systems (VAS), and Vector Addition Systems with control States (VASS) are equivalent well-known classes of counter systems for which the reachability problem is decidable [30, 27, 29], even if its complexity is still open. On the other hand, testing equality of the reachability sets of two such systems is undecidable [4, 22]. For this reason, one cannot compute a canonical finite representation of the reachability set that would make it possible to

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test for equality of two reachability sets. However, there is such an effective finite representation for the *cover*, a useful over-approximation of the reachability set which is connected to various verification problems. Therefore, one can decide not only the coverability problem (that is, membership to the cover), but also whether two VAS have the same cover.

Vector Addition Systems are powerful models for the verification of networks of identical finite-state machines communicating by rendez-vous, with dynamic creation and destruction. Intuitively, a global configuration of such a system is abstracted by nonnegative counters, one for each possible location of the finite-state machine. A counter value denotes the number of machines in the corresponding location (see for instance [12]). Notice that dynamic creation makes the number of processes, and therefore the values of counters, possibly unbounded. For modeling client-server systems where clients are identical finite-state machines, and the server is another finite-state machine that can check that no process is in a critical section, the VAS model is no longer sufficient. Indeed, one must be able to check that a particular counter is equal to zero, namely the one counting processes in the critical section. This is a first practical motivation for adding to VAS the ability to test a counter for 0.

Another reason to consider such a model is that it constitutes a first step towards the verification of VAS equipped with a stack, a model borrowing features both to pushdown automata and to VAS, and that abstracts recursive programs manipulating constrained counters. However, these systems are difficult to analyze. Abstracting away the actual stack alphabet transforms the stack into a counter that can be tested to zero. In this paper, we study verification problems for VAS with one zero test.

If one adds to VAS the ability to test at least two counters for zero, one obtains a model equivalent to Minsky machines, for which all nontrivial properties (in the sense of Rice's theorem) of the language they recognize are undecidable, and many properties of their behavior, such as reachability of a control state or termination, are also undecidable. The study of VAS with *a single* zero-test is recent, and only few results are known for this model. Reinhardt [33] has shown that the reachability problem is decidable for VAS with one zero-test transition (as well as for hierarchical zero-tests), and an alternate, simpler proof of this result was recently given by the first author [7]. Abdulla and Mayr have shown that the coverability problem is decidable in [2], by using both the backward procedure of Well Structured Transition Systems [1] (see [20] for a survey on Well Structured Transition Systems), and the decidability of forward-reachability of ordinary VASS as an oracle. The boundedness problem (whether the reachability set is finite), the termination and the reversal-boundedness problem (whether the counters can alternate infinitely often between the increasing and the decreasing modes) are all decidable by using a forward procedure, computing a finite, yet *incomplete*, Karp-Miller tree [19].

LTL specifications. Linear time temporal logic is a widely used specification logic, which can express safety and liveness properties. Emerson [12] has designed an algorithm based on a covering graph to check LTL properties on Well Structured Transition Systems, but which may not terminate. Esparza [13, 14] has shown that LTL specifications on the actions of a VAS is decidable (contrary to CTL) and that LTL becomes undecidable if one adds state predicates. Habermehl [21] completed this proof by showing EXPSPACE-completeness of LTL satisfiability, by generalizing Rackoff's proof [32]. These results have been unified in [5].

Our contribution. We give an algorithm for computing a *finite representation* of the cover for a VAS with one zero-test. This result makes it possible to decide the place-boundedness problem, which is in general undecidable for VAS extensions (such as VAS with resets [11] or lossy counter machines, *i.e.*, lossy VAS with zero-test transitions [9, 31]).

Our proof first introduces a new notion of cover, called *refined cover*, where the usual ordering on vectors is replaced by one that insists on keeping equality on certain components. The refined cover is a set hybrid between the reachability set and the classical cover. We show that equality of two refined covers is undecidable, even for usual VAS (with no zero-test). However, one can show that such a refined cover is recursive for a VAS. We then introduce *filtered covers*, the main technical tool of our algorithm. A filtered cover is defined wrt. some specific values attached to some components. It consists in retaining only these vectors from the reachability set that agree with these values, before taking the usual downward closure. By transferring decidability results from refined covers to filtered covers, we are able to compute a finite representation of any filtered cover. We use this representation to propose an algorithm *à la* Karp and Miller, which builds a tree to compute the cover of a VAS with one zero-test. This allows us to obtain new decidability results for such systems, namely for the classical problems of place-boundedness. Finally, we show that the repeated control state reachability for vector addition systems with states and one zero-test is decidable, as well as LTL model-checking, by reducing these problems to the reachability problem. Note that, for VASS (with no zero-test), both problems can be reduced to the computation of the cover set. We do not know whether there is such a reduction between the corresponding problems for VASS with one zero test, and we leave it as an open problem.

Thus, this work can be viewed as a contribution to understanding the limits of decidability, taking into account two parameters: the models (VAS and VAS with one zero-test) and the problems (reachability, cover, refined and filtered cover).

The difficulty. The central problem is to compute the cover of a VAS *with one zero-test*. Let us explain why the usual Karp-Miller algorithm is not sufficient for that purpose. A crucial property of VAS used by this algorithm is *monotony*: actions fireable from a state are still fireable from any larger state. This property is clearly broken by the zero-test.

A natural idea appearing in [19] is to adapt the classical Karp-Miller construction [25], first building the Karp-Miller tree, but *without* firing the zero test. To continue the construction after this first stage, we need to fire the zero test from the leaves of the Karp-Miller tree carrying a 0 value on the component that is tested to 0. The problem is that accelerations performed while building the Karp-Miller tree may have produced, on this component in the label of such a leaf, an ω value that represents arbitrarily large values, and that abstracts actual values. For this reason, one may not be able to determine if the zero test succeeds or not. We therefore want a more accurate information for the labeling of the leaves, for the component tested to 0. This is what the filtered cover actually captures.

To be more precise, let us illustrate this difficulty with some short examples (assuming basic knowledge on VAS/VASS, see Sec. 3/7). The Karp-Miller algorithm [25, 15] computes a finite representation of the cover of a VASS, *i.e.*, the downward closure of its reachability set (for the usual ordering over $\mathbb{N}^d$, where d is the dimension of the VASS). It builds a finite tree, whose nodes are labeled by elements of $(\mathbb{N} \cup \{\omega\})^d$, where intuitively ω represents arbitrary large values. At the end of the algorithm, the cover is exactly the set of vectors of $\mathbb{N}^d$ belonging to the downward closure of the set of labels. The tree is obtained by unwinding the system, and by performing acceleration when possible, in order to guarantee termination:

if one finds two nodes on the same branch, such that the lowest one in the branch is labeled by a greater element, one replaces by ω all components that have grown (this captures the iteration of the firing sequence between the two nodes, and this is where monotony is used). We aim at generalizing this algorithm for VASS with one zero-test.

As a first example, consider in dimension 1 the two VASS with one zero-test represented in Fig. 1. They only differ by the transition from p to q. The transition from q to r is the

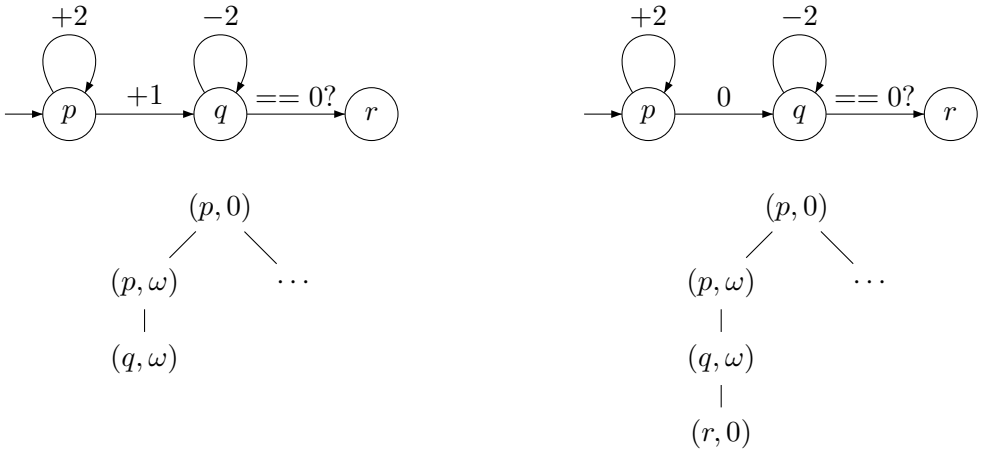

Fig. 1: Two VASS with one zero-test, and their Karp-Miller trees

zero-test, fireable only when the counter is 0, and which does not affect the counter. Starting from the initial state $(p, 0)$ and firing the loop from p to itself, the algorithm first computes as left child of the root a node labeled $(p, 2)$, which then gets accelerated as (p, ω). Then, firing the transition from p to q yields the node (q, ω). Now, the zero-test is not fireable in the first case, while it is fireable in the second case. Therefore, the Karp-Miller trees we want to compute should differ (see Fig. 1, which shows two such partial Karp-Miller trees). However, this cannot be detected with the information available on the branch from $(p, 0)$ to (q, ω), because this information is identical for both systems: it consists of the nodes $(p, 0)$, (p, ω), (q, ω). This example illustrates the fact that the ω component, in (q, ω), hides the actual reachable values, and therefore also hides the ability or inability to fire the zero-test.

The next example (Fig. 2) is in dimension 2. The zero-test occurs on the first component. It shows that even if one could determine when to fire the zero-test, one might be unable to compute the relevant node labeling using only information provided by classical Karp-Miller trees. Indeed, the Karp-Miller trees for both systems before firing the zero-test are identical.

Fig. 2: Two VASS with one zero-test

However, firing the zero-test from (q, ω, ω) should produce a node labeled $(r, 0, 0)$ in the first

case, and $(r, 0, \omega)$ in the second one. Here, ω values in (q, ω, ω) hide relevant relationships between components (namely, that both components remain equal in the first system).

The schema of our proof.

(1) We start in Section 4 with usual VAS: we extend the decidability of the reachability problem for VAS, by proving that the set $\mathsf{Lim\,Reach}$ of *limits* of sequences of reachable states is also recursive. This set $\mathsf{Lim\,Reach}$ contains the reachability set, and captures more information, in general. Actually, it is more sophisticated than both the cover and the reachability set: it allows one to know whether an element in $(\mathbb{N} \cup \{\omega\})^d$ is a reachable state or if it is the limit of a sequence of reachable states. This information is not given by the reachability set, neither by the cover (using the pointwise ordering over $(\mathbb{N} \cup \{\omega\})^d$, and the natural ordering over $\mathbb{N} \cup \{\omega\}$: $n \leqslant \omega$ for all n). The proof carries on by using Higman's Lemma, using a nontrivial ordering.

(2) In Section 5, we refine the definition of cover in which the first component of the vectors has now to be known exactly (and not only bounded by some maximal value). We prove that, for VAS, the fact that $\mathsf{Lim\,Reach}$ is recursive implies that one can *compute* the finite basis of this filtered cover.

(3) In Section 6, we compute the finite basis of the cover of a VAS with one zero-test by using a variation of the Karp-Miller algorithm that uses the previously defined filtered covers in order to convey enough information to go through the zero-test.

(4) We add control states to our VAS with one zero-test in Section 7, and we show that one can detect reachable increasing loops on a given control state, by reducing this problem to the reachability problem for VASS with one zero-test, a decidable problem [33, 7]. This allows us to decide repeated control state reachability. We also note that this makes it possible to solve model checking against LTL or ω-regular specifications. However, contrary to the situation without any zero-test, this is obtained by reducing this problem to the reachability problem, and not to the computation of the cover. Whether a reduction to this simpler problem exists is left open.

2. Preliminaries

Words. We denote by A^* the set of finite words over A. A word $u \in A^*$ is written $a_1 \cdots a_n$, with $a_i \in A$. The concatenation of two words u and v is simply written uv and the empty word is denoted ε, with $\varepsilon a = a\varepsilon = a$. We let $A^+ = A^* \setminus \{\varepsilon\}$ be the set of nonempty words.

Orderings. An *ordering* $\preccurlyeq$ on a set X is a reflexive, transitive and antisymmetric binary relation over X. Given $x, y \in X$, we write $x \prec y$ for $x \preccurlyeq y$ and $x \neq y$. For $Y \subseteq X$, let

$$\downarrow_{\preccurlyeq} Y = \{x \in X \mid \exists y \in Y,\ x \preccurlyeq y\}$$

denote the *downward closure* of Y with respect to $\preccurlyeq$. The set Y is said *downward closed* if $Y = \downarrow_{\preccurlyeq} Y$. When working in $\mathbb{N}^d$ or $\mathbb{N}_\omega^d$ with the usual ordering $\leqslant$ (see below), we shorten the corresponding downward closure operator $\downarrow_{\leqslant}$ as $\downarrow$. Symmetrically, the *upward closure* of $Y \subseteq X$, denoted $\uparrow_{\preccurlyeq} Y$ is defined by

$$\uparrow_{\preccurlyeq} Y = \{x \in X \mid \exists y \in Y,\ y \preccurlyeq x\}.$$

The set Y is said to be *upward closed* if $\uparrow_{\preccurlyeq} Y = Y$.

Vectors. For $d \geqslant 1$, we write any vector $\boldsymbol{x} \in X^d$ as $\boldsymbol{x} = (\boldsymbol{x}(1), \ldots, \boldsymbol{x}(d))$, with $\boldsymbol{x}(i) \in X$. Given an ordering $\preccurlyeq$ over X, the *pointwise ordering* over X^d, still denoted $\preccurlyeq$, is defined by $\boldsymbol{x} \preccurlyeq \boldsymbol{y}$ if $\boldsymbol{x}(i) \preccurlyeq \boldsymbol{y}(i)$ for all i. For $X = \mathbb{N}$, we let $\mathbf{0}$ be the vector whose components are all 0, and we say that $\boldsymbol{x}$ is *nonnegative* if $\boldsymbol{x} \geqslant \mathbf{0}$. For $i \in \{1, \ldots, d\}$, we let $\boldsymbol{e}_i$ be the vector such that $\boldsymbol{e}_i(i) = 1$ and $\boldsymbol{e}_i(k) = 0$ if $k \neq i$.

Limits in $\mathbb{N}_\omega^d$. We introduce an element $\omega \notin \mathbb{N}$ and the set $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$. A sequence $(\ell_n)_{n \geqslant 0}$ (also written $(\ell_n)_n$) of elements of $\mathbb{N}_\omega$ *converges to* $\ell \in \mathbb{N}_\omega$, if either it is ultimately constant with value ℓ, or its subsequence of integer values is infinite, tends to infinity, and $\ell = \omega$. We then say that ℓ is *the limit* of $(\ell_n)_n$, noted $\lim_n \ell_n = \ell$, or $\ell_n \overset{n \to \infty}{\dashrightarrow} \ell$. A sequence $(\boldsymbol{x}_n)_n$ of vectors of $\mathbb{N}_\omega^d$ has limit $\boldsymbol{x} \in \mathbb{N}_\omega^d$, noted $\lim_n \boldsymbol{x}_n = \boldsymbol{x}$, if $\lim_n \boldsymbol{x}_n(i) = \boldsymbol{x}(i)$ for all $i \in \{1, \ldots, d\}$.

