Characterising Ocone local martingales with reflections

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CHARACTERISING OCONE LOCAL MARTINGALES WITH REFLECTIONS

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Abstract

Let $M = (M_t)_{t \geqslant 0}$ be any continuous real-valued stochastic process such that $M_0 = 0$. Chaumont and Vostrikova proved that if there exists a sequence $(a_n)_{n \geqslant 1}$ of positive real numbers converging to 0 such that M satisfies the reflection principle at levels 0, a_n and $2a_n$, for each $n \geqslant 1$, then M is an Ocone local martingale. They also asked whether the reflection principle at levels 0 and a_n only (for each $n \geqslant 1$) is sufficient to ensure that M is an Ocone local martingale.

We give a positive answer to this question, using a slightly different approach, which provides the following intermediate result. Let a and b be two positive real numbers such that $a/(a+b)$ is not dyadic. If M satisfies the reflection principle at the level 0 and at the first passage-time in $\{-a, b\}$, then M is close to a local martingale in the following sense: $|\mathbb{E}[M_{S \circ M}]| \leqslant a + b$ for every stopping time S in the canonical filtration of $\mathbb{W} = \{w \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}) : w(0) = 0\}$ such that the stopped process $M_{\cdot \wedge (S \circ M)}$ is uniformly bounded.

1 Introduction

Let $(M_t)_{t \geqslant 0}$ denote a continuous local martingale, defined on some probability space $(\Omega, \mathcal{A}, P)$, such that $M_0 = 0$. Let $\mathcal{F}^M$ denote its natural filtration and $\mathcal{H}$ the set of all predictable processes with respect to $\mathcal{F}^M$ with values in $\{-1, 1\}$. Then for every $H \in \mathcal{H}$, the local martingale

$$H \cdot M = \int_0^{\cdot} H_s \, \mathrm{d}M_s$$

has the same quadratic variation as M. In particular, if M is a Brownian motion, then $H \cdot M$ is still a Brownian motion.

A natural problem is to determine when $H \cdot M$ has the same law as M for every $H \in \mathcal{H}$. Ocone proved in [4] that a necessary and sufficient condition is that M is a Gaussian martingale conditionally on its quadratic variation $\langle M \rangle$. Such processes are called *Ocone local martingales*. Various characterisations of these processes have been given, by Ocone himself, by Dubins, Émery and Yor in [3], by Vostrikova and Yor in [6]. We refer to [2] for a more complete presentation.

The following characterisation, given by Dubins, Émery and Yor, is particularly illuminating: M is an Ocone local martingale if and only if there exists a Brownian motion β (possibly defined on a larger probability space) which is independent of $\langle M \rangle$ and such that $M_t = \beta_{\langle M \rangle_t}$ for every t. Loosely speaking, Ocone local martingales are the processes obtained by the composition of a Brownian motion and an independant time-change.

Another characterisation of Ocone local martingales is based on their invariance with respect to reflections. For every positive real r, call h_r the map from $\mathbb{R}_+$ to $\{-1, 1\}$ defined by

$$h_r(t) = \mathbf{1}_{[t \leqslant r]} - \mathbf{1}_{[t > r]}.$$

Then $h_r \cdot M = \varrho_r \circ M$, where ϱ_r is the *reflection at time r*. Let $\mathbb{W}$ denote the set of all continuous functions $w : \mathbb{R}_+ \to \mathbb{R}$ such that $w(0) = 0$. The transformation ϱ_r maps $\mathbb{W}$ into itself and is defined by

$$\varrho_r(w)(t) = \begin{cases} w(t) & \text{if } t \leqslant r, \\ 2w(r) - w(t) & \text{if } t \geqslant r. \end{cases}$$

The functions h_r are sufficient to characterise Ocone local martingales: Theorem A of [4] states that if $h_r \cdot M$ has the same law as M for every positive r, then $H \cdot M$ has the same law as M for every $H \in \mathcal{H}$. In other words, if the law of M is invariant by the reflections at fixed times, then M is an Ocone local martingale. Note that it is not necessary to assume that M is a local martingale since the invariance by the reflections at fixed times implies that for every $t \geqslant s \geqslant 0$, the law of the increment $M_t - M_s$ is symmetric conditionally on $\mathcal{F}_s^M$.

The celebrated reflection principle due to André [1] shows that it may be worthwhile to consider reflections at first-passage times, which we now define. For every real a and $w \in \mathbb{W}$, note $T_a(w)$ the first-passage time of w

at level a. The reflection at time T_a transforms w into $\varrho_{T_a}(w)$ where

$$\varrho_{T_a}(w)(t) = \begin{cases} w(t) & \text{if } t \leqslant T_a(w), \\ 2a - w(t) & \text{if } t \geqslant T_a(w). \end{cases}$$

Note that $\varrho_{T_a}(w) = w$ if $T_a(w)$ is infinite.

Chaumont and Vostrikova recently established in [2] that any continuous process whose law is invariant by the reflections at first-passage times is an Ocone local martingale. Actually, their result is even stronger.

