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DISORDERED BOSE EINSTEIN CONDENSATES WITH INTERACTION IN ONE DIMENSION

ROBERT SEIRINGER, JAKOB YNGVASON, AND VALENTIN A.ZAGREBNOV

Abstract. We study the effects of random scatterers on the ground state of the one-dimensional Lieb-Liniger model of interacting bosons on the unit interval in the Gross-Pitaevskii regime. We prove that Bose Einstein condensation survives even a strong random potential with a high density of scatterers. The character of the wave function of the condensate, however, depends in an essential way on the interplay between randomness and the strength of the two-body interaction. For low density of scatterers or strong interactions the wave function extends over the whole interval. High density of scatterers and weak interaction, on the other hand, leads to localization of the wave function in a fragmented subset of the interval.

1. Introduction

While the effects of random potentials on single particle Schrödinger operators [1] and ideal Bose gases [2]–[4] are rather well explored, the present understanding of such effects on many-body systems of interacting particles is much less complete. In recent years, however, many papers concerning the interplay of Bose-Einstein Condensation (BEC) and disorder have appeared of which references [5]–[16] are but a sample.

In this paper we present results on a model that is in a sense the simplest one imaginable where this interplay can be studied by rigorous mathematical means. This is the one-dimensional Lieb-Liniger model [17] of bosons with contact interaction in a 'flat' trap, augmented by an external random potential that is generated by Poisson distributed point scatterers of equal strength. We study the ground state and prove that, no matter the strength of the random potential, BEC is not destroyed by the random potential in the Gross-Pitaevskii (GP) limit where the particle number tends to infinity while the coupling parameter in a mean-field scaling stays fixed. The character of the wave function of the condensate, however, depends in an essential way on the relative size of the three parameters involved. These are the scaled coupling parameter γ for the interaction among the particles, the density of the scatterers ν, and the strength σ of the scattering potential. All these parameters are assumed to be large in suitable units.

The investigation consists of three parts. In the following Section 2 we prove Bose-Einstein condensation in the ground state of our model in a GP limit. This result holds, in fact, for quite general external potentials, provided they are bounded below. Specific properties of the random potential enter only in the estimate of the energy gap of an

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effective mean-field Hamiltonian that sets a bound to the depletion of the condensate for finite particle number.

Section 3 is concerned with the properties of the condensate in dependence of the parameters. In particular, for $\sigma \gg 1$ we identify three different "phases" of the condensate. For large $\gamma \gg \nu^2$ the condensate is extended over the whole trap (the unit interval in our model). A transition from a delocalized to a localized state takes place when γ is of the order ν^2, in the sense that for $\gamma \ll \nu^2$ the density is essentially distributed among a fraction $\lambda \ll 1$ of the $\nu \gg 1$ intervals between the obstacles. For $\gamma \gg \nu/(\ln \nu)^2$ we still have $\lambda\nu \gg 1$, but for $\gamma \sim \nu/(\ln \nu)^2$ the fraction of intervals that are significant occupied shrinks to $O(\nu^{-1})$. We stress, however, that in all cases there is complete BEC into a single state in the limit when the particle number tends to infinity.

In an appendix, we prove an estimate on the energy gap of one-dimensional Schrödinger Hamiltonians. This estimate, applied to a mean field Hamiltonian, is needed to establish a lower bound for the condensate fraction that depends only on the parameters of the problem and not on the individual realizations of the random potential. The estimate on the energy gap is a generalization of a result from [20] and is of independent interest. Finally, for convenience of the reader, we collect in a second appendix some facts about the Poisson distribution of obstacles.

Our study of the Gross-Pitaevskii energy functional in Section 3 can be regarded as complementary to the investigations in [5, 7, 9, 10, 11, 13, 15] where transitions between delocalization and localization for solutions of the GP equation (also the time-dependent one) are considered from different points of view. Among these papers [7] is the one closest to ours as far as the questions asked are concerned, and it is appropriate to make a brief comparison.

One difference is that the random potentials considered in [7] are intended to model experimentally generated laser speckles and somewhat different from the present model. In both cases, however, there is a length scale associated with the random potential (ν^{-1} in our case, σ_R in [7]), and the strength of the random potential that is denoted by V_R in [11] is the analogue of our $\nu\sigma$. Moreover, our coupling parameter γ may be compared with the chemical potential μ in [7].

The main difference, however, is that in [7] BEC is stated to hold only for large μ (analogous to large γ in our model), while for weaker coupling a breakdown of BEC through a transition to a fragmented condensate and finally to a Lifschitz glass phase is claimed. We, however, *prove rigorously* that complete BEC holds in the whole parameter range considered when the particle number tends to infinity. Some further comments on this issue will be made at the end of Section 2.

2. The Many-Body Model and BEC

The model we consider is the Lieb-Liniger model of bosons with contact interaction on the unit interval, with an additional external potential V. In the next section we shall take V to be a sum of delta functions at random points in the unit interval, but for the

proof of the energy bounds and BEC in the present section the only assumption used is that V is a *nonnegative* sum of delta functions and a locally integrable function. The Hamiltonian on the Hilbert space $L^2([0,1],dz)^{\otimes^N_{\rm symm}}$ is

$$H := \sum_{i=1}^{N} \left(-\partial_{z_i}^2 + V(z_i)\right) + \frac{\gamma}{N}\sum_{i<j}\delta(z_i - z_j) \tag{1}$$

with $\gamma \geq 0$, and we assume Dirichlet boundary conditions at the end points of the unit interval. The Hamiltonian has a unique ground state and we denote the ground state energy by $E_0^{\rm QM}$.

The Gross-Pitaevskii energy functional is defined by

$$\mathcal{E}^{\rm GP}[\psi] := \int_0^1 \left\{|\psi'(z)|^2 + V(z)|\psi(z)|^2 + (\gamma/2)|\psi(z)|^4\right\} dz. \tag{2}$$

Again we assume Dirichlet boundary conditions. By standard methods one proves that there is a unique nonnegative minimizer $\psi_0 \in H_0^1([0,1])$, normalized so that $\int \psi_0^2(z)dz = 1$, with a corresponding energy $e^{\rm GP}$. This is equal to the ground state energy e_0 of the mean-field Hamiltonian

$$h := -\partial_z^2 + V(z) + \gamma\psi_0(z)^2 - \frac{\gamma}{2}\int \psi_0^4 \tag{3}$$

with ψ_0 as the corresponding eigenfunction.

THEOREM 2.1 (Energy bounds)**.**

$$e_0 \geq \frac{E_0^{\rm QM}}{N} \geq e_0\left(1 - (\text{const.})N^{-1/3}\min\{\gamma^{1/2},\gamma\}\right). \tag{4}$$

Note that the error term depends only on γ besides N and is independent of V (as long as $V \geq 0$).

We denote the second eigenvalue of h by e_1 and by N_0 the average occupancy of ψ_0 in the many-body ground state Ψ_0 of H, i.e.,

$$N_0 := \operatorname{tr}\rho^{(1)}|\psi_0\rangle\langle\psi_0| \tag{5}$$

where $\rho^{(1)}$ is the one-particle reduced density matrix of Ψ_0. The depletion of the condensate is $\left(1 - \frac{N_0}{N}\right)$.

THEOREM 2.2 (BEC)**.**

$$\left(1 - \frac{N_0}{N}\right) \leq (\text{const.})\frac{e_0}{e_1 - e_0}N^{-1/3}\min\{\gamma^{1/2},\gamma\}. \tag{6}$$

Theorem 2.2 implies in particular complete BEC in the limit $N \to \infty$ with γ and V fixed. In fact, Eq. (6) implies that the reduced one-particle density matrix divided by N converges in trace norm to the projector on ψ_0. The energy gap $e_1 - e_0$ is always strictly positive since h has a unique ground state. Its dependence on γ and V is discussed in the Appendix. For the random external potentials introduced in (20) below, Eq. (103)

implies that complete BEC holds not only for almost every random sample of scatterers, but the depletion of the condensate goes to zero even in L^p norm on the probability space for every $p < \infty$.

Our proof of BEC in the GP limit is simpler than the corresponding proof in three and two dimensions [18] because the one-dimensional case considered here corresponds to a high-density, mean-field limit. In contrast, the work in [18] deals with a dilute limit that requires quite different tools.

2.1. **Proof of Theorems 2.1 and 2.2.** The upper bound in (4) is immediately obtained by using $\psi_0^{\otimes N}$ as a trial function for the expectation value of the Hamiltonian (1).