For $\boldsymbol{M} \subseteq \mathbb{N}_\omega^d$, let $\mathsf{Lim}\, \boldsymbol{M}$ be the set of limits of sequences of elements of $\boldsymbol{M}$. Notice that

$$\boldsymbol{M} \subseteq \mathsf{Lim}\, \boldsymbol{M}, \tag{2.1}$$

and

$$\text{if } \boldsymbol{M} \subseteq \mathbb{N}^d, \text{ then } \boldsymbol{M} = \mathbb{N}^d \cap \mathsf{Lim}\, \boldsymbol{M}. \tag{2.2}$$

Topologically speaking, $\mathsf{Lim}\, \boldsymbol{M}$ is the least limit closed set containing $\boldsymbol{M}$. It is called the *limit closure* of $\boldsymbol{M}$. The set $\boldsymbol{M}$ is said to be *limit closed* if $\boldsymbol{M} = \mathsf{Lim}\, \boldsymbol{M}$.

Downward closed sets of $\mathbb{N}^d$ and $\mathbb{N}_\omega^d$. Given an ordered set, one may under suitable hypotheses construct a topological completion of this set, to recover a *finite description* of its downward closed subsets [16, 17]. The completion of $(\mathbb{N}^d, \leqslant)$ is $(\mathbb{N}_\omega^d, \leqslant)$ where we extend the ordering $\leqslant$ over $\mathbb{N}$ by $n \leqslant \omega$ for all $n \in \mathbb{N}_\omega$.

A *basis* of a set $\boldsymbol{D} \subseteq \mathbb{N}_\omega^d$ is a *finite* set $\boldsymbol{B} \subseteq \mathbb{N}_\omega^d$ such that

$$\mathsf{Lim}\, \boldsymbol{D} = {\downarrow}\boldsymbol{B}. \tag{2.3}$$

Such a set $\boldsymbol{B}$ is a finite representation of $\mathsf{Lim}\, \boldsymbol{D}$. One verifies that the maximal elements of any basis $\boldsymbol{B}$ of $\boldsymbol{D}$ still form a basis, which only depends on $\boldsymbol{D}$. It is minimal for inclusion among all bases, and is called *the minimal basis* of $\boldsymbol{D}$. Of course, not all sets admit a basis. By [16, 17], any downward closed set $\boldsymbol{D} \subseteq \mathbb{N}^d$ admits a basis. This extends to any downward closed set $\boldsymbol{D}$ of $\mathbb{N}_\omega^d$. Indeed, one can check that

$$\mathsf{Lim}\, \boldsymbol{D} = \mathsf{Lim}(\boldsymbol{D} \cap \mathbb{N}^d), \tag{2.4}$$

so that a basis $\boldsymbol{B}$ of the downward closed set $\boldsymbol{D} \cap \mathbb{N}^d$ satisfies $\mathsf{Lim}\, \boldsymbol{D} = {\downarrow}\boldsymbol{B}$. Note that conversely, if $\boldsymbol{B} \subseteq \mathbb{N}_\omega^d$ is *finite*, then ${\downarrow}\boldsymbol{B}$ is limit closed (this may fail if $\boldsymbol{B}$ is infinite). Finally, the limit and downward closure operators commute:

$${\downarrow}\mathsf{Lim}\, \boldsymbol{M} = \mathsf{Lim}\, {\downarrow}\boldsymbol{M} \tag{2.5}$$

Upward closed sets. If $\preccurlyeq$ is a well ordering over X (see Sec. 4 page 10), then for any upward closed set $Y \subseteq X$, there exists a finite set $B \subseteq Y$ such that $Y = {\uparrow_{\preccurlyeq}}B$. Such a set is again called a basis (as for downward sets, but there will be no ambiguity). Observe that contrary to the case of downward closed sets, no topological completion is needed here.

Example 2.1. Consider the set $\boldsymbol{D} = \big\{(x, y) \in \mathbb{N}^2 \mid x \leqslant 3 \vee y \leqslant 1\big\} \cup \big\{(4,2), (4,3), (5,2)\big\}$, which is downward closed. It is represented by the greyed grayed area in Fig. 3. Its limit closure is $\mathsf{Lim}\, \boldsymbol{D} = \boldsymbol{D} \cup \big(\{0,1,2,3\} \times \{\omega\}\big) \cup \{\omega\} \times \{0,1\}$. A non-minimal basis of $\boldsymbol{D}$ is $(\mathsf{Lim}\, \boldsymbol{D} \setminus \boldsymbol{D}) \cup \{(4,3), (5,2)\}$, shown with dots $\bullet$ and $\odot$ in Fig. 3, where elements involving ω

fall beyond the grid. Its minimal basis is $\{(3,\omega),(4,3),(5,2),(\omega,1)\}$ (circled $\odot$ in Fig. 3). The minimal basis of its (upward closed) complement in $\mathbb{N}^d$ is $\{(4,4),(5,3),(6,2)\}$.

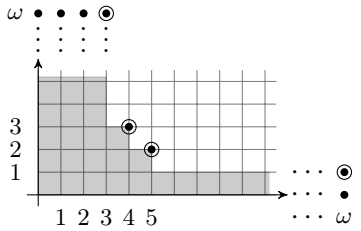

Fig. 3: A set $\boldsymbol{D}$ (grayed), elements of a basis ($\bullet$ and $\odot$) and of its minimal basis ($\odot$)

3. Vector Addition Systems

Definition 3.1. A *Vector Addition System with one zero-test* (shortly VAS_z) of dimension d is a tuple $\mathcal{V} = \langle A, a_z, \delta, \boldsymbol{x}_{in}\rangle$, where A is a finite alphabet of *actions*, $a_z \notin A$ is called the *zero-test*, $\delta : A \cup \{a_z\} \to \mathbb{Z}^d$ is a mapping, and $\boldsymbol{x}_{in} \in \mathbb{N}^d$ is the *initial state*.

Other equivalent formalisms exist, for instance with states, or with multiple zero-tests transitions that test the same counter for zero. For now, we stick to the simplest version, and we shall introduce states in Section 7.

Intuitively, a VAS_z works with d counters, one for each component, whose initial values are given by $\boldsymbol{x}_{in}$. Executing action $a \in A \cup \{a_z\}$ translates the counter values according to $\delta(a) \in \mathbb{Z}^d$. The mapping δ extends to a monoid morphism $\delta : (A \cup \{a_z\})^* \to \mathbb{Z}^d$, so that $\delta(\varepsilon) = \mathbf{0}$ and $\delta(uv) = \delta(u) + \delta(v)$ for $u, v \in (A \cup \{a_z\})^*$. More formally, a VAS_z $\mathcal{V} = \langle A, a_z, \delta, \boldsymbol{x}_{in}\rangle$ of dimension d induces a transition relation $\to \subseteq \mathbb{N}^d \times A \times \mathbb{N}^d$ with:

$$\begin{cases} \boldsymbol{x} \xrightarrow{a} \boldsymbol{y} & \text{if} \quad \delta(a) = \boldsymbol{y} - \boldsymbol{x} \qquad\qquad \text{for all } a \in A \\ \boldsymbol{x} \xrightarrow{a_z} \boldsymbol{y} & \text{if} \quad \delta(a_z) = \boldsymbol{y} - \boldsymbol{x}, \text{ and } \boldsymbol{x}(1) = 0. \end{cases} \tag{3.1}$$

We extend this relation to words by $\boldsymbol{x} \xrightarrow{\varepsilon} \boldsymbol{x}$ and $\boldsymbol{x} \xrightarrow{uv} \boldsymbol{z}$ if there exists $\boldsymbol{y}$ such that $\boldsymbol{x} \xrightarrow{u} \boldsymbol{y} \xrightarrow{v} \boldsymbol{z}$. We say that $u \in (A \cup \{a_z\})^*$ is *fireable* from $\boldsymbol{x}$ if there exists $\boldsymbol{y}$ such that $\boldsymbol{x} \xrightarrow{u} \boldsymbol{y}$. When there may be ambiguity on the VAS_z, we will write $\xrightarrow{u}_{\mathcal{V}}$ instead of $\xrightarrow{u}$.

Definition 3.2. A *Vector Addition System (VAS) of dimension* d is a tuple $\langle A, \delta, \boldsymbol{x}_{in}\rangle$, where A is a finite alphabet, $\delta : A \to \mathbb{Z}^d$ is a mapping and $\boldsymbol{x}_{in} \in \mathbb{N}^d$ is the *initial state*.

A VAS is a particular VAS_z: choosing $a_z \notin A$, this VAS is formally equivalent to the VAS_z $\langle A, a_z, \delta', \boldsymbol{x}_{in}\rangle$, where δ' extends δ by $\delta'(a_z) = (-1, 0, ..., 0)$ (*i.e.*, a_z can never be fired).

For a VAS_z or a VAS $\mathcal{V}$ of dimension d, the *reachability set* $\mathsf{Reach}(\mathcal{V})$ and the *cover* $\mathsf{Cover}(\mathcal{V})$ of $\mathcal{V}$ are the following subsets of $\mathbb{N}^d$:

$$\mathsf{Reach}(\mathcal{V}) = \left\{\boldsymbol{y} \in \mathbb{N}^d \mid \exists u \in (A \cup \{a_z\})^*, \boldsymbol{x}_{in} \xrightarrow{u} \boldsymbol{y}\right\},$$
$$\mathsf{Cover}(\mathcal{V}) = {\downarrow}\mathsf{Reach}(\mathcal{V}).$$

We call elements of $\mathsf{Reach}(\mathcal{V})$ *reachable states* (also called reachable markings in related work). The reachability (resp. coverability) problem consists in deciding membership in $\mathsf{Reach}(\mathcal{V})$ (resp. in $\mathsf{Cover}(\mathcal{V})$). Reachability is decidable for VAS [30, 27, 29] and VAS_z [33, 7].

Theorem 3.1. *Given a VAS or VAS_z $\mathcal{V}$, the reachability problem for $\mathcal{V}$ is decidable.*

Testing membership in the cover set is much easier, and one even gets a more precise result [25, 20, 17]:

Theorem 3.2. *Given a VAS $\mathcal{V}$, one can effectively compute a (finite) basis of $\mathsf{Cover}(\mathcal{V})$.*

Observe that given a (finite) basis $\boldsymbol{B}$ of a downward closed set $\boldsymbol{D} \subseteq \mathbb{N}^d$, one can effectively test membership in $\boldsymbol{D}$, since $\boldsymbol{D} = \mathbb{N}^d \cap {\downarrow}\boldsymbol{B}$ by (2.2) and (2.3). Therefore, Theorem 3.2 implies that one can effectively decide membership in $\mathsf{Cover}(\mathcal{V})$.

Computing a finite basis of the cover makes it also possible to decide whether two VAS have the same cover, since from a finite basis, one can also compute the minimal basis, which is canonical. Likewise, one can decide inclusion of covers. Finally, Theorem 3.2 implies that one can decide *place-boundedness*, that is, whether the projection of $\mathsf{Reach}(\mathcal{V})$ on some given component is bounded. In the next three sections, we shall show that one can also effectively compute a finite basis for the cover of a VAS_z.

4. Limits of reachable states of a VAS

As observed above, for $\boldsymbol{M} \subseteq \mathbb{N}^d$, one can immediately construct an algorithm deciding membership in $\boldsymbol{M}$ from an algorithm deciding membership in $\mathsf{Lim}\,\boldsymbol{M}$, since $\boldsymbol{M} = \mathbb{N}^d \cap \mathsf{Lim}\,\boldsymbol{M}$ by (2.2). However, the converse is not true. Let us explain two reasons for this.

a. First, even if $\boldsymbol{M}$ is recursive, it may happen that $\mathsf{Lim}\,\boldsymbol{M}$ is not. We recall here an example from [18, Prop. 2.4]. Let $T_0, T_1, \ldots$ be an effective enumeration of Turing machines. Let $\alpha(k,\ell) = \big|\{j \leqslant k \mid T_j \text{ halts in at most } \ell \text{ steps on } \varepsilon\}\big|$ and $\boldsymbol{M} = \big\{(k,\ell,\alpha(k,\ell)) \mid k,\ell \geqslant 0\big\}$. It is easy to describe an algorithm computing $\alpha(k,\ell)$ given $k,\ell \in \mathbb{N}$, and therefore also an algorithm to decide membership in $\boldsymbol{M}$. However, $\mathsf{Lim}\,\boldsymbol{M}$ is not recursive, since the halting problem reduces to it. Indeed, $(k,\omega,m) \in \mathsf{Lim}\,\boldsymbol{M}$ means that exactly m machines among $T_0, \ldots, T_k$ halt on the empty word. Therefore, T_k halts on ε if and only if there exists $m \leqslant k+1$ such that $(k,\omega,m) \in \mathsf{Lim}\,\boldsymbol{M}$ and $(k-1,\omega,m-1) \in \mathsf{Lim}\,\boldsymbol{M}$.

b. Second, even if $\mathsf{Lim}\,\boldsymbol{M}$ is recursive, one may not be able to effectively derive an algorithm deciding membership in $\mathsf{Lim}\,\boldsymbol{M}$ from a description of $\boldsymbol{M}$ (such as a data structure, or an algorithm deciding membership in $\boldsymbol{M}$). As an example, consider the reachability set $\boldsymbol{M}$ of a lossy counter machine (see again [31], or [34] for a survey). An algorithm to decide membership of $\boldsymbol{x}$ in $\boldsymbol{M}$ is to compute the bases of the upward closed sets $\mathrm{Pre}^i({\uparrow}\boldsymbol{x})$ for $i = 0, 1, 2, \ldots$, where $\mathrm{Pre}(X)$ denotes the set of predecessors of X. The sequence stabilizes, since it consists only of upward closed sets. Moreover, due to the lossy behavior, $\boldsymbol{M}$ is downward closed. Therefore, it admits a finite basis $\boldsymbol{B}$, so that $\mathsf{Lim}\,\boldsymbol{M} = {\downarrow}\boldsymbol{B}$ is recursive. However, there is no algorithm taking as input a lossy counter machine and a vector $\boldsymbol{x} \in \mathbb{N}_\omega^d$, and deciding membership of $\boldsymbol{x}$ in $\mathsf{Lim}\,\boldsymbol{M}$, where $\boldsymbol{M}$ is the reachability set. Indeed, the set $\boldsymbol{M}$ is infinite if and only if $\mathsf{Lim}\,\boldsymbol{M}$ contains some vector of $\mathbb{N}_\omega^d$ having at least an ω-component. Therefore, the existence of such an algorithm would imply that the boundedness problem (*i.e.*, whether the reachability set is finite) is co-recursively enumerable, which is not the case: boundedness for lossy counter machines is Σ_1^0-complete.

The main result of this section considers the case where $\boldsymbol{M}$ is the reachability set of a VAS $\mathcal{V}$. Since $\mathsf{Cover}(\mathcal{V}) = {\downarrow}\mathsf{Reach}(\mathcal{V}) = \mathbb{N}^d \cap \mathsf{Lim}\,{\downarrow}\mathsf{Reach}(\mathcal{V}) = \mathbb{N}^d \cap {\downarrow}\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$ (where the last two equalities follow from (2.2) and (2.5)), one can by Theorem 3.2 effectively compute a basis of ${\downarrow}\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$. However, since $\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$ is not necessarily downward closed, this does not directly entail an algorithm for deciding membership in this set.

Theorem 4.1. *Given a VAS $\mathcal{V}$ and $\boldsymbol{x} \in \mathbb{N}_\omega^d$, one can decide whether $\boldsymbol{x} \in \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$.*

We establish Theorem 4.1 by describing two semi-algorithms proving that $\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$ and its complement in $\mathbb{N}_\omega^d$ are both recursively enumerable sets. Let us start with the most interesting direction. We shall prove that $\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$ is recursively enumerable, by introducing *productive sequences*, a notion inspired by Hauschildt [23].

