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THE ALGEBRA OF THE PARALLEL ENDOMORPHISMS OF A PSEUDO-RIEMANNIAN METRIC

CHARLES BOUBEL¹

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Abstract. On a (pseudo-)Riemannian manifold (\mathcal{M}, g) , some fields of endomorphisms *i.e.* sections of $\text{End}(T\mathcal{M})$ may be parallel for g . They form an associative algebra \mathfrak{e} , which is also the commutant of the holonomy group of g . As any associative algebra, \mathfrak{e} is the sum of its radical and of a semi-simple algebra \mathfrak{s} . We show the following: \mathfrak{s} may be of eight different types, including the generic type $\mathfrak{s} = \mathbb{R}\text{Id}$, and the Kähler and hyperkähler types $\mathfrak{s} \simeq \mathbb{C}$ and $\mathfrak{s} \simeq \mathbb{H}$. This is a result on real, semi-simple algebras with involution. Then, for any self adjoint nilpotent element N of the commutant of such an \mathfrak{s} in $\text{End}(T\mathcal{M})$, the set of germs of metrics such that $\mathfrak{e} \supset \mathfrak{s} \cup \{N\}$ is non-empty. We parametrise it. Generically, the holonomy algebra of those metrics is the full commutant $\mathfrak{o}(g)^{\mathfrak{s} \cup \{N\}}$. Apart from some “degenerate” cases, the algebra \mathfrak{e} is then $\mathfrak{s} \oplus (N)$, where (N) is the ideal spanned by N . To prove it, we introduce an analogy with complex Differential Calculus, the ring $\mathbb{R}[X]/(X^n)$ replacing the field \mathbb{C} . This describes totally the local situation when the radical of \mathfrak{e} is principal and consists of self adjoint elements. We add a glimpse on the case where this radical is not principal, and give the constraints imposed to the Ricci curvature when $\mathfrak{e} \neq \mathbb{R}\text{Id}$.

Keywords: Pseudo-Riemannian, Kähler, hyperkähler, parakähler metrics, holonomy group, parallel endomorphism, nilpotent endomorphism, “nilomorphic” functions, commutant, Ricci curvature, real algebra with involution, semi-simple associative algebra.

M.S.C. 2010: 53B30, 53C29, 16K20, 16W10 secondary 53B35, 53C10, 53C12, 15A21.

We investigate here what are the possible algebras of parallel endomorphism fields, for a germ of (pseudo-)Riemannian metric. Our motivation is the following.

Motivation. A Kähler metric g on some manifold \mathcal{M} may be defined as a Riemannian metric admitting an almost complex structure J which is parallel: $DJ = 0$ with D the Levi-Civita connection of g . A natural question is to ask whether other fields of endomorphisms, *i.e.* sections of $\text{End}(T\mathcal{M})$, may be parallel for a Riemannian metric. The answer is nearly immediate. First, one restricts the study to metrics that do not split into a non trivial Riemannian product, called here “indecomposable”. Otherwise, any parallel endomorphism field is the direct sum of parallel such fields on each factor (considering as a unique factor the possible flat factor). Then a brief reasoning ensures that only three cases occur: g may be generic *i.e.* admit only the homotheties as parallel endomorphisms, be Kähler, or be hyperkähler *i.e.* admit two (hence three) anticommuting parallel complex structures. The brevity of this list is due to a simple fact: the action of the holonomy group H of an indecomposable Riemannian metric is irreducible *i.e.* does not stabilise any proper subspace. In particular, this compels any parallel endomorphism field to be of the form $\lambda\text{Id} + \mu J$ with J some parallel, skew adjoint almost complex structure. Now, such irreducibility fails in general for an indecomposable pseudo-Riemannian metric, so that a miscellany of other parallel endomorphism fields may appear. This gives rise to the question handled here:

Which (algebra of) parallel endomorphism fields may a pseudo-Riemannian metric admit ?

Its first natural step, treated here, is local *i.e.* concerns germs of metrics.

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The interest of this question lies also in the following. When studying the holonomy of indecomposable pseudo-Riemannian metrics, the irreducible case may be exhaustively treated: the full list of possible groups, together with the corresponding spaces of germs of metrics (and possibly compact examples) may be provided. After a long story that we do not recall here, this has been done, even for germs of arbitrary torsion free affine connections, see *e.g.* the surveys [12, 25]. Yet, in the general case, the representation of H may be non-semi-simple — see the survey [15] of this field, and [14] for the Lorentzian case — and such an exhaustive answer is out of reach, except perhaps in very low dimension, see *e.g.* the already long list of possible groups in dimension four in [2, 16]. Thus, intermediate questions are needed: not aiming at the full classification, but still significant. Investigating the commutant $\text{End}(T_m\mathcal{M})^H$ of H at some point m of \mathcal{M} , instead of H itself — that is to say studying the algebra of parallel endomorphisms — is such a question. It has been partially treated, namely for an individual self adjoint endomorphism with a minimal polynomial of degree 2, by G. Kručkovič and A. Solodovnikov [20]. I thank V. S. Matveev for this reference. One may also notice that determining all the parallel tensors, not only the endomorphisms, would mean determining the algebraic closure of the holonomy group H . So this work is a step towards this.

Finally, the metrics sharing the same Levi-Civita connection as a given metric g are exactly the $g(\cdot, U\cdot)$ with U self adjoint, invertible and parallel. So describing $\text{End}(T_m\mathcal{M})^H$ enables to describe those metrics, which is also a useful work. The skew adjoint, invertible and parallel endomorphisms are similarly linked with the parallel symplectic forms.

So we investigate here the algebra of the title, the interest of the work being that:

We deal with indecomposable metrics the holonomy group of which is never supposed to be irreducible or totally reducible.

As it is classical in holonomy problems, the question is twofold: **(i)** Which algebras are possible ? **(ii)** By which sets of germs of metrics are they produced ? We will handle both, we say just below to what extent.

We give a last motivation, linked with **(ii)**. With our notation, $U \in \text{End}(T_m\mathcal{M})^H$ means $H \subset \text{O}(g)^U$. We will prove that for a generic metric in this case, $H = \text{O}(g)^U$. This produces new holonomy groups in general, notably when U is nilpotent, and builds the corresponding set of germs of metrics. Examples of such metrics, with U nilpotent, have been recently built by A. Bolsinov and D. Tsonev [6]. We also do the same work with classical holonomy groups H_0 replacing $\text{O}(g)$.

Contents and structure of the article. Let (\mathcal{M}, g) be a (pseudo-)Riemannian manifold, H its holonomy group, H^0 the neutral component of H and $m \in \mathcal{M}$. As any associative algebra, $\text{End}(T_m\mathcal{M})^H$ classically splits into a sum $\mathfrak{s} \oplus \mathfrak{n}$ with \mathfrak{s} a semi-simple subalgebra — in general not canonical, only its isomorphism class is — and $\mathfrak{n} := \text{Rad}(\text{End}(T_m\mathcal{M})^H)$ a nilpotent ideal, its radical. Once again, only the semi-simple part \mathfrak{s} allows an exhaustive treatment, provided here, whereas no list of possible forms for \mathfrak{n} may be given. Recall that, purely algebraically, the classification of nilpotent associative algebras is presently out of reach. Even the case of *pairs of commuting* nilpotent matrices is an active subject; we did not find any explicit review of it, but *e.g.* [1] and its bibliography may be consulted. Besides, our geometric context does not seem to simplify significantly the algebraic nature of \mathfrak{n} . So we treat here a natural first step, the case where $\text{End}(T_m\mathcal{M})^H$ contains:

- some given semi-simple subalgebra \mathfrak{s} of $\text{End}(T_m\mathcal{M})$,
- and possibly also some given nilpotent endomorphism N not in \mathfrak{s} .

More precisely, our main statement may be summed up as follows.

Theorem (i) *The semi-simple part \mathfrak{s} of the algebra $\text{End}(T_m\mathcal{M})^H$ may be of eight different types, including the generic type $\mathfrak{s} = \mathbb{R}\text{Id}$.*

(ii) *Take one of those \mathfrak{s} , and N any self adjoint nilpotent element — possibly null — of the commutant of \mathfrak{s} in $\text{End}(T_m\mathcal{M})$. The set of germs of metrics such that $\text{End}(T_m\mathcal{M})^H$ contains $\mathfrak{s} \cup \{N\}$ i.e. such that the elements of $\mathfrak{s} \cup \{N\}$ extend as parallel endomorphisms fields, admits a parametrisation (explicit, or obtained via Cartan-Kähler theory).*

On this set, generically, equality $H^0 = (\text{O}(g)^{\mathfrak{s} \cup \{N\}})^0$ holds.

Apart from some cases that “degenerate” for reasons of Linear Algebra, the algebra of parallel endomorphisms of those metrics is $\mathfrak{s} \oplus (N)$, where (N) is the ideal spanned by N in the algebra $\langle \mathfrak{s} \cup \{N\} \rangle$.

Item (i) is plain linear algebra: the classification of some semi simple, g -self adjoint subalgebras of $\mathfrak{gl}(\mathbb{R}^d)$, see Remark 1.12. The theorem allows a glimpse on the general case for \mathfrak{n} . Finally we give the consequences of the existence of parallel endomorphisms on the Ricci curvature.

The article is divided into five parts.

– Part 1 is essentially devoted to \mathfrak{s} . We introduce the decomposition $\text{End}(T_m\mathcal{M})^H = \mathfrak{s} \oplus \mathfrak{n}$ in §1.1 and some natural objects associated with a *reducible* holonomy representation in §1.2; Proposition 1.8 is a simple but remarkable commutation property in $\text{End}(T_m\mathcal{M})^H$. We list in §1.3, Theorem 1.10 and Tables 1 and 2, the eight possible forms of \mathfrak{s} , with corollaries. We give the corresponding spaces of germs of metrics in §1.4.

– Part 2 is devoted to \mathfrak{n} , in the sense that we parametrise the set \mathcal{G} of germs of metrics g admitting a parallel nilpotent endomorphism field N . More exactly, it deals with the case where N is g -self adjoint. Indeed if some $N \in \mathfrak{n}$ is parallel, so are its self- and skew-adjoint parts $\frac{1}{2}(N \pm N^*)$, so it is natural to study first the cases $N^* = \pm N$. A natural local description of \mathcal{G} follows from an analogy with complex differential calculus, N and $\mathbb{R}[X]/(X^n)$ replacing J and $\mathbb{C} = \mathbb{R}[X]/(X^2 + 1)$. Counterparts of holomorphic functions, of their power series expansion, appear. So we introduce this analogy in §2.1 and §2.2, notably with Definition 2.8 and Proposition 2.16. Then §2.3 provides the announced parametrisation of \mathcal{G} in Proposition 2.29 and Theorem 2.31, and investigates in detail how it works in some particular cases: low nilpotence indices *etc.*

The case $N^* = -N$ demands some more theory — introducing counterparts of $\partial, \bar{\partial}$, of the Dolbeault lemma *etc.* We hope to publish it later.