For the proof of the lower bound in (4), as well as the proof of (6), we begin by splitting off part of the kinetic energy and write, with $\varepsilon > 0$ and $p_i := -\mathrm{i}\partial_{z_i}$,

$$H = \sum_{i=1}^{N} \left\{ \left(1 - \frac{N-1}{2N}\varepsilon\right) p_i^2 + V(z_i) \right\} + \frac{1}{N} \sum_{i<j} \left[\frac{\varepsilon}{2}(p_i^2 + p_j^2) + \gamma\delta(z_i - z_j) \right]. \tag{7}$$

Next we estimate the term in square brackets from below by a regular interaction potential using the bound

$$-\varepsilon\partial_z^2 + \gamma\delta(z) \geq w(z) \tag{8}$$

with

$$w(z) := \frac{\gamma}{1 + (b\gamma/2\varepsilon)} \frac{1}{2b} \exp(-|z|/b) \tag{9}$$

for any $b > 0$, cf. [19, Lemma 6.3]. Note that

$$\delta_b(z) := \frac{1}{2b} \exp(-|z|/b) \to \delta(z) \tag{10}$$

as $b \to 0$, and $\int \delta_b(z)dz = 1$ for all b. From (8) we obtain the following inequality (in the sense of quadratic forms on wave functions that satisfy Dirichlet boundary conditions at the boundary of $[0,1]^N$ and are identically zero outside of $[0,1]^N$)

$$H \geq \sum_{i=1}^{N} \left\{ \left(1 - \frac{\varepsilon}{2}\right) p_i^2 + V(z_i) \right\} + \frac{\tilde{\gamma}}{N} \sum_{i<j} \delta_b(z_i - z_j) \tag{11}$$

where we have denoted $\gamma/[1 + (b\gamma/2\varepsilon)]$ by $\tilde{\gamma}$ for short, and bounded $-(N-1)\varepsilon/2N$ from below by $-\varepsilon/2$. Since δ_b is of positive type we can, for any density $\rho(z)$, bound the last term by a sum of one-body terms:

$$\frac{\tilde{\gamma}}{N} \sum_{i<j} \delta_b(z_i - z_j) \geq \sum_{i=1}^{N} \left\{ \tilde{\gamma}\, \delta_b * \rho(z_i) - \frac{\gamma}{2} \int (\delta_b * \rho)\rho - \frac{\gamma}{2N}\delta_b(0) \right\} \tag{12}$$

where we have used that $\gamma \geq \tilde{\gamma}$. We take ρ to be the GP density ψ_0^2 (and zero outside of $[0,1]$). For the comparison of the mean field Hamiltonian (3) with the right side of (12)

we use the following simple estimates (recall that all integrals are effectively over the unit interval):

$$\|\delta_b * \rho\|_\infty \leq \|\psi_0\|_\infty^2 \leq \|\psi_0\|_2 \|\psi_0'\|_2 \leq e_0^{1/2} \tag{13}$$

and

$$\|\delta_b * \rho - \rho\|_\infty = \sup_x \left| \int [\rho(x-y) - \rho(x)]\delta_b(y)dy \right| \leq \sup_x \left| \int \int_{x-y}^{x} |\rho'(u)|du\, \delta_b(y)dy \right|$$
$$\leq 2\|\psi_0\|_\infty\, e_0^{1/2} \int |y|^{1/2}\delta_b(y)dy \leq 2^{1/2}\pi^{1/2}\, e_0^{3/4}\, b^{1/2}. \tag{14}$$

Using $\tilde{\gamma} \geq \gamma - b\gamma^2/(2\varepsilon)$, Eqs. (12)–(14) and (3) lead to

$$H \geq \sum_{i=1}^{N} \left\{ h^{(i)} - \frac{\varepsilon}{2}p_i^2 - \frac{b\gamma^2}{2\varepsilon}e_0^{1/2} - \frac{3\pi^{1/2}}{2^{1/2}}\gamma\, e_0^{3/4}b^{1/2} - \frac{\gamma}{4Nb} \right\}. \tag{15}$$

In the second term we estimate p_i^2 by e_0 leading to an optimal $\varepsilon \sim b^{1/2}\gamma e_0^{-1/4}$. Optimizing the resulting expression

$$(\text{const.})\gamma \left[b^{1/2}e_0^{3/4} + (Nb)^{-1} \right] \tag{16}$$

over b gives $b \sim N^{-2/3}e_0^{-1/2}$ and

$$\frac{E_0^{\mathrm{QM}}}{N} \geq e_0 \left(1 - (\text{const.})N^{-1/3}\gamma e_0^{-1/2} \right). \tag{17}$$

For large γ we may use the bound $e_0 \geq (\gamma/2)\int \rho^2 \geq (\gamma/2)$ while for $\gamma \lesssim 1$ we use that $e_0 \geq \pi^2$ since π^2 is the lowest eigenvalue of $-\partial_z^2$ on the unit interval with Dirichlet boundary conditions. Altogether we obtain (4).

To prove Theorem 2.2 we trace the one-particle operator on the right side of (15) with the one-particle reduced density matrix of the ground state of H and obtain the more precise lower bound

$$\frac{E_0^{\mathrm{QM}}}{N} \geq \frac{N_0}{N}e_0 + \left(1 - \frac{N_0}{N}\right) e_1 - (\text{const.})\, e_0\, N^{-1/3} \min\{\gamma^{1/2}, \gamma\} \tag{18}$$

where e_1 is the energy of the first excited state of the mean field Hamiltonian h. Combined with the upper bound $E_0^{\mathrm{QM}} \leq Ne_0$ this leads to the claimed estimate (6) for the depletion of the condensate.

2.2. **Remarks.** 1. If $N_{\leq k}$ denotes the occupation of the k lowest eigenvalues of h with energies up to e_k, then the following generalization of (6) holds:

$$\left(1 - \frac{N_{\leq k}}{N}\right) \leq C\frac{e_0}{e_k - e_0}N^{-1/3} \min\{\gamma^{1/2}, \gamma\}. \tag{19}$$

The proof is the same way as for (6), using again the bound (15). For *finite* N the bounds (6) and (19) are, of course, only useful if their respective right-hand sides are less than 1. This may hold for (19) even if it does not for (6).

2. As remarked in the Introduction, the authors of [7] claim a transition from a BEC to a fragmented condensate and finally to a Lifschitz glass phase in their model. Since no rigorous limit of large particle numbers is considered in [7] there is no contradiction with the complete BEC that is proved in the present paper in the whole parameter range. In fact, for a given *finite* particle number N a fragmented condensate, i.e., $(1-N_{\leq k}/N)$ small for some $1 \ll k \ll N$, may be a reasonable substitute for the fully condensed state that emerges according to (6) in the large N limit. We also point out that as far as the density in the position variable is concerned, a fragmented condensate with non-overlapping single-particle wave functions is indistinguishable from a fully condensed state where the wave function of the condensate is a coherent superposition of the non-overlapping functions. The difference shows up, however, in the momentum distribution.

3. The Gross-Pitaevskii Theory

Having established the GP minimizer as the wave function of the condensate in the Gross-Pitaevskii limit of the many-body problem we now turn to the study of the dependence of this minimizer on a random external potential. Specifically, we shall take V in (2) to be

$$V(z) = \sigma V_\omega(z) \tag{20}$$

with $\sigma \geq 0$ and

$$V_\omega(z) := \sum_i \delta(z - z_i/\nu) \tag{21}$$

where $\nu > 0$ and ω is a variable in a probability space generating the random points z_i in $\mathbb{R}$ which are assumed to be Poisson distributed with density $\lambda = 1$ (see Appendix 2). The distances ℓ_i between neighboring points z_i/ν and z_{i+1}/ν are then independent random variables distributed according to the exponential law (cf. Eq.(107))

$$dp_\nu(\ell) = \nu e^{-\ell\nu} d\ell\,. \tag{22}$$

Thus, with probability one only finitely many of these points are in $[0,1]$, and on average there are ν such points. We start by considering the energy in an interval of length ℓ between two of the points, which after a translation and scaling

$$z \to x := z/\ell, \quad \sigma \to \alpha := \ell\sigma, \quad \gamma \to \kappa := \ell\gamma \tag{23}$$

can be conveniently taken to be the unit interval.