Theorem 1 (Theorem 1 of [2]). *Let M be any continuous stochastic process such that $M_0 = 0$. If there exists a sequence $(a_n)_{n \geqslant 1}$ of positive real numbers converging to 0 such that the law of M is invariant by the reflections at times $T_0 = 0$, T_{a_n} and T_{2a_n}, then M is an Ocone local martingale. Moreover, if $T_{a_1} \circ M$ is almost surely finite, then M is almost surely divergent.*

We note that the assumption that the law of M is invariant by the reflection ϱ_0 is missing in [2] and that it cannot be omitted: consider for example the deterministic process defined by $M_t = -t$. However, if $\inf\{t \geqslant 0 : M_t > 0\}$ is 0 almost surely, the invariance by ϱ_0 is a consequence of the invariance by the reflections $\varrho_{T_{a_n}}$.

To prove Theorem 1 above, Chaumont and Vostrikova establish a discrete version of the theorem and they apply it to some discrete approximations of M. The discrete version (Theorem 3 in [2]) states that if $(M_n)_{n \geqslant 0}$ is a discrete time skip-free process (this means that $M_0 = 0$ and $M_n - M_{n-1} \in \{-1, 0, 1\}$ for every $n \geqslant 1$) whose law is invariant by the reflections at times T_0, T_1 and T_2, then $(M_n)_{n \geqslant 0}$ is a discrete Ocone martingale (this means that $(M_n)_{n \geqslant 0}$ is obtained by the composition of a symmetric Bernoulli random walk with an independent skip-free time change).

The fact that the three invariances by the reflections at times T_0, T_1, and T_2 are actually useful (two of them would not be sufficient) explains the surprising requirement that the law of $(M_t)_{t \geqslant 0}$ is invariant by reflections at times T_{a_n} and T_{2a_n} in Theorem 1 of [2]. Chaumont and Vostrikova ask whether the assumption on T_{2a_n} can be removed. Their study of the discrete case could lead to believe that it cannot. Yet, we give in this paper a positive answer to this question. Here is our main result.

Theorem 2. *Let M be any continuous stochastic process such that $M_0 = 0$. If there exists a sequence $(a_n)_{n \geqslant 1}$ of positive real numbers converging to 0 such that the law of M is invariant by the reflections at times $T_0 = 0$ and*

T_{a_n}, then M is an Ocone local martingale. Moreover, if $T_{a_1} \circ M$ is almost surely finite, then M is almost surely divergent.

We provide a simpler proof of this stronger statement (the final steps in the approximation method of [2] were rather technical). Let us now indicate the steps of the proof and the plan of the paper.

Our proof first uses some stability properties of the set of all stopping times T such that ϱ_T preserves the law of M. These properties are established in section 2.

In section 3, we show that for any positive real numbers a and b such that $a/(a+b)$ is not dyadic, if the reflections ϱ_0 and $\varrho_{T_{-a} \wedge T_b}$ preserve the law of M, then M is close to a local martingale in the following sense: for every stopping time S in the canonical filtration of $\mathbb{W}$ such that the stopped process $M_{\cdot \wedge (S \circ M)}$ is uniformly bounded, $|\mathbb{E}[M_{S \circ M}]| \leqslant a + b$. To prove this, we build a nondecreasing sequence $(\tau_n)_{n \geqslant 0}$ of stopping times, increasing while finite ($\tau_n < \tau_{n+1}$ if $\tau_n < +\infty$), starting with $\tau_0 = 0$, such that the reflections ϱ_{τ_n} preserve the law of M and such that the increments of M on each interval $[\tau_n, \tau_{n+1}]$ are bounded by $a + b$.

The proof that the reflections ϱ_{τ_n} actually preserve the law of M is given in section 4. The final step of the proof of theorem 2 is in section 5.

To prove these results, it is more convenient to work in the canonical space. From now on, $\mathbb{W}$ denotes the σ-field on $\mathbb{W}$ generated by the canonical projections, $X = (X_t)_{t \geqslant 0}$ the coordinate process on $(\mathbb{W}, \mathcal{W})$, and $\mathcal{F}^0$ its natural filtration of the space $\mathbb{W}$ (without any completion). Moreover, Q denotes the law of M and $\mathbb{E}_Q$ is the expectation whith respect to Q.

2   Stability properties

Call $\mathcal{T}_Q$ the set of all stopping times T of the filtration $\mathcal{F}^0$ such that the reflection ϱ_T preserves Q. In this section, we establish some stability properties of $\mathcal{T}_Q$. Let us begin with a preliminary lemma.

Lemma 1. *Let S and T be $\mathcal{F}^0$-stopping times. If $w_1, w_2 \in \mathbb{W}$ coincide on $[0, T(w_1) \wedge T(w_2)]$, then*

- $T(w_1) = T(w_2)$;

- *either $S(w_1) = S(w_2)$ or $S(w_1) \wedge S(w_2) > T(w_1) = T(w_2)$.*

4

Thus, the random times S and T are in the same order on w_1 as on w_2.

Proof. The first point is an application of Galmarino's test (see [5], chapter I, exercise 4.21). The second follows by the same argument, since the inequality $S(w_1) \wedge S(w_2) \leqslant T(w_1) = T(w_2)$ would imply that $w_1, w_2 \in \mathbb{W}$ coincide on $[0, S(w_1) \wedge S(w_2)]$. $\square$

Corollary 1. *Let T be an $\mathcal{F}^0$-stopping time. Then*

1. *$T \circ \varrho_T = T$*

2. *ϱ_T is an involution.*

3. *for every $A \in \mathcal{F}^0_T$, $\varrho_T^{-1}(A) = A$. In particular, if S is another stopping time, the events $\{S < T\}$, $\{S = T\}$ and $\{S > T\}$ are invariant by ϱ_T.*

Proof. The first point is a consequence of the application of the application of lemma 1 to the paths w and $\varrho_T(w)$. The secund point follows. The third point is another application of Galmarino's test (see [5], chapter I, exercise 4.21) since w and $\varrho_T(w)$ coincide on $[0, T(w)]$. $\square$

The next lemma states that $\mathcal{T}_Q$ is stable by the optional mixtures.