Definition 4.1. Let $\mathcal{V} = \langle A, \delta, \boldsymbol{x}_{in}\rangle$ be a VAS, and let $\pi = (u_i)_{0\leqslant i\leqslant k}$ be a sequence of words over A. We say that π is *productive* in $\mathcal{V}$ for a word $v = a_1 \cdots a_k \in A^*$ if the words

$$u_0^n a_1 u_1^n \cdots a_k u_k^n, \qquad n \geqslant 1$$

are all fireable from $\boldsymbol{x}_{in}$.

In particular, if π is productive for v, the state $\boldsymbol{x}_{in}+\delta(v)+n\delta(\pi)$ is a reachable state in $\mathcal{V}$, where $\delta(\pi) = \sum_{i=0}^k \delta(u_i)$. Definition 4.1 shows that the set $\big\{(\pi, v) \mid \pi \text{ productive in } \mathcal{V} \text{ for } v\big\}$ is co-recursively enumerable. The following characterization immediately gives an algorithm to decide membership in this set, showing that it is actually recursive.

Lemma 4.2. *A sequence $\pi = (u_i)_{0\leqslant i\leqslant k}$ is* productive *in $\mathcal{V}$ for a word $a_1 \cdots a_k$ if and only if*

1. *the partial sums $\delta(u_0) + \cdots + \delta(u_j)$ are nonnegative for every $j \in \{0, \ldots, k\}$, and*
2. *the word $u_0 a_1 u_1 \cdots a_k u_k$ is fireable from $\boldsymbol{x}_{in}$.*

Proof. Let us introduce the states $\boldsymbol{y}_0 = \boldsymbol{x}_{in}$ and $\boldsymbol{y}_j = \boldsymbol{x}_{in} + \delta(a_1 \cdots a_j)$ for $j \in \{1, \ldots, k\}$, and the partial sums $\boldsymbol{x}_{-1} = \boldsymbol{0}$ and $\boldsymbol{x}_j = \delta(u_0) + \cdots + \delta(u_j)$ for $j \in \{0, \ldots, k\}$. We put $u[-1, n] = \varepsilon$, $u[j, n] = u_0^n a_1 u_1^n \cdots a_j u_j^n$ for $j \geqslant 0$, and $v[j, n] = u[j, n]a_{j+1}$.

If π is productive for $v = a_1 \cdots a_k$, then $u[k, n]$ is fireable from $\boldsymbol{x}_{in}$ for all $n \geqslant 1$. Therefore, $u[j, n]$ is also fireable from $\boldsymbol{x}_{in}$ for $j \leqslant k$. We deduce that $\boldsymbol{x}_{in}+\delta(u[j, n]) = \boldsymbol{y}_j+n\boldsymbol{x}_j$ is nonnegative for every $n \in \mathbb{N}$. In particular $\boldsymbol{x}_j \geqslant \boldsymbol{0}$. We have proved (1), and (2) is obvious.

Conversely, assume that (1) and (2) both hold. For all $n \geqslant 1$, we have to show that $u[k, n]$ is fireable from $\boldsymbol{x}_{in}$, *i.e.*, that $\boldsymbol{x}_{in} + \delta(w) \geqslant \boldsymbol{0}$ for any nonempty prefix w of $u[k, n]$. Such a prefix is of the form $v[j-1, n]u_j^p u'_j$ for some $0 \leqslant j \leqslant k$, $0 \leqslant p < n$, and some prefix u'_j of u_j. By rearranging terms, we obtain

$$\begin{aligned}
\boldsymbol{x}_{in} + \delta(v[j-1, n]u_j^p u'_j) &= \boldsymbol{x}_{in} + \delta(u_0^n a_1 u_1^n \cdots a_{j-1} u_{j-1}^n a_j u_j^p u'_j) \\
&= \boldsymbol{x}_{in} + \delta(u_0 a_1 u_1 \cdots a_j u'_j) + (n-1)\boldsymbol{x}_{j-1} + \delta(u_j^p) \\
&= \boldsymbol{x}_{in} + \delta(u_0 a_1 u_1 \cdots a_j u'_j) + (n-p-1)\boldsymbol{x}_{j-1} + p\boldsymbol{x}_j.
\end{aligned}$$

By (1), we have $\boldsymbol{x}_{j-1}, \boldsymbol{x}_j \geqslant \boldsymbol{0}$. By (2), the word $u_0 a_1 u_1 \cdots a_k u_k$ is fireable from $\boldsymbol{x}_{in}$, and in particular, $\boldsymbol{x}_{in}+\delta(u_0 a_1 u_1 \cdots a_j u'_j) \geqslant \boldsymbol{0}$. Therefore, $\boldsymbol{x}_{in}+\delta(v[n, j-1]u_j^p u'_j) \geqslant \boldsymbol{0}$, which proves that $u_0^n a_1 u_1^n \cdots a_k u_k^n$ is fireable from $\boldsymbol{x}_{in}$. We have shown that π is productive for $a_1 \cdots a_k$. □

We will now show in Proposition 4.4 below that limits of reachable states are witnessed by productive sequences. Its essential argument is Higman's Lemma. We recall that an ordering $\preccurlyeq$ is *well* if every infinite sequence $(\ell_n)_{n\in\mathbb{N}}$ admits an infinite increasing subsequence $(\ell_{n_k})_{k\in\mathbb{N}}$: $\ell_{n_0} \preccurlyeq \ell_{n_1} \preccurlyeq \ell_{n_2} \preccurlyeq \cdots$. The pointwise ordering over $\mathbb{N}^d$ or over $\mathbb{N}_\omega^d$ is well (Dickson's Lemma).

Higman's Lemma. Let Σ be a (possibly infinite) set. Given an ordering $\preccurlyeq$ over Σ, let $\preccurlyeq^*$ be the ordering over Σ^* defined as follows: for $u, v \in \Sigma^*$, we have $u \preccurlyeq^* v$ if $u = a_1 \cdots a_n$ with $a_i \in \Sigma$, $v = v_0 b_1 v_1 \cdots v_{n-1} b_n v_n$, with $v_i \in \Sigma^*$, $b_j \in \Sigma$, and for all $i = 1, \ldots, n$, we have $a_i \preccurlyeq b_i$. In other words, u is obtained from v by removing some letters, and then replacing some of the remaining letters by smaller ones. Higman's Lemma is the following result. See for instance [10] for a proof.

Lemma 4.3 (Higman)**.** *If $\preccurlyeq$ is a well ordering over Σ, then $\preccurlyeq^*$ is a well ordering over Σ^*.*

We extend the multiplication over $\mathbb{N}_\omega$ by $\omega \cdot 0 = 0 = 0 \cdot \omega$ and $\omega \cdot k = \omega = k \cdot \omega$ if $k \neq 0$. This multiplication then extends componentwise to the scalar multiplication of $\mathbb{N}_\omega^d$ by $\mathbb{N}_\omega$.

Proposition 4.4. *Let $\mathcal{V} = \langle A, \delta, \boldsymbol{x}_{in}\rangle$ be a VAS. Then*

$$\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}) = \big\{\boldsymbol{x}_{in} + \delta(v) + \omega\delta(\pi) \mid v \in A^* \textit{and } \pi \textit{ productive in } \mathcal{V} \textit{ for } v\big\}.$$

Proof. For the inclusion from right to left, if π is a productive sequence for a word v, then $\boldsymbol{x}_{in}+\delta(v)+\omega\delta(\pi)$ is the limit of the sequence $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ with $\boldsymbol{x}_n = \boldsymbol{x}_{in}+\delta(v)+n\delta(\pi)$, which is a reachable state by Definition 4.1. We prove the reverse inclusion thanks to Higman's lemma. We follow the approach of Jančar introduced in [24, Section 6].

Let us first introduce a well ordering $\sqsubseteq$ over $\mathsf{Reach}(\mathcal{V})$, using a temporary ordering $\preccurlyeq$. Consider the infinite set $\Sigma = A \times \mathbb{N}_\omega^d$. This set is well ordered by $\preccurlyeq$, defined by:

$$(a, \boldsymbol{y}) \preccurlyeq (b, \boldsymbol{z}) \text{ if and only if } a = b \text{ and } \boldsymbol{y} \leqslant \boldsymbol{z}.$$

Since $\preccurlyeq$ is a well ordering, Higman's lemma shows that $\preccurlyeq^*$ is a well ordering over Σ^*. We associate to every reachable state $\boldsymbol{y} \in \mathsf{Reach}(\mathcal{V})$ a word $\alpha_{\boldsymbol{y}}$ in Σ^* as follows: since $\boldsymbol{y}$ is reachable, the set $V_{\boldsymbol{y}} = \{v \in A^* \mid \boldsymbol{x}_{in} \xrightarrow{v} \boldsymbol{y}\}$ is nonempty. Let us choose arbitrarily some $v_{\boldsymbol{y}}$ in $V_{\boldsymbol{y}}$ (the actual choice is irrelevant, one can choose for instance the minimal element of $V_{\boldsymbol{y}}$ wrt. the lexicographic ordering). Let $v_{\boldsymbol{y}} = a_1 \cdots a_k$, with $k \geqslant 0$ and $a_i \in A$. We introduce the sequence $(\boldsymbol{y}_i)_{0\leqslant i\leqslant k}$ of states defined by $\boldsymbol{y}_0 = \boldsymbol{x}_{in}$, and $\boldsymbol{y}_i = \boldsymbol{x}_{in} + \delta(a_1 \cdots a_i)$ for $i \geqslant 1$. We let

$$\alpha_{\boldsymbol{y}} = (a_1, \boldsymbol{y}_1) \cdots (a_k, \boldsymbol{y}_k).$$

We define the ordering $\sqsubseteq$ over $\mathsf{Reach}(\mathcal{V})$ by $\boldsymbol{y} \sqsubseteq \boldsymbol{z}$ if $\alpha_{\boldsymbol{y}} \preccurlyeq^* \alpha_{\boldsymbol{z}}$ and $\boldsymbol{y} \leqslant \boldsymbol{z}$. Since the orderings $\preccurlyeq^*$ over Σ^* and $\leqslant$ over $\mathbb{N}^d$ are well, we deduce that $\sqsubseteq$ is a well ordering over $\mathsf{Reach}(\mathcal{V})$.

Now, let us pick $\boldsymbol{x} \in \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$: $\boldsymbol{x}$ is the limit of a sequence $(\boldsymbol{x}_k)_{k\in\mathbb{N}}$ of reachable states. By extracting a subsequence if necessary, one can assume that for every index i:

(*i*) if $\boldsymbol{x}(i) < \omega$, then $\boldsymbol{x}_k(i)$ is constant, equal to $\boldsymbol{x}(i)$, and
(*ii*) if $\boldsymbol{x}(i) = \omega$, then $(\boldsymbol{x}_k(i))_{k\in\mathbb{N}}$ is strictly increasing.

Denote by α_j the word $\alpha_{\boldsymbol{x}_j}$ associated to the reachable state $\boldsymbol{x}_j$. Since $\sqsubseteq$ is a well ordering, there exist $m < n$ such that $\boldsymbol{x}_m \sqsubseteq \boldsymbol{x}_n$. By construction of α_m, there exists a word $v = a_1 \cdots a_k$ with $a_j \in A$ such that the sequence $(\boldsymbol{y}_j)_{1\leqslant j\leqslant k}$ defined by $\boldsymbol{y}_j = \boldsymbol{x}_{in} + \delta(a_1 \cdots a_j)$ for every $j \in \{1, \ldots, k\}$ satisfies:

$$\alpha_m = (a_1, \boldsymbol{y_1}) \cdots (a_k, \boldsymbol{y}_k)$$

Since $\alpha_m \preccurlyeq^* \alpha_n$ and by definition of $\preccurlyeq^*$, there exist a sequence $(\boldsymbol{z}_j)_{1\leqslant j\leqslant k}$ of states with $\boldsymbol{y}_j \leqslant \boldsymbol{z}_j$, and a sequence $(\beta_j)_{0\leqslant j\leqslant k}$ of words in Σ^* such that the following equality holds:

$$\alpha_n = \beta_0(a_1, \boldsymbol{z}_1)\beta_1 \cdots (a_k, \boldsymbol{z}_k)\beta_k$$

We call *label* of a word $(b_1, \boldsymbol{t}_1)\cdots(b_\ell, \boldsymbol{t}_\ell)$ over Σ the word $b_1\cdots b_\ell$ over A. Consider the sequence $\pi = (u_j)_{0\leqslant j\leqslant k}$ where u_j is the label of β_j. Since $\boldsymbol{x}_m$ and $\boldsymbol{x}_n$ are reachable, we have by definition of α_m and α_n:

$$\begin{aligned} \boldsymbol{x}_{in} &\xrightarrow{a_1} \boldsymbol{y}_1 \cdots \xrightarrow{a_k} \boldsymbol{y}_k = \boldsymbol{x}_m \\ \boldsymbol{x}_{in} &\xrightarrow{u_0 a_1} \boldsymbol{z}_1 \cdots \xrightarrow{u_{k-1}a_k} \boldsymbol{z}_k \xrightarrow{u_k} \boldsymbol{x}_n \end{aligned} \tag{4.1}$$

From (4.1), we obtain in particular

$$\boldsymbol{z}_j = \boldsymbol{y}_j + \delta(u_0) + \cdots + \delta(u_{j-1}) \text{ for every } j \in \{1, \ldots, k\} \tag{4.2}$$

and in the same way,

$$\boldsymbol{x}_n = \boldsymbol{x}_m + \delta(\pi) \tag{4.3}$$

Using (4.2) with $\boldsymbol{y}_j \leqslant \boldsymbol{z}_j$ for $1 \leqslant j \leqslant k$, and (4.3) with $\boldsymbol{x}_m \leqslant \boldsymbol{x}_n$, we deduce that π satisfies property (1) of Lemma 4.2. Since, by (4.1), it also satisfies (2), it is productive for v.

It remains to prove that $\boldsymbol{x} = \boldsymbol{y}$ where $\boldsymbol{y} = \boldsymbol{x}_{in} + \delta(v) + \omega\delta(\pi)$. Let $i \in \{1, \ldots, d\}$.

- If $\boldsymbol{x}(i) < \omega$ then by (i), we get $\boldsymbol{x}_m(i) = \boldsymbol{x}(i) = \boldsymbol{x}_n(i)$, so using (4.3), we obtain $\delta(\pi)(i) = 0$. Since we have $\boldsymbol{x}_n = \boldsymbol{x}_{in} + \delta(v) + \delta(\pi)$ by (4.1), we deduce that $\boldsymbol{x}(i) = \boldsymbol{x}_n(i) = \boldsymbol{x}_{in}(i) + \delta(v)(i) = \boldsymbol{y}(i)$.
- If $\boldsymbol{x}(i) = \omega$, then by (ii) $\boldsymbol{x}_m(i) < \boldsymbol{x}_n(i)$. We deduce from (4.3) that $\delta(\pi)(i) > 0$. Therefore, $\boldsymbol{x}(i) = \omega = \boldsymbol{y}(i)$.

Finally, $\boldsymbol{x} = \boldsymbol{y}$, and we have proved that there exists a productive sequence π for a word v such that $\boldsymbol{x} = \boldsymbol{x}_{in} + \delta(v) + \omega\delta(\pi)$. □

Proposition 4.4 and Lemma 4.2 provide a semi-algorithm to test whether a given vector $\boldsymbol{x} \in \mathbb{N}^d_\omega$ belongs to $\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$: it suffices to enumerate the pairs (π, v), where π is productive for v, and to check whether $\boldsymbol{x} = \boldsymbol{x}_{in} + \delta(v) + \omega\delta(\pi)$.