3.1. An Auxiliary Problem.

For $\kappa \geq 0$ and $\alpha \geq 0$, let $e(\kappa,\alpha)$ denote the auxiliary GP energy

$$e(\kappa,\alpha) := \inf_{\|\phi\|_2=1} \mathcal{E}_{\kappa,\alpha}[\phi]\ , \tag{24}$$

where

$$\mathcal{E}_{\kappa,\alpha}[\phi] := \int_0^1 dx \ \left(|\phi'(x)|^2 + \frac{\kappa}{2}|\phi(x)|^4\right) + \frac{\alpha}{2}\left(|\phi(0)|^2 + |\phi(1)|^2\right) \tag{25}$$

for $\phi \in H^1([0,1])$. Standard methods show that there exists a minimizer for (24), which we denote by $\phi_{\kappa,\alpha}$, i.e.,

$$e(\kappa,\alpha) = \mathcal{E}_{\kappa,\alpha}[\phi_{\kappa,\alpha}] \ . \tag{26}$$

The minimizer is unique up to a constant phase factor, which can be chosen such that $\phi_{\kappa,\alpha}$ is non-negative.

Note that, as an infimum over linear functions, $e(\kappa,\alpha)$ is jointly concave in κ and α. For us it will also be important that $\kappa \mapsto \kappa\, e(\kappa,\alpha)$ is strictly convex. This follows from the fact that

$$\kappa\, e(\kappa,\alpha) = \inf\left\{\mathcal{E}_{1,\alpha}(|\phi|) \ : \ \int_0^1 |\phi|^2 = \kappa\right\} \tag{27}$$

together with the strict convexity of $|\phi|^2 \mapsto \mathcal{E}_{1,\alpha}(|\phi|)$.

We have the simple bounds

$$e(\kappa,\alpha) = \mathcal{E}_{0,\alpha}(\phi_{\kappa,\alpha}) + \frac{\kappa}{2}\int_0^1 dx \ |\phi_{\kappa,\alpha}(x)|^4 \geq e(0,\alpha) + \frac{\kappa}{2} \tag{28}$$

and

$$e(\kappa,\alpha) \leq \mathcal{E}_{\kappa,\alpha}(\phi_{0,\alpha}) = e(0,\alpha) + \frac{\kappa}{2}\int_0^1 dx \ |\phi_{0,\alpha}(x)|^4 \ , \tag{29}$$

where we used the Schwarz inequality to obtain the first inequality.

Obviously $e(0,0) = 0$ and $e(0,\infty) = \pi^2$. The minimizer $\phi_{0,\alpha}$ for $\kappa = 0$ is of the form $\phi_{0,\alpha}(x) = a_\alpha \cos[b_\alpha(x - 1/2)]$, where $b_\alpha \geq 0$ increases from 0 to π as α increases from 0 to ∞, and $a_\alpha > 0$ is determined by the normalization condition $\|\phi_{0,\alpha}\|_2 = 1$. A close inspection shows that $\|\phi_{0,\alpha}\|_4$ is monotone increasing in α, and hence

$$\int_0^1 dx \ |\phi_{0,\alpha}(x)|^4 \leq \int_0^1 dx \ |\phi_{0,\infty}(x)|^4 = \frac{3}{2} \ . \tag{30}$$

In particular, from (28)–(30) we obtain

$$\frac{1}{2} \leq \frac{e(\kappa,\alpha) - e(0,\alpha)}{\kappa} \leq \frac{3}{4} \tag{31}$$

for all $\kappa > 0$ and $\alpha \geq 0$.

Lemma 3.1. *For some constant C independent of κ and α we have*

$$e(\kappa,\infty) \geq e(\kappa,\alpha) \geq e(\kappa,\infty)\left(1 - C\alpha^{-1/2}\right) \ . \tag{32}$$

Proof. The first inequality follows clearly from monotonicity of $e(\kappa,\alpha)$ in α. To prove the second, it is enough to consider $\alpha > C^2$ (for otherwise the right side of (32) is negative).

Using a piecewise linear function which is constant for $x \in (\varepsilon, 1-\varepsilon)$ as a trial function, we obtain (after optimizing over ε) that

$$e(\kappa, \infty) \leq \frac{\kappa}{2}\left(1 + C\kappa^{-1/2}\right) \tag{33}$$

for some $C > 0$ and κ large enough. In particular, since obviously $e(\kappa, \alpha) \geq \kappa/2$, the second inequality in (32) follows right away if $\kappa > \alpha$.

We are left with the case $\alpha \geq \kappa$. Since

$$e(\kappa, \infty) \geq e(\kappa, \alpha) \geq \frac{\kappa}{2} + \alpha\phi_{\kappa,\alpha}(0)^2 \tag{34}$$

(using the symmetry of $\phi_{\kappa,\alpha}$ with respect to reflections at 1/2), we conclude from (33) that

$$\phi_{\kappa,\alpha}(0)^2 \leq C\frac{\kappa^{1/2}}{\alpha} \leq C\alpha^{-1/2} \tag{35}$$

in this case. To obtain the desired bound, we use use $\lambda(\phi_{\kappa,\alpha}(x) - \phi_{\kappa,\alpha}(0))$ as a trial function for $e(\kappa, \infty)$, where λ is an appropriate normalization factor. A simple calculation then yields (32). □

For later use we shall also introduce the Legendre transform of the map $\kappa \mapsto \kappa\, e(\kappa, \alpha)$,

$$g(\mu, \alpha) := \inf_{n \geq 0} \left(n\, e(n, \alpha) - \mu n\right) . \tag{36}$$

From the concavity of $\kappa \mapsto e(\kappa, \alpha)$ and the uniqueness of the minimizer of (25) one can easily conclude differentiability of the map $\kappa \mapsto e(\kappa, \alpha)$. Since $\kappa \mapsto \kappa\, e(\kappa, \alpha)$ is strictly convex, the infimum in (36) is uniquely attained. If it is attained at some $n > 0$, this n satisfies the equation

$$e(n, \alpha) + n\, e'(n, \alpha) = \mu \ , \tag{37}$$

where we denote $e'(\kappa, \alpha) := \partial_\kappa e(\kappa, \alpha)$. We subtract the energy $e(0, \alpha)$ on both sides of (37) for convenience. Consequently, we observe that the infimum in (36) is attained at n satisfying

$$n = [\mu - e(0, \alpha)]_+ \ \frac{1}{e'(n, \alpha) + \frac{1}{n}(e(n, \alpha) - e(0, \alpha))} \ . \tag{38}$$

Here $[t]_+$ stands for the positive part of a real number t. The unique solution of (38) will be denoted by $\bar{n}(\mu, \alpha)$.

From (31) and concavity of $e(\kappa, \alpha)$ is follows that $1/2 \leq e'(\kappa, \alpha) \leq 3/4$. Moreover, the second term in the denominator in (38) lies in $[1/2, 3/4]$, again by (31). We thus conclude that the optimizing n satisfies

$$\frac{2}{3}\,[\mu - e(0, \alpha)]_+ \leq \bar{n}(\mu, \alpha) \leq [\mu - e(0, \alpha)]_+ \ . \tag{39}$$

In particular, it is non-zero if and only if $\mu > e(0, \alpha)$, and we can write

$$\bar{n}(\mu, \alpha) \sim [\mu - e(0, \alpha)]_+ \ , \tag{40}$$

where $a \sim b$ means that a/b is bounded from above and below by positive constants.

Finally, we note that since $e(0,\alpha)$ is increasing and concave in α, there exists a constant $C>0$ such that

$$e(0,\alpha) \geq \frac{C\alpha}{1+\alpha}\,. \tag{41}$$

This bound will be useful later.