Lemma 2. *Let (S_n) be a (finite or infinite) sequence of $\mathcal{F}^0$-stopping times and (A_n) a measurable partition of $(\mathbb{W}, \mathcal{W})$ such that $A_n \in \mathcal{F}_{S_n}$ for every n. Then*

$$T := \sum_n S_n \mathbf{1}_{A_n}$$

is an $\mathcal{F}^0$-stopping time. If $S_n \in \mathcal{T}_Q$ for every n, then $T \in \mathcal{T}_Q$.

Proof. Note that T is an $\mathcal{F}^0$-stopping time since for every $t \in \mathbb{R}_+$,

$$\{T \leqslant t\} = \bigcup_n (A_n \cap \{S_n \leqslant t\}) \in \mathcal{F}_t.$$

Fix any bounded measurable function ϕ from $\mathbb{W}$ to $\mathbb{R}$. Since for each n, the

event A_n and the probability Q are invariant by ϱ_{S_n}, one has

$$
\begin{aligned}
\mathbb{E}_Q[\phi \circ \varrho_T] &= \sum_n \mathbb{E}_Q[(\phi \circ \varrho_{S_n})\mathbf{1}_{A_n}] \\
&= \sum_n \mathbb{E}_Q[(\phi\mathbf{1}_{A_n}) \circ \varrho_{S_n}] \\
&= \sum_n \mathbb{E}_Q[\phi\mathbf{1}_{A_n}] \\
&= \mathbb{E}_Q[\phi].
\end{aligned}
$$

Hence ϱ_T preserves Q.

$\square$

Corollary 2. *For every S and T in $\mathcal{T}_Q$, $S \wedge T$ and $S \vee T$ are in $\mathcal{T}_Q$.*

Proof. As the events $\{S < T\}$, $\{S = T\}$ and $\{S > T\}$ belong to $\mathcal{F}_S \cap \mathcal{F}_T$, the result is a direct application of lemma 2. $\square$

The following lemmas will be used to prove a subtler result: if S and T are in $\mathcal{T}_Q$, then $S \circ \varrho_T$ is in $\mathcal{T}_Q$.

Lemma 3. *Let S and T be $\mathcal{F}^0$-stopping times. Then the following holds.*

- *For every $t \geqslant 0$, $\varrho_T^{-1}(\mathcal{F}_t) = \mathcal{F}_t$.*

- *$S \circ \varrho_T$ is an $\mathcal{F}^0$-stopping time.*

Proof. Fix $t \geqslant 0$. Then $\varrho_T^{-1}(\mathcal{F}_t)$ is the σ-field generated by the random variables $X_s \circ \varrho_T$ for $s \in [0, t]$, and the equality

$$
X_s \circ \varrho_T = (2X_T - X_s)\mathbf{1}_{[T \leqslant s]} + X_s\mathbf{1}_{[T > s]}
$$

shows that these random variables are measurable for $\mathcal{F}_t$. Thus $\varrho_T^{-1}(\mathcal{F}_t) \subset \mathcal{F}_t$. Since ϱ_T is an involution, the reverse inclusion follows, which proves the first statement.

For each $t \geqslant 0$, $\{S \circ \varrho_T \leqslant t\} = \varrho_T^{-1}(\{S \leqslant t\}) \in \mathcal{F}_t$, which proves the second statement. $\square$

Lemma 4. *Let S and T be $\mathcal{F}^0$-stopping times and $w \in \mathbb{W}$.*
If $S(w) \leqslant T(w)$, then $S(\varrho_T(w)) = S(w)$ and $\varrho_{S \circ \varrho_T}(w) = \varrho_S(w)$.
If $S(w) \geqslant T(w)$, then $T(\varrho_S(\varrho_T(w))) = T(w)$ and $\varrho_{S \circ \varrho_T}(w) = \varrho_T(\varrho_S(\varrho_T(w)))$. ∎

Proof. If $S(w) \leqslant T(w)$, then w and $\varrho_T(w)$ coincide on $[0, S(w)]$, thus $S(\varrho_T(w)) = S(w)$ and $\varrho_{S \circ \varrho_T}(w) = \varrho_S(w)$.

If $S(w) \geqslant T(w)$, then $S(\varrho_T(w)) \geqslant T(\varrho_T(w)) = T(w)$ by corollary 1, thus $\varrho_S(\varrho_T(w))$, $\varrho_T(w)$ and w coincide on $[0, T(w)]$, thus $T(\varrho_S(\varrho_T(w))) = T(w)$. But, to get $\varrho_T \circ \varrho_S \circ \varrho_T(w)$ from w, one must successively:

- multiply by -1 the increments after $T(w)$;

- multiply by -1 the increments after $S(\varrho_T(w))$;

- multiply by -1 the increments after $T(\varrho_S(\varrho_T(w)))$.