It is easier to prove that the *complement* of $\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$ is recursively enumerable. Consider $\boldsymbol{y} \in \mathbb{N}^d_\omega$. We introduce d distinct additional elements $b_1, \ldots, b_d \notin A$. Let $B = \{b_1, \ldots, b_d\}$. We now introduce the VAS $\mathcal{V}_{\boldsymbol{y}} = \langle A \uplus B, \delta_{\boldsymbol{y}}, \boldsymbol{x}_{in}\rangle$, where $\delta_{\boldsymbol{y}}$ extends δ by:

$$\delta_{\boldsymbol{y}}(b_i) = \begin{cases} \mathbf{0} & \text{if } \boldsymbol{y}(i) < \omega, \\ -\boldsymbol{e}_i & \text{if } \boldsymbol{y}(i) = \omega. \end{cases}$$

Finally, we define from $\boldsymbol{y}$ a sequence $(\boldsymbol{y}_\ell)_\ell$ converging to $\boldsymbol{y}$, by $\boldsymbol{y}_\ell(i) = \begin{cases} \boldsymbol{y}(i) & \text{if } \boldsymbol{y}(i) < \omega, \\ \ell & \text{if } \boldsymbol{y}(i) = \omega. \end{cases}$

Lemma 4.5. *Let $\mathcal{V}_{\boldsymbol{y}}$ and $(\boldsymbol{y}_\ell)_\ell$ constructed from $\boldsymbol{y}$ as above. Then,*

$$\boldsymbol{y} \notin \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}) \iff \exists \ell \in \mathbb{N},\ \boldsymbol{y}_\ell \notin \mathsf{Reach}(\mathcal{V}_{\boldsymbol{y}}). \tag{4.4}$$

In particular, the complement of $\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$ is effectively recursively enumerable.

Proof. We prove the following, which is equivalent to (4.4):

$$\boldsymbol{y} \in \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}) \iff \forall \ell \in \mathbb{N},\ \boldsymbol{y}_\ell \in \mathsf{Reach}(\mathcal{V}_{\boldsymbol{y}}).$$

Assume that $\boldsymbol{y} \in \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$. Fix $\ell \in \mathbb{N}$. There exists a sequence $(\boldsymbol{z}_n)_n$ of elements of $\mathsf{Reach}(\mathcal{V})$ such that $\lim_n \boldsymbol{z}_n = \boldsymbol{y}$, so for n large enough, we have for all $i = 1, \dots, d$:

- $\boldsymbol{z}_n(i) = \boldsymbol{y}(i) = \boldsymbol{y}_\ell(i)$ if $\boldsymbol{y}(i) < \omega$,
- $\boldsymbol{z}_n(i) \geqslant \ell = \boldsymbol{y}_\ell(i)$ if $\boldsymbol{y}(i) = \omega$.

Then $\boldsymbol{z}_n \xrightarrow{u} \boldsymbol{y}_\ell$ in $\mathcal{V}_{\boldsymbol{y}}$, with $u = \prod_{i=1}^{d} b_i^{\boldsymbol{z}_n(i) - \boldsymbol{y}_\ell(i)}$. Since $\boldsymbol{z}_n$ is reachable from $\boldsymbol{x}_{in}$ (already in $\mathcal{V}$), we deduce that $\boldsymbol{y}_\ell \in \mathsf{Reach}(\mathcal{V}_{\boldsymbol{y}})$.

Conversely, assume that $\boldsymbol{y}_\ell \in \mathsf{Reach}(\mathcal{V}_{\boldsymbol{y}})$ for all ℓ, and let $u_\ell \in (A \cup B)^*$ such that $\boldsymbol{x}_{in} \xrightarrow{u_\ell} \boldsymbol{y}_\ell$, in $\mathcal{V}_{\boldsymbol{y}}$. Consider the word v_ℓ obtained from u_ℓ by erasing all letters of B. Since $\delta(b) \leqslant \mathbf{0}$ for $b \in B$, the word v_ℓ is still fireable from $\boldsymbol{x}_{in}$, so that $\boldsymbol{z}_\ell = \boldsymbol{x}_{in} + \delta(v_\ell) \in \mathsf{Reach}(\mathcal{V})$. Moreover, by definition of $\mathcal{V}_{\boldsymbol{y}}$, $\boldsymbol{z}_\ell(i) = \boldsymbol{y}_\ell(i)$ if $\boldsymbol{y}(i) < \omega$ and $\boldsymbol{y}_\ell(i) \leqslant \boldsymbol{z}_\ell(i)$ otherwise. Therefore, $\lim_\ell \boldsymbol{z}_\ell = \lim_\ell \boldsymbol{y}_\ell = \boldsymbol{y}$, and it follows that $\boldsymbol{y} \in \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$.

This shows (4.4). Hence, we can enumerate vectors $\boldsymbol{y}_\ell$ and test, for each $\boldsymbol{y}_\ell$, its membership in $\mathsf{Reach}(\mathcal{V}_{\boldsymbol{y}})$. This proves that $\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$ is co-recursively enumerable. □

Theorem 4.1 now follows from Proposition 4.4 and Lemma 4.5.

5. Refined and filtered covers

In this section, we introduce two new notions of covers: refined and filtered covers. Both are parameterized, and the following inclusions will hold, regardless of the parameters:

$$\mathsf{Reach}(\mathcal{V}) \subseteq \mathsf{RefinedCover}(\mathcal{V}) \subseteq \mathsf{Cover}(\mathcal{V}), \qquad \text{and} \qquad \mathsf{FilteredCover}(\mathcal{V}) \subseteq \mathsf{Cover}(\mathcal{V})$$

Let us first introduce the refined cover, a set hybrid between the reachability and cover sets, that to our knowledge has not yet been considered. Instead of the downward closure $\mathsf{Cover}(\mathcal{V})$ of $\mathsf{Reach}(\mathcal{V})$ wrt. the pointwise ordering $\leqslant$, we consider

$$\mathsf{Cover}_{\leqslant_P}(\mathcal{V}) = {\downarrow_{\leqslant_P}}\mathsf{Reach}(\mathcal{V}),$$

that is, we replace $\leqslant$ with an ordering $\leqslant_P$ over $\mathbb{N}_\omega^d$ parameterized by a set of "positions" $P \subseteq \{1, \dots, d\}$:

$$\boldsymbol{x} \leqslant_P \boldsymbol{y} \quad \text{if} \quad \begin{cases} \boldsymbol{x}(i) = \boldsymbol{y}(i) & \text{for } i \in P, \\ \boldsymbol{x}(i) \leqslant \boldsymbol{y}(i) & \text{for } i \notin P. \end{cases}$$

The set P contains the components for which we insist on keeping equality. Thus, $\leqslant_\varnothing$ is the usual pointwise ordering $\leqslant$, while $\leqslant_{\{1,\dots,d\}}$ boils down to equality. Notice that $\leqslant_P$ is *not* a well ordering, except if $P = \varnothing$ (*e.g.*, $\mathbb{N}$ ordered by $\leqslant_{\{1\}}$ consists only of incomparable elements, since in this case, $\leqslant_{\{1\}}$ is just equality).

The ordering $\leqslant_{\{1\}}$ will be abbreviated as $\leqslant_1$. It is a natural order to study for a VAS_z (recall that the zero-test occurs on the first component). Indeed, the transition relation of a VAS_z is monotonic with respect to this order: if $\boldsymbol{x} \xrightarrow{u} \boldsymbol{x}'$ and $\boldsymbol{x} \leqslant_1 \boldsymbol{y}$, then there exists $\boldsymbol{y}'$ with $\boldsymbol{y} \xrightarrow{u} \boldsymbol{y}'$ and $\boldsymbol{x}' \leqslant_1 \boldsymbol{y}'$. In words, from a $\leqslant_1$-larger state than $\boldsymbol{x}$, one can perform the same transitions as from $\boldsymbol{x}$, and reach a state $\leqslant_1$-above that the one reached from $\boldsymbol{x}$. This is clearly not the case if one uses the pointwise ordering $\leqslant$ instead of $\leqslant_1$: some zero-tests may fail from the largest state and succeed from the smallest one.

More precisely, testing if $\mathsf{Cover}_{\leqslant_1}(\mathcal{V})$ contains a vector whose first component is 0 is what we need to design our algorithm computing the cover of a VAS with one zero test. Unfortunately, the set $\mathsf{Cover}_{\leqslant_1}(\mathcal{V})$ cannot be represented by a finite set of $\leqslant_1$-maximal elements, since it may well have infinitely many of them. Actually, the following theorem shows that we cannot find a sensible way to compute a representation of this set, as any representation would not allow to test for equality.

Theorem 5.1. *Given two VAS $\mathcal{V}_1$, $\mathcal{V}_2$, it is undecidable whether* $\mathsf{Cover}_{\leqslant_1}(\mathcal{V}_1) = \mathsf{Cover}_{\leqslant_1}(\mathcal{V}_2)$.

Proof. We reduce the equality problem $\mathsf{Reach}(\mathcal{V}_1) = \mathsf{Reach}(\mathcal{V}_2)$, which is known to be undecidable [4, 22], to the problem of the statement. Let us first consider a VAS $\mathcal{V} = \langle A, \delta, \boldsymbol{x}_{in}\rangle$ of dimension d. We introduce a VAS $\mathcal{V}' = \langle A, \delta', \boldsymbol{x}'_{in}\rangle$ of dimension $d+1$ that counts in the first component the sum of the other components. Formally, $\boldsymbol{x}'_{in} = \big(\sum_{i=1}^{d} \boldsymbol{x}_{in}(i), \boldsymbol{x}_{in}\big)$ and $\delta'(a) = \big(\sum_{i=1}^{d} \delta(a)(i), \delta(a)\big)$ for every $a \in A$. Observe that the following equivalence holds:

$$(n, \boldsymbol{x}) \in \mathsf{Reach}(\mathcal{V}') \iff \boldsymbol{x} \in \mathsf{Reach}(\mathcal{V}) \text{ and } n = \sum_{i=1}^{d} \boldsymbol{x}(i).$$

Finally, consider two VAS $\mathcal{V}_1$ and $\mathcal{V}_2$, and just observe that $\mathsf{Reach}(\mathcal{V}_1) = \mathsf{Reach}(\mathcal{V}_2)$ if and only if $\mathsf{Cover}_{\leqslant_1}(\mathcal{V}'_1) = \mathsf{Cover}_{\leqslant_1}(\mathcal{V}'_2)$. □

So, we cannot hope for a useful representation of the sets $\mathsf{Cover}_{\leqslant_P}(\mathcal{V})$. However, one can capture the needed information differently, by replacing the downward closure $\downarrow_{\leqslant_P}$ in $\mathsf{Cover}_{\leqslant_P}(\mathcal{V}) = \downarrow_{\leqslant_P}\mathsf{Reach}(\mathcal{V})$ with another operator $\Downarrow_{\boldsymbol{f}}$, parameterized by a vector $\boldsymbol{f}$ of $\mathbb{N}_\omega^d$ (the letter $\boldsymbol{f}$ stands for *filter*). Informally, $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ is a downward closure taking into account only elements of $\boldsymbol{M}$ that agree with $\boldsymbol{f}$ on its finite components. Other elements will just be discarded. Formally, for $\boldsymbol{f} \in \mathbb{N}_\omega^d$ and $\boldsymbol{M} \subseteq \mathbb{N}_\omega^d$, we define the *filtered cover* $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ by:

$$\mathrm{Filter}(\boldsymbol{M}, \boldsymbol{f}) = \Big\{\boldsymbol{x} \in \boldsymbol{M} \mid \bigwedge_{i=1}^{d} \big[\, \boldsymbol{f}(i) < \omega \Longrightarrow \boldsymbol{x}(i) = \boldsymbol{f}(i)\big]\Big\},$$
$$\Downarrow_{\boldsymbol{f}}\boldsymbol{M} = \Big\downarrow \mathrm{Filter}(\boldsymbol{M}, \boldsymbol{f}).$$

Observe that $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ is a downward closed subset of $\downarrow\boldsymbol{M}$, and that $\Downarrow_{(\omega,\omega,\ldots,\omega)}\boldsymbol{M} = \downarrow\boldsymbol{M}$. Elements of the minimal basis of $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ agree with $\boldsymbol{f}$ on components i where $\boldsymbol{f}(i) < \omega$. One can check that the limit and filter operators commute:

$$\mathrm{Filter}(\mathsf{Lim}\,\boldsymbol{M}, \boldsymbol{f}) = \mathsf{Lim}\,\mathrm{Filter}(\boldsymbol{M}, \boldsymbol{f}).$$

Since the limit and the downward closure operators also commute (see (2.5)), we obtain

$$\Downarrow_{\boldsymbol{f}}\mathsf{Lim}\,\boldsymbol{M} = \mathsf{Lim}\Downarrow_{\boldsymbol{f}}\boldsymbol{M}. \tag{5.1}$$

The motivation for considering filtered covers is that, for $\boldsymbol{f} = (0, \omega, \ldots, \omega) \in \mathbb{N}_\omega^d$ and $\boldsymbol{M} = \mathsf{Reach}(\mathcal{V})$ where $\mathcal{V}$ is a VAS of dimension d, the set $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ captures all information we need to overcome the difficulty described on page 3. Moreover, contrary to the refined cover of a VAS, all its filtered covers are computable, as stated in Theorem 5.2 below. Our goal in this section is to describe an algorithm computing a filtered cover of a VAS. Our algorithm both refines Karp and Miller's one to compute the usual cover, and generalizes Theorem 4.1.

Theorem 5.2. *Let $\mathcal{V}$ be a VAS. Given $\boldsymbol{f} \in \mathbb{N}_\omega^d$, one can compute a basis of* $\Downarrow_{\boldsymbol{f}}\mathsf{Reach}(\mathcal{V})$.

Karp and Miller's algorithm computing $\mathsf{Cover}(\mathcal{V})$ corresponds to the case $\boldsymbol{f} = (\omega, \dots, \omega)$. Since $\boldsymbol{M}$ and $\mathsf{Lim}\,\boldsymbol{M}$ have the same bases (by definition (2.3) of a basis), computing a basis of $\Downarrow_{\boldsymbol{f}}\mathsf{Reach}(\mathcal{V})$ is the same as computing a basis of $\mathsf{Lim}\Downarrow_{\boldsymbol{f}}\mathsf{Reach}(\mathcal{V})$, *i.e.*, by (5.1), of $\Downarrow_{\boldsymbol{f}}\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$. We first reduce this computation to a decision problem, as in [35, Th. 2.10].

For $\boldsymbol{M} \subseteq \mathbb{N}^d_\omega$, let us introduce the following set:

$$\mathcal{F}(\boldsymbol{M}) = \big\{(\boldsymbol{f}, \boldsymbol{y}) \in \mathbb{N}^d_\omega \times \mathbb{N}^d_\omega \mid \boldsymbol{y} \in \Downarrow_{\boldsymbol{f}}\boldsymbol{M}\big\}.$$

Lemma 5.3. *Let $\boldsymbol{M} \subseteq \mathbb{N}^d_\omega$ be a limit closed set. From an algorithm solving the membership problem for $\mathcal{F}(\boldsymbol{M})$, one can construct an algorithm which, given an input vector $\boldsymbol{f} \in \mathbb{N}^d_\omega$, outputs a basis of $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$.*

Proof. Observe that $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ has the same bases as $\mathbb{N}^d \cap \Downarrow_{\boldsymbol{f}}\boldsymbol{M}$. Now, if $\boldsymbol{D} \subseteq \mathbb{N}^d$ is downward closed, one can compute a basis $\boldsymbol{B_D} \subseteq \mathbb{N}^d_\omega$ of $\boldsymbol{D}$ from a basis $\boldsymbol{B_U}$ of its (upward closed) complement $\boldsymbol{U} = \mathbb{N}^d \setminus \boldsymbol{D}$: an algorithm generates all candidates $\boldsymbol{B_D}$ for a basis of $\boldsymbol{D}$, (*i.e.*, all finite subsets of the countable set $\mathbb{N}^d_\omega$), and checks for each candidate whether it is indeed a basis of $\boldsymbol{D}$, *i.e.*, that the union of the sets $\mathbb{N}^d \cap {\downarrow}\boldsymbol{B_D}$ and $\mathbb{N}^d \cap {\uparrow}\boldsymbol{B_U}$ is $\mathbb{N}^d$, and that their intersection is empty. This property is Presburger definable, whence decidable.