3.1.1. *The Average Energy.* We shall consider the average energy for a particle distributed over intervals of side length ℓ, whose distribution is given by the exponential law (22) with $\nu>0$. More precisely, we define

$$e_0(\gamma,\nu) = \inf\left\{\nu\int_0^\infty dp_\nu(\ell)\,\frac{n(\ell)}{\ell^2}e(n(\ell)\ell\gamma,\infty)\;:\;\nu\int_0^\infty dp_\nu(\ell)\,n(\ell)=1\right\}\,, \tag{42}$$

where the infimum as over all $n:\mathbb{R}_+\to\mathbb{R}_+$ satisfying the normalization constraint. With the aid of (36), we can alternatively write

$$e_0(\gamma,\nu) = \sup_{\mu>0}\left\{\mu+\nu\int_0^\infty dp_\nu(\ell)\,\frac{1}{\ell^3\gamma}g(\mu\ell^2,\infty)\right\}\,. \tag{43}$$

From simple scaling it follows that

$$e_0(\gamma,\nu) = \gamma\, e_0(1,\nu/\sqrt{\gamma})\,. \tag{44}$$

The infimum in (42) is attained for $n(\ell)=(\ell\gamma)^{-1}\bar{n}(\mu\ell^2,\infty)$ for suitable $\mu>0$, being equal to the optimal μ in (43). Using (40) as well as the fact that

$$1 \leq e^x\int_x^\infty e^{-t}\left(t-\frac{x^2}{t}\right)dt \leq 2 \tag{45}$$

we see that μ satisfies the relation

$$1 \sim \frac{\mu}{\gamma}e^{-\pi\nu/\sqrt{\mu}}\,. \tag{46}$$

In other words, $\mu\sim\gamma\, f(\nu^2/\gamma)$, where $f:\mathbb{R}_+\to\mathbb{R}_+$ denotes the function

$$f(x) = \begin{cases} 1 & \text{for } x\leq 1 \\ \frac{x}{(1+\ln x)^2} & \text{for } x\geq 1. \end{cases} \tag{47}$$

Also $e_0(\gamma,\nu)\sim\gamma\, f(\nu^2/\gamma)$.

3.2. **The Gross-Pitaevskii Energy.** We now turn to the actual GP energy functional we want to consider,

$$\mathcal{E}_\omega^{\mathrm{GP}}[\psi] := \int_0^1 dz\,\left(|\psi'(z)|^2+\sigma V_\omega(z)|\psi(z)|^2+\frac{\gamma}{2}|\psi(z)|^4\right) \tag{48}$$

with V_ω as in (21), and define the GP energy as

$$e_\omega(\gamma,\sigma,\nu) := \inf_{\|\phi\|_2=1}\mathcal{E}_\omega^{\mathrm{GP}}[\phi]\,. \tag{49}$$

THEOREM 3.1 (Convergence of the energy)**.** *Assume that* $\nu \to \infty$*,* $\sigma \to \infty$ *and* $\gamma \to \infty$ *in such a way that*

$$\gamma \gg \frac{\nu}{(\ln \nu)^2} \quad \text{and} \quad \sigma \gg \frac{\nu}{1 + \ln\left(1 + \nu^2/\gamma\right)} \,. \tag{50}$$

Then, for almost every ω*,*

$$\lim \frac{e_\omega(\gamma, \sigma, \nu)}{e_0(\gamma, \nu)} = 1 \tag{51}$$

where $e_0(\gamma, \nu)$ *is the* deterministic *function defined in (43), which satisfies* $e_0(\gamma, \nu) = \gamma\, e_0(1, \nu/\sqrt{\gamma}) \sim \gamma\, f(\nu^2/\gamma)$*.*

3.2.1. *Remark.* Without the interaction, i.e., for $\gamma = 0$, the energy is not deterministic but is simply the ground state energy in the random potential σV_ω. For $\sigma \to \infty$ it becomes equal to the kinetic energy of a particle localized in the largest subinterval free of obstacles and hence, with probability 1, of the order $\nu^2/(\ln \nu)^2$, cf. Appendix 2.

We shall now investigate the optimal particle number distribution in (42). Since $\bar{n}(\mu\ell^2, \infty) > 0$ if and only if $\mu\ell^2 > \pi^2$, the average number of intervals with non-zero occupation numbers is given by

$$\nu \int_{\pi/\sqrt{\mu}}^{\infty} dp_\nu(\ell) = e^{-\pi\nu/\sqrt{\mu}}\, \nu =: \lambda\, \nu \,. \tag{52}$$

Since ν is the total number of available intervals, $\lambda \leq 1$ in (52) defines the *fraction* of them which are occupied. With the help of the definition of λ in (52) the relation (46) can also be written as

$$\frac{\pi^2 \nu^2}{\left(\ln \lambda^{-1}\right)^2} = \mu \sim \frac{\gamma}{\lambda} \,. \tag{53}$$

In other words, λ is determined by γ and ν via relation

$$\gamma \sim \frac{\lambda\, \nu^2}{\left(\ln \lambda^{-1}\right)^2} \,. \tag{54}$$

We can distinguish the following **limiting cases**:

- If $\gamma \gg \nu^2$ then by (54) we get $\lambda \to 1$, i.e., *all* the intervals are occupied. The chemical potential satisfies $\mu \sim \gamma$ in this regime.
- If $\gamma \sim \nu^2$ than $\lambda \sim 1$, but λ is strictly less than 1. Again we have $\mu \sim \gamma$.
- If $\gamma \ll \nu^2$ then $\lambda \ll 1$, i.e., only a small fraction of the intervals are occupied. The relation (53) implies $\mu \sim (\nu/\ln(\nu^2/\gamma))^2$ for the chemical potential.
- If $\gamma \sim \nu/(\ln \nu)^2$ then by (54) the fraction λ becomes $O(1/\nu)$, i.e., only *finitely many* intervals are occupied. In this latter case, $\mu \sim \gamma\nu \sim \nu^2/(\ln \nu)^2$, which corresponds exactly to the inverse of the square of the size of the *largest* interval, cf. Appendix 2.

In particular, $\lambda \gg 1/\nu$ only if $\gamma \gg \nu/(\ln \nu)^2$, and hence this condition guarantees that many intervals are occupied. In this case the law of large numbers applies and hence the energy becomes deterministic in the limit. If $\lambda\nu = O(1)$, on the other hand, the value of $e_\omega(\gamma, \sigma, \nu)$ is random. This shows, in particular, that our condition (50) on γ is optimal.

Also the second condition in (50) on σ can be expected to be optimal. It can be rephrased as $\bar{\ell}\sigma \gg 1$, where $\bar{\ell}$ is the (weighted) average interval length

$$\bar{\ell} = \nu \int_0^\infty dp_\nu(\ell)\, \ell\, n(\ell) \tag{55}$$

with $n(\ell) = (\ell\gamma)^{-1}\bar{n}(\mu\ell^2, \infty)$ the optimizer in (42). A simple calculation shows that $\bar{\ell} \sim \nu^{-1}(1 + \ln(1 + \nu^2/\gamma))$.

3.2.2. *Remark.* The conclusions above apply to the minimizing particle distribution in (42). In the parameter regime defined in (50), they can be shown to apply also to the actual minimizer of the GP functional (48). This follows in a standard way by perturbing the energy functional, and then taking a derivative with respect to the perturbation. Specifically, one can add a term $\beta \int_{\mathbb{R}} f(|\psi(x)|)dx$ for suitable functions f to the GP functional, and change the definition of e_0 in (42) accordingly. One then concludes that the energy asymptotics (51) continues to hold in this perturbed case. Taking a derivative with respect to β at $\beta = 0$ yields the desired information on the GP minimizer.

3.3. **Proof of Theorem 3.1.**

3.3.1. *Preliminaries.* Let $z_1 \leq z_2 \leq \cdots \leq z_m$ denote those points in (21) which lie in the open interval $(0, \nu)$. Let also $z_0 = 0$ and $z_{m+1} = \nu$, and denote $\ell_i = |z_{i+1} - z_i|/\nu$ for $0 \leq i \leq m$. Then $\sum_{j=0}^m \ell_j = 1$. For any $\psi \in H_0^1([0, 1])$, we have

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \sum_{j=0}^m \frac{n_j}{\ell_j^2} \mathcal{E}_{n_j\ell_j\gamma, \ell_j\sigma}[\psi_j] \tag{56}$$

with

$$n_j = \int_{z_j}^{z_{j+1}} dx\, |\psi(x)|^2 \tag{57}$$

and

$$\psi_j(x) = \sqrt{\frac{\ell_j}{n_j}} \psi\left(z_j + \ell_j x\right) \quad \text{for } x \in [0, 1]. \tag{58}$$

We can choose $\psi \in H_0^1([0, 1])$ such that $\psi_j \propto \phi_{\ell_j\gamma,\infty}$ for all j and $\{n_j\}$ is an arbitrary collection of non-negative numbers adding up to 1. In particular, we obtain the upper bound

$$e_\omega(\gamma, \sigma, \nu) \leq \inf_{\{n_j\}} \sum_{j=0}^m \frac{n_j}{\ell_j^2} e(n_j\ell_j\gamma, \infty)\ , \tag{59}$$

where the infimum is over all collections of non-negative numbers $\{n_0, n_1, \ldots, n_m\}$ with $\sum_{j=0}^m n_j = 1$.