Since $T(\varrho_S(\varrho_T(w))) = T(w)$, one gets $\varrho_{S \circ \varrho_T}(w) = \varrho_T \circ \varrho_S \circ \varrho_T(w)$. $\qquad\square$

Lemma 5. *For every S and T in $\mathfrak{T}_Q$, $S \circ \varrho_T$ belongs to $\mathfrak{T}_Q$.*

Proof. By lemma 4 and corollary 1, one has, for every $B \in \mathcal{W}$,

$$
\begin{aligned}
Q[\varrho_{S \circ \varrho_T}^{-1}(B)] &= Q[\varrho_{S \circ \varrho_T}^{-1}(B) \; ; \; S \leqslant T] + Q[\varrho_{S \circ \varrho_T}^{-1}(B) \; ; \; S > T] \\
&= Q[\varrho_S^{-1}(B \cap \{S \leqslant T\})] + Q[(\varrho_T \circ \varrho_S \circ \varrho_T)^{-1}(B \cap \{S > T\})] \\
&= Q[B \cap \{S \leqslant T\}] + Q[B \cap \{S > T\}] = Q[B].
\end{aligned}
$$

Thus $S \circ \varrho_T$ belongs to $\mathfrak{T}_Q$. $\qquad\square$

Here is a simple application of our last lemmas.

Corollary 3. *For every $a \in \mathbb{R}$, $T_{-a} = T_a \circ \varrho_0$ and $\varrho_{T_{-a}} = \varrho_0 \circ \varrho_{T_a} \circ \varrho_0$. Thus, if $0 \in \mathfrak{T}_Q$ and $T_a \in \mathfrak{T}_Q$, then $T_{-a} \in \mathfrak{T}_Q$.*

Proof. The first equality is obvious and the second equality follows from lemma 4. One can deduce the last point either from the first equality by lemma 5 or directly from the second equality. $\qquad\square$

3 Reflections at 0 and at the hitting time of $\{-a, b\}$

We keep the notations of the previous section and we fix two positive real numbers a, b such that $a/(a+b)$ is not dyadic. Note that $T = T_{-a} \wedge T_b$ is the hitting time of $\{-a, b\}$. This section is devoted to the proof of the following result.

7

Proposition 1. *Let Q be a probabiliy measure on $(\mathbb{W}, \mathcal{W})$. If $0 \in \mathfrak{T}_Q$ and $T \in \mathfrak{T}_Q$, then, for every finite stopping time S in the canonical filtration of $\mathbb{W}$ such that the stopped process $X_{\cdot \wedge S}$ is uniformly bounded, one has*

$$|\mathbb{E}_Q[X_S]| \leqslant a + b.$$

Note that the process X may not be a local martingale. The law of any process which stops when its absolute value hits $\min(a, b)$ fulfills the assumptions provided it is invariant by T_0.

The requirement that $a/(a + b)$ is not dyadic may seem surprising, and one could think that it is just a technicality provided by the method used to prove the result. In fact, proposition 1 becomes false if this assumption is removed. A simple counterexample is given by the continuous stochastic process $(M_t)_{t \geqslant 0}$ defined by

$$M_t = \begin{cases} t\xi & \text{if } t \leqslant 1, \\ \xi + (t - 1)\eta & \text{if } t > 1, \end{cases}$$

where ξ and η are independent symmetric Bernoulli random variables. Indeed, the law Q of M is invariant by reflections at times 0 and $T_{-1} \wedge T_1$ since $T_{-1} \wedge T_1 = 1$ Q-almost surely. Yet, for every $c > 1$, the random variable $X_{T_{-2} \wedge T_c}$ is uniform on $\{-2, c\}$ and its expectation $(c - 2)/2$ can be made as large as one wants.

The proof of proposition 1 uses an increasing sequence of stopping times defined as follows. Call D the set of $c \in \,]-a, b[$ such that $(c + a)/(b + a)$ is not dyadic. For every $x \in D$, set

$$f(x) = \begin{cases} 2x + a & \text{if } x < (b - a)/2, \\ 2x - b & \text{if } x > (b - a)/2. \end{cases}$$

This defines a map f from D to D. Conjugating f by the affine map which sends $-a$ on 0 and b on 1 gives the classical map $x \mapsto 2x \mod 1$ restricted to the non-dyadic elements of $]0, 1[$.

By hypothesis, $0 \in D$, so one can define an infinite sequence $(c_n)_{n \geqslant 0}$ of elements of D by $c_0 = 0$, and $c_n = f(c_{n-1})$ for $n \geqslant 1$. By definition, c_{n-1} is the middle point of the subinterval $[c_n, d_n]$ of $[-a, b]$, where

$$d_n = \begin{cases} -a & \text{if } c_{n-1} < (b - a)/2, \\ b & \text{if } c_{n-1} > (b - a)/2. \end{cases}$$

Note that $|c_n - c_{n-1}| = d(c_{n-1}, \{-a, b\})$.