Hence, to compute a basis of $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ given $\boldsymbol{f}$, it suffices to compute a basis of $\mathbb{N}^d \setminus \Downarrow_{\boldsymbol{f}}\boldsymbol{M}$. Now, [35, Th. 2.10] describes an algorithm computing such a basis from an algorithm deciding, given $\boldsymbol{y} \in \mathbb{N}^d_\omega$, whether $(\mathbb{N}^d \setminus \Downarrow_{\boldsymbol{f}}\boldsymbol{M}) \cap {\downarrow}\boldsymbol{y} = \varnothing$, or equivalently, whether $\mathbb{N}^d \cap {\downarrow}\boldsymbol{y} \subseteq \Downarrow_{\boldsymbol{f}}\boldsymbol{M}$. Note that $\boldsymbol{y}$ may have some components whose value is ω, so, if $\boldsymbol{M}$ were an arbitrary set, it might happen that ${\downarrow}\boldsymbol{y} \not\subseteq \Downarrow_{\boldsymbol{f}}\boldsymbol{M}$. However, $\boldsymbol{M}$ is limit closed, and therefore $\mathbb{N}^d \cap {\downarrow}\boldsymbol{y} \subseteq \Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ is equivalent to $\boldsymbol{y} \in \Downarrow_{\boldsymbol{f}}\boldsymbol{M}$, that is, to $(\boldsymbol{f}, \boldsymbol{y}) \in \mathcal{F}(\boldsymbol{M})$. □

Notice that Lemma 5.3 requires as input an algorithm solving the membership problem in $\mathcal{F}(\boldsymbol{M})$, *i.e.*, a unique algorithm solving the membership of $\boldsymbol{y}$ in $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ where $\boldsymbol{f}$ is an input parameter. This hypothesis cannot be weakened by just assuming that for each $\boldsymbol{f}$ we have an algorithm deciding the membership of $\boldsymbol{y}$ in $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$. In fact this hypothesis is a tautology, since the set $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ is recursive, as every downward closed set. The lemma becomes clearly wrong without any condition on $\boldsymbol{M}$.

We will now reduce membership in $\mathcal{F}(\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}))$ to a similar problem involving refined covers. The next lemma provides a relationship between the sets $\Downarrow_{\boldsymbol{f}}\boldsymbol{M}$ and ${\downarrow_{\leqslant_P}}\boldsymbol{M}$.

Lemma 5.4. *Let $\boldsymbol{M} \subseteq \mathbb{N}^d_\omega$, $P \subseteq \{1, \dots, d\}$, and $\boldsymbol{y} \in \mathbb{N}^d_\omega$. Define $\boldsymbol{f} \in \mathbb{N}^d_\omega$ by*

$$\boldsymbol{f}(i) = \begin{cases} \boldsymbol{y}(i) & \text{if } i \in P, \text{ and} \\ \omega & \text{otherwise.} \end{cases} \tag{5.2}$$

Then we have:

$$\boldsymbol{y} \in {\downarrow_{\leqslant_P}}\boldsymbol{M} \iff \boldsymbol{y} \in \Downarrow_{\boldsymbol{f}}\boldsymbol{M}. \tag{5.3}$$

Proof. Assume first that $\boldsymbol{y} \in {\downarrow_{\leqslant_P}}\boldsymbol{M}$. Then, there exists $\boldsymbol{x} \in \boldsymbol{M}$ such that $\boldsymbol{y} \leqslant_P \boldsymbol{x}$. We prove that $\boldsymbol{x} \in \mathrm{Filter}(\boldsymbol{M}, \boldsymbol{f})$ by observing that if i is an index such that $\boldsymbol{f}(i) < \omega$, then $i \in P$ and $\boldsymbol{f}(i) = \boldsymbol{y}(i) < \omega$. From $i \in P$ we get $\boldsymbol{x}(i) = \boldsymbol{y}(i)$. Hence $\boldsymbol{x}(i) = \boldsymbol{f}(i)$ and we have proved that $\boldsymbol{x} \in \mathrm{Filter}(\boldsymbol{M}, \boldsymbol{f})$. Since $\boldsymbol{y} \leqslant \boldsymbol{x}$, we get $\boldsymbol{y} \in \Downarrow_{\boldsymbol{f}}\boldsymbol{M}$.

Conversely, assume that $\boldsymbol{y} \in \Downarrow_{\boldsymbol{f}}\boldsymbol{M}$: there exists $\boldsymbol{x} \in \mathrm{Filter}(\boldsymbol{M}, \boldsymbol{f})$ such that $\boldsymbol{y} \leqslant \boldsymbol{x}$. Let $i \in P$. If $\boldsymbol{y}(i) = \omega$ then from $\boldsymbol{y}(i) \leqslant \boldsymbol{x}(i)$ we get $\boldsymbol{y}(i) = \boldsymbol{x}(i)$. If $\boldsymbol{y}(i) < \omega$ then $\boldsymbol{f}(i) = \boldsymbol{y}(i)$ and form $\boldsymbol{x} \in \mathrm{Filter}(\boldsymbol{M}, \boldsymbol{f})$ we get $\boldsymbol{x}(i) = \boldsymbol{f}(i)$. Hence in both cases, we have $\boldsymbol{x}(i) = \boldsymbol{y}(i)$. We have proved that $\boldsymbol{y} \leqslant_P \boldsymbol{x}$. Therefore $\boldsymbol{y} \in {\downarrow_{\leqslant_P}}\boldsymbol{M}$. □

Let us now introduce another set, again for a set $\boldsymbol{M} \subseteq \mathbb{N}_\omega^d$:

$$\mathcal{P}(\boldsymbol{M}) = \left\{(P, \boldsymbol{y}) \in 2^{\{1,\dots,d\}} \times \mathbb{N}_\omega^d \mid \boldsymbol{y} \in {\downarrow_{\leqslant_P}}\boldsymbol{M}\right\}$$

Corollary 5.5 (of Lemma 5.4)**.** *The membership problems in $\mathcal{P}(\boldsymbol{M})$ and in $\mathcal{F}(\boldsymbol{M})$ are inter-reducible. Both reductions are effective: from an algorithm solving the first problem, we construct an algorithm solving the second one.*

Proof. From $P \subseteq \{1, \dots, d\}$ and $\boldsymbol{y} \in \mathbb{N}_\omega^d$, define $\boldsymbol{f} \in \mathbb{N}_\omega^d$ by (5.2). From (5.3), we deduce that $(P, \boldsymbol{y}) \in \mathcal{P}(\boldsymbol{M})$ if and only if $(\boldsymbol{f}, \boldsymbol{y}) \in \mathcal{F}(\boldsymbol{M})$.
Conversely, let $\boldsymbol{f} \in \mathbb{N}_\omega^d$ and $\boldsymbol{y} \in \mathbb{N}_\omega^d$. Observe that if $\boldsymbol{y} \not\leqslant \boldsymbol{f}$ then $\boldsymbol{y} \notin {\Downarrow_{\boldsymbol{f}}}\boldsymbol{M}$. So we can assume that $\boldsymbol{y} \leqslant \boldsymbol{f}$. We introduce the set $P = \left\{i \in \{1, \dots, d\} \mid \boldsymbol{f}(i) < \omega\right\}$ and the vector $\boldsymbol{z} \in \mathbb{N}_\omega^d$ defined by $\boldsymbol{z}(i) = \boldsymbol{f}(i)$ if $i \in P$ and $\boldsymbol{z}(i) = \boldsymbol{y}(i)$ otherwise. We have $\boldsymbol{y} \in {\Downarrow_{\boldsymbol{f}}}\boldsymbol{M}$ if and only if $\boldsymbol{z} \in {\Downarrow_{\boldsymbol{f}}}\boldsymbol{M}$. Moreover, from Lemma 5.4 we deduce that $\boldsymbol{z} \in {\Downarrow_{\boldsymbol{f}}}\boldsymbol{M}$ if and only if $\boldsymbol{z} \in {\downarrow_{\leqslant_P}}\boldsymbol{M}$. In summary, $(\boldsymbol{y}, \boldsymbol{f}) \in \mathcal{F}(\boldsymbol{M})$ if and only if $\boldsymbol{y} \leqslant \boldsymbol{f}$ and $(\boldsymbol{z}, P) \in \mathcal{P}(\boldsymbol{M})$. □

To establish Theorem 5.2, it remains, in view of Lemma 5.3 and Corollary 5.5, to find an algorithm solving membership to $\mathcal{P}(\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}))$. This is obtained by first proving that, for a VAS $\mathcal{V}_P$ suitably constructed from $\mathcal{V}$ and P, we have

$$\mathsf{Reach}(\mathcal{V}_P) = {\mathsf{Cover}_{\leqslant_P}}(\mathcal{V}) \tag{5.4}$$

which implies $\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_P) = \mathsf{Lim}\,{\mathsf{Cover}_{\leqslant_P}}(\mathcal{V}) = \mathsf{Lim}{\downarrow_{\leqslant_P}}\mathsf{Reach}(\mathcal{V}) = {\downarrow_{\leqslant_P}}\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$. Then, Theorem 4.1 applied to $\mathcal{V}_P$ will give an algorithm to decide membership in this set. Since there is a finite number of subsets P of $\{1, \dots, d\}$, this yields an algorithm to decide membership in $\mathcal{P}(\mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}))$.

So let $\mathcal{V} = \langle A, \delta, \boldsymbol{x}_{in}\rangle$ be a VAS and $P \subseteq \{1, \dots, d\}$, and let us define a VAS $\mathcal{V}_P$ satisfying (5.4). We consider d distinct additional elements $b_1, \dots, b_d \notin A$. Let $B = \{b_1, \dots, b_d\}$. We consider the VAS $\mathcal{V}_P = \langle A \uplus B, \delta_P, \boldsymbol{x}_{in}\rangle$, where δ_P extends δ by:

$$\delta_P(b_i) = \begin{cases} \boldsymbol{0} & \text{if } i \in P \\ -\boldsymbol{e_i} & \text{if } i \notin P. \end{cases}$$

Lemma 5.6. *Let $\mathcal{V}_P$ constructed from $\mathcal{V}$ and P as above. Then $\mathsf{Cover}_{\leqslant_P}(\mathcal{V}) = \mathsf{Reach}(\mathcal{V}_P)$.*

Proof. Let $\boldsymbol{x} \in {\mathsf{Cover}_{\leqslant_P}}(\mathcal{V})$. By definition, there exists $\boldsymbol{y} \in \mathsf{Reach}(\mathcal{V})$ such that $\boldsymbol{x} \leqslant_P \boldsymbol{y}$. Note that $\boldsymbol{y} \in \mathsf{Reach}(\mathcal{V}_P)$, and that $\boldsymbol{y} \xrightarrow{u} \boldsymbol{x}$ in $\mathcal{V}_P$ with $u = \prod_{i=1}^d b_i^{\boldsymbol{y}(i)-\boldsymbol{x}(i)}$, so $\boldsymbol{x} \in \mathsf{Reach}(\mathcal{V}_P)$. Conversely let $\boldsymbol{x} \in \mathsf{Reach}(\mathcal{V}_P)$, and $u \in (A \cup B)^*$ such that $\boldsymbol{x}_{in} \xrightarrow{u}_{\mathcal{V}_P} \boldsymbol{x}$. Let v be obtained from u by erasing all letters of B. Since $\delta_P(b) \leqslant \boldsymbol{0}$ for $b \in B$, the word v is fireable from $\boldsymbol{x}_{in}$. Thus $\boldsymbol{y} = \boldsymbol{x}_{in} + \delta(v) \in \mathsf{Reach}(\mathcal{V})$. By definition of $\mathcal{V}_P$ we have $\boldsymbol{x} \leqslant_P \boldsymbol{y}$, so $\boldsymbol{x} \in {\mathsf{Cover}_{\leqslant_P}}(\mathcal{V})$. □

As explained above, Theorem 5.2 is now established, by combining Lemmas 5.3 and Corollary 5.5 applied to $\boldsymbol{M} = \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V})$, as well as Lemma 5.6.

6. Computing the cover of a VAS with one zero-test

This section describes an algorithm computing a basis of the cover of a VAS_z given as input.

It will be convenient to consider VAS or VAS_z whose initial state belongs to $\mathbb{N}_\omega^d$. The semantics given by (3.1) is generalized by extending addition to $\mathbb{N}_\omega$, letting $\omega + n = n + \omega = \omega$ for all $n \in \mathbb{Z}$. Notice that all results obtained so far for a VAS, and in particular Theorem 5.2,

extend to VAS with such generalized initial states. Indeed, an ω value in some component of $\boldsymbol{x}_{in}$ remains frozen to ω, whatever action is executed, and can therefore be safely ignored.

We introduce a notation to change the initial state of a VAS/VAS$_z$ $\mathcal{V}$. For $\boldsymbol{x} \in \mathbb{N}_\omega^d$, we let $\mathcal{V}(\boldsymbol{x})$ be the VAS/VAS$_z$ obtained from $\mathcal{V}$ by replacing the initial state $\boldsymbol{x}_{in}$ by $\boldsymbol{x}$.

In this section, we fix a VAS$_z$ $\mathcal{V}_z = \langle A, a_z, \delta, \boldsymbol{x}_{in}\rangle$. To simplify the presentation, we assume without loss of generality that $\boldsymbol{x}_{in} \in \{0\} \times \mathbb{N}^{d-1}$, and that $\delta(a_z) \in \{0\} \times \mathbb{Z}^{d-1}$. In the sequel, we denote by $\mathcal{V} = \langle A, \delta, \boldsymbol{x}_{in}\rangle$ the VAS obtained from $\mathcal{V}_z$ by removing the zero test. We shall work with a single filter throughout the section: we introduce $\boldsymbol{f} = (0, \omega, \ldots, \omega)$.

Input/output of the algorithm. Our algorithm is inspired by Karp and Miller's one for a VAS [25]. Given as input a VAS$_z$ $\mathcal{V}_z$, it builds a finite tree with nodes labeled by vectors in $\{0\} \times \mathbb{N}_\omega^{d-1}$, such that when the algorithm terminates:

$$\text{The set } \boldsymbol{R} \text{ of node labels is a basis of } \Downarrow_{\boldsymbol{f}}\mathsf{Reach}(\mathcal{V}_z). \qquad (*)$$

Observe that, at the end of the algorithm, $\boldsymbol{R}$ is not a basis of the whole cover of $\mathcal{V}_z$, but only a basis of an $\boldsymbol{f}$-filtered cover of $\mathcal{V}_z$.