From (56) we also have the lower bound

$$e_\omega(\gamma, \sigma, \nu) \geq \inf_{\{n_j\}} \sum_{j=0}^m \frac{n_j}{\ell_j^2} e(n_j \ell_j \gamma, \ell_j \sigma) \ . \tag{60}$$

From Lemma 3.1 we conclude that the two bounds (60) and (59) agree, to leading order, as long as $\ell_j \sigma$ is large for all the intervals of length ℓ_j. In particular, the study of the GP energy reduces to a study of independent intervals in this case. More precisely, it suffices that $\ell_j \sigma$ be large for those intervals containing most of the particles.

3.3.2. *Poisson Distribution.* In this subsection we shall collect a few useful facts about the Poisson distribution, we will be used later on. A more detailed discussion of some of the points can be found in Appendix 2. Let $\chi_m(\omega)$ denote the characteristic function of the set corresponding to having exactly m scatterers inside the interval $(0, \nu)$. Its expectation value equals

$$\langle \chi_m \rangle = e^{-\nu} \frac{\nu^m}{m!} \ . \tag{61}$$

We shall frequently use the bounds

$$e^{-\nu} \sum_{m \geq \lambda\nu} \frac{\nu^m}{m!} \leq e^{-\nu} \sum_{m \geq 0} \frac{\nu^m}{m!} \lambda^{m - \lambda\nu} = e^{-\nu(1 - \lambda + \lambda \ln \lambda)} \quad \text{for } \lambda \geq 1 \tag{62}$$

and

$$e^{-\nu} \sum_{m \leq \lambda\nu} \frac{\nu^m}{m!} \leq e^{-\nu} \sum_{m \geq 0} \frac{\nu^m}{m!} \lambda^{m - \lambda\nu} = e^{-\nu(1 - \lambda + \lambda \ln \lambda)} \quad \text{for } \lambda \leq 1. \tag{63}$$

Note that $1 - \lambda + \lambda \ln \lambda \geq 0$ for $\lambda \geq 0$, with equality only for $\lambda = 1$.

For any function depending only on ℓ_j for some fixed $1 \leq j \leq m-1$, we have

$$\langle f(\ell_j) \rangle = \int_0^\infty dp_\nu(\ell)\, f(\ell) \tag{64}$$

with dp_ν defined in (22). This fact (cf. Eq. (107) in Appendix 2) will be used repeatedly below. Moreover, the variables ℓ_j and ℓ_k for $k \neq j$ are independent, and hence $\langle f_1(\ell_j) f_2(\ell_k) \rangle = \langle f_1(\ell_j) \rangle \langle f_2(\ell_k) \rangle$ in this case.

3.3.3. *Upper Bound.* Let $\mu = \mu(\gamma, \nu)$ be the optimizer in (43). With $\bar{n}(\mu, \alpha)$ defined above, we then have

$$\nu \int_0^\infty dp_\nu(\ell)\, \frac{1}{\ell\gamma} \bar{n}(\mu \ell^2, \infty) = 1 \ . \tag{65}$$

In every interval of length ℓ_j, $1 \leq j \leq m-1$, we shall place $n_j = (\ell_j \gamma)^{-1} \bar{n}(\mu \ell_j^2, \infty)$ particles. For simplicity, we shall not place any particles in the first and last interval. The total number of particles in the system is then $N = \sum_{j=1}^{m-1} n_j$.

Lemma 3.2. If $\nu \to \infty$ and $\gamma \to \infty$ with $\gamma \gg \nu/(\ln \nu)^2$ then

$$\lim N = 1 \tag{66}$$

for almost every ω.

Proof. We shall show that $\langle N \rangle \to 1$ and $\langle N^2 \rangle \to 1$ in the limit under consideration, which implies the result.

With $\chi_m(\omega)$ denoting the characteristic function of the set corresponding to having exactly m scatterers inside the interval $(0, \nu)$, we have, by symmetry,

$$\langle N \rangle = \frac{1}{\gamma} \sum_{m \geq 2} (m-1) \langle \chi_m \ell_1^{-1} \bar{n}(\mu \ell_1^2, \infty) \rangle \,. \tag{67}$$

To obtain a lower bound, we can restrict the sum to $m \geq 1 + (1-\varepsilon)\nu$, for some $0 < \varepsilon < 1$. We thus obtain

$$\begin{aligned} \langle N \rangle &\geq \frac{(1-\varepsilon)\nu}{\gamma} \sum_{m \geq 1+(1-\varepsilon)\nu} \langle \chi_m \ell_1^{-1} \bar{n}(\mu \ell_1^2, \infty) \rangle \\ &= (1-\varepsilon) \left(1 - \frac{\nu}{\gamma} \sum_{m < 1+(1-\varepsilon)\nu} \langle \chi_m \ell_1^{-1} \bar{n}(\mu \ell_1^2, \infty) \rangle \right) , \end{aligned} \tag{68}$$

where we used (65) the obtain the last equality. From (39) it follows that $\bar{n}(\mu \ell_1^2, \infty) \leq \mu \ell_1^2$, and hence

$$\langle N \rangle \geq (1-\varepsilon) \left(1 - \frac{\nu \mu}{\gamma} \langle P \ell_1 \rangle \right) \quad , \quad P(\omega) = \sum_{m < 1+(1-\varepsilon)\nu} \chi_m(\omega) \,. \tag{69}$$

A Schwarz inequality yields

$$\langle P \ell_1 \rangle \leq \langle P \rangle^{1/2} \langle \ell_1^2 \rangle^{1/2} = \langle P \rangle^{1/2} \frac{\sqrt{2}}{\nu} \,, \tag{70}$$

where we have evaluated the expectation value of ℓ_1^2 using (64). Moreover, it follows from (63) that $\langle P \rangle$ is bounded by a factor that decays exponentially in ν. In particular, using (46), it follows that

$$\liminf \langle N \rangle \geq 1 - \varepsilon \,. \tag{71}$$

Since ε was arbitrary, this proves the desired lower bound.

For an upper bound, we can proceed similarly. From (67), (65) and $\bar{n}(\mu \ell_1^2, \infty) \leq \mu \ell_1^2$ we have

$$\langle N \rangle \leq 1 + \varepsilon + \frac{\mu}{\gamma} \sum_{m > 1+(1+\varepsilon)\nu} (m-1) \langle \chi_m \ell_1 \rangle \tag{72}$$

for $\varepsilon > 0$. An similar analysis as above, using the Schwarz inequality as in (70), and (62) instead of (63), then yields

$$\limsup \langle N \rangle \leq 1 + \varepsilon \,. \tag{73}$$

This shows that $\langle N \rangle \to 1$, as claimed.

Next we consider $\langle N^2\rangle$. Similarly to (67), we can write it as

$$\langle N^2\rangle = \frac{1}{\gamma^2}\sum_{m\geq 3}(m-1)(m-2)\langle\chi_m\ell_1^{-1}\bar{n}(\mu\ell_1^2,\infty)\ell_2^{-1}\bar{n}(\mu\ell_2^2,\infty)\rangle + \frac{1}{\gamma^2}\sum_{m\geq 2}(m-1)\langle\chi_m\ell_1^{-2}\bar{n}(\mu\ell_1^2,\infty)^2\rangle\,. \tag{74}$$

An analysis as above shows that the first sum can be restricted to an interval $m \in [(1-\varepsilon)\nu,(1+\varepsilon)\nu]$, and therefore yields 1 in the limit considered. Hence it remains to show that last term in (74) goes to zero. If we restrict the sum to $m-1\leq 2\nu$, say, then we have

$$\sum_{m=2}^{1+2\nu}(m-1)\langle\chi_m\bar{\ell}_1^{-2}n(\mu\ell_1^2,\infty)^2\rangle \leq 2\nu\langle\bar{\ell}_1^{-2}n(\mu\ell_1^2,\infty)^2\rangle \leq 2^4\frac{\mu^2}{\nu}e^{-\pi\nu/\sqrt{\mu}} \tag{75}$$

where we used the upper bound in (39) as well as the bound

$$e^x\int_x^\infty e^{-t}\left(t-\frac{x^2}{t}\right)^k dt \leq k!\,2^k \tag{76}$$

for $k=2$. It follows from (46) that (75), when multiplied by γ^{-2}, goes to zero as $\nu\to\infty$ if $\lambda\gg\nu/(\ln\nu)^2$.