– Part 3 deals with \mathfrak{s} and \mathfrak{n} together: it uses parts 1 and 2 to show the theorem stated above, which gathers and generalises the most part of the preceding results. See Theorem 3.2 for the metrics and corresponding holonomy groups, and Corollary 3.5 for the form of $\text{End}(T_m\mathcal{M})^H$. If g is hyperkähler *i.e.* $\text{O}(g)^{\mathfrak{s}} = \text{Sp}(p, q)$, or is of a similar type, this is done by solving an exterior differential system exactly as is done by R. Bryant in [12], but in the framework of “ $\mathbb{R}[X]/(X^n)$ -differential calculus” introduced in part 2. To show Corollary 3.5, we need to compute the general matrix of the elements of all the algebras involved here: commutants, bicommutants *etc.* Displaying those matrices is also of practical interest, so we gathered all this at the end of Part 3, pp. 45 *sq.*

– Part 4, a lot shorter, uses parts 1 and 2 to give a glimpse, through a simple example, on the case where the holonomy group is the commutant $\text{O}(g)^{\{N, N'\}}$ of *two* algebraically independent nilpotent endomorphisms.

– Part 5 consists of one single Theorem 5.1, studying the Ricci curvature of metrics admitting non trivial parallel endomorphisms.

General setting and some general notation. Here \mathcal{M} is a simply connected manifold of dimension d and g a Riemannian or pseudo-Riemannian metric on it, whose holonomy representation does not stabilise any nondegenerate subspace, that is to say does not split in an orthogonal sum of subrepresentations. In particular, g does not split into a Riemannian product. We set $H \subset \mathrm{SO}^0(T_m\mathcal{M}, g|_m)$ the holonomy group of g at m and \mathfrak{h} its Lie algebra. As \mathcal{M} is supposed to be simply connected, we deal everywhere with \mathfrak{h} , forgetting H . Let \mathfrak{e} be the algebra $\mathrm{End}(T_m\mathcal{M})^{\mathfrak{h}}$ of the parallel endomorphisms of g — to commute with \mathfrak{h} amounts to extend as a parallel field —; it is isomorphic to some subalgebra of $M_d(\mathbb{R})^{\mathfrak{h}}$. Notice that \mathfrak{e} is stable by g -adjunction, which we denote by $\sigma : a \mapsto a^*$. If A is an algebra and $B \subset A$, we denote by $\langle B \rangle$, (B) , and A^B the algebra, respectively the ideal, spanned by B , and the commutant of B in A . When lower case letters: x_i, y_i etc. stand for local coordinates, the corresponding upper case letters: X_i, Y_i etc. stand for the corresponding coordinate vector fields. Viewing vector fields X as derivations, we denote Lie derivatives $\mathcal{L}_X u$ also by $X.u$.

The matrix $\mathrm{diag}(I_p, -I_q) \in M_{p+q}(\mathbb{R})$ is denoted by $I_{p,q}$, $\begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix} \in M_{2p}(\mathbb{R})$ by J_p and $\begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix} \in M_{2p}(\mathbb{R})$ by L_p . If V is a vector space of even dimension d , we recall that an $L \in \mathrm{End}(V)$ is called *paracomplex* if $L^2 = \mathrm{Id}$ with $\dim \ker(L - \mathrm{Id}) = \dim \ker(L + \mathrm{Id}) = \frac{d}{2}$.

Finally, take $A \in \Gamma(\mathrm{End}(T\mathcal{M}))$, paracomplex or nilpotent. If it is integrable *i.e.* if its matrix is constant in well-chosen local coordinates, we call it a “paracomplex structure” or a “nilpotent structure”, like a complex structure, as opposed to an almost complex one.

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1 The algebra $\mathfrak{e} = \mathrm{End}(T\mathcal{M})^H$ and its semi-simple part \mathfrak{s}

1.1 The decomposition $\mathfrak{e} = \mathfrak{s} \oplus \mathfrak{n}$ of \mathfrak{e} in a semi-simple part and its radical

First we need to recall some facts and set some notation. All the results invoked are classical for finite dimensional associative algebras; we state them for a unital real algebra A .

1.1 Notation If A is a subset of an algebra, $A^* \subset A$ denotes here the subset of its invertible elements. If σ is an involutive anti morphism of A , then $A^\pm = \{U \in A; \sigma(U) = \pm U\}$ denotes the subspace of its self adjoint or skew adjoint elements.

1.2 Reminder An algebra A is said to be *nilpotent* if A^k , the algebra spanned by the products of k elements of A , is $\{0\}$ for some k . In particular, the elements of a nilpotent subalgebra of $M_n(\mathbb{R})$ are simultaneously strictly upper triangular in some well-chosen basis.

1.3 Definition (See [13] §25 or [18]) *The radical $\mathrm{Rad} A$ of A is the intersection of its maximal ideals. It is a nilpotent ideal. Equivalently, it is the sum of the nilpotent ideals of*

A. The algebra A is said to be simple if its only proper ideal is $\{0\}$, and semi-simple if its radical is $\{0\}$ — so a simple algebra is semi-simple, and $A/\text{Rad}(A)$ is semi-simple.

The decomposition $\mathfrak{e} = \mathfrak{s} \oplus \mathfrak{n}$ is provided by the following classical result. The last assertion is a refinement due to Taft [27, 28]. I thank P. Baumann for this reference.

1.4 Theorem [Wedderburn – Malčev] (see [13] §72) *Let A be a finite dimensional \mathbb{R} -algebra. Then there exists a semi-simple algebra A_S in A such that $A = A_S \oplus \text{Rad}(A)$. If moreover A is endowed with an involutive anti-morphism σ , then A_S may be chosen σ -stable.*

1.5 Notation We set $\mathfrak{n} = \text{Rad } \mathfrak{e}$. Being the unique maximal nilpotent ideal of \mathfrak{e} , \mathfrak{n} is self adjoint *i.e.* stable by g -adjunction. We take $\mathfrak{s} \simeq \mathfrak{e}/\mathfrak{n}$ some self adjoint semi-simple subalgebra of \mathfrak{e} provided by Theorem 1.4.

1.2 Some natural objects associated with a reducible holonomy representation; a “quasi-commutation” property

1.6 Remark/Notation We denote by $E_0 = \cap_{W \in \mathfrak{h}} \ker W$ the (possibly trivial) maximal subspace of $T_m \mathcal{M}$ on which the holonomy group H acts trivially. As $T_m \mathcal{M}$ is H -orthogonally indecomposable, E_0 is totally isotropic. We set $\mathfrak{n}_0 = \{N \in \mathfrak{e}; \text{Im } N \subset E_0\}$; as the actions of H and \mathfrak{e} on $T_m \mathcal{M}$ commute, \mathfrak{n}_0 is an ideal of \mathfrak{e} , moreover self adjoint. So, for any $x, y \in T_m \mathcal{M}$, and any $N, N' \in \mathfrak{n}_0$, $g(N'Nx, y) = g(Nx, N'^*y) \in g(E_0, E_0) = \{0\}$, so $N'N = 0$ *i.e.* $\mathfrak{n}_0^2 = \{0\}$.

1.7 Remark/Notation The algebra \mathfrak{e} is naturally endowed with the bilinear symmetric form $\langle U, V \rangle = \frac{1}{d} \text{tr}(U^*V)$. By Reminder 1.2, $\mathfrak{n} \subset \ker(\langle \cdot, \cdot \rangle)$. If moreover \mathfrak{e} admits some self adjoint complex structure \underline{J} , and denoting by $\mathfrak{e}_{\underline{J}}$ the \underline{J} -complex algebra $\{U \in \mathfrak{e}; U\underline{J} = \underline{J}U\}$, then $\mathfrak{e}_{\underline{J}}$ is endowed with the complex form $\langle U, V \rangle_{\underline{J}} = \frac{1}{d}(\text{tr}(U^*V) - i \text{tr}(U^*\underline{J}V))$.

The following proposition is the key of most steps of the classification 1.10. As it is also worth to be noticed by itself, we state it apart, here.

1.8 Proposition *Let U, V be in \mathfrak{e} and m be any point of \mathcal{M} . Then, if U is self adjoint, then for any $x, y \in T_m \mathcal{M}$, $R(x, y)(UV - VU) = 0$. Consequently, $UV - VU \in \mathfrak{n}_0$. In particular, in case $E_0 = \cap_{W \in \mathfrak{h}} \ker W = \{0\}$, all self adjoint elements of \mathfrak{e} are central in \mathfrak{e} .*

Proposition 1.8 rests on the following remark.

1.9 Reminder/Remark We will need the following remark. The Bianchi identity implies that, at any point $m \in \mathcal{M}$ and for any $x, y, z, t \in T_m \mathcal{M}$, $g(R(x, y)z, t) = g(R(z, t)x, y)$. This holds also for any, possibly degenerate, bilinear form g' , parallel with respect to the Levi Civita connection of g . The proof does not need nondegeneracy, see *e.g.* Lemma 9.3 in [23]. So if U is a parallel self adjoint endomorphism, $R(Ux, y)z = R(x, Uy)z = R(x, y)Uz$. The first equality is classical. For the second one, take t any fourth vector and denote by g_U the bilinear form $= g(\cdot, U \cdot)$, which is parallel, as U is, and symmetric, as $U^* = U$. Then:

$$\begin{aligned}
 g(R(x, Uy)z, t) &= g(R(z, t)x, Uy) && \text{applying the relation to } g, \\
 &= g_U(R(z, t)x, y) \\
 &= g_U(R(x, y)z, t) && \text{applying the relation to } g_U, \\
 &= g(R(x, y)Uz, t) && \text{as } U^* = U, \text{ being parallel,} \\
 &&& \text{commutes with } R(x, y). \quad \text{q.e.d.}
 \end{aligned}$$

Proof of Proposition 1.8. Take $U, V \in \mathfrak{e}$ with $U^* = U$ and $x, y, z, t \in T_m\mathcal{M}$. The bilinear form $g_U := g(\cdot, U\cdot)$ is parallel, as U is.

$$\begin{aligned}
 & g(R(x, y)z, VUt) \\
 = & g(R(x, y)V^*z, Ut) && \text{as, } V^*, \text{ parallel, commutes with } R(x, y), \\
 = & g(R(x, Uy)V^*z, t) && \text{by Remark 1.9, applied to } U, \\
 = & g(R(x, Uy)z, Vt) && \text{as, } V^* \text{ commutes with } R(x, y), \\
 = & g(R(x, y)z, UVt) && \text{again by Remark 1.9, so the result.} \quad \text{q.e.d.}
 \end{aligned}$$

1.3 The eight possible forms of \mathfrak{s}

The types given by Theorem 1.10 are known, but not in full generality for type **(3')** *i.e.* with the corresponding set of germs of metrics clearly stated, and except **(3^C)** which I never encountered explicitly. So 1.10 closes the list, may the action of H be totally reducible or not. The proof rests on the classical Wedderburn-Artin and Skolem Noether theorems, and then is elementary. Remark 1.15 below gives the generic holonomy group corresponding to each case of the theorem.

1.10 Theorem *The algebra \mathfrak{s} is of one of the following types, where \underline{J} , J , and L denote respectively self adjoint complex structures and skew adjoint complex and paracomplex structures. Each case is precisely described in Tables 1 p. 8 and 2 p. 9, which are part of the theorem.*

(1) generic, $\mathfrak{s} = \text{vect}(\text{Id})$.