Let us first explain how to compute from $\boldsymbol{R}$ a basis of $\mathsf{Cover}(\mathcal{V}_z)$. If $\boldsymbol{x} \in \mathsf{Cover}(\mathcal{V}_z)$, then there exist $u \in A^*$ and $\boldsymbol{y} \in \mathbb{N}^d$ such that $\boldsymbol{x}_{in} \xrightarrow{u} \boldsymbol{y} \geqslant \boldsymbol{x}$. Let us factorize $u = u_1u_2$, where u_1 ends with the last zero test a_z, or is empty if there is no zero-test. Then, we have $\boldsymbol{x}_{in} \xrightarrow{u_1} \boldsymbol{r} \xrightarrow{u_2} \boldsymbol{y} \geqslant \boldsymbol{x}$, with $\boldsymbol{r} \in \{0\} \times \mathbb{N}^{d-1}$ (if u_1 is empty, we use the assumption $\boldsymbol{x}_{in} \in \{0\} \times \mathbb{N}^{d-1}$). In particular, $\boldsymbol{r} \in \Downarrow_{\boldsymbol{f}}\mathsf{Reach}(\mathcal{V}_z) = \downarrow\boldsymbol{R} \cap \mathbb{N}^d$ by $(*)$. Since no zero-test occurs in u_2, the state $\boldsymbol{y}$ reached after firing u belongs to $\mathsf{Reach}(\mathcal{V}(\boldsymbol{r}))$, and therefore, $\boldsymbol{x} \in \downarrow\mathsf{Reach}(\mathcal{V}(\boldsymbol{r}))$. This simple remark yields the following result:

Lemma 6.1. *If $\boldsymbol{R}$ is a basis of $\Downarrow_{\boldsymbol{f}}\mathsf{Reach}(\mathcal{V}_z)$, then $\mathsf{Cover}(\mathcal{V}_z) = \bigcup_{\boldsymbol{r}\in\boldsymbol{R}} \downarrow\mathsf{Reach}(\mathcal{V}(\boldsymbol{r}))$.*

In words, we obtain a basis of $\mathsf{Cover}(\mathcal{V}_z)$ as the union of all bases output by the usual Karp-Miller algorithm run on inputs $\mathcal{V}(\boldsymbol{r})$, for $\boldsymbol{r} \in \boldsymbol{R}$. Let us now explain how to compute $\boldsymbol{R}$.

Outline of the algorithm. To build a tree whose set of labels is $\boldsymbol{R} \subseteq \{0\} \times \mathbb{N}_\omega^d$, the algorithm works top-down from the root labeled by the initial state $\boldsymbol{x}_{in} \in \{0\} \times \mathbb{N}^{d-1}$. Its main loop is similar to that of the Karp-Miller algorithm: for each leaf of the tree,

(1) if the label of the leaf already occurs above it along the path to the root, then the leaf is not expanded, and will remain a leaf during the execution of the algorithm.
(2) Otherwise, we try to expand the tree from the leaf. As in the Karp-Miller algorithm:
   a. we perform some standard acceleration, which is explained below,
   b. we then expand the leaf, adding new children to it. However, unlike the Karp-Miller algorithm, which fires all original transitions of the VAS from the label of the leaf, we add two kinds of children to the current leaf labeled $\boldsymbol{x} \in \{0\} \times \mathbb{N}_\omega^{d-1}$:
      (*i*) one child corresponding to firing the zero-test from the leaf label, if possible,
      (*ii*) several children representing a basis of $\Downarrow_{\boldsymbol{f}}\mathsf{Reach}(\mathcal{V}(\boldsymbol{x}))$.

Note that Step (*ii*) involves $\mathcal{V}$ and not $\mathcal{V}_z$, *i.e.*, the zero-test is not considered during this step. It is a macro-step computing itself a basis of a cover, to be used in the whole computation. In the particular case where the VAS$_z$ is obtained by just adding to states of a VAS an extra first component, left untouched (therefore remaining 0 forever) and where the zero-test is never fired, step (*ii*) actually computes in one shot the cover of the original VAS (completed with the first component, left to 0). Theorem 5.2 shows that Step (*ii*) is effective.

We now enter the details of the algorithm. At any step of the execution, in the tree built by the algorithm, every ancestor node $n_{\boldsymbol{x}}$ of a node $n_{\boldsymbol{y}}$ satisfies the invariant $\boldsymbol{x} \stackrel{*}{\Rightarrow} \boldsymbol{y}$ where $\boldsymbol{x}, \boldsymbol{y}$ are the labels of $n_{\boldsymbol{x}}, n_{\boldsymbol{y}}$ and where $\stackrel{*}{\Rightarrow}$ is the binary relation defined over $\{0\} \times \mathbb{N}_\omega^{d-1}$ by:

$$\boldsymbol{x} \stackrel{*}{\Rightarrow} \boldsymbol{y} \;\text{ if }\; \boldsymbol{y} \in \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x})).$$

By the next lemma, it is sufficient to maintain this invariant along each parent-child edge.

Lemma 6.2. *The binary relation $\stackrel{*}{\Rightarrow}$ over $\{0\} \times \mathbb{N}_\omega^{d-1}$ is reflexive and transitive.*

The proof of Lemma 6.2 is itself based on the following intermediate statement. To shorten notation, for a set $\boldsymbol{M} \subseteq \mathbb{N}_\omega^d$, we let $\mathsf{Reach}\,\boldsymbol{M} = \bigcup_{\boldsymbol{x} \in \boldsymbol{M}} \mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x}))$ denote the set of states that can be reached in $\mathcal{V}_z$ from any initial vector chosen in $\boldsymbol{M}$ (in this notation used only in Lemmas 6.2 and 6.3, the VAS_z will always be $\mathcal{V}_z$, and is therefore omitted).

Lemma 6.3. *Let $\boldsymbol{M} \subseteq \mathbb{N}_\omega^d$. Then, we have* $\mathsf{Lim}\,\mathsf{Reach}\,\mathsf{Lim}\,\boldsymbol{M} = \mathsf{Lim}\,\mathsf{Reach}\,\boldsymbol{M}$.

Proof. Since $\boldsymbol{M} \subseteq \mathsf{Lim}\,\boldsymbol{M}$, we have $\mathsf{Lim}\,\mathsf{Reach}\,\boldsymbol{M} \subseteq \mathsf{Lim}\,\mathsf{Reach}\,\mathsf{Lim}\,\boldsymbol{M}$. For the other inclusion, pick $\boldsymbol{x} \in \mathsf{Lim}\,\mathsf{Reach}\,\mathsf{Lim}\,\boldsymbol{M}$. This means that we have the following situation

$$\boldsymbol{y}_n \overset{n\to\infty}{\dashrightarrow} \boldsymbol{y} \xrightarrow{u_n} \boldsymbol{x}_n \overset{n\to\infty}{\dashrightarrow} \boldsymbol{x},$$

with $\boldsymbol{y}_n \in \boldsymbol{M}$, $\boldsymbol{y}, \boldsymbol{x}_n \in \mathbb{N}_\omega^d$ and $u_n \in A^*$ for all n.

Since $\lim_n \boldsymbol{y}_n = \boldsymbol{y}$, we may assume that $\boldsymbol{y}_n(i) = \boldsymbol{y}(i)$ for all n if $\boldsymbol{y}(i) < \omega$, and that $(\boldsymbol{y}_n(i))_n$ is strictly increasing if $\boldsymbol{y}(i) = \omega$. Let k_n be a strictly increasing sequence such that $k_n \geqslant n + \max_{1 \leqslant i \leqslant d} |\delta(u_n)(i)|$, and let $\boldsymbol{y}'_n = \boldsymbol{y}_{k_n}$. Clearly, $\lim_n \boldsymbol{y}'_n = \boldsymbol{y}$. By construction, u_n is fireable from $\boldsymbol{y}'_n$: let $\boldsymbol{y}'_n \xrightarrow{u_n} \boldsymbol{x}'_n$. We then have $\boldsymbol{x}'_n(i) = \boldsymbol{x}_n(i)$ if $\boldsymbol{y}(i) < \omega$, and $\boldsymbol{x}'_n(i) \geqslant n$ if $\boldsymbol{y}(i) = \omega$. So, $\boldsymbol{x} = \lim_n \boldsymbol{x}'_n \in \mathsf{Lim}\,\mathsf{Reach}\,\boldsymbol{M}$. □

Proof of Lemma 6.2. Reflexivity is obvious. For transitivity, assume that $\boldsymbol{x} \stackrel{*}{\Rightarrow} \boldsymbol{y} \stackrel{*}{\Rightarrow} \boldsymbol{z}$. Then by definition of $\stackrel{*}{\Rightarrow}$, we have $\boldsymbol{z} \in \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_z(\boldsymbol{y}))$ and $\boldsymbol{y} \in \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x}))$. Since $\boldsymbol{f} = (0, \omega, \ldots, \omega)$, we can use monotony to obtain $\Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x})) = \mathsf{Reach} \Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x}))$. We deduce from this equality that

$$\begin{aligned} \Downarrow_{\boldsymbol{f}} \mathsf{Lim} \Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x})) &= \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach} \Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x})) && \text{by applying the monotonous operator } \Downarrow_{\boldsymbol{f}} \mathsf{Lim}, \\ &= \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}\,\mathsf{Lim} \Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x})) && \text{by Lemma 6.3.} \end{aligned}$$

Since Lim and $\Downarrow_{\boldsymbol{f}}$ commute (see (5.1)), and since the operator $\Downarrow_{\boldsymbol{f}}$ is obviously idempotent, we finally get $\Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x})) = \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach} \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x}))$. Now, the hypotheses imply that $\boldsymbol{z} \in \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach} \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x}))$. We deduce that $\boldsymbol{z} \in \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x}))$, that is, $\boldsymbol{x} \stackrel{*}{\Rightarrow} \boldsymbol{z}$. □

Assume now that $\boldsymbol{x} \in \{0\} \times \mathbb{N}_\omega^{d-1}$ labels a leaf. We create a child of this leaf if the vector $\boldsymbol{y} = \boldsymbol{x} + \delta(a_z)$ is nonnegative. Note that in this case $\boldsymbol{y} \in \{0\} \times \mathbb{N}_\omega^{d-1}$, since $\delta(a_z)(1) = 0$. We do not violate the invariant when creating the child labeled $\boldsymbol{y}$ since $\boldsymbol{x} \stackrel{*}{\Rightarrow} \boldsymbol{y}$. We also add new children labeled by elements of the minimal basis $\boldsymbol{B}(\boldsymbol{x})$ of $\Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}(\boldsymbol{x}))$. Since $\mathbb{N}^d \cap \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}(\boldsymbol{x}))$ is equal to $\mathbb{N}^d \cap \Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}(\boldsymbol{x}))$, by Theorem 5.2, one can compute $\boldsymbol{B}(\boldsymbol{x})$. Observe that $\boldsymbol{x} \stackrel{*}{\Rightarrow} \boldsymbol{b}$ for every $\boldsymbol{b} \in \boldsymbol{B}(\boldsymbol{x})$, so that the invariant is still fulfilled after adding elements of $\boldsymbol{B}(\boldsymbol{x})$.

The termination of the algorithm is obtained by introducing an *acceleration operator* ∇. For $\boldsymbol{x}, \boldsymbol{y} \in \{0\} \times \mathbb{N}_\omega^{d-1}$ such that $\boldsymbol{x} \leqslant \boldsymbol{y}$, we define the vector $\boldsymbol{x} \nabla \boldsymbol{y} \in \{0\} \times \mathbb{N}_\omega^{d-1}$ by:

$$(\boldsymbol{x} \nabla \boldsymbol{y})(i) = \begin{cases} \omega & \text{if } \boldsymbol{x}(i) < \boldsymbol{y}(i) \\ \boldsymbol{x}(i) & \text{if } \boldsymbol{x}(i) = \boldsymbol{y}(i). \end{cases}$$

Let us first verify that performing acceleration cannot violate the invariant.

Lemma 6.4. *If $\boldsymbol{x} \stackrel{*}{\Rightarrow} \boldsymbol{y}$ with $\boldsymbol{x} \leqslant \boldsymbol{y}$ then $\boldsymbol{x} \stackrel{*}{\Rightarrow} (\boldsymbol{x} \nabla \boldsymbol{y})$.*

Proof. If $\boldsymbol{x} \stackrel{*}{\Rightarrow} \boldsymbol{y}$, then $\boldsymbol{y} \in \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x}))$, and we obtain the following situation

$$\boldsymbol{x} \xrightarrow{u_n} \boldsymbol{z}_n \overset{n\to\infty}{\dashrightarrow} \boldsymbol{z} \geqslant \boldsymbol{y},$$

with $u_n \in (A \cup \{a_z\})^*$ and $\boldsymbol{z}, \boldsymbol{z}_n \in \{0\} \times \mathbb{N}_\omega^{d-1}$. Since $\boldsymbol{z} \geqslant \boldsymbol{y} \geqslant \boldsymbol{x}$, there exists ℓ such that $\boldsymbol{z}_\ell(i) \geqslant \boldsymbol{x}(i)$ for all indices i satisfying $\boldsymbol{x}(i) < \omega$, and further $\boldsymbol{z}_\ell(i) > \boldsymbol{x}(i)$ if $\boldsymbol{x}(i) < \boldsymbol{y}(i)$. Therefore, $\boldsymbol{z}_\ell \geqslant \boldsymbol{x}$, and as we have $\boldsymbol{z}_\ell(1) = \boldsymbol{x}(1) = 0$, we deduce that u_ℓ^k is fireable from $\boldsymbol{x}$ for all k. Call $\boldsymbol{t}_k$ the state reached from $\boldsymbol{x}$ after firing u_ℓ^k. Then we have $\boldsymbol{t}_k \in \{0\} \times \mathbb{N}_\omega^{d-1}$ and $\lim_{k\to\infty} \boldsymbol{t}_k \geqslant \boldsymbol{x} \nabla \boldsymbol{y}$, which proves $\boldsymbol{x} \nabla \boldsymbol{y} \in \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\,\mathsf{Reach}(\mathcal{V}_z(\boldsymbol{x}))$. □

Algorithm 1 An algorithm to compute a basis of $\Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z)$

- Inputs: A VAS$_z$ $\mathcal{V}_z$ such that $\boldsymbol{x}_{in} \in \{0\} \times \mathbb{N}^{d-1}$ and $\delta(a_z) \in \{0\} \times \mathbb{Z}^{d-1}$.
- Outputs: $\boldsymbol{R}$, a finite subset of $\{0\} \times \mathbb{N}_\omega^{d-1}$.
- Internal Variables:
  - $\mathcal{T}$, a tree labeled by elements of $\mathbb{N}_\omega^d$.
  - $\mathcal{N}$, a set of nodes.
- Algorithm:

```
1: Initialize T as a single root n_in, labeled by x_in
2: N ← {n_in}
3: while N ≠ ∅ do
4:     Choose a node n from N
5:     N ← N \ {n}
6:     x ← label(n)
7:     if no strict ancestor of n has label x then
8:         for all strict ancestor n_0 of n do          ▷ Acceleration, step 2.a
9:             x_0 ← label(n_0)
10:            if x_0 ⩽ x then
11:                x ← x_0 ∇ x
12:        Replace the label of n by x
13:        if x + δ(a_z) ⩾ 0 then                       ▷ Expand by zero-test, step 2.b (i)
14:            Create a new node in T labeled by x + δ(a_z), as a child of n
15:            Add this node to N
16:        for all b ∈ B(x) do                          ▷ Expand by B(x), step 2.b (ii)
17:            Create a new node in T labeled by b, as a child of n
18:            Add this node to N
19: R ← {label(n) | n ∈ nodes(T)}
20: return R
```

Algorithm 1 computes $\boldsymbol{R}$. If every leaf has a (strict) ancestor with the same label, then it terminates and returns the current set of node labels. If it finds some leaf n whose ancestors carry different labels than that of n, it performs acceleration at n (step 2.a of the outline): while n has an ancestor n_0 labeled by a vector $\boldsymbol{x_0}$ such that $\boldsymbol{x_0} \leqslant \boldsymbol{x} < \boldsymbol{x_0} \nabla \boldsymbol{x}$, it replaces the label $\boldsymbol{x}$ of the leaf n with $\boldsymbol{x_0} \nabla \boldsymbol{x}$.