For the remaining terms corresponding to $m>1+2\nu$, we use again the Schwarz inequality, which implies that

$$\sum_{m>1+2\nu}(m-1)\langle\chi_m\bar{\ell}_1^{-2}n(\mu\ell_1^2,\infty)^2\rangle \leq \nu\sqrt{2}\,e^{(1-\ln 4)\nu}\langle\ell_1^{-4}\bar{n}(\mu\ell_1^2,\infty)^4\rangle^{1/2}\,. \tag{77}$$

Because of the upper bound in (39) and (76) for $k=4$ the latter expectation value is above bounded by

$$\langle\ell_1^{-4}\bar{n}(\mu\ell_1^2,\infty)^4\rangle \leq 4!\,2^4\left(\frac{\mu}{\nu}\right)^4 e^{-\pi\nu/\sqrt{\mu}}\,. \tag{78}$$

With the aid of (46) one now readily checks that the right side of (77), when divided by γ^2, goes to zero as $\nu\to\infty$. This completes the proof. □

We can take $n_j=(\ell_j\gamma)^{-1}\bar{n}(\mu\ell_j^2,\infty)$ for $1\leq j\leq m-1$ as above, and divide each n_j by $N=\sum_{j=1}^{m-1}n_j$ in order to have the right particle number. Then (59) implies the upper bound

$$e_\omega(\gamma,\sigma,\nu)\leq E\equiv\sum_{j=1}^{m-1}\frac{n_j}{N\ell_j^2}e(n_j\ell_j\gamma/N,\infty)\,. \tag{79}$$

The following lemma concludes the proof of the upper bound in Theorem 3.1.

Lemma 3.3. If $\nu\to\infty$ and $\gamma\to\infty$ with $\gamma\gg\nu/(\ln\nu)^2$ then

$$\lim\frac{E}{e_0(\gamma,\nu)}=1 \tag{80}$$

for almost every ω.

Proof. We proceed as in the proof of Lemma 3.2, and show that both $\langle E\rangle/e_0(\gamma,\nu)$ and $\langle E^2\rangle/e_0(\gamma,\nu)^2$ converge to 1 in the limit considered. Because of the result in Lemma 3.2, we can replace N by 1 without loss of generality. Using (40) (for $\alpha=\infty$) and (31) we see that

$$\frac{n_j}{\ell_j^2}e(n_j\ell_j\gamma,\infty)\sim\mu n_j\,. \tag{81}$$

Since also $\mu\sim e_0(\gamma,\nu)$ we can proceed exactly as in the proof of Lemma 3.2 to conclude the result. □

We thus conclude that, for almost every ω,

$$\limsup\frac{e_\omega(\gamma,\sigma,\nu)}{e_0(\gamma,\nu)}\leq 1 \tag{82}$$

in the limit $\nu\to\infty$, $\gamma\to\infty$ with $\gamma\gg\nu/(\ln\nu)^2$.

3.3.4. *Lower Bound.* We start from (60). Recall the definition of $g(\mu,\alpha)$ in (36). We have

$$e_\omega(\gamma,\sigma,\nu)\geq\inf_{\{n_j\}}\sum_{j=0}^m\frac{n_j}{\ell_j^2}e(n_j\ell_j\gamma,\ell_j\sigma)\geq\mu+\sum_{j=0}^m\frac{1}{\gamma\ell_j^3}g(\mu\ell_j^2,\ell_j\sigma) \tag{83}$$

for any $\mu\in\mathbb{R}$.

Using the lower bound in (31), as well as (41), have

$$g(\mu\ell^2,\ell\sigma)\geq-\frac{1}{2}\left[\mu\ell^2-e(0,\ell\sigma)\right]_+^2\geq-\frac{\mu^2\ell^4}{2}\theta(\ell-\widetilde{\ell})\,, \tag{84}$$

where

$$\widetilde{\ell}=-\frac{1}{2\sigma}+\sqrt{\frac{1}{4\sigma^2}+\frac{C}{\mu}} \tag{85}$$

and θ denotes the Heaviside step function. For the first and last interval, corresponding to $j=0$ and $j=m$, respectively, we shall simply use that

$$\frac{1}{\gamma\ell_j^3}g(\mu\ell_j^2,\ell_j\sigma)\geq-\frac{\mu^2\ell_j}{2\gamma}\,. \tag{86}$$

When divided by $\mu\sim e_0(\gamma,\nu)$, this goes to zero almost surely in the limit considered, since $\ell_j\lesssim\nu^{-1}$ and $\mu\ll\nu\gamma$.

Pick some small $\varepsilon>0$, and consider the contribution to the sum in (83) coming from intervals with length $\ell_j<(\varepsilon\sigma)^{-1}$, for $1\leq j\leq m-1$. If $\varepsilon\sigma\geq 1/\widetilde{\ell}$, this contribution is zero, hence we can assume $\varepsilon\sigma<1/\widetilde{\ell}$ from now on, which is equivalent to

$$\sigma^2<\frac{\mu(1+\varepsilon)}{C\varepsilon^2}\,. \tag{87}$$

Because of our assumption (50) on σ, this means that

$$\frac{\nu}{1+\ln\left(1+\nu^2/\gamma\right)} \ll \sqrt{\mu} \sim \sqrt{\gamma\, f(\nu^2/\gamma)}\,, \tag{88}$$

i.e., $\nu^2 \ll \gamma$. We have

$$\frac{1}{\gamma\ell_j^3} g(\mu\ell_j^2, \ell_j\sigma) \geq -\frac{\mu^2}{2\gamma\varepsilon\sigma} \sim -\frac{\mu}{\nu}\frac{\nu}{\sigma(1+\ln(1+\nu^2/\gamma))} \tag{89}$$

in this case, and the last fraction goes to zero in the limit considered.

Finally consider the intervals $1 \leq j \leq m-1$ with length $\ell_j \geq (\varepsilon\sigma)^{-1}$. In this case, we can use Lemma 3.1 to bound

$$g(\mu\ell_j^2, \ell_j\sigma) \geq \left(1 - C\varepsilon^{1/2}\right) g(\mu\ell_j^2(1-C\varepsilon^{1/2})^{-1}, \infty)\,. \tag{90}$$

An either case, we thus have the lower bound

$$\frac{1}{\gamma\ell_j^3} g(\mu\ell_j^2, \ell_j\sigma) \geq \frac{\left(1 - C\varepsilon^{1/2}\right)}{\gamma\ell_j^3} g(\mu\ell_j^2(1-C\varepsilon^{1/2})^{-1}, \infty) - \frac{C\mu}{\sigma(1+\ln(1+\nu^2/\gamma))} \tag{91}$$

for $1 \leq j \leq m-1$, for some $C > 0$.

Let $\bar{\mu} = \mu(1-C\varepsilon^{1/2})^{-1}$. With $\bar{n}(\mu,\infty)$ defined in Subsection 3.1 as the optimizer in (36), we can write

$$g(\bar{\mu}\ell^2, \infty) = \bar{n}(\bar{\mu}\ell^2, \infty) e(\bar{n}(\bar{\mu}\ell^2, \infty), \infty) - \bar{\mu}\ell^2 \bar{n}(\bar{\mu}\ell^2, \infty)\,. \tag{92}$$

We shall choose $\bar{\mu}$ as in Subsection 3.3.3, i.e., such that

$$\nu \int_0^\infty dp_\nu(\ell)\, \frac{1}{\ell\gamma} \bar{n}(\bar{\mu}\ell^2, \infty) = 1\,. \tag{93}$$

Then $\bar{\mu} \sim e_0(\gamma,\nu)$. Let $n_j = (\ell_j\gamma)^{-1}\bar{n}(\bar{\mu}\ell_j^2, \infty)$ and $N = \sum_{j=1}^{m-1} n_j$. From (83), (86), (91) and (92) we have

$$\begin{aligned} e_\omega(\gamma,\sigma,\nu) \geq \left(1 - C\varepsilon^{1/2}\right) &\left(\bar{\mu}\,(1-N) + \sum_{j=1}^{m-1} \frac{n_j}{\ell_j^2} e(n_j\ell_j\gamma, \infty)\right) \\ &- \frac{C\mu m}{\sigma(1+\ln(1+\nu^2/\gamma))} - \frac{\mu^2(\ell_0+\ell_m)}{2\gamma}\,. \end{aligned} \tag{94}$$

We have already shown in Lemma 3.2 that $\lim N = 1$ almost surely, and in Lemma 3.3 that the ratio of the sum in (94) and $e_0(\gamma,\nu)$ converges to 1 almost surely. Moreover, after division by $e_0(\gamma,\nu) \sim \mu$ the terms on the last line of (94) vanish almost surely. This proves that

$$\liminf \frac{e_\omega(\gamma,\sigma,\nu)}{e_0(\gamma,\nu)} \geq 1 - C\varepsilon^{1/2} \tag{95}$$

for almost every ω. This holds for all $\varepsilon > 0$, hence the proof of the theorem is complete.