(1^C) “complex Riemannian”, $\mathfrak{s} = \text{vect}(\text{Id}, \underline{J})$. Here $d \geq 4$ is even, $\text{sign}(g) = (\frac{d}{2}, \frac{d}{2})$, $(\mathcal{M}, J, g(\cdot, \cdot) - \text{ig}(\cdot, \underline{J}\cdot))$ is complex Riemannian for a unique complex structure in \mathfrak{s} , up to sign.

(2) (pseudo-)Kähler, $\mathfrak{s} = \text{vect}(\text{Id}, J)$. Here d is even and (\mathcal{M}, J, g) is (pseudo-)Kähler, for a unique complex structure in \mathfrak{s} , up to sign.

(2') **parakähler**, $\mathfrak{s} = \text{vect}(\text{Id}, L)$. Here d is even, $\text{sign}(g) = (\frac{d}{2}, \frac{d}{2})$, (\mathcal{M}, L, g) is parakähler, for a unique paracomplex structure in \mathfrak{s} , up to sign.

(2^C) “complex Kähler”, $\mathfrak{s} = \text{vect}(\text{Id}, \underline{J}, L, J)$. Here $d \in 4\mathbb{N}^*$, $\text{sign}(g) = (\frac{n}{2}, \frac{n}{2})$ and $(\mathcal{M}, \underline{J}, J, L, g)$ is at once complex Riemannian, pseudo-Kähler, and parakähler, on a unique way in \mathfrak{s} , up to sign of each structure.

(3) (pseudo-)hyperkähler, $\mathfrak{s} = \text{vect}(\text{Id}, J_1, J_2, J_3)$. Here $d \in 4\mathbb{N}^*$, $(\mathcal{M}, J_1, J_2, g)$ is (pseudo-)hyperkähler, the set of Kähler structures in \mathfrak{s} being a 2-dimensional submanifold.

(3') **“para-hyperkähler”**, $\mathfrak{s} = \text{vect}(\text{Id}, J, L_1, L_2)$. Here $d \in 4\mathbb{N}^*$, $\text{sign}(g) = (\frac{d}{2}, \frac{d}{2})$ and (\mathcal{M}, J, L_1, g) is at once pseudo-Kähler and parakähler, the set of complex and of paracomplex structures in \mathfrak{s} being each a 2-dimensional submanifold.

(3^C) “complex hyperkähler”, $\mathfrak{s} = \text{vect}(\text{Id}, \underline{J}, J, L_1, L_2, \underline{J}J, \underline{J}L_1, \underline{J}L_2)$ Here $d \in 8\mathbb{N}^*$, $\text{sign}(g) = (\frac{d}{2}, \frac{d}{2})$ and (\mathcal{M}, g) is at once complex Riemannian (on a unique way up to sign in \mathfrak{s}), and pseudo-Kähler and parakähler. The sets of pseudo- or parakähler structures are 2-dimensional \underline{J} -complex submanifolds of \mathfrak{s} .

Each type is produced by a non-empty set of germs of metrics. On a dense open subset of them, for the C^2 topology, the holonomy group of the metric is the commutant $\text{SO}^0(g)^{\mathfrak{s}}$ of \mathfrak{s} in $\text{SO}^0(g)$.

1.11 Remark The fact that the set of germs of metrics in each case is non-empty is well-known, except perhaps for types **(3')** and **(3^C)**. In all cases, §1.4 gives their parametrisation.

1.12 Remark In fact, we proved the following result in plain linear algebra. If g is a (pseudo-)Euclidean product on \mathbb{R}^d and A a semi-simple, g -self adjoint subalgebra of $\mathfrak{gl}(\mathbb{R}^d)$, whose action on \mathbb{R}^d is indecomposable (in an orthogonal sum), then A is one of the eight algebras of Theorem 1.10 or the algebra $A \simeq \mathbb{H} \oplus \mathbb{H}$ of Remark 1.21.

1.13 Notation If G is a subgroup of $GL_d(\mathbb{K})$, we denote here by \mathbf{V} its standard representation in \mathbb{K}^d . We denote then by $\mathbf{V}^* : g \mapsto (\lambda \mapsto \lambda \circ g^{-1})$ its contragredient representation in $(\mathbb{K}^d)^*$ and, if $\mathbb{K} = \mathbb{C}$, by $\overline{\mathbf{V}}^*$ the complex conjugate of it.

1.14 Remark In cases **(2')**, **(2^C)**, **(3')** and **(3^C)**, the existence of a paracomplex structure L splits $\mathbf{T}\mathcal{M} = \ker(L - \text{Id}) \oplus \ker(L + \text{Id}) = V \oplus V'$ into a sum of two totally isotropic factors, and the morphism b given by the metric identifies V' with V^* . Then, H is isomorphic to a subgroup $[H]$ of $GL_{d/2}(\mathbb{K})$, the holonomy representation being $\mathbf{V} \oplus \mathbf{V}^*$, if $\mathbb{K} = \mathbb{R}$, or $\mathbf{V} \oplus \overline{\mathbf{V}}^*$, if $\mathbb{K} = \mathbb{C}$, on $\ker(L - \text{Id}) \oplus \ker(L + \text{Id})$. Matricially:

$$H := \left\{ \begin{pmatrix} U & 0 \\ 0 & {}^t U^{-1} \end{pmatrix}, U \in [H] \right\};$$

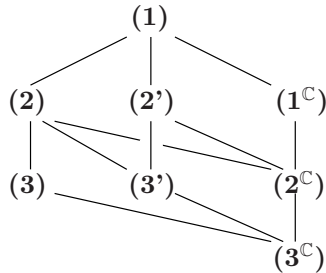
so if $\mathbb{K} = \mathbb{R}$, $H \subset SO^0(\frac{d}{2}, \frac{d}{2})$ and if $\mathbb{K} = \mathbb{C}$, $H \subset U(\frac{d}{2}, \frac{d}{2})$.

1.15 Remark For \mathfrak{s} of each type, we sum up here: the possible signature(s) of g , the group in which H (possibly identified with $[H]$, see Rem. 1.14) is included, and to which it is generically equal (proof in §1.4), and the representation of H or $[H]$. Notice that each time, this group is also the commutant of \mathfrak{s} in $SO^0(g)$. See Notation 1.13 for \mathbf{V} .

(1)	(1^C)	(2)	(2')	(2^C)	(3)	(3')	(3^C)
(p, q)	(p, p)	$(2p, 2q)$	(p, p)	$(2p, 2p)$	$(4p, 4q)$	$(2p, 2p)$	$(4p, 4p)$
$SO^0(p, q)$	$SO(p, \mathbb{C})$	$U(p, q)$	$GL^0(p, \mathbb{R})$	$GL(p, \mathbb{C})$	$Sp(p, q)$	$Sp(2p, \mathbb{R})$	$Sp(2p, \mathbb{C})$
\mathbf{V}	\mathbf{V}	\mathbf{V}	$\mathbf{V} \oplus \mathbf{V}^*$	$\mathbf{V} \oplus \overline{\mathbf{V}}^*$	\mathbf{V}	$\mathbf{V} \oplus \mathbf{V}^*$	$\mathbf{V} \oplus \overline{\mathbf{V}}^*$

1.16 Remark In Theorem 1.10, the new cases with respect to the Riemannian framework occur only for metrics g of signature $(\frac{d}{2}, \frac{d}{2})$.

1.17 Remark [Justification of the labels in Theorem 1.10] The generic holonomy groups corresponding to \mathfrak{s} of types **(1^C)**, **(2^C)** and **(3^C)** are complexification of those corresponding to \mathfrak{s} of respective types **(1)**, **((2) or (2'))**, and **((3) or (3'))**. Besides, if you consider the different types in a comprehensive sense, type **(2)** *e.g.* meaning only “ $H \subset U(p, q)$ ”, and so on, you obtain the following inclusion diagram:



where the strokes denote the inclusion of the set of metrics below into the one above.

This justifies our notation. Another point of view is the following. Suppose that g is a real analytic germ of metric at m . Then \mathfrak{h} is generated by $\{D^k R(u_1, \dots, u_{k+2}), (u_i)_{i=1}^{k+2} \in$

$T_m\mathcal{M}$ }, the curvature tensors at m and their covariant derivatives at all orders. So the complexification $g^{\mathbb{C}}$ of the germ g has $H \otimes \mathbb{C}$ as holonomy group. Thus here, if g is “of type **(1)**”, respectively **((2)** or **(2')**), or **((3)** or **(3')**), its complexification is “of type **(1^C)**”, respectively **(2^C)** and **(3^C)**.

(1)	(1^C)	(2)	(2')	(2^C)
$\mathbb{R} \subset \mathbb{R},$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{1,0}$	$\mathbb{C} \subset \mathbb{C},$ $\mathbb{R}^{1,1} \subset \mathbb{R}^{1,1}$	$\mathbb{R} \subset \mathbb{C},$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{2,0}$	$\mathbb{R} \cdot (1,1) \subset \mathbb{R} \oplus \mathbb{R},$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{1,1}$	$\mathbb{C} \cdot (1,1) \subset \mathbb{C} \oplus \mathbb{C},$ $\mathbb{R}^{1,1} \subset \mathbb{R}^{2,2}$
$\langle \rangle$	$\langle \underline{J} \rangle$	$\langle J \rangle$	$\langle L \rangle$	$\langle \underline{J}, L, J \mid$ $\underline{J} \in Z(\mathfrak{s}),$ $LJ = JL = \underline{J} \rangle$
\emptyset	$\{i\}$	$\{i\}$	$\{(1, -1)\}$	$\{(i, i), (1, -1),$ $(i, -i)\}$
Id	Id $_{\mathbb{C}}$	$z \mapsto \bar{z}$	$(a, b) \mapsto (b, a)$	$(a, b) \mapsto (b, a)$
In case (2^C) , viewed as a \underline{J} -complex algebra, $\mathfrak{s} = \langle L \rangle$ is given as in (2') .				
(3)	(3')	(3^C)	(3^C) , \mathfrak{s} given as	
$\mathbb{R} \subset \mathbb{H},$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{4,0}$	$\mathbb{R} \cdot I_2 \subset M_2(\mathbb{R}),$ $\mathbb{R}^{1,0} \subset \mathbb{R}^{2,2}$	$\mathbb{C} \cdot I_2 \subset M_2(\mathbb{C}),$ $\mathbb{R}^{1,1} \subset \mathbb{R}^{4,4}$	a \underline{J} -complex algebra	
$\langle J_1, J_2, J_3 \mid$ $J_{[i]}J_{[i+1]} = J_{[i+2]},$ $J_iJ_{i'} = -J_{i'}J_i \rangle$	$\langle L_1, L_2, J \mid$ $J = -L_1L_2$ $= L_2L_1,$ $L_1 = L_2J = -JL_2,$ $L_2 = JL_1 = -L_1J \rangle$	$\langle \underline{J}, L_1, L_2, J \mid$ $\underline{J} \in Z(\mathfrak{s}),$ $J = -L_1L_2$ $= L_2L_1,$ $L_1 = L_2J = -JL_2,$ $L_2 = JL_1 = -L_1J \rangle$	$\langle L_1, L_2, L_3 \mid$ $iL_{[i]}L_{[i+1]}$ $= L_{[i+2]},$ $L_{[i]}L_{[i']}$ $= -L_{[i']}L_{[i]} \rangle$	
the canonical $\{i, j, k\} \subset \mathbb{H}$	$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$	$\left\{ iI_2, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ i & 0 \end{pmatrix} \right\}$	
$z \mapsto \bar{z}$ (quaternionic conjugation)	$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ i.e. transpose of the comatrix	idem	idem	

Table 1: *Theorem 1.10 summed up in a table.* We give on each line:

- $\mathfrak{s}^+ \subset \mathfrak{s}$ as an inclusion of algebras, then of (pseudo-)euclidian spaces for $\langle \cdot, \cdot \rangle$; $\mathbb{R}^{a,b}$ means $(\mathbb{R}^{a+b}, \langle \cdot, \cdot \rangle)$ with $\text{sign}(\langle \cdot, \cdot \rangle) = (a, b)$,
- \mathfrak{s} as a unital \mathbb{R} -algebra generated by $\langle \cdot, \cdot \rangle$ -orthogonal complex and paracomplex structures,
- the corresponding generators of A ,
- the antimorphism of A conjugated to the adjunction of \mathfrak{s} .