We define a sequence $(\tau_n)_{n \geqslant 0}$ of stopping times on $\mathbb{W}$ by setting $\tau_0 = 0$, and for every $n \geqslant 1$,

$$\tau_n(w) = \inf\{t \geqslant \tau_{n-1}(w) : |w(t) - w(\tau_{n-1}(w))| = |c_n - c_{n-1}|\}.$$

Note that $c_n \neq c_{n-1}$ for every $n \geqslant 1$, hence the sequence $(\tau_n(w))_{n \geqslant 0}$ is increasing. Moreover, since $(|c_n - c_{n-1}|)_{n \geqslant 1}$ does not converge to 0, the continuity of w forces the sequence $(\tau_n(w))_{n \geqslant 0}$ to be unbounded. By convention, we set $\tau_\infty = +\infty$.

Note that if $a = -1$ and $b = 2$, then $c_n = 0$ for every even n and $c_n = 1$ for every odd n and the sequence $(\tau_n)_{n \geqslant 0}$ is similar to the sequences used in [2].

The proof of proposition 1 relies on the following key statement.

Proposition 2. *If $0 \in \mathcal{T}_Q$ and $T \in \mathcal{T}_Q$, then $\tau_n \in \mathcal{T}_Q$ for every $n \geqslant 0$.*

This statement, that will be proved in the next section, has a remarkable consequence.

Corollary 4. *If $0 \in \mathcal{T}_Q$ and $T \in \mathcal{T}_Q$, then the sequence $(Y_n)_{n \geqslant 0}$ of random variables defined on the probability space $(\mathbb{W}, \mathcal{W}, Q)$ by*

$$Y_n(w) = X_{\tau_{D_n}}(w) \text{ where } D_n(w) = \max\{k \leqslant n : \tau_k(w) < +\infty\}.$$

is a martingale in the filtration $(\mathcal{F}^0_{\tau_n})_{n \geqslant 0}$.

Proof. Fix $n \geqslant 0$. The equality

$$Y_n(w) = \sum_{k=0}^{n-1} \mathbf{1}_{[\tau_k(w) < +\infty \ ; \ \tau_{k+1}(w) = +\infty]} X_{\tau_k}(w) + \mathbf{1}_{[\tau_n(w) < +\infty]} X_{\tau_n}(w),$$

shows that Y_n is measurable for $\mathcal{F}^0_{\tau_n}$. Moreover, from the equality

$$Y_{n+1}(w) - Y_n(w) = (X_{\tau_{n+1}}(w) - X_{\tau_n}(w)) \mathbf{1}_{[\tau_{n+1}(w) < +\infty]},$$

we deduce that $(Y_{n+1} - Y_n) \circ \varrho_{\tau_n} = -(Y_{n+1} - Y_n)$. Take $A \in \mathcal{F}^0_{\tau_n}$. Then $\varrho^{-1}_{\tau_n}(A) = A$, since every $w \in \mathbb{W}$ coincide with $\varrho_{\tau_n}(w)$ on $[0, \tau_n(w)]$. Since the reflection ϱ_{τ_n} preserves Q, we get

$$\mathbb{E}_Q[(Y_{n+1} - Y_n)\mathbf{1}_A] = \mathbb{E}_Q\big[((Y_{n+1} - Y_n)\mathbf{1}_A) \circ \varrho_{\tau_n}\big] = -\mathbb{E}_Q[(Y_{n+1} - Y_n)\mathbf{1}_A],$$

which shows that $\mathbb{E}_Q[Y_{n+1} - Y_n | \mathcal{F}^0_{\tau_n}] = 0$. $\qquad\square$

We are now ready to prove proposition 1.

Proof. Fix $C \in \mathbb{R}_+$ such that $|X_{t \wedge S}(w)| \leqslant C$ for every $t \in \mathbb{R}_+$ and $w \in \mathbb{W}$. For each $w \in \mathbb{W}$, set $N(w) = \inf\{n \geqslant 1 : \tau_n(w) \geqslant S(w)\}$. Since $S(w)$ is finite and $\tau_n(w)$ is unbounded as $n \to +\infty$, $N(w)$ is finite.

For every $n \geqslant 0$, $\{N \leqslant n\} = \{\tau_n \geqslant S\} \in \mathcal{F}_{\tau_n}^0$. Thus N is a stopping time and $(Y_{n \wedge N})_{n \geqslant 0}$ is a martingale in the filtration $(\mathcal{F}_{\tau_n}^0)_{n \geqslant 0}$. Note that:

- for all $n < N(w)$, one has $\tau_{D_n}(w) < S(w)$ hence $|Y_n(w)| = |X_{D_n(w)}| \leqslant C$,

- and $|Y_N(w)| \leqslant |Y_{N-1}(w)| + |Y_N(w) - Y_{N-1}(w)| \leqslant C + (a+b)/2$.

This shows that the martingale $(Y_{n \wedge N})_{n \geqslant 0}$ is uniformly bounded, hence it converges in $L^1(Q)$ to Y_N and $\mathbb{E}_Q[Y_N] = \mathbb{E}_Q[Y_0] = 0$.