4. Conclusions

The main conclusions that can be drawn from the preceding discussion are as follows:

1. BEC in the ground state of the interacting gas in the Gross-Pitaevskii regime can survive even in a strong random potential with a high density of scatterers. As far as BEC is concerned the interacting gas in this regime thus behaves in a similar way as an ideal gas at zero temperature. The character of the wave function of the condensate, however, is strongly affected by the interaction.

2. A random potential may lead to localization of the wave function of the condensate, even though the density of obstacles is much less than the particle density. The interparticle interaction counteracts this effect and can lead to complete delocalization if the interaction is strong enough.

3. In terms of the interaction strength, γ, and density of scatterers, ν, the transition between localization and delocalization occurs in the model considered when $\gamma \sim \nu^2$. In the localized regime $\gamma \ll \nu^2$ the chemical potential is $\mu \sim (\nu/\ln(\nu^2/\gamma))^2$. For $\gamma \lesssim \nu/(\ln \nu)$ the condensate is localized in a small number of random intervals.

5. Appendix 1: Energy Gaps

We consider the Schrödinger operator $H = -\partial_z^2 + W(z)$ on $L^2([0,1])$, with Dirichlet boundary conditions. Assume, for the moment, that W is a bounded and continuous function. Without loss of generality, we may assume $W \geq 0$. Let $0 < e_0 < e_1$ denote the lowest two eigenvalues of H.

Lemma 5.1. Define $\eta > 0$ by

$$\eta^2 = \pi^2 + 3\int_0^1 W(z)dz \tag{96}$$

Then

$$e_1 - e_0 \geq \eta \ln\left(1 + \pi e^{-2\eta}\right) \tag{97}$$

For the proof, we follow closely [20].

Proof. For general $E > 0$, let $u(z,E)$ denote the solution of

$$-u'' + Wu = Eu \tag{98}$$

with $u(0,E) = 0$ and $u'(0,E) = 1$ (the prime denoting ∂_z). Pick an $\eta > 0$, and introduce the Prüfer variables $r(z,E)$ and $\theta(z,E)$ via

$$u = r\cos\theta\ , \quad u' = -\eta r \sin\theta$$

The variable θ is only determined modulo 2π. We fix it uniquely be requiring it to be continuous and $\theta(0,E) = -\pi/2$. We note that θ is increasing in E for fixed z, i.e., $\varphi(z,E) = \partial_E\theta(z,E) \geq 0$. This follows from

$$\varphi = \dot{\theta} = \eta\frac{\dot{u}u' - u\dot{u}'}{\eta^2 u^2 + (u')^2}$$

where the dot stands for ∂_E. The function $\chi = \dot{u}u' - u\dot{u}'$ satisfies $\chi' = u^2$ and is thus positive since it vanishes at $z = 0$.

From (98) it follows that

$$\theta' = \eta^{-1}\,(E - W)\cos^2\theta + \eta\sin^2\theta \tag{99}$$

In particular, $\theta' > 0$ near points $z \in [0,1]$ where u vanishes. Since $u(z, e_0)$ has no zeroes for $z \in (0,1)$, we conclude that $\theta(1, e_0) = \pi/2$. Moreover, $u(z, e_1)$ has exactly one zero in $(0,1)$, hence $\theta(1, e_1) = 3\pi/2$. In particular,

$$\theta(1, e_1) - \theta(1, e_0) = \pi \tag{100}$$

We have $\varphi(0, E) = 0$ for all E. From (99) it follows that

$$\begin{aligned}\varphi' &= \big(\eta + \eta^{-1}\,(W - E)\big)\sin(2\theta)\varphi + \eta^{-1}\cos^2(\theta)\\ &\le \big|\eta + \eta^{-1}\,(W - E)\big|\,\varphi + \eta^{-1}\end{aligned}$$

From Gronwall's inequality [21, Thm. III.1.1], we thus conclude that

$$\varphi(z, E) \le \frac{z}{\eta}\exp\left(\int_0^z \big|\eta + \eta^{-1}\,(W(y) - E)\big|\,dy\right)$$

In particular,

$$\varphi(1, E) \le \frac{1}{\eta}\exp\left(\eta + \eta^{-1}E + \eta^{-1}\int_0^1 W(y)dy\right)$$

for $E > 0$.

This implies that

$$\begin{aligned}\pi = \theta(1, e_1) - \theta(1, e_0) &= \int_{e_0}^{e_1}\varphi(1, E)dE\\ &\le \exp\left(\eta + \eta^{-1}\int_0^1 W + \eta^{-1}e_0\right)\big(\exp\big(\eta^{-1}(e_1 - e_0)\big) - 1\big)\end{aligned} \tag{101}$$

Using the trial function $\sqrt{2}\sin(\pi z)$, we see that $e_0 \le \pi^2 + 2\int_0^1 W$. From (101) we thus have

$$e_1 - e_0 \ge \eta\ln\left(1 + \pi\exp\left(-\eta - \frac{3}{\eta}\int_0^1 W(y)dy - \frac{\pi^2}{\eta}\right)\right)$$

We choose η in order to make the exponent as large as possible, which leads to the choice (96), and concludes the proof. □

By a simple approximation argument, the bound (97) extends to our case of interest, where

$$W(z) = \sigma\sum_{i=1}^m \delta(z - z_i) + \gamma|\psi_0(z)|^2 \tag{102}$$

for $\sigma \ge 0$ and $\gamma \ge 0$. The result is then

$$e_1 - e_0 \ge \eta\ln\big(1 + e^{-2\eta}\big) \quad \text{with } \eta = \sqrt{\pi^2 + 3m\sigma + 3\gamma}\,. \tag{103}$$

5.0.5. *Remark.* While the bound (103) on the gap is uniform in the location of the scatterers, we do not expect it to be optimal for large η. In fact, the gap is presumably smallest when all the z_i are equal to $1/2$. In this case, a simple calculation (for $\gamma = 0$) shows that the gap is asymptotically equal to $32\pi^2/(\sigma m)$ for large σm.

6. Appendix 2: Remarks on the Poisson Random Point Field (RPF) on the Real Axis $\mathbb{R}$.

6.1. **Remark 1.** (Poisson and exponential distributions, see, e.g., [3].)
The following theorem is a consequence of the hypothesis of an independent uniform distribution of m point "impurities" on the interval $\Lambda_l = [-l/2, l/2] \subset \mathbb{R}$ (homogeneous m-point binomial RPF) and the limit: $\Lambda_l \to \mathbb{R}, m \to \infty$, for a fixed density

$$\lambda = \lim_{m,\, l\to\infty} \frac{m}{l} \ . \tag{104}$$

Theorem 6.1. (a) In the limit (104) the finite-volume binomial RPF of impurities $\{X_l^\omega\}$ converges (*in distribution*) to the Poisson RPF $\{X^\omega\}$ with the Poisson distribution

$$\mathbb{P}\{\omega : |X^\omega \cap \Lambda_a| = n\} = \frac{(a\lambda)^n}{n!}\, e^{-\lambda a} \ , \tag{105}$$

of the number of impurities in any interval Λ_a of the length a.
(b) The uniform distribution of $m-1$ independent points of impurities split up the box Λ_l into a set of sub-intervals $\left\{I_j^\omega\right\}_{j=1}^m$ with $\cup_{j=1}^m I_j^\omega = \Lambda_l$ and their random lengths $\{|I_j^\omega| = L_j^\omega\}_{j=1}^m$, $\omega \in \Omega$ have the joint probability distribution

$$dP_{l,m}(L_1, ..., L_m) = \frac{(m-1)!}{l^{m-1}}\, \delta(L_1 + ... + L_m - l)\ \ dL_1 dL_2 \dots dL_m \ . \tag{106}$$