All letters J denote complex structures, and L paracomplex ones. All are g -skew adjoint, except the g -self adjoint underlined \underline{J} . Bracketed indices $[i]$ denote indices modulo 3.

1.18 Lemma Take $U \in \mathfrak{e}$ and $N \in \mathfrak{n}$, μ the minimal polynomial of U and μ' that of $U + N$. Then any irreducible factor of μ is also in μ' , and vice versa.

Proof. $\mu(U + N) = \mu(U) + NV = NV$ with V some polynomial in U and N . As \mathfrak{n} is an ideal, $NV \in \mathfrak{n}$ and by Proposition 1.2, NV is nilpotent. So for some $k \in \mathbb{N}$, $(\mu^k)(U + N) = 0$ i.e. $\mu^k | \mu'$. Symmetrically, $\exists l \in \mathbb{N}^* : \mu | \mu'^l$, so the result. q.e.d.

Proof of Theorem 1.10. We denote $T_m\mathcal{M}$ by E in this proof. We first state the announced classical results in associative algebra.

$\dim \mathcal{M}, \text{sign}(g)$	Matrices, in a well-chosen basis
(1) any, any	$\text{Mat}(g) = I_{p,q}$
(1 ^c) $d = 2p, (p, p)$	$\text{Mat}(g) = I_{p,p}, \text{Mat}(\underline{J}) = J_p$
(2) $d = 2p + 2q, (2p, 2q)$	$\text{Mat}(g) = \text{diag}(I_{p,q}, I_{p,q}), \text{Mat}(J) = J_{d/2}$
(2') $d = 2p, (p, p)$	$\text{Mat}(g) = I_{p,p}, \text{Mat}(L) = L_p$ ^[1]
(2 ^c) $d = 4p, (2p, 2p)$	$\text{Mat}(g) = L_{2p}, \text{Mat}(L) = I_{2p,2p}, \text{Mat}(\underline{J}) = \text{diag}(J_p, -J_p)$
(3) $d \in 4\mathbb{N}^*, (4p, 4q), p + q = \frac{d}{4}$	$\text{Mat}(g) = \text{diag}(I_{p,q}, I_{p,q}, I_{p,q}, I_{p,q}), \text{Mat}(J_2) = J_{d/2},$ $\text{Mat}(J_1) = \text{diag}(-J_{d/4}, J_{d/4}), \text{Mat}(J_3) = \begin{pmatrix} 0 & J_{d/4} \\ J_{d/4} & 0 \end{pmatrix}$
(3') $d = 4p, (2p, 2p)$	$\text{Mat}(g) = I_{2p,2p}, \text{Mat}(L_1) = L_{2p}$ $\text{Mat}(J) = \text{diag}(-J_p, J_p)$ ^[2] , $\text{Mat}(L_2) = \begin{pmatrix} 0 & -J_p \\ J_p & 0 \end{pmatrix}$
(3 ^c) $d = 8p, (4p, 4p)$	$\text{Mat}(g) = \text{diag}(I_{2p,2p}, -I_{2p,2p}), \text{Mat}(\underline{J}) = \text{diag}(J_{2p}, J_{2p}),$ $\text{Mat}(J) = \text{diag}(J_p, J_p, -J_p, -J_p), \text{Mat}(L_1) = L_{4p}$ $\text{Mat}(L_2) = \begin{pmatrix} 0 & \text{diag}(J_p, J_p) \\ -\text{diag}(J_p, J_p) & 0 \end{pmatrix}$
^[1] or e.g.: $\text{Mat}(g) = L_p, \text{Mat}(L) = I_{p,p}$.	
^[2] or e.g.: $\text{Mat}(g) = L_{2p}, \text{Mat}(L_1) = I_{2p,2p}, \text{Mat}(J) = \begin{pmatrix} 0 & J_p \\ J_p & 0 \end{pmatrix}$.	
structures (below):	(1) (1 ^c) (2) (2') (2 ^c) (3) (3') (3 ^c)
“Complex Riemannian”	\emptyset $\{\pm \underline{J}\}$ \emptyset \emptyset $\{\pm \underline{J}\}$ \emptyset \emptyset $\{\pm \underline{J}\}$
Kähler	\emptyset \emptyset $\{\pm J\}$ \emptyset $\{\pm J\}$ [3] [4] [6]
para Kähler	\emptyset \emptyset \emptyset $\{\pm L\}$ $\{\pm L\}$ \emptyset [5] [7]
^[3] the 2-sphere $\{\alpha J_1 + \beta J_2 + \gamma J_3; \alpha^2 + \beta^2 + \gamma^2 = 1\} = \{U; \langle U, U \rangle = 1\}$	
^[4] the two-sheet hyperboloid $\{\alpha L_1 + \beta L_2 + \gamma J; \alpha^2 + \beta^2 - \gamma^2 = -1\} = \{U; \langle U, U \rangle = 1\}$	
^[5] the one-sheet hyperboloid $\{\alpha L_1 + \beta L_2 + \gamma J; \alpha^2 + \beta^2 - \gamma^2 = 1\} = \{U; \langle U, U \rangle = -1\}$	
^[6] the proper quadric with centre $\{U; \langle U, U \rangle_{\underline{J}} = 1\} = \{\alpha L_1 + \beta L_2 + \gamma J; \alpha = \alpha' + \alpha'' \underline{J} \text{ etc. } \alpha^2 + \beta^2 - \gamma^2 = -1\}$	
^[7] the proper quadric with centre $\{U; \langle U, U \rangle_{\underline{J}} = -1\} = \{\alpha L_1 + \beta L_2 + \gamma J; \alpha = \alpha' + \alpha'' \underline{J} \text{ etc. } \alpha^2 + \beta^2 - \gamma^2 = 1\}$	
Cases (3), (3'), and (3 ^c) imply that the metric is Ricci-flat.	

 Table 2: *Th. 1.10 in a matricial form, with the (para)complex structures.*

1.19 Theorem [Wedderburn – Artin] (see [18] §3, p. 40). *Let A be a finite dimensional semi-simple \mathbb{R} -algebra. Then A is isomorphic to a direct sum of matrix algebras: $A \simeq \bigoplus_{i=1}^k M_{d_i}(\mathbb{K}_i)$ with for each i , $d_i \in \mathbb{N}^*$ and $\mathbb{K}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.*

1.20 Theorem [corollary of a theorem of Skolem – Noether] (see [9] §10, no. 1). *Let θ be an automorphism of a finite dimensional semi-simple \mathbb{R} -algebra A . If θ acts trivially on the center of A , θ is interior.*

As g is orthogonally indecomposable, then if it is flat, $\dim \mathcal{M} = 1$ and $\mathfrak{s} = \mathfrak{e} = \mathbb{R} \text{Id}$ is of type (1). We now suppose that g is not flat. The list 1.10 follows from the orthogonal indecomposability of the action of \mathfrak{h} , through the claim below. The elimination of only one possible form for \mathfrak{s} will also require, through Proposition 1.8, the fact that \mathfrak{h} is a holonomy algebra *i.e.* from the Bianchi identity satisfied by the curvature tensor.

Claim 1. If $p \in \mathfrak{e}$ is self adjoint, its minimal polynomial $\mu_p(X)$ is of the form Q^α with Q irreducible — so if p is not invertible, it is nilpotent.

Proof. The minimal polynomial reads $\mu_p(X) = \prod_{i=1}^N Q_i^{\alpha_i}$ with irreducible Q_i 's. As p is self adjoint, the direct sum $E = \bigoplus_{i=1}^N \ker Q_i^{\alpha_i}(p)$ is g -orthogonal. As $p \in \text{End}(E)^{\mathfrak{h}}$, each $\ker Q_i^{\alpha_i}(p)$ is \mathfrak{h} -stable. Now E is indecomposable, so $N = 1$ and the claim.

By 1.4, $\mathfrak{e} = \mathfrak{s} \oplus \mathfrak{n}$ where $\mathfrak{n} = \text{Rad}(\mathfrak{e})$ and \mathfrak{s} is a semi-simple, self adjoint subalgebra of \mathfrak{e} . As \mathfrak{n} is the intersection of the maximal ideals of \mathfrak{e} and as the adjunction σ is an anti-morphism, \mathfrak{n} is also self adjoint. So 1.19 gives an isomorphism $\varphi : \mathfrak{s} \xrightarrow{\sim} A$ with $A = \bigoplus_{i=1}^k I_i$ and $I_i = M_{\delta_i}(\mathbb{K}_i)$, $\mathbb{K}_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. By a slight abuse, we also denote by σ the conjugate action $\varphi \circ \sigma \circ \varphi^{-1}$ of σ on A .

Claim 2. $k \leq 2$. If $k = 2$, then $\sigma(I_1) = I_2$. We then denote the δ_i by δ and the \mathbb{K}_i by \mathbb{K} .

Proof. Let us denote by 1 the unit matrix of any factor of A . As an (anti) automorphism of A , σ acts on the factors I_i of A , permuting them. Take $p = (1, 0, \dots, 0) \in A$. As $p^2 = p$, $\varphi^{-1}(p)$ is a (non zero) projection, so by Claim 1, either $\varphi^{-1}(p) = 1_{\mathfrak{e}}$ and thus $k = 1$, or $\sigma(p) \neq p$. In the latter case, $\sigma(I_1) \neq I_1$. Take $p' = p + \sigma(p)$. It is self adjoint by construction, and $p'^2 = p^2 + \sigma(p)^2 = p + \sigma(p^2) = p'$ so it is a (non zero) projection. By Claim 1, $\varphi^{-1}(p') = 1_{\mathfrak{e}}$ so $A = I_1 \oplus \sigma(I_1)$ and then $k = 2$.