From Lemma 6.4, we deduce that the invariant still holds. Since this loop just replaces some components by ω, it terminates. Finally, once the label $\boldsymbol{x}$ of n has been updated, the algorithm creates a new child labeled by $\boldsymbol{x} + \delta(a_z)$ if this vector is nonnegative (step 2.$b(i)$), and it creates a new child of n labeled by $\boldsymbol{b}$ for each $\boldsymbol{b} \in \boldsymbol{B}(\boldsymbol{x})$ (step 2.$b(ii)$). Note that all labels belong to $\{0\} \times \mathbb{N}_\omega^d$, since $\{\boldsymbol{x}_{in}, \delta(a_z)\} \cup \boldsymbol{B}(\boldsymbol{x}) \subseteq \{0\} \times \mathbb{N}_\omega^d$.

Proposition 6.5. *Algorithm 1 terminates, and it returns a finite set $\boldsymbol{R}$ such that*

$$\downarrow \boldsymbol{R} = \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\, \mathsf{Reach}(\mathcal{V}_z). \tag{6.1}$$

Proof. The termination of the algorithm follows from König's lemma. If the algorithm does not terminate, then it would generate an infinite tree. Because this tree has a finite branching degree, by König's lemma, there is an infinite branch. Since $\leqslant$ is a well-ordering over $\{0\} \times \mathbb{N}_\omega^{d-1}$, this implies that we can extract from this infinite branch an infinite increasing subsequence. However, since we add children to a leaf only if there does not exist a strict ancestor labeled by the same vector, this sequence cannot contain the same vector twice, and must therefore be *strictly* increasing. But, due to the use of the operator ∇, a component with an integer is replaced by ω at every acceleration step. Because the number of ω's in the vectors labeling a branch cannot decrease, we obtain a contradiction. Let us now prove (6.1).

- $\subseteq$ Let n be a node of $\mathcal{T}$, whose label is $\boldsymbol{x}$. By Lemmas 6.2 and 6.4, we have $\boldsymbol{x}_{in} \overset{*}{\Rightarrow} \boldsymbol{x}$. By definition of $\overset{*}{\Rightarrow}$, we conclude that $\boldsymbol{x} \in \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\, \mathsf{Reach}(\mathcal{V}_z)$.
- $\supseteq$ We shall show $\Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z) \subseteq \downarrow \boldsymbol{R}$. The desired inclusion follows by taking limits of both sides, since $\mathsf{Lim} \Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z) = \Downarrow_{\boldsymbol{f}} \mathsf{Lim}\, \mathsf{Reach}(\mathcal{V}_z)$ and $\mathsf{Lim} \downarrow \boldsymbol{R} = \downarrow \boldsymbol{R}$ (since $\boldsymbol{R}$ is finite). So let $(0, \boldsymbol{\alpha}) \in \Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z)$: there exist $\boldsymbol{\alpha}' \in \mathbb{N}^{d-1}$ with $\boldsymbol{\alpha} \leqslant \boldsymbol{\alpha}'$ and $u \in (A \cup \{a_z\})^*$ such that $\boldsymbol{x}_{in} \xrightarrow{u} (0, \boldsymbol{\alpha}')$. We will show by induction on the length of u that $(0, \boldsymbol{\alpha}') \in \downarrow \boldsymbol{R}$. If u is empty, just observe that $\boldsymbol{x}_{in}$ labels the root, hence $\boldsymbol{x}_{in} \in \boldsymbol{R}$. Otherwise, $u = va$ and we have:

  $$\boldsymbol{x}_{in} \xrightarrow{v} (0, \boldsymbol{\beta}) \xrightarrow{a} (0, \boldsymbol{\alpha}')$$

  The induction hypothesis yields $(0, \boldsymbol{\beta}) \in \downarrow \boldsymbol{R}$. Hence, there is in the tree a node labeled $\boldsymbol{\gamma} \geqslant \boldsymbol{\beta}$. Since a node label cannot be modified after acceleration (lines 8 to 11), this means that instructions at lines 13 and 16 have been executed when the variable $\boldsymbol{x}$ was set to $\boldsymbol{\gamma}$, and this ensures that $\boldsymbol{\alpha}' \in \downarrow \boldsymbol{R}$.

We have proved that Algorithm 1 computes a basis $\boldsymbol{R}$ of $\Downarrow_{\boldsymbol{f}} \mathsf{Reach}(\mathcal{V}_z)$. □

Proposition 6.5 and Lemma 6.1 finally imply the central theorem of this paper:

Theorem 6.6. *Given a VAS$_z$ $\mathcal{V}_z$, one can effectively compute the minimal basis of* $\mathsf{Cover}(\mathcal{V}_z)$.

This theorem solves the place-boundedness problem for VAS$_z$. For vector addition systems, it can be transferred to obtain model-checking algorithms. We investigate model-checking problems in the presence of one zero-test in the next section. However, we shall use the decidability of the reachability problem instead of Theorem 6.6.

7. Repeated Control State Reachability is decidable for VASS_z

Vector addition systems can be extended with control flow graphs. Such a control flow graph is given by a finite set of *control states* and a finite set of *transitions* labeled by *actions*. This model is called *Vector Addition Systems with States* (VASS for short). If instead of a VAS, we enrich a VAS_z with a control flow graph, we obtain a *Vector Addition System with States and one zero-test* (VASS_z for short). These models are formally defined in the sequel.

For these systems, the repeated control state reachability consists in deciding whether a given control state can be visited infinitely often along some run. This problem is interesting since a number of model-checking problems, such as LTL model-checking, are reducible to it. For the class of VASS, the repeated control state reachability problem is known to be decidable thanks to a reduction to the *computation of the cover set*. In this section, we extend this decidability result for the class of VASS_z. However, our proof relies on a reduction to the *reachability problem* for VASS_z [33, 7]. We leave as an open question whether the repeated control state reachability for VASS_z can be reduced to the computation of the cover.

Let us first recall the classical extensions of VAS and VAS_z with States, respectively written VASS and VASS_z. States can be seen as mutually-exclusive, 1-bounded counters, and hence are only used as a syntactic convenience.

Definition 7.1. ($\mathbf{VASS}_z$) A *Vector Addition System with States and one zero-test ($VASS_z$) of dimension d* is a tuple $\mathcal{V} = \langle A, a_z, \delta, \boldsymbol{x}_{in}, Q, T, q_{in}\rangle$, where $\langle A, a_z, \delta, \boldsymbol{x}_{in}\rangle$ is a VAS_z of dimension d, Q is a non-empty finite set of *control states*, $T \subseteq Q \times (A \cup \{a_z\}) \times Q$ is a finite set of *transitions*, and $q_{in} \in Q$ is the *initial control state*.

A *Vector Addition System with States (VASS)* is defined similarly from a VAS $\langle A, \delta, \boldsymbol{x}_{in}\rangle$, with $T \subseteq Q \times A \times Q$, and can be thought of as a VASS_z where the action a_z is not used. The VASS_z semantics is defined as follows. Let us call *state* any pair $(q, \boldsymbol{x}) \in Q \times \mathbb{N}^d$. A VASS_z of dimension d induces a transition system over the set of states, given for every $a \in A \cup \{a_z\}$ by:

$$(p, \boldsymbol{x}) \xrightarrow{a} (q, \boldsymbol{y}) \text{ if } (p, a, q) \in T \text{ and } \boldsymbol{x} \xrightarrow{a} \boldsymbol{y}$$

These relations extend uniquely into relations $\xrightarrow{w}$ over the set of states, for $w \in (A \cup \{a_z\})^*$, by requiring that $\xrightarrow{\varepsilon}$ is the identity relation and $\xrightarrow{w_1 w_2}$ is the composition $\xrightarrow{w_1} \circ \xrightarrow{w_2}$, for $w_1, w_2 \in (A \cup \{a_z\})^*$. The *reachability relation*, denoted by $\xrightarrow{*}$ is defined as the union of all relations $\xrightarrow{w}$, when w ranges over $(A \cup \{a_z\})^*$. We also introduce the relation $\xrightarrow{+}$ defined as the union of all relations $\xrightarrow{w}$ when w ranges over $(A \cup \{a_z\})^+$.

A control state $q_f \in Q$ is said to be *visited infinitely often* if there exists an infinite sequence $(\boldsymbol{x}_j)_{j>0}$ of vectors $\boldsymbol{x}_j \in \mathbb{N}^d$ such that $(q_{in}, \boldsymbol{x}_{in}) \xrightarrow{*} (q_f, \boldsymbol{x}_1)$ and such that $(q_f, \boldsymbol{x}_j) \xrightarrow{+} (q_f, \boldsymbol{x}_{j+1})$ for all $j > 0$. The *repeated control state reachability* consists in deciding whether a given control state q_f is visited infinitely often.

We first reduce the repeated control state reachability to a simpler property.

Lemma 7.1. *Let $\mathcal{V} = \langle A, a_z, \delta, \boldsymbol{x}_{in}, Q, T, q_{in}\rangle$ be a $VASS_z$ of dimension d. A control state q_f is visited infinitely often if and only if there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^d$ such that $(q_{in}, \boldsymbol{x}_{in}) \xrightarrow{*} (q_f, \boldsymbol{x}) \xrightarrow{w} (q_f, \boldsymbol{y})$, and one of the following conditions is satisfied:*

(i) we have $\boldsymbol{x} \leqslant \boldsymbol{y}$ and $w \in A^+$, or

(ii) we have $\boldsymbol{x} \leqslant_1 \boldsymbol{y}$ and $w \in (A \cup \{a_z\})^+$.

Proof. Naturally, if (i) or (ii) holds, then q_f is visited infinitely often by monotony of $\xrightarrow{w}$. Conversely, assume that q_f is visited infinitely often. There exists an infinite sequence $(\boldsymbol{x}_j)_{j>0}$ of vectors $\boldsymbol{x}_j \in \mathbb{N}^d$, a word $w_0 \in (A \cup \{a_z\})^*$ such that $(q_{in}, \boldsymbol{x}_{in}) \xrightarrow{w_0} (q_f, \boldsymbol{x}_1)$, and an infinite sequence $(w_j)_{j>0}$ of words $w_j \in (A \cup \{a_z\})^+$ such that $(q_f, \boldsymbol{x}_j) \xrightarrow{w_j} (q_f, \boldsymbol{x}_{j+1})$ for every $j > 0$. We introduce the set J of indexes $j > 0$ such that a_z occurs in w_j. We distinguish two cases according to whether J is finite or infinite.

Assume first that J is finite. By replacing w_0 with $w_0 \cdots w_m$, where $m = \max J$, and w_ℓ with $w_{m+\ell}$ for $\ell > 0$, we may assume without loss of generality that $J = \varnothing$, *i.e.*, that $w_j \in A^+$ for all $j > 0$. By Dickson's lemma, there exist positive integers $j < k$ such that $\boldsymbol{x}_j \leqslant \boldsymbol{x}_k$. We deduce that (i) holds, by observing that $(q_{in}, \boldsymbol{x}_{in}) \xrightarrow{v} (q_f, \boldsymbol{x}) \xrightarrow{w} (q_f, \boldsymbol{y})$ with $v = w_0 \dots w_{j-1}$, $w = w_j \dots w_{k-1}$, $\boldsymbol{x} = \boldsymbol{x}_j$ and $\boldsymbol{y} = \boldsymbol{x}_k$.

Assume now that J is infinite. By suitably concatenating some words w_j, we can assume without loss of generality that a_z occurs in w_j for every $j > 0$. This means that w_j can be decomposed into $w_j = u_j a_z v_j$ for some words $u_j, v_j \in (A \cup \{a_z\})^*$. Hence there exists a state $(q_j, \boldsymbol{y}_j)$ such that $(q_f, \boldsymbol{x}_j) \xrightarrow{u_j a_z} (q_j, \boldsymbol{y}_j) \xrightarrow{v_j} (q_f, \boldsymbol{x}_{j+1})$. Dickson's lemma shows that there exist $j < k$ such that $\boldsymbol{y}_j \leqslant \boldsymbol{y}_k$ and $q_j = q_k$. Since the vectors $\boldsymbol{y}_j$ and $\boldsymbol{y}_k$ appear just after the zero test a_z, we deduce that $\boldsymbol{y}_j(0) = \boldsymbol{y}_k(0)$, so $\boldsymbol{y}_j \leqslant_1 \boldsymbol{y}_k$. Let $\boldsymbol{z} = \boldsymbol{y}_k - \boldsymbol{y}_j$. Note that we have:

$$(q_{in}, \boldsymbol{x}_{in}) \xrightarrow{w_0 \dots w_{j-1} u_j a_z} (q_j, \boldsymbol{y}_j) \xrightarrow{v_j} (q_f, \boldsymbol{x}_{j+1}) \xrightarrow{w_{j+1} \dots w_{k-1} u_k a_z} (q_k, \boldsymbol{y}_k)$$

Now we use monotony: since $(q_j, \boldsymbol{y}_j) \xrightarrow{v_j} (q_f, \boldsymbol{x}_{j+1})$, $\boldsymbol{y}_j \leqslant_1 \boldsymbol{y}_k$, and $q_k = q_j$, we get $(q_k, \boldsymbol{y}_k) \xrightarrow{v_j} (q_f, \boldsymbol{x}_{j+1} + \boldsymbol{z})$. Therefore $(q_{in}, \boldsymbol{x}_{in}) \xrightarrow{v} (q_f, \boldsymbol{x}) \xrightarrow{w} (q_f, \boldsymbol{y})$ with $v = w_0 \dots w_j$, $w = w_{j+1} \dots w_{k-1} u_k a_z v_j$, $\boldsymbol{x} = \boldsymbol{x}_{j+1}$, and $\boldsymbol{y} = \boldsymbol{x}_{j+1} + \boldsymbol{z}$. ☐

Theorem 7.2. *The repeated control state reachability problem is decidable for VASS$_z$.*

Proof. Consider a VASS$_z$ $\mathcal{V} = \langle A, a_z, \delta, \boldsymbol{x}_{in}, Q, T, q_{in} \rangle$ of dimension d and a control state $q_f \in Q$. Without loss of generality, by introducing some extra control states and actions, we can assume that $\delta(a_z)$ is the zero vector.

We construct from $\mathcal{V}$ a VASS$_z$ $\mathcal{V}' = \langle A', a_z, \delta', Q', T', q_{in} \rangle$ of dimension $2d$ as follows. We duplicate the set of control states Q into two additional copies for simulating conditions (i) and (ii) of Lemma 7.1. These copies are denoted by $Q_{(i)}$ and $Q_{(ii)}$, and the copies of a control state $q \in Q$ are denoted by $q_{(i)}$ and $q_{(ii)}$. We define $Q' = Q \cup Q_{(i)} \cup Q_{(ii)}$. We duplicate the set of actions A into two additional copies $A_{(i)}$ and $A_{(ii)}$. The copies of an action $a \in A$ are denoted by $a_{(i)}$ and $a_{(ii)}$. We introduce the set of transitions

$$\begin{aligned} T_{(i)} &= \{(p_{(i)}, a_{(i)}, q_{(i)}) \mid (p, a, q) \in T \wedge a \in A\} \cup \{(q_f, a_{(i)}, q_{(i)}) \mid (q_f, a, q) \in T \wedge a \in A\}, \\ T_{(ii)} &= \{(p_{(ii)}, a_{(ii)}, q_{(ii)}) \mid (p, a, q) \in T\} \qquad\quad \cup \{(q_f, a_{(ii)}, q_{(ii)}) \mid (q_f, a, q) \in T\}, \end{aligned}$$

where $(a_z)_{(ii)}$ denotes a_z. Observe that transitions in $T_{(i)}$ are not labeled by the zero-test a_z. The set of transitions of $\mathcal{V}'$ is $T' = T \cup T_{(i)} \cup T_{(ii)}$. The displacement function δ' is defined by $\delta'(a) = (\delta(a), \delta(a))$, and $\delta'(a_{(i)}) = \delta'(a_{(ii)}) = (\delta(a), \mathbf{0})$ for every $a \in A$, and $\delta'(a_z) = (\mathbf{0}, \mathbf{0})$. Now just observe that for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^d$, we have:

- (i) There exists a run in $\mathcal{V}$ of the form $(q_{in}, \boldsymbol{x}_{in}) \xrightarrow{*} (q_f, \boldsymbol{x}) \xrightarrow{w} (q, \boldsymbol{y})$ such that $w \in A^+$ if and only if $(q_{(i)}, \boldsymbol{y}, \boldsymbol{x})$ is reachable in $\mathcal{V}'$.
- (ii) There exists a run in $\mathcal{V}$ of the form $(q_{in}, \boldsymbol{x}_{in}) \xrightarrow{*} (q_f, \boldsymbol{x}) \xrightarrow{w} (q, \boldsymbol{y})$ such that $w \in (A \cup \{a_z\})^+$ if and only if $(q_{(ii)}, \boldsymbol{y}, \boldsymbol{x})$ is reachable in $\mathcal{V}'$.