Therefore, the random variables $\{L_j^\omega\}_{j=1}^m$ are *dependent.*
(c) In the thermodynamic limit: $\lambda = \lim_{m,\, l\to\infty}\ m/l$, the intervals $\left\{L_j^\omega\right\}_{j\geq 1}$ form an infinite set of *independent* random variables and the distribution corresponding to (106) converges (*weakly*) to the product-measure distribution σ_λ defined by the set of consistent marginals:

$$d\sigma_{\lambda, j_1, \dots, j_k}(L_{j_1}, \dots, L_{j_k}) = \lambda^k \prod_{s=1}^k e^{-\lambda L_{j_s}} dL_{j_s} \ , \ k \in \mathbb{N} \ . \tag{107}$$

6.2. **Remark 2.** (Log intervals [3],[4].)
We can estimate the maximal length, $(\max_{1\leqslant j\leqslant m}\ L_j^\omega)_l$, of an interval in the family $\left\{I_j^\omega\right\}_{j=1}^m$ of random sub-intervals of $\Lambda_l = [-l/2, l/2]$ when $l \to \infty$. To this end we define the events

$$A_l^\delta := \{\omega : \frac{(\max_{1\leqslant j\leqslant m}\ L_j^\omega)_l}{\ln l} \ \leqslant\ \frac{\delta}{\lambda}\} \ , \ l \in \mathbb{N} \text{ and } l > 2 \ , \tag{108}$$

as well as the following "tail" event:

$$A^{\delta}_{\infty} := \limsup_{l\to\infty} A^{\delta}_{l} = \bigcap_{n=1}^{\infty}\bigcup_{l=n}^{\infty} A^{\delta}_{l} \ . \tag{109}$$

By definition, (109) is the event that for increasing l the length of the largest subinterval in Λ_l is bounded from above by the nonrandom number $(\delta \ln l)/\lambda$ for all but finitely many $l \in \mathbb{N}$. The following theorem holds:

Theorem 6.2. Let $\lambda > 0$ be a mean concentration of the point Poisson "impurities" on $\mathbb{R}$. Then
(i) for any $\delta > 4$, one has:

$$\mathbb{P}(A^{\delta}_{\infty}) = 1 \ , \tag{110}$$

(ii) for any $\delta \leq 2$, one has:

$$\mathbb{P}(A^{\delta}_{\infty}) = 0 \ . \tag{111}$$

Proof: (i) The complementary event to (108) is

$$(A^{\delta}_{l})^{c} := \big\{\omega : (\max_{1\leqslant j\leqslant m} L^{\omega}_{j})_{l} \ > \ \frac{\delta}{\lambda}\ \ln l\big\}.$$

Let $N_l := \big[\lambda l/(\delta \ln l)\big] + 1$, where $[x]$ denotes the integer part of $x \geq 0$, and define a new interval:

$$\widetilde{\Lambda}_l := [-N_l(\frac{\delta}{2\lambda}\ln l)\ ,\ N_l(\frac{\delta}{2\lambda}\ln l)] \ \supset \ \Lambda_l \ . \tag{112}$$

Split this bigger interval into $2N_l$ identical disjoint intervals $\{J^l_i\}_{i=1}^{2N_l}$ of size $\delta(2\lambda)^{-1}\ln l$. If the event $(A^{\delta}_{l})^{c}$ occurs, then there exists at least one empty interval J^l_i (interval without any impurities), i.e.,

$$(A^{\delta}_{l})^{c} \ \subset \bigcup_{1\leqslant i\leqslant 2N_l} \{\omega : J^l_i \text{ is empty}\} \ . \tag{113}$$

Note that for a given number of $m-1$ impurities on the finite interval Λ_l the RPF $\{X^{\omega}_{l}\}$ is *binomial* for any two sets: J^l_i and $\Lambda_l \setminus J^l_i$. Since the probability for the interval J^l_i to be empty depends only on its size $\delta(2\lambda)^{-1}\ln l$ and the size of Λ_l, one obtains by (112) and (113) the estimate

$$\mathbb{P}^{bin}_{m-1,l}((A^{\delta}_{l})^{c}) \ \leq 2N_l\left(1-\frac{\delta}{2\lambda}\ln l/|\widetilde{\Lambda}_l|\right)^{m-1} \leq 2N_l\left(1-\frac{1}{2N_l}\right)^{(N_l-1)\,\delta\ln l} \ . \tag{114}$$

To get the last inequality we put $m-1 = [\lambda\, l] + 1 \geq \lambda\, l$, see (104), and we use that $(N_l - 1)\,\delta \ln l \leq \lambda\, l$. Since the estimate (114) yields

$$\lim_{l\to\infty} \mathbb{P}^{bin}_{[\lambda\, l]+1,l}((A^{\delta}_{l})^{c})/\left[2([\lambda l/(\delta\ln l)]+1)\ l^{-\delta/2}\right] \leq 1 \ , \tag{115}$$

we obtain that for $\delta > 4$

$$\sum_{l\geqslant 1} \mathbb{P}^{bin}_{[\lambda\, l]+1,l}((A^{\delta}_{l})^{c}) \ < \ \infty \ . \tag{116}$$

By Theorem 6.1 binomial distribution $\mathbb{P}^{bin}_{[\lambda\, l]+1,l}(\cdot)$ converges to the Poisson distribution $\mathbb{P}(\cdot)$ on $\mathbb{R}$, when $l \to \infty$. Then estimate (115) and (116) imply

$$\sum_{l\geqslant 1} \mathbb{P}((A^{\delta}_{l})^{c}) \; < \; \infty.$$

Therefore, by the Borel-Cantelli lemma $\mathbb{P}((A^{\delta}_{\infty})^{c}) = 0$, which is equivalent to (110). This also implies that there exists a subset $\widetilde{\Omega} \subset \Omega$ of full measure, $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for each $\omega \in \widetilde{\Omega}$ one can find $l_0(\omega) < \infty$ with

$$\mathbb{P}\,\{\omega : (\max_{1\leqslant j\leqslant m} L^{\omega}_{j})_{l} \;\leqslant\; \frac{\delta}{\lambda}\,\ln l\} = 1\;,\;\text{ for } \delta > 4\;,$$

for all $l \geqslant l_0(\omega)$ and $m \geq [\lambda\, l]$.
(ii) Consider the event $(A^{\delta}_{l})^{c}$ and note that one can estimate its binomial probability from below taking only one term in the union (113). Then

$$\mathbb{P}^{bin}_{m-1,l}((A^{\delta}_{l})^{c}) \;\geq \left(1 - \frac{\delta}{2\lambda}\ln l/|\widetilde{\Lambda}_{l}|\right)^{m-1} \geq \left(1 - \frac{\delta}{2\lambda}\ln l/[\lambda\, l]\right)^{[\lambda\, l]}\;, \tag{117}$$

where we put $m - 1 = [\lambda\, l]$, see (104), and we use (112). Then (117) yields

$$\lim_{l\to\infty} \mathbb{P}^{bin}_{[\lambda\, l],l}((A^{\delta}_{l})^{c})/(l^{-\delta/2}) \geq 1\;, \tag{118}$$

By the same arguments as in (i) one concludes from (118) that for $\delta \leq 2$

$$\sum_{l\geqslant 1} \mathbb{P}((A^{\delta}_{l})^{c}) \;=\; \infty.$$

Since the events $\{(A^{\delta}_{l})^{c}\}_{l>2}$ are independent, by the Borel-Cantelli lemma we obtain: $\mathbb{P}((A^{\delta}_{\infty})^{c}) = 1$, or equivalently (111). This again can be translated as

$$\mathbb{P}\,\{\omega : (\max_{1\leqslant j\leqslant m} L^{\omega}_{j})_{l} \;\geq\; \frac{\delta}{\lambda}\,\ln l\} = 1\;,\;\text{ for } \delta \leq 2\;,$$

for all $l \geqslant l_0(\omega)$ and $m > [\lambda\, l]$. □

Since the Poisson RPF on $\mathbb{R}$ conditional to the interval $[-l/2, l/2]$ and to $m-1$ impurities *coincides* with the binomial RPF, instead of increasing boxes with finite binomial RPF configurations one can consider *restrictions* of infinite Poisson RPF configurations to the increasing set of windows $[-l/2, l/2]$, cf [4]. Then Theorem 6.2 claims that *almost surely* (with respect to the Poisson RPF distribution) the maximal length of impurity-free sub-intervals, $(\max_{1\leqslant j\leqslant m} L^{\omega}_{j})_{l}$, in this windows is bounded from below and above by $(\text{const})\ln l/\lambda$.

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