Claim 3. If $k = 2$, then $\delta = 1$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Proof. Suppose $k = 2$. Take $p = (\text{diag}(1, 0, \dots, 0), 0) \in A$ and $p' = p + \sigma(p)$. By the same reasoning as above, $\varphi^{-1}(p')$ is a non zero self adjoint projection so $p' = 1_A$ by Claim 1. As the I_1 -component of $\sigma(p)$ is zero, then in fact $p = (1, 0)$ and $\sigma(p) = (0, 1)$; in particular $\text{diag}(1, 0, \dots, 0) = 1_{I_1}$ i.e. $\delta = 1$. Now Proposition 1.8 implies $\mathbb{K} \neq \mathbb{H}$. Indeed, suppose that $\mathbb{K} = \mathbb{H}$, denote by i and j two of the three canonical roots of -1 in \mathbb{H} , take $m \in \mathcal{M}$ and $x, y \in T_m \mathcal{M}$. Set $I = \varphi^{-1}(i, 0)$ and $J = \varphi^{-1}(j, 0)$ in $\mathfrak{e} = \varphi^{-1}(\mathbb{H} \oplus \mathbb{H})$. Notice that the I_1 component of $\sigma((i, 0))$ is zero, so $I^* J = 0$, similarly $I J^* = 0$. By construction, $I + I^*$ is self adjoint, so:

$$\begin{aligned} R(x, y).(I + I^*)(J + J^*) &= R(x, y).(J + J^*)(I + I^*) \text{ by Proposition 1.8,} \\ &= R(x, y).(JI + J^* I^*) \\ &= -R(x, y).(IJ + I^* J^*) \quad \text{as in } \mathbb{H}, \text{ } ji = -ij, \\ &= -R(x, y).(I + I^*)(J + J^*). \end{aligned}$$

So $R(x, y).(I + I^*)(J + J^*) = 0$. Now $(I + I^*)(J + J^*) = IJ + (IJ)^* = \varphi^{-1}((ij, 0) + \sigma((ij, 0)))$ is invertible, so for any $m \in \mathcal{M}$ and any $x, y \in T_m \mathcal{M}$, $R(x, y) = 0$ i.e. (\mathcal{M}, g) is flat, in contradiction with $\mathfrak{s} \simeq \mathbb{H} \oplus \mathbb{H}$.

Let us suppose $k = 1$ and finish the proof. Let τ be the transposition $u \mapsto {}^t u$ in $A \simeq M_\delta(\mathbb{K})$, and $\bar{\tau}$ its composition $u \mapsto {}^t \bar{u}$ with the conjugation, in case $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$. Then for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, respectively $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$, τ , respectively $\bar{\tau}$, is an anti-morphism (of \mathbb{R} -algebra) of A . So either $\tau \circ \sigma$ or $\bar{\tau} \circ \sigma$ is an automorphism of A and, for $\mathbb{K} \in \{\mathbb{R}, \mathbb{H}\}$, it acts trivially on the center $Z(A)$ as $Z(A) = \mathbb{K}.I_\delta$. If $\mathbb{K} = \mathbb{C}$, either $\sigma \circ \tau$ or $\sigma \circ \bar{\tau}$ acts trivially on the center $Z(A) = \mathbb{C}.I_\delta$. Then Theorem 1.20 gives a $v \in A$ such that $\sigma : u \mapsto v^t \tilde{u} v^{-1}$ with $\tilde{u} = u$ if $\mathbb{K} = \mathbb{R}$, $\tilde{u} = \bar{u}$ if $\mathbb{K} = \mathbb{H}$ and $\tilde{u} = u$ or $\tilde{u} = \bar{u}$ if $\mathbb{K} = \mathbb{C}$. As $\sigma^2 = \text{Id}_{\mathfrak{e}}$, $v^t \tilde{v}^{-1} \in Z(A)$ i.e. ${}^t \tilde{v} = \lambda v$ with $\lambda \in \mathbb{R}$ if $\mathbb{K} \in \{\mathbb{R}, \mathbb{H}\}$ and $\lambda \in \mathbb{C}$ if $\mathbb{K} = \mathbb{C}$. Applying $\bar{\tau}$ on both sides, we

get that $\lambda = \pm 1$ (in the case $\mathbb{K} = \mathbb{C}$ and $\tilde{u} = \bar{u}$, we get only $|\lambda| = 1$, but replacing v by an adequate element of $\mathbb{C}.v$ achieves even $\lambda = 1$).

If we replace φ by $c_w \circ \varphi$ with $c_w : u \mapsto w^{-1}uw$, then v is replaced by $wv^t\tilde{w}$ i.e. v undergoes a basis change like the matrix of a bilinear or \sim -sesquilinear form. So using a suitable c_w , and recalling that ${}^t\tilde{v} = \lambda v$ with $\lambda = \pm 1$, we may suppose:

– in case $\lambda = 1$, that $v = \text{diag}(I_{\delta'}, -I_{\delta''})$ with $\delta' + \delta'' = \delta$ if $\mathbb{K} = \mathbb{R}$ or ($\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$ and $\tilde{u} = \bar{u}$), and that $v = I_\delta$ if ($\mathbb{K} = \mathbb{C}$ and $\tilde{u} = u$),

– in case $\lambda = -1$, that δ is even and $v = \begin{pmatrix} 0 & -I_{\delta/2} \\ I_{\delta/2} & 0 \end{pmatrix}$ if ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\tilde{u} = u$), and that $v = I_\delta$ if $\mathbb{K} = \mathbb{H}$.

Now all cases where v is diagonal imply $\delta = 1$. Indeed, if $v = \text{diag}(I_{\delta'}, -I_{\delta''})$ or $v = I_\delta$, set $p = \text{diag}(1, 0, \dots, 0)$, and if $\mathbb{K} = \mathbb{H}$ and $v = I_\delta$, set $p = \text{diag}(j, 0, \dots, 0)$. Then p is self adjoint, non nilpotent, so $p = 1_A$ or $p = 1_{A,j}$ by Claim 1 i.e. $\delta = 1$. So if $\delta \geq 2$, then $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\lambda = -1$, $\tilde{u} = u$, δ is even and $v = \begin{pmatrix} 0 & -I_{\delta/2} \\ I_{\delta/2} & 0 \end{pmatrix}$. Setting $p' = \text{diag}(1, 0, \dots, 0) \in M_{\delta/2}(\mathbb{K})$ we get $p = \text{diag}(p', p')$ a self adjoint non nilpotent element of A , so p is invertible by Claim 1 i.e. $\delta' = 1$ i.e. $\delta = 2$. So the only allowed cases are those listed in 1.10:

– if $k = 1$ and $\delta = 1$, $\mathbb{K} = \mathbb{R}$ and $\sigma : u \mapsto {}^t u = u$, or $\mathbb{K} = \mathbb{C}$ and $\sigma : u \mapsto {}^t u = u$, or $\mathbb{K} = \mathbb{C}$ and $\sigma : u \mapsto {}^t \bar{u} = \bar{u}$, or $\mathbb{K} = \mathbb{H}$ and $\sigma : u \mapsto {}^t \bar{u} = \bar{u}$,

– if $k = 1$ and $\delta = 2$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and $\sigma : u \mapsto v^t u v^{-1}$ with $v = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ i.e. σ is as described in Table 1,

– if $k = 2$ and $\delta = 1$ i.e. $A = I_1 \oplus I_2$ with $I_1 \simeq I_2 \simeq \mathbb{K}$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and σ permutes I_1 and I_2 . Composing possibly φ with a suitable automorphism of A , we get simply $\sigma : (a, b) \mapsto (b, a)$.

The remaining informations given in Tables 1 and 2 follow from standard calculations. We give only the following details.

In Table 1, the given generators are a (pseudo-)orthogonal family of $(\mathfrak{s}, \langle \cdot, \cdot \rangle)$, indeed $\frac{1}{d} \text{tr}(L^*L) = \frac{1}{d} \text{tr}(-L^2) = \frac{1}{d} \text{tr}(-\text{Id}) = -1$ or, in case $(\mathbf{2}^{\mathbb{C}})$, $\frac{1}{d} \text{tr}(L^*J) = \frac{1}{d} \text{tr}(-J) = 0$ as J is a complex structure.

For the three last columns of Table 2, we must check that the different (para)complex structures U announced are indeed the only ones. Notice that if $U \in \mathfrak{s}^-$, $U^2 = \pm \text{Id} \Leftrightarrow U^*U = \mp \text{Id} \Rightarrow \langle U, U \rangle = \mp 1$.

In cases $(\mathbf{3})$, $(\mathbf{3}')$, and $(\mathbf{3}^{\mathbb{C}})$, after Proposition 1.23, the (pseudo-)Kähler manifold (\mathcal{M}, g, J) admits a non zero complex volume form so is Ricci flat. See also another brief proof in Theorem 5.1.

Finally, in §1.4 are built the (non-empty) sets of germs of metrics inducing each case, and Proposition 1.27 and Remark 1.34 show Remark 1.15 above and hence the last assertion of the theorem. q.e.d.

1.21 Remark In Claim 3 above, the use of the Bianchi identity, through Proposition 1.8, is necessary. Consider the case $E = \mathbb{R}^{8p} \simeq \mathbb{H}^p \oplus \mathbb{H}^p$ and $H' := \{u \in \text{GL}_p(\mathbb{H})^2 : u = (u_1, {}^t \bar{u}_1)\} \subset \text{GL}_{8p}(\mathbb{R})$. Then H' preserves the non degenerate real quadratic form $(x_1, x_2) \mapsto {}^t \bar{x}_1 . x_2$ on E , and its action is orthogonally indecomposable. Now $\mathfrak{gl}(E)^{h'} = (\text{Id}_{\mathbb{H}^p} . \mathbb{H})^2 \subset \text{GL}_p(\mathbb{H})^2 \subset \text{GL}_{8p}(\mathbb{R})$ and thus $\mathfrak{gl}(E)^{h'} \simeq \mathbb{H} \oplus \mathbb{H}$.

The following corollary of Theorem 1.10 may be noticed.

1.22 Corollary *A metric g admits parallel self adjoint complex structures exactly in cases (1^C) , (2^C) and (3^C) , and they are: $\{\pm \underline{J} + N; N \in \mathfrak{n}_0 \text{ and } N\underline{J} = -\underline{J}N\}$.*

Proof. Suppose that some $\underline{J}_0 \in \mathfrak{e}^+$ satisfies $\underline{J}_0^2 = -\text{Id}$. Take the decomposition $\underline{J}_0 = S + N$ with $S \in \mathfrak{s}^+$ and $N \in \mathfrak{n}^+$. By Lemma 1.18, and as the minimal polynomial of \underline{J}_0 is $X^2 + 1$, irreducible, $S^2 = -\text{Id}$ so we are in case (1^C) , (2^C) or (3^C) and $S = \pm \underline{J}$. Now $-\text{Id} = \underline{J}_0^2 = (\underline{J} + N)^2 = -\text{Id} + JN + NJ + N^2$. By Proposition 1.8, $JN - NJ \in \mathfrak{n}_0$, so $N(2J + N) = JN + NJ + N^2 - (JN - NJ) = -(JN - NJ) \in \mathfrak{n}_0$. By Lemma 1.18, $((2J + N)^2 + 4\text{Id})^k = 0$ for some k , so $2J + N$ is invertible, so $N \in \mathfrak{n}_0$, and as then $N^2 = 0$, $N \in \{U \in \mathfrak{n}_0; JU = -UJ\}$. q.e.d.