Note that $\tau_{N-1} < S < +\infty$, hence $Y_N = X_{\tau_N}$ or $Y_N = X_{\tau_{N-1}}$. The inequalities $\tau_{N-1} < S \leqslant \tau_N$ and the fact that the increments of X are bounded by $a+b$ on each interval $[\tau_{n-1}, \tau_n[$ yield $|X_S - Y_N| \leqslant a+b$. This completes the proof. $\qquad\square$

4   Proof of proposition 2

We keep the notations of the previous section, and we introduce for every $n \geqslant 1$,
$$\varepsilon_n = \frac{Y_n - Y_{n-1}}{c_n - c_{n-1}} = \frac{X_{\tau_n} - X_{\tau_{n-1}}}{c_n - c_{n-1}} \mathbf{1}_{[\tau_n < +\infty]}.$$
For every $e = (e_n)_{n \geqslant 1} \in \{-1, 0, 1\}^\infty$, set
$$m_0(e) = \inf\{n \geqslant 1 : e_n = 0\}, \quad m(e) = \inf\{n \geqslant 1 : e_n = -1\}.$$

Call Σ the set of all sequences $e = (e_n)_{n \geqslant 1} \in \{-1, 0, 1\}^\infty$ such that $e_n = 0$ for all $n \geqslant m_0(e)$. Then $\varepsilon = (\varepsilon_n)_{n \geqslant 1}$ can be seen as a random variable with values in Σ.

The first key point is that T is always one of the times $(\tau_n)_{n \geqslant 1}$.

Lemma 6.   *One has $T = \tau_{m \circ \varepsilon}$ (remind the convention $\tau_\infty = +\infty$). Thus, for every $n \geqslant 1$, $\{m \circ \varepsilon = n\} = \{T = \tau_n < +\infty\}$.*

Proof. Fix $w \in \mathbb{W}$ and set $m = m(\varepsilon(w))$ and $m_0 = m_0(\varepsilon(w))$.

For every $n \geqslant 1$, by definition of τ_n and ε_n, one has

$$\varepsilon_n(w) = \pm 1 \quad \text{if } \tau_n(w) < +\infty,$$
$$\varepsilon_n(w) = 0 \quad \text{if } \tau_n(w) = +\infty.$$

In particular, $\tau_n(w) = +\infty$ for every $n \geqslant m_0$ since the sequence $(\tau_n)_{n \geqslant 0}$ is non decreasing. Thus, whether $m \leqslant m_0$ or $m \geqslant m_0$, one has $\tau_{m \wedge m_0}(w) = \tau_m(w)$.

For every $k < m \wedge m_0$, $\varepsilon_k(w) = 1$ hence $w(\tau_k(w)) - w(\tau_{k-1}(w)) = c_k - c_{k-1}$. A recursion then gives $w(\tau_k(w)) = c_k \in]-a, b[$. Moreover, for $\tau_k(w) \leqslant t < \tau_{k+1}(w)$,

$$|w(t) - c_k| = |w(t) - w(\tau_k(w))| < |c_{k+1} - c_k| = d(c_k, \{-a, b\}).$$

Hence for every $t \in [0, \tau_m(w)[$, $w(t) \notin \{-a, b\}$. This proves that $T(w) \geqslant \tau_m(w)$.

If m is infinite, then $T(w)$ is infinite.

If m is finite, the equality

$$w(\tau_m(w)) - w(\tau_{m-1}(w)) = -(c_m - c_{m-1}) = d_m - c_{m-1}$$

implies $w(\tau_m(w)) = d_m \in \{-a, b\}$, hence $T(w) = \tau_m(w)$.

The proof of the first statement is complete. Since the sequence $(\tau_n(w))_{n \geqslant 0}$ is increasing and unbounded, the second statement follows. $\square$

We can now describe the effect of the reflection ϱ_T on the sequence $\varepsilon = (\varepsilon_n)_{n \geqslant 1}$. For every $e = (e_n)_{n \geqslant 1} \in \Sigma$, define $r(e) = (f_n)_{n \geqslant 1} \in \Sigma$ by

$$f_n = \begin{cases} e_n & \text{if } n \leqslant m(e), \\ -e_n & \text{if } n > m(e). \end{cases}$$

Set $g(e) = r(-e)$ and $\gamma = \varrho_T \circ \varrho_0$. Note that g and γ are bijective maps.

Corollary 5. *With the notation above, the following properties hold.*

1. *The reflections ϱ_0, ϱ_T and their composition $\gamma = \varrho_T \circ \varrho_0$ preserve the stopping times τ_n.*

2. *One has $\varepsilon \circ \varrho_0 = -\varepsilon$, $\varepsilon \circ \varrho_T = r \circ \varepsilon$ and $\varepsilon \circ \gamma = g \circ \varepsilon$.*

Proof. Let $w \in \mathbb{W}$. The trajectories w and $\varrho_T(w)$ have the same increments on $[0, T(w)]$ and have opposite increments on $[T(w), +\infty[$. Since $T(w) = \tau_{m \circ \varepsilon}(w)$, the results on ϱ_T follow immediatly. The other statements are obvious. $\qquad \square$

For $n \in \mathbb{N}$, note $\mathbf{1}_n = (1, \ldots, 1) \in \{-1, 1\}^n$. For $(e_1, \ldots, e_n) \in \{-1, 1\}^n$ and $\sigma \in \Sigma$, note $(e_1, \ldots, e_n, \sigma) \in \Sigma$ the sequence obtained by concatenation. The next formula will play the same role as lemma 1 of [2].