From Lemma 7.1 we deduce that q_f is a repeated control state in $\mathcal{V}$ if and only there exists for $\mathcal{V}'$ a reachable state of the form $((q_f)_{(i)}, \boldsymbol{y}, \boldsymbol{x})$ with $\boldsymbol{x} \leqslant \boldsymbol{y}$, or a reachable state of the form $((q_f)_{(ii)}, \boldsymbol{y}, \boldsymbol{x})$ with $\boldsymbol{x} \leqslant_1 \boldsymbol{y}$.

We reduce these two problems to the reachability problem for a VASS_z $\mathcal{V}''$ obtained from $\mathcal{V}'$ by adding two extra states $r_{(i)}$ and $r_{(ii)}$, two extra transitions $((q_f)_{(i)}, (\mathbf{0}, \mathbf{0}), r_{(i)})$ and $((q_f)_{(ii)}, (\mathbf{0}, \mathbf{0}), r_{(ii)})$, and two extra cycles on $r_{(i)}$ and $r_{(ii)}$ that suitably decrease the counters, in such a way that

– $((q_f)_{(i)}, \boldsymbol{y}, \boldsymbol{x})$ with $\boldsymbol{x} \leqslant \boldsymbol{y}$ is reachable in $\mathcal{V}'$ if and only if $(r_{(i)}, \mathbf{0}, \mathbf{0})$ is reachable in $\mathcal{V}''$, and
– $((q_f)_{(ii)}, \boldsymbol{y}, \boldsymbol{x})$ with $\boldsymbol{x} \leqslant_1 \boldsymbol{y}$ is reachable in $\mathcal{V}'$ if an only if $(r_{(ii)}, \mathbf{0}, \mathbf{0})$ is reachable in $\mathcal{V}''$.

We have reduced the repeated control state reachability problem to the reachability problem for VASS_z, which is decidable [33, 7]. □

A classical application of the decidability of the repeated control state reachability for VASS is the decidability of LTL model-checking, and more generally of model-checking against ω-regular specifications (it is well-known that LTL specifications can be effectively compiled into ω-regular specifications, see [37] for some original results, or [36] for a survey). Let us informally describe this problem (see [14, 5] for formal presentations). Its inputs are a Σ-labeled VASS_z $\mathcal{V}$ and an ω-regular language L over Σ. By a Σ-labeled VASS_z, we mean a VASS_z $\mathcal{V}$ with transition set T, equipped with a labeling function $\ell : T \to \Sigma$. The *trace* of an infinite run of $\mathcal{V}$ is the infinite word over Σ obtained as the image under ℓ of the run. The question is whether all traces of $\mathcal{V}$ belong to L.

For VASS, the standard technique to solve this problem is to build the product $\mathcal{V} \times \mathcal{A}$ of the VASS $\mathcal{V}$ with a Büchi automaton $\mathcal{A}$ recognizing L, synchronized on Σ. The problem then reduces to the repeated control state reachability in $\mathcal{V} \times \mathcal{A}$, which is a VASS. This also works in our case, since the class of VASS_z is closed under direct product with a finite-state automaton. We deduce the following statement.

Theorem 7.3. *Model-checking a labeled vector addition system with states and one zero-test against an ω-regular property (and in particular against an LTL specification) is decidable.*

8. Conclusion and perspectives

Summary. Our main result is a forward algorithm, *à la* Karp and Miller, to compute the downward closure of the reachability set of a non-monotonic transition system: VAS_z. The proof first goes by strengthening the decidability of the reachability set of a VAS: we show that the *limit closure* of this set is decidable. We have then introduced new sets, sitting between the cover and the reachability set. We have shown that the decidability of the limit closure of the reachability set entails the decidability of filtered covers for a usual VAS. This tool has then be used to perform accurate macro-steps in an adapted Karp-Miller algorithm for VAS_z. Finally, we have shown how to use this result to decide place boundedness for VAS_z, as well as the repeated control state reachability problem, and LTL model-checking.

VAS vs. VAS$_z$. Classical decidable problems for VAS are still decidable for VAS$_z$: reachability, coverability, boundedness, place boundedness, LTL model-checking, repeated control state state reachability. One may want to investigate which logical properties remain decidable for VAS$_z$ (see *e.g.* [5] for properties on VAS solvable using Karp-Miller trees). Note that VAS$_z$ cannot be simulated by VAS. For instance the prefix-closure of the language $\{a^n b^n \mid n \geqslant 1\}^*$ can be recognized by a VAS$_z$, but not by a VAS [26].

Complexity and dependency to the reachability problem. Unfortunately, we cannot say anything about the complexity of the computation of the cover for VAS$_z$, because our proof uses the decidability of the reachability problem for VAS as an oracle, whose complexity is still open. Observe that, more precisely, we have used the decidability of the reachability problem for VAS in Section 4, and this cannot be avoided to get Theorem 4.1. However, to decide the repeated control state reachability problem in Section 7, we have also used a reduction to the decidability of the reachability problem, this time for VAS$_z$. It is not clear whether one can avoid it: we leave it as an open problem.

Future work. Our results cannot be trivially extended to the more general class of VAS with hierarchical zero-tests [33]. In fact, for this class, the coverability problem and the reachability problem are mutually reducible with immediate log-space reductions. The reachability problem was proved to be decidable by Reinhardt in [33]. Recently, the model of VAS with hierarchical zero-tests was proved to be equivalent to VAS with one stack encoding bounded-index context-free languages [3]. As future work, we are interested in the decidability of the reachability problem for VAS equipped with an unrestricted stack. With this class, it becomes possible to model client-server systems where clients are dynamically created and destructed, identical finite-states machines, and the server is a recursive finite-state machine communicating by rendez-vous. The reachability problem for this class is open. For tackling this problem, we recently investigated a simplification of Reinhardt's decidability proof of the reachability problem for VAS with hierarchical zero-tests [33]: for the subclass of VAS$_z$, the first author published a simplified proof in [7], based on the work of the third author [28].

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References

[1] P. A. Abdulla, K. Cerans, B. Jonsson, and Y.-K. Tsay. General decidability theorems for infinite-state systems. In *11th IEEE Symp. on Logic in Computer Science, LICS'96*, pages 313–321, New Brunswick, New Jersey, 1996. IEEE Computer Society Press.

[2] P. A. Abdulla and R. Mayr. Minimal cost reachability/coverability in priced timed Petri nets. In L. de Alfaro, editor, *3rd Int. Conf. on Foundations of Software Science and Computation Structures, FoSSaCS'09*, volume 5504 of *Lect. Notes Comp. Sci.*, pages 348–363, York, UK, 2009. Springer.

[3] M. F. Atig and P. Ganty. Approximating Petri net reachability along context-free traces. In *31th IARCS Conf. on Foundations of Software Technology and Theoretical Computer Science, FSTTCS'11*, volume 13 of *LIPIcs*, pages 152–163. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2011.

[4] H. Baker. Rabin's proof of the undecidability of the reachability set inclusion problem of vector addition systems. M.I.T., Project MAC, CSGM 1979, 1973.

[5] M. Blockelet and S. Schmitz. Model checking coverability graphs of vector addition systems. In F. Murlak and P. Sankowski, editors, *Mathematical Foundations of Computer Science, MFCS'11*, volume 6907 of *Lect. Notes Comp. Sci.*, pages 108–119. Springer, 2011.

[6] R. Bonnet. Decidability of LTL model checking for vector addition systems with one zero-test. In G. Delzanno and I. Potapov, editors, *5th Workshop on Reachability Problems, RP'11*, volume 6945 of *Lect. Notes Comp. Sci.*, pages 85–95, Genova, Italy, 2011. Springer.

[7] R. Bonnet. The reachability problem for vector addition systems with one zero-test. In F. Murlak and P. Sankowski, editors, *36th Mathematical Foundations of Computer Science, MFCS'11*, volume 6907 of *Lect. Notes Comp. Sci.*, pages 145–157, Warsaw, Poland, 2011. Springer.

[8] R. Bonnet, A. Finkel, J. Leroux, and M. Zeitoun. Place-boundedness for vector addition systems with one zero-test. In K. Lodaya and M. Mahajan, editors, *30th IARCS Conf. on Foundations of Software Technology and Theoretical Computer Science, FSTTCS'10*, volume 8 of *Leibniz International Proceedings in Informatics*, pages 192–203, Chennai, India, 2010. Leibniz-Zentrum für Informatik.

[9] A. Bouajjani and R. Mayr. Model checking lossy vector addition systems. In Ch. Meinel and S. Tison, editors, *16th Symp. on Theoretical Aspects of Computer Science, STACS'99*, volume 1563 of *Lect. Notes Comp. Sci.*, pages 323–333. Springer, 1999.

[10] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, fourth edition, 2010.

[11] C. Dufourd. *Réseaux de Petri avec Reset/Transfert : décidabilité et indécidabilité*. Thèse de doctorat, Laboratoire Spécification et Vérification, ENS Cachan, France, Oct. 1998.

[12] E. A. Emerson and K. S. Namjoshi. On model checking for non-deterministic infinite-state systems. In *13th IEEE Symp. on Logic in Computer Science, LICS'98*, pages 70–80, Washington, DC, USA, 1998. IEEE Computer Society Press.

[13] J. Esparza. On the decidability of model checking for several μ-calculi and Petri Nets. In S. Tison, editor, *Trees in Algebra and Programming, CAAP'94*, volume 787 of *Lect. Notes Comp. Sci.*, pages 115–129. Springer, 1994.

[14] J. Esparza. Decidability and complexity of Petri Net problems: An introduction. In W. Reisig and G. Rozenberg, editors, *Lectures on Petri Nets I: Basic Models*, volume 1491 of *Lect. Notes Comp. Sci.*, pages 374–428. Springer, 1998.

[15] A. Finkel. The minimal coverability graph for Petri nets. In G. Rozenberg, editor, *Advances in Petri Nets 1993*, volume 674 of *Lect. Notes Comp. Sci.*, pages 210–243. Springer, 1993.

[16] A. Finkel and J. Goubault-Larrecq. Forward analysis for WSTS, part I: Completions. In S. Albers and J.-Y. Marion, editors, *26th Symp. on Theoretical Aspects of Computer Science, STACS'09*, pages 433–444. Springer, 2009.

[17] A. Finkel and J. Goubault-Larrecq. Forward analysis for WSTS, Part II: Complete WSTS. In S. Albers, A. Marchetti-Spaccamela, Y. Matias, S. E. Nikoletseas, and W. Thomas, editors, *36th Int. Colloquium on Automata, Languages and Programming, ICALP'09*, volume 5556 of *Lect. Notes Comp. Sci.*, pages 188–199. Springer, 2009.

[18] A. Finkel, P. McKenzie, and C. Picaronny. A well-structured framework for analysing Petri Net extensions. *Inf. Comput.*, 195(1-2):1–29, 2004.

[19] A. Finkel and A. Sangnier. Mixing coverability and reachability to analyze VASS with one zero-test. In D. Peleg and A. Muscholl, editors, *SOFSEM'10*, volume 5901 of *Lect. Notes Comp. Sci.*, pages 394–406. Springer, 2010.

[20] A. Finkel and Ph. Schnoebelen. Well-structured transition systems everywhere! *Theoret. Comput. Sci.*, 256(1–2):63–92, 2001.

[21] P. Habermehl. On the complexity of the linear-time μ-calculus for Petri nets. In P. Azéma and G. Balbo, editors, *Application and Theory of Petri Nets 1997*, volume 1248 of *Lect. Notes Comp. Sci.*, pages 102–116. Springer, 1997.

[22] M. Hack. The equality problem for vector addition systems is undecidable. *Theoret. Comput. Sci.*, 2(1):77–95, 1976.

[23] D. Hauschildt. *Semilinearity of the Reachability Set is Decidable for Petri Nets.* PhD thesis, University of Hamburg, 1990.

[24] P. Jančar. Decidability of a temporal logic problem for Petri nets. *Theoret. Comput. Sci.*, 74(1):71–93, 1990.

[25] R. M. Karp and R. E. Miller. Parallel program schemata. *J. Comput. System Sci.*, 2:147–195, 1969.

[26] S. R. Kosaraju. Limitations of Dijkstra's Semaphore Primitives and Petri Nets. In *4th Symp. on Operating System Principles, SOSP'73*, pages 122–136, Yorktown Heights, New York, USA, 1973. ACM.

[27] S. R. Kosaraju. Decidability of reachability in vector addition systems (preliminary version). In *14th ACM Symp. on Theory of Computing, STOC'82*, pages 267–281, New York, NY, USA, 1982. ACM.

[28] J. Leroux. The general vector addition system reachability problem by Presburger inductive invariants. In *24th IEEE Symp. on Logic in Computer Science, LICS'09*, pages 4–13. IEEE Computer Society Press, 2009.
[29] J. Leroux. Vector addition system reachability problem: a short self-contained proof. In *38th ACM Symp. on Principles of Programming Languages, POPL'11*, pages 307–316. ACM, 2011.
[30] E. W. Mayr. An algorithm for the general Petri net reachability problem. In *13th ACM Symp. on Theory of Computing, STOC'81*, pages 238–246, New York, NY, USA, 1981. ACM.
[31] R. Mayr. Undecidable problems in unreliable computations. *Theoret. Comput. Sci.*, 297(1-3):337–354, 2003.
[32] Ch. Rackoff. The covering and boundedness problems for vector addition systems. *Theoret. Comput. Sci.*, 6(2):223–231, 1978.
[33] K. Reinhardt. Reachability in Petri Nets with inhibitor arcs. *Electr. Notes Theor. Comput. Sci.*, 223:239–264, 2008.
[34] Ph. Schnoebelen. Lossy counter machines decidability cheat sheet. In A. Kucera and I. Potapov, editors, *4th Workshop on Reachability Problems, RP'10*, Lect. Notes Comp. Sci., pages 51–75, Brno, Czech Republic, 2010. Springer.
[35] R. Valk and M. Jantzen. The residue of vector sets with applications to decidability problems in Petri Nets. *Acta Informatica*, 21:643–674, 1985.
[36] M. Y. Vardi. An automata-theoretic approach to linear temporal logic. In *Proceedings of the VIII Banff Higher order workshop conference on Logics for concurrency: structure versus automata*, pages 238–266, Secaucus, NJ, USA, 1996. Springer.
[37] M. Y. Vardi and P. Wolper. Reasoning about infinite computations. *Inform. Comput.*, 115(1):1–37, 1994.