Finally, it may be useful to list the different possible parallel tensors.

1.23 Proposition *In each case of Theorem 1.10, the metric admits the nondegenerate parallel multi- or sesquilinear forms of Table 3 p. 12.*

parallel tensor/exists in cases	parametrised by	given as
Pseudo-Riemannian metric/all cases	$U \in \mathfrak{e}^+ \setminus \mathfrak{n}^+$	$g(\cdot, U\cdot)$
Symplectic form/all except (1) and (1^C)	$U = V + N,$ $V \in (\mathfrak{s}^-)^*, N \in \mathfrak{n}^-$	$g(\cdot, U\cdot)$
“Complex Riemannian” metric/ (1^C) , (2^C) , (3^C)	$U \in \mathfrak{e}^+ \setminus \mathfrak{n}^+$ such that $U\underline{J} = \underline{J}U$	$\underline{g}_U =$ $g(\cdot, U\cdot) + ig(\cdot, \underline{J}U\cdot)$
Hermitian (pseudo-)Kähler metric w. r. to some $J \in \mathfrak{s}^-$ (2) , (2^C) , (3) , $(3')$, (3^C)	$U \in \mathfrak{e}^+ \setminus \mathfrak{n}^+$ such that $UJ = JU$	$h_U =$ $g(\cdot, U\cdot) + ig(\cdot, JU\cdot)$
\underline{J} -complex symplectic form (2^C) , (3^C)	$U = V + N,$ $V \in (\mathfrak{s}^-)^*, N \in \mathfrak{n}^-,$ such that $N\underline{J} = \underline{J}N$	$\underline{\omega}_U =$ $g(\cdot, U\cdot) + ig(\cdot, \underline{J}U\cdot)$
J -complex symplectic form (3) , $(3')$, (3^C)	$U = V + N,$ $V \in (\mathfrak{s}^-)^*, N \in \mathfrak{n}^-,$ such that $UJ = -JU$	$\omega_U =$ $g(\cdot, U\cdot) + ig(\cdot, JU\cdot)$
Non null \underline{J} -complex volume form/ (1^C) , (2^C) , (3^C)	associated with \underline{g}_U above	
Non null J -complex volume form/ (3) , $(3')$, (3^C)	equal to $\omega_U^{\wedge(d/4)}$ with ω_U as above	

Table 3: *The real and complex parallel tensors existing in the different cases.* In cases (3) , $(3')$ and (3^C) , $(\mathfrak{s}^-)^*$ is the complement of the isotropic cone in \mathfrak{s}^- . The real part of h_U is a (pseudo-)Riemannian metric, its imaginary part is a 2-form of type (1,1).

Proof. Some lines of Table 3 require a brief checking.

(1) Any $U \in \mathfrak{e}^+ \setminus \mathfrak{n}^+$ is nondegenerate. Indeed, any $U \in \mathfrak{s}^+ \setminus \{0\}$ is (see Table 1), so its minimal polynomial μ is not divisible by X ; by Lemma 1.18, neither is the minimal polynomial of $U + N$ for any $N \in \mathfrak{n}$.

(2) If some nondegenerate alternate form is parallel for a torsion-free connection, it is closed, thus symplectic. Then proceed as in (1) above.

(3) If J is a parallel complex structure (self- or skew-adjoint), nondegenerate complex bilinear forms are the $g(\cdot, U\cdot) - ig(\cdot, V\cdot)$ such that (check it) $\ker U \cap \ker V = \{0\}$, $V = UJ$, $U^* = U$ and $V^* = V$. By Proposition 1.2, the first condition implies that $U \notin \mathfrak{n}$ or $V \notin \mathfrak{n}$, so by Lemma 1.18 and the reasoning of (1), that U or V is nondegenerate, hence both. Now if

$J^* = -J$, the relations give that $UJ = -JU$ and $VJ = -JV$. As $U^* = U$, by Proposition 1.8, everywhere, $R(\cdot, \cdot)(UJ - JU) = 0$. As $UJ - JU = 2UJ$ is nondegenerate, \mathcal{M} would be flat. So $J^* = J$, we denote it by \underline{J} . This time $U\underline{J} = \underline{J}U$. After Table 1 and Lemma 1.18, the existence of such a \underline{J} leads to the announced form of \mathfrak{s}^+ . Conclude by the same reasoning as in (1); (4)-(6) are entirely similar.

(7) If some parallel \underline{J} exists, so some complex Riemannian metric $\underline{g}_{\underline{J}}$ as on line 3 of Table 3, take $(e_i)_{i=1}^{d/2}$ some $\underline{g}_{\underline{J}}$ -orthonormal complex frame field, and $\nu = e_1^* \wedge \dots \wedge e_{d/2}^*$. As $\underline{g}_{\underline{J}}$ is parallel, so is ν . q.e.d.

1.4 The space of germs of metrics realising each form of \mathfrak{s}

1.24 Reminder Metrics with \mathfrak{s} of type (1^C) are the real parts of complex Riemannian metrics *i.e.* of holomorphic, non degenerate \mathbb{C} -bilinear forms on complex manifolds $(\mathcal{M}, \underline{J})$. It is well known and easy to check.

As it is also well known, germs of (pseudo-)Kähler metrics (type (2)) are parametrised by a Kähler potential u , which is a real function:

$$g\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right) = \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}. \quad (\mathbf{a})$$

Similarly, germs of para Kähler metrics (type (2')) are parametrised by a para Kähler potential (see *e.g.* §2 of [2]). The supplementary distributions $\ker(L \pm \text{Id})$ are integrable. Take $((x_i)_{i=1}^{d/2}, (y_i)_{i=1}^{d/2})$ coordinates adapted to the corresponding pair of integral (g -isotropic) foliations. Then the metrics of type (2') depend on a real potential u through:

$$g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right) = \frac{\partial^2 u}{\partial x_i \partial y_j}. \quad (\mathbf{b})$$

A metric of type (2^C) is given by the complexification of (a) or (b), indifferently: take u complex and replace the real and imaginary parts of the z_i , in case (a), or $(x_i)_i$ and $(y_i)_i$, case (b), by complex variables.

1.25 Remark Be careful however that a manifold (\mathcal{M}, g) of type (2) or (2^C) has to be complex, hence in particular real analytic, whereas one of type (2') may be only smooth.

1.26 Remark We recall also that the ‘‘complex Riemannian’’ metrics defined in Table 3 in cases (1^C), (2^C) and (3^C) are holomorphic with respect to the self adjoint complex structure \underline{J} . Check that, if $z_j = x_j + iy_j$ are complex coordinates, $\frac{\partial}{\partial y_j} \underline{g}_{k,l} = i \frac{\partial}{\partial x_j} \underline{g}_{k,l}$ for all k, l .

1.27 Proposition *A generic metric of type (1), (2), (2'), (1^C) or (2^C) has the holonomy algebra given in Remark 1.15. More precisely, if the 2-jet at the origin of some metric of the wished type satisfies some dense open condition among such 2-jets, then its holonomy algebra is as in Remark 1.15. In particular, those holonomy groups are obtained on a dense open subset, for the C^2 topology, of the corresponding metrics.*

Proof. It is standard, but as we will generalise it in Theorem 3.2, we recall it. At the origin, take normal coordinate vectors $(X_i)_{i=1}^d$, moreover such that $X_{i+1} = JX_i$ or $X_{i+1} = LX_i$ for i odd, in case (2) or (2'). So for any coordinate vectors U, V , $D_U V = 0$ at 0. For any coordinate vectors A, B, U, V at the origin, $g(R(A, B)U, V)$ is equal to (check it):

$$\frac{1}{2}(A.U.(g(B, V)) - B.U.(g(A, V)) - A.V.(g(B, U)) + B.V.(g(A, U))).$$

In case **(1)**, $g(R(X_i, X_j)|_0 \cdot, \cdot)$ is the alternate part of the bilinear form:

$$\beta_{i,j} : (U, V) \mapsto X_i.U.g(X_j, V) - X_j.U.g(X_i, V).$$

The $\beta_{i,j}$ depend on the second derivatives of the coefficients of g at 0, which are free in normal coordinates. So, on a dense open subset of the 2-jets of metrics, their alternate parts are linearly independent and span a $\frac{d(d-1)}{2}$ -dimensional space in $\dim \mathfrak{o}_d(\mathbb{R})$, hence $\mathfrak{o}_d(\mathbb{R})$ itself.

In case **(2)** we set, for j odd, $Z_{\frac{j+1}{2}} = X_j - iX_{j+1}$ and $\bar{Z}_{\frac{j+1}{2}} = X_j + iX_{j+1}$ in $T^{\mathbb{C}}\mathcal{M}$. The $R(Z_i, Z_j)$ and $R(\bar{Z}_i, \bar{Z}_j)$ vanish, and the $R(Z_i, \bar{Z}_j)$ vanish when evaluated on $\Lambda^2 T^{1,0}\mathcal{M}$ or $\Lambda^2 T^{0,1}\mathcal{M}$. So R is determined at 0 by the $\beta_{i,j}$: $(Z_k, \bar{Z}_l) \mapsto g(R(Z_i, \bar{Z}_j), Z_k, \bar{Z}_l)$. As:

$$g(R(Z_i, \bar{Z}_j), Z_k, \bar{Z}_l) = \frac{1}{2}(-\bar{Z}_j.Z_k.(g(Z_i, \bar{Z}_l)) - Z_i.\bar{Z}_l.(g(\bar{Z}_j, Z_k))),$$

$R|_0$ is given by the fourth derivatives of the Kähler potential u . Those are free in normal coordinates, so on a dense open subset of the 2-jets of metrics, the $(\beta_{i,j})_{i,j=1}^{d/2}$ are linearly independent hence span a $(\frac{d}{2})^2$ -dimensional space in $\mathfrak{u}_{d/2}$, hence $\mathfrak{u}_{d/2}$.

For **(2')**, replace $(Z_i, \bar{Z}_i)_{i=1}^{d/2}$ by $(X_i, Y_i)_{i=1}^{d/2}$, and $\mathfrak{u}_{d/2}$ by $\mathfrak{gl}_{d/2}(\mathbb{R})$.

For types **(1^C)** and **(2^C)**, R is \underline{J} -complex; repeat the proofs in complex coordinates. q.e.d.

Now we describe the space of germs of metrics of type **(3)**, **(3')** and **(3^C)**. It is classical for type **(3)** (hyperkähler), the other cases are only an adaptation of the argument.

1.28 Notation Take $\varepsilon \in \{-1, 1\}$ and $\delta \in \mathbb{N}^*$. We denote by \mathcal{G}_ε the set of germs at 0 of triples (g, J, U) with g a (pseudo-)Riemannian metric on $\mathbb{R}^d = \mathbb{R}^{4\delta}$ and J and U two g -skew adjoint parallel endomorphisms fields such that $\varepsilon U^2 = -J^2 = \text{Id}$, and $JU = -UJ$. We define $\mathcal{G}_{\mathbb{C}}$ similarly, with g a complex Riemannian metric on $\mathbb{C}^{4\delta}$ and similar J and U (with *e.g.* $\varepsilon = -1$, but this makes no difference on \mathbb{C}).