Lemma 7. *Let $N = a_0 + 2a_1 + \cdots + 2^{n-1}a_{n-1}$ be a natural integer written in base 2 with n digits (the digit a_{n-1} may be 0). Then for every $\sigma \in \Sigma$,*

$$g^N(\mathbf{1}_n, \sigma) = ((-1)^{a_0}, \ldots, (-1)^{a_{n-1}}, \sigma).$$

Moreover, if $n \geqslant 1$,

$$g^{N-2^{n-1}}(\mathbf{1}_{n-1}, -1, \sigma) = ((-1)^{a_0}, \ldots, (-1)^{a_{n-1}}, \sigma).$$

Proof. The first formula will be proved by induction on the number of digits. If $n = 0$, then $N = 0$ and the formula is obvious.

Assume the formula holds for all integers written with n digits. Let $N = a_0 + 2a_1 + \cdots + 2^n a_n$ be an integer written with $n + 1$ digits.

If $a_n = 0$, then it suffices to write N with n digits and to apply the induction hypothesis to the sequence $(1, \sigma)$.

If $a_n = 1$, let us apply the induction hypothesis to the integer $2^n - 1 = 1 + 2 + \cdots + 2^{n-1}$ and to the sequence $(1, \sigma)$. We get

$$g^{2^n-1}(\mathbf{1}_{n+1}, \sigma) = (-\mathbf{1}_n, 1, \sigma).$$

Applying g once more yields

$$g^{2^n}(\mathbf{1}_{n+1}, \sigma) = (\mathbf{1}_n, -1, \sigma).$$

Applying the induction hypothesis to the integer

$$N - 2^n = a_0 + 2a_1 + \cdots + 2^{n-1}a_{n-1}$$

and to the sequence $(-1, \sigma)$ yields

$$g^N(\mathbf{1}_{n+1}, \sigma) = ((-1)^{a_0}, \ldots, (-1)^{a_{n-1}}, -1, \sigma),$$

which achieves the proof of the first formula.

In particular, if $n \geqslant 1$, $g^{2^{n-1}}(\mathbf{1}_n, \sigma) = (\mathbf{1}_{n-1}, -1, \sigma)$, hence

$$g^{-2^{n-1}}(\mathbf{1}_{n-1}, -1, \sigma) = (\mathbf{1}_n, \sigma).$$

The second formula follows. $\qquad\square$

Introduce $\Sigma_n \subset \{-1, 0, 1\}^n$ the subset of n-uples such that each component after a 0 is 0. Define the map g from Σ_n to itself as before.

Corollary 6. *For every $n \geqslant 1$ and $e = (e_1, \ldots, e_n) \in \Sigma_n$, there exists an integer $M(e)$ such that the event $A_e = \{(\varepsilon_1, \ldots, \varepsilon_n) = (e_1, \ldots, e_n)\}$ belongs to $\mathcal{F}^0_{T \circ \gamma^{M(e)}}$ and $\tau_n = T \circ \gamma^{M(e)}$ on A_e.*

Proof. Set $(e_1, \ldots, e_n) = ((-1)^{a_0}, \ldots, (-1)^{a_{d-1}}, 0, \ldots, 0)$ with $0 \leqslant d \leqslant n$ and $a_0, \ldots, a_{d-1} \in \{0, 1\}$.

If $d = n$, set $M(e) = 2^{n-1} - a_0 - \cdots - 2^{n-1} a_{n-1}$. Then by lemmas 7 and 6,

$$
\begin{aligned}
A_e &= \{g^{M(e)} \circ (\varepsilon_1, \ldots, \varepsilon_n) = (\mathbf{1}_{n-1}, -1)\} \\
&= \{(\varepsilon_1, \ldots, \varepsilon_n) \circ \gamma^{M(e)} = (\mathbf{1}_{n-1}, -1)\} \\
&= \{m \circ \varepsilon \circ \gamma^{M(e)} = n\} \\
&= \{T \circ \gamma^{M(e)} = \tau_n \circ \gamma^{M(e)} < +\infty\}
\end{aligned}
$$

Thus $A_e \in \mathcal{F}^0_{T \circ \gamma^{M(e)}}$, and $\tau_n = \tau_n \circ \gamma^{M(e)} = T \circ \gamma^{M(e)}$ on A_e.

If $d \leqslant n-1$, set $M(e) = -a_0 - \cdots - 2^{d-1} a_{d-1}$. Then by lemmas 7 and 6,

$$
\begin{aligned}
A_e &= \{g^{M(e)} \circ (\varepsilon_1, \ldots, \varepsilon_n) = (\mathbf{1}_d, 0, \ldots, 0)\} \\
&= \{(\varepsilon_1, \ldots, \varepsilon_n) \circ \gamma^{M(e)} = (\mathbf{1}_d, 0, \ldots, 0)\} \\
&= \{m_0 \circ \varepsilon \circ \gamma^{M(e)} = d+1 \; ; \; m \circ \varepsilon \circ \gamma^{M(e)} = +\infty\} \\
&= \{\tau_d \circ \gamma^{M(e)} < +\infty \; ; \; T \circ \gamma^{M(e)} = \tau_{d+1} \circ \gamma^{M(e)} = +\infty\}
\end{aligned}
$$

Thus $A_e \in \mathcal{F}^0_{T \circ \gamma^{M(e)}}$, and $\tau_n = \tau_n \circ \gamma^{M(e)} = +\infty = T \circ \gamma^{M(e)}$ on A_e. $\qquad\square$

The last corollary and the stability properties given in lemmas 5 and 2 show that if $0 \in \mathcal{T}_Q$ and $T \in \mathcal{T}_Q$, then $\tau_n \in \mathcal{T}_Q$ for all $n \geqslant 0$ (recall that $\tau_0 = 0$). This ends the proof of proposition 2.