Using Cartan-Kähler theory (see [11, 17]), we parametrise \mathcal{G}_ε and $\mathcal{G}_{\mathbb{C}}$ in the real analytic category. We proceed as R. Bryant did in [12] §2.5 pp. 122–126 for hyperkähler metrics *i.e.* for $\varepsilon = -1$, detailing slightly the calculations to show that the case $\varepsilon = 1$ works alike, and to allow a generalisation in Theorem 3.2. The complex case $\mathcal{G}_{\mathbb{C}}$ follows.

1.29 Remark/Notation Let ω_0 be some complex symplectic form on some open set \mathcal{O} of $\mathbb{C}^{2\delta}$. Then any 2-form ω of type (1,1), real, may be written as $\omega = \Im(\omega_0(\cdot, U_\omega \cdot))$, with U_ω an ω_0 -self adjoint complex antimorphism field. The correspondence is bijective between such forms ω and such U_ω , so we use this notation U_ω in the following.

1.30 Remark The set \mathcal{G}_ε is in bijection with the set \mathcal{G}'_ε of germs of couples (ω_0, ω) , with ω_0 a complex symplectic form on $\mathbb{C}^{2\delta}$ and ω a closed 2-form of type (1,1), real, such that $U_\omega^2 = \varepsilon \text{Id}$, through the following.

– Let (g, J, U) be given. Then on $\mathbb{C}^{2\delta} := (\mathbb{R}^{4\delta}, J)$ set:

$$\omega_0 := g(\cdot, U \cdot) + ig(\cdot, JU \cdot) \quad \text{and} \quad \omega := \varepsilon g(\cdot, J \cdot) = \Im(\omega_0(\cdot, U \cdot)).$$

As $DJ = DU = 0$, immediately $d\omega_0 = d\omega = 0$.

– Let (ω_0, ω) be given. Then on $(\mathbb{R}^{4\delta}, J) := (\mathbb{C}^{2\delta}, i)$ set:

$$g := -\varepsilon \omega(\cdot, i \cdot) \quad \text{and} \quad U := U_\omega.$$

As $d\omega_0 = d\omega = 0$, $DJ = DU = 0$. This is standard, see *e.g.* [24] §11.2.

In this new point of view, up to a biholomorphism of $\mathbb{C}^{2\delta}$, ω_0 may be considered, by the Darboux theorem, as the canonical symplectic form:

$$\omega_0 = \sum_{j=1}^{\delta} dz_j \wedge dz_{\delta+j} = \frac{1}{2} {}^t dz \wedge \Omega_0 \wedge dz \text{ with } \Omega_0 = \begin{pmatrix} 0 & I_\delta \\ -I_\delta & 0 \end{pmatrix},$$

dz denoting the column $(dz_i)_{i=1}^{2\delta}$. From now on, we consider that ω_0 is this canonical form. Then the elements of \mathcal{G}_ε , seen up to diffeomorphism of $\mathbb{R}^{4\delta}$, are in bijection with those of \mathcal{G}'_ε , seen up to symplectomorphism of $(\mathbb{C}^{2\delta}, \omega_0)$. Now we use Cartan-Kähler theory to describe \mathcal{G}'_ε .

1.31 Notation Set $V := \text{Mat}(U)$, U is an antimorphism so $U(z) = V\bar{z}$. As $\omega_0(U\cdot, \cdot) = -\omega_0(\cdot, U\cdot)$, we get $\Omega_0 V = -{}^t \bar{V} \Omega_0$. A 2-form ω is in \mathcal{G}'_ε if and only if it is closed and:

$$\omega = \Im(\omega_0(\cdot, U\cdot)) = \frac{1}{2} {}^t dz \wedge \Omega_0 V \wedge d\bar{z} \text{ with } V\bar{V} = \varepsilon \text{Id}$$

i.e., setting $H := -\Omega_0 V$, if and only if:

$$\omega = \frac{1}{2} {}^t dz \wedge H \wedge d\bar{z} \text{ with } {}^t H = \bar{H} \text{ and } \bar{H} \Omega_0 H = -\varepsilon \Omega_0.$$

Let $\mathcal{H}_\varepsilon \subset M_{2\delta}(\mathbb{C})$ be the space of such matrices H . The (1,1)-forms ω such that $U_\omega^2 = \varepsilon \text{Id}$ are exactly given by the functions $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_\varepsilon$, through: $\omega_H := \frac{1}{2} {}^t dz \wedge H(z) \wedge d\bar{z}$. Denoting by (z, H) the points in $\mathbb{C}^{2\delta} \times \mathcal{H}_\varepsilon$, such an ω_H is *closed* if and only if the 3-form $\lambda := {}^t dz \wedge dH \wedge d\bar{z}$ vanishes along the graph \mathcal{S} of H . So we are looking for the integral manifolds \mathcal{S} of the exterior differential system $\mathbf{I} = (\lambda)$ on $\mathbb{C}^{2\delta} \times \mathcal{H}_\varepsilon$, with the independence condition that $dz_1 \wedge \dots \wedge dz_{2\delta}$ never vanishes (*i.e.* \mathcal{S} is the graph of some $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_\varepsilon$). Then the Cartan-Kähler theorem parametrises \mathcal{G}'_ε , hence \mathcal{G}_ε , providing:

1.32 Proposition *The elements of \mathcal{G}_ε , considered up to diffeomorphism, are parametrised by $\frac{d}{2} = 2\delta$ real analytic functions of $2\delta + 1$ real variables. Those of $\mathcal{G}_\mathbb{C}$, up to biholomorphism, are parametrised by $\frac{d}{4} = 2\delta$ holomorphic functions of $2\delta + 1$ complex variables.*

1.33 Remark The generality of the elements of \mathcal{G}_ε and $\mathcal{G}_\mathbb{C}$ ensures that their corresponding algebra \mathfrak{s} is indeed, generically, in cases **(3)**, **(3')** or **(3^C)** (and *e.g.* not the full $\text{End}(T\mathcal{M})$). In fact, their holonomy group itself is generically that of Remark 1.15, see Remark 1.34.

Proof. The writing of \mathbf{I} in $\mathbb{C}^{2\delta} \times \mathcal{H}_\varepsilon$ does not depend on z , so we have only to perform Cartan's test on some arbitrary fibre $\{z_0\} \times H$, say with $z_0 = 0$. Moreover, over that point z_0 , the symplectic group $\text{Sp}(2\delta, \mathbb{C})$ acts transitively on $\{\frac{1}{2} {}^t dz \wedge H \wedge d\bar{z}; H \in \mathcal{H}_\varepsilon\}$, preserving \mathbf{I} , so we have only to perform Cartan's test at some specific element $H_0 \in \mathcal{H}_\varepsilon$, say:

- if $\varepsilon = -1$, $H_0 = I_{p,q,p,q} = \text{diag}(I_p, -I_q, I_p, -I_q)$ with $p + q = n$,
- if $\varepsilon = 1$, $H_0 = iI_{n,n}$.

Remark. As it appears in [12], the connected component $\mathcal{H}_{-1}^{p,q}$ of $I_{p,q,p,q}$ in $\mathcal{H}_{-1} = \sqcup_{p+q=n} \mathcal{H}_{-1}^{p,q}$ is canonically isomorphic to $\text{Sp}(n, \mathbb{C}) / \text{Sp}(p, q)$. So choosing some function $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_{-1}^{p,q}$ amounts to choosing a reduction to $\text{Sp}(p, q)$, which is a real form of $\text{Sp}(n, \mathbb{C})$, of the principal bundle $\text{Sp}(n, \mathbb{C}) \times \mathbb{C}^{2\delta}$. Similarly here, $\mathcal{H}_1 \simeq \text{Sp}(n, \mathbb{C}) / \text{Sp}(n, \mathbb{R})$ so choosing some $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_1$ is choosing a reduction of it to $\text{Sp}(n, \mathbb{R})$, which is another real form of $\text{Sp}(n, \mathbb{C})$.

Let us set $\partial z_j = \partial x_j + i\partial y_j$. If a subspace E of $T_{m_0}\mathcal{M}$ is horizontal *i.e.* tangent to the factor $\mathbb{C}^{2\delta}$, $\lambda|_E = 0$ so E is an integral element of \mathbf{I} . Let us define $(E_k)_{k=0}^{4\delta}$ by:

$$E_k = \text{span} \left((e_j)_{j=1}^k \right) \quad \text{with, for } 1 \leq j \leq \delta: \quad e_j = (\partial x_j, 0) \quad \text{and} \\ e_{\delta+j} = \left(\partial x_{\delta+j} + \frac{j-1}{\delta} \partial y_{\delta+j}, 0 \right), \quad \text{and for } 1 \leq j \leq 2\delta: \quad e_{2\delta+j} = (\partial y_j, 0).$$

Each E_k is horizontal so $(E_k)_{k=0}^{4\delta}$ is an integral flag of \mathbf{I} at m_0 . We classically set $H(E_k) := \{v; \text{span}(v, E_k) \text{ is an integral element of } \mathbf{I}\}$, and $s_k := \text{codim}_{H(E_{k-1})} H(E_k)$ the k th. character of \mathbf{I} (indeed this flag is ordinary, as we will see). We will check:

(1) for all k , $s_k = k - 1$, and $s_k = 0$ for $k > 2\delta + 1$,

(2) $\dim V_{4\delta}(\mathbf{I}) \geq 2C_{2\delta+2}^3$, with $V_{4\delta}(\mathbf{I})$ the variety of integral elements of \mathbf{I} in the grassmannian $G_{4\delta}(T(\mathbb{C}^{2\delta} \times \mathcal{H}_\varepsilon))$.

After Cartan's criterion, $\dim V_{4\delta}(\mathbf{I}) \leq \sum_{k=1}^{4\delta} k s_k$, and if equality holds then $E_{4\delta}$ is ordinary. So here:

$$\dim V_{4\delta}(\mathbf{I}) \leq \sum_{k=1}^{4\delta} k s_k = \sum_{k=1}^{2\delta+1} k(k-1) = \frac{8}{3}\delta^3 + 4\delta^2 + \frac{4}{3}\delta = 2C_{2\delta+2}^3.$$

As $2C_{2\delta+2}^3 \leq \dim V_{4\delta}(\mathbf{I})$, equality holds, hence $E_{4\delta}$ is ordinary and after the Cartan-Kähler theorem, \mathbf{I} admits an integral manifold \mathcal{S} through $(0, H_0)$ with $T\mathcal{S} = E_{4\delta}$, and the space of germs of integral manifolds passing by z_0 depends on s_k functions of k variables, with s_k the last non vanishing character, so here 2δ functions of $2\delta + 1$ variables. This parametrisation of the set \mathcal{G}_ε is done up to the choice of complex Darboux coordinates for ω_0 , and such coordinates depend on one symplectic generating function of 2δ variables. As $2\delta < 2\delta + 1$, this does not interfere and \mathcal{G}_ε itself is parametrised by 2δ functions of $2\delta + 1$ variables, the proposition. We are left with showing (1) and (2).