5   Proof of the main theorem

Let us now prove theorem 2.

Proof. Call Q the law of M as before. The first step of the proof is the observation that for every integer $n \geqslant 1$, $T_{-a_n} \in \mathfrak{T}_Q$ by corollary 3. Hence for all integers $m, n \geqslant 1$, $T_{-a_n} \wedge T_{a_m} \in \mathfrak{T}_Q$ by corollary 2. Lemma 8, which will be stated and proved below, ensures that the ratios $a_n/(a_n + a_m)$ are not dyadic for arbitrarily large m and n. For such m and n, proposition 1 applies and yields $\mathbb{E}_Q[X_S] \leqslant a_n + a_m$ for every finite stopping time S (in the canonical filtration $\mathcal{W}$) such that the stopped process $X_{\cdot \wedge S}$ is uniformly bounded. Since $(a_n)_{n \geqslant 1}$ converges to 0, this proves that $\mathbb{E}_Q[X_S] = 0$, hence X is a local martingale under Q.

The next arguments are the same as in [2] and we now summarize them.

Q-almost surely, the process X admits a quadratic variation $\langle X \rangle$ (defined as a limit in probability of sums of squared increments), which is preserved by the reflections ϱ_0 and $\varrho_{T_{a_n}}$. Consider a regular version of the conditional law of X with respect to $\langle X \rangle$. For any continuous non-decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$, call Q_f the law of X conditionally on $\langle X \rangle = f$. Then for almost every f (for the law of $\langle X \rangle$), the probability Q_f is invariant by the reflections ϱ_0 and $\varrho_{T_{a_n}}$.

By the part of the theorem which is already proven, X is a local martingale under Q_f. But $\langle X \rangle = f$ almost surely under Q_f. Calling ϕ the right-continuous inverse of f, one gets that the process $B = (X_{\phi(s)})_{0 \leqslant s < f(+\infty)}$ is a Brownian motion with lifetime $f(+\infty)$.

Consider, in some suitable enlargement of the probability space $(\mathbb{W}, \mathcal{W}, Q)$, a Brownian motion W, independent of X. For almost every f, the Brownian motion W is still independent of X under Q_f. Since the local martingale X converges Q-almost surely to a random variable X_∞ on the event $\{\langle X \rangle_\infty < +\infty\}$, one gets a Brownian motion B defined on the whole interval $[0, +\infty[$ and independent of $\langle X \rangle$ by setting

$$B_s = X_\infty + W_{s - \langle X \rangle_\infty} \text{ on the event } \{\langle X \rangle_\infty \leqslant s\}.$$

Since $X_t = B_{\langle X \rangle_t}$ almost surely for all $t \geqslant 0$, this shows that X is an Ocone local martingale under Q.

Assume now that $\langle X \rangle_\infty$ is finite with positive probability. Then for some $s \in \mathbb{R}_+$, $\langle X \rangle_\infty \leqslant s$ with positive probability. But with positive probability,

B does not visit a_1 before time s. By independence of B and $\langle X \rangle$,

$$Q[T_{a_1} = +\infty] \geqslant Q[T_{a_1} \circ B > s] \, Q[\langle X \rangle_\infty \leqslant s] > 0.$$

This shows that if T_{a_1} is finite Q-almost surely, then $\langle X \rangle_\infty$ is infinite Q-almost surely, hence X is almost surely divergent. $\square$

Note that the proof of the last statement (if T_{a_1} is finite Q-almost surely, then X is almost surely divergent) given in the discrete case by Chaumont and Vostrikova (lemma 2 of [2]) is not correct because they prove the implication

$$T_a(M) \vee T_{-a}(M) < +\infty \text{ a.s. } \implies T_{a+2}(M) \wedge T_{-a-2}(M) < +\infty \text{ a.s.,}$$

which is not sufficient to perform an induction. Yet, the same arguments that Chaumont and Vostrikova used to prove their lemma 1 are sufficient to prove their lemma 2. Our lemma 7 generalises these arguments, and the case in which some stopping time T_a is infinite is covered by the possibility for the sequence of signs $\sigma \in \Sigma$ to be eventually 0.

Lemma 8. *If $c > b > a > 0$, then at least one of the three following ratios $a/(a + b)$, $b/(b + c)$ and $a/(a + c)$ is not dyadic.*

Proof. The three ratios above belong to $]0, 1/2[$. Assume that they are dyadic. Then

$$\frac{a}{a+b} = \frac{i}{2^p}, \quad \frac{b}{b+c} = \frac{j}{2^q}, \quad \frac{a}{a+c} = \frac{k}{2^r},$$

where i, j and k are odd positive integers and p, q and r are integers greater or equal to 2. Thus

$$\frac{2^r - k}{k} = \frac{c}{a} = \frac{b}{a} \times \frac{c}{b} = \frac{2^p - i}{i} \times \frac{2^q - j}{j},$$

$$ij(2^r - k) = k(2^p - i)(2^q - j),$$

$$2^r ij + 2^q ik + 2^p jk - 2^{p+q}k = 2ijk.$$

This is a contradiction since the left-hand side is a multiple of 4 whereas the right-hand side is not. $\square$

15

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