We introduce $W_\varepsilon := T_{H_0}\mathcal{H}_\varepsilon$, then:

$$W_1 = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}; a, b \in M_\delta(\mathbb{C}), {}^t a = \bar{a}, {}^t b = b \right\} \\ \text{and: } W_{-1} = \left\{ \begin{pmatrix} a & I_{p,q}b \\ \bar{b}I_{p,q} & -I_{p,q}\bar{a}I_{p,q} \end{pmatrix}; a, b \in M_n(\mathbb{C}), {}^t a = \bar{a}, {}^t b = b \right\}.$$

Then $T_{m_0}\mathcal{M}_\varepsilon = T_0\mathbb{C}^{2\delta} \oplus W_\varepsilon \simeq \mathbb{C}^{2\delta} \oplus W_\varepsilon$ and the subset of the grassmannian $G_{4\delta}(T_{m_0}\mathcal{M})$ on which the independence condition holds is canonically identified with $(\mathbb{C}^{2\delta})^* \otimes W_\varepsilon$.

(1) follows from the fact that for $k > 2\delta$, $H(E_k) = \mathbb{C}^{2\delta} \oplus \{0\}$, and for $1 \leq k \leq \delta$:

- $H(E_k) = \mathbb{C}^{2\delta} \oplus \{\Im a_{i,j} = 0 \text{ for } 1 \leq i < j \leq k\} \subset \mathbb{C}^{2\delta} \oplus W_\varepsilon$, so $\text{codim}_{H(E_{k-1})} H(E_k) = k - 1$,

- $H(E_{n+k}) = \mathbb{C}^{2\delta} \oplus \{\Re a_{i,j} = \Re b_{i,j} = 0 \text{ for } 1 \leq i < j \leq k \text{ and } \Im b_{k,j} + \frac{k-1}{\delta} \Re b_{k,j} = 0 \text{ for } k \leq j \leq \delta\} \subset \mathbb{C}^{2\delta} \oplus W_\varepsilon$, so $\text{codim}_{H(E_{\delta+k-1})} H(E_{\delta+k}) = \delta + k - 1$.

To check (2), we introduce some notation. We denote the basis vectors $(\partial x_i)_{i=1}^{2\delta}$ of $\mathbb{C}^{2\delta}$ by $((u_i)_{i=1}^k, (u'_i)_{i=1}^k)$ (the u_i and u'_i are ω_0 -dual), then $(\partial y_i)_{i=1}^{2\delta} = ((Ju_i)_{i=1}^k, (Ju'_i)_{i=1}^k)$. We denote by $H^{(1)}$ a generic element of $(\mathbb{C}^{2\delta})^* \otimes W_\varepsilon$. If a function $H : \mathbb{C}^{2\delta} \rightarrow \mathcal{H}_\varepsilon$ with $H(0) = H_0$ is such that $dH|_0 = H^{(1)}$, then, at 0, $d\omega_H$ is determined by $d\omega_H = \lambda|_{m_0}(H^{(1)} \cdot, H^{(1)} \cdot, H^{(1)} \cdot)$,

that we denote by $\lambda_{H^{(1)}}$. In concrete terms, for the calculations below, $\lambda_{H^{(1)}}(u, v, w)$ is equal to:

$$\omega_0 \left(u, H^{(1)}(v) \cdot \bar{w} \right) + \omega_0 \left(v, H^{(1)}(w) \cdot \bar{u} \right) + \omega_0 \left(w, H^{(1)}(u) \cdot \bar{v} \right).$$

At $(0, H_0)$, $V_{4\delta}(\mathbf{I})$ is the set of the 1-jets of *closed* 2-forms ω_H as wanted. An $H^{(1)}$ is in $V_{4\delta}(\mathbf{I})$ if and only if $\lambda_{H^{(1)}} = 0$, which may be written as the two following conditions:

- (a) for any three $\{u, v, w\} \subset \{u_i, Ju_i\}_{i=1}^k$, $\lambda_{H^{(1)}}(u^{(l)}, v^{(l)}, w^{(l)}) = 0$,
- (b) for any two $\{u, v\} \subset \{u_i, Ju_i\}_{i=1}^k$,
 $\lambda_{H^{(1)}}(u, u', v^{(l)}) = \lambda_{H^{(1)}}(v, v', u^{(l)}) = 0$.

The parenthesised primes enable to denote several equations at once, so (a) consists of $8C_{2\delta}^3$ equations and (b) of $4C_{2\delta}^2$. Now the equations of (a) are redundant. Indeed the reader may check the following. Take any $H^{(1)}$ and any $\{i, j, k\} \subset \llbracket 1, \delta \rrbracket$ and $\{\alpha, \beta, \gamma\} \subset \{0, 1\}$ such that $\sharp\{J^\alpha u_i, J^\beta u_j, J^\gamma u_k\} = 3$ (so, $C_{2\delta}^3$ possibilities). Set $(u, v, w) := (J^\alpha u_i, J^\beta u_j, J^\gamma u_k)$ and, in case $\varepsilon = 1$, $\eta_1 := (-1)^{\gamma-\beta}$, $\eta_2 := (-1)^{\alpha-\gamma}$ and $\eta_3 := (-1)^{\beta-\alpha}$, and in case $\varepsilon = -1$, $\eta_1 := (-1)^{\gamma-\beta}(-1)^{\chi_{\{k \leq p\}} + \chi_{\{j \leq p\}}}$, $\eta_2 := (-1)^{\alpha-\gamma}(-1)^{\chi_{\{i \leq p\}} + \chi_{\{k \leq p\}}}$, $\eta_3 := (-1)^{\beta-\alpha}(-1)^{\chi_{\{j \leq p\}} + \chi_{\{i \leq p\}}}$. We denote by χ_P the characteristic function of the set P , equal to 1 on P and null elsewhere. Explicitly, $\chi_{\{i \leq p\}} + \chi_{\{j \leq p\}}$ is even if and only if $(i, j) \subset \llbracket 1, p \rrbracket^2 \cup \llbracket p+1, \delta \rrbracket^2$. Then the following sets of relations (say respectively (i), (ii), (iii) and (iv)) hold.

$$\left\{ \begin{array}{l} \eta_1 \lambda_{H^{(1)}}(u', v, w) + \eta_2 \lambda_{H^{(1)}}(u, v', w) \\ \quad + \eta_3 \lambda_{H^{(1)}}(u, v, w') + \varepsilon \lambda_{H^{(1)}}(u', v', w') = 0 \\ \eta_1 \lambda_{H^{(1)}}(Ju', v, w) + \eta_2 \lambda_{H^{(1)}}(u, Jv', w) \\ \quad + \eta_3 \lambda_{H^{(1)}}(u, v, Jw') + \varepsilon \lambda_{H^{(1)}}(Ju', Jv', Jw') = 0 \\ \eta_1 \lambda_{H^{(1)}}(u, v', w') + \eta_2 \lambda_{H^{(1)}}(u', v, w') \\ \quad + \eta_3 \lambda_{H^{(1)}}(u', v', w) + \varepsilon \lambda_{H^{(1)}}(u, v, w) = 0 \\ \eta_1 \lambda_{H^{(1)}}(u, Jv', Jw') + \eta_2 \lambda_{H^{(1)}}(Ju', v, Jw') \\ \quad + \eta_3 \lambda_{H^{(1)}}(Ju', Jv', w) + \varepsilon \lambda_{H^{(1)}}(u, v, w) = 0. \end{array} \right.$$

So the $8C_{2\delta}^3$ linear forms of the type $H^{(1)} \mapsto \lambda_{H^{(1)}}((J)u^{(l)}, (J)v^{(l)}, (J)w^{(l)})$ are linearly dependent, through the $4C_{2\delta}^3$ equations above. In turn, those equations are linearly independent. Counting the number of primes appearing in them, one sees that equations of types (i)–(ii) on the one hand, and types (iii)–(iv) on the other hand, span subspaces in direct sum. Now any dependence relation would involve some fixed triple (i, j, k) . For such a triple, equations of type (i) may be seen as expressing the forms $H^{(1)} \mapsto \lambda_{H^{(1)}}((J)u'_i, (J)u'_j, (J)u'_k)$ as combination of the other ones, and then equations of type (i)–(ii), doing the same with the forms $H^{(1)} \mapsto \lambda_{H^{(1)}}((J)u'_i, (J)u_j, (J)u_k)$. Equations of types (iii)–(iv) are similar, so all the $4C_{2\delta}^3$ equations are independent, and the $8C_{2\delta}^3$ forms span a space of dimension $\leq 8C_{2\delta}^3 - 4C_{2\delta}^3 = 4C_{2\delta}^3$. So (a) and (b) consist of not more than $4C_{2\delta}^3 + 4C_{2\delta}^2 = 4C_{2\delta+1}^3$ independent equations, so $\dim V_{4\delta}(\mathbf{I}) \geq \dim[\mathbb{C}^{2\delta} \otimes W_\varepsilon] - 4C_{2\delta+1}^3 = (4\delta) \cdot (2\delta^2 + \delta) - 4C_{2\delta+1}^3 = 2C_{2\delta+2}^3$. This is (2).

We finally treat $\mathcal{G}_{\mathbb{C}}$. In all that precedes, see all complex variables $x + iy$ as real matrices $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. Then, complexifying everything *i.e.* replacing the real entries x, y by complex numbers amounts to parametrise $\mathcal{G}_{\mathbb{C}}$; so the same reasoning gives the proposition for $\mathcal{G}_{\mathbb{C}}$. q.e.d.

1.34 Important Remark Among real analytic germs of metrics with holonomy H included in $H_0 = \text{Sp}(p, q)$, $H_0 = \text{Sp}(2\delta, \mathbb{R})$ or $H_0 = \text{Sp}(2\delta, \mathbb{C})$, corresponding to cases (3), (3')

$$\begin{array}{ccccccc}
 F^{*,0} = F^{n,*} = \text{Im } N^n = \{0\} & & & & & & \\
 \parallel & \cap & & & & & \\
 \ker N^0 & F^{n-1,1} & = \text{Im } N^{n-1} & & & & \\
 & \cap & & & & & \\
 & F^{n-2,1} & \subset & F^{n-2,2} & = \text{Im } N^{n-2} & & \\
 & \cap & & \cap & & & \\
 & F^{n-3,1} & \subset & F^{n-3,2} & \subset & F^{n-3,3} & = \text{Im } N^{n-3} \\
 & \cap & & \cap & & & \\
 & \vdots & & \vdots & & \ddots & \\
 & \cap & & \cap & & & \\
 & F^{0,1} & \subset & F^{0,2} & \subset & \dots & \subset & F^{0,n} & = \text{Im } N^0 \\
 \parallel & \parallel & & \parallel & & & \parallel & \parallel & \\
 \ker N & \ker N^2 & & \ker N^n & = \mathbb{R}^d & & & &
 \end{array}$$

Table 4: The $F^{a,b} = \text{Im } N^a \cap \ker N^b$, defined for $(a, b) \in \mathbb{N}^2$. Those for $a \notin \llbracket n - b, n \rrbracket$, seemingly absent, are equal to some $F^{a',b'}$ present here.

and $(\mathbf{3}^C)$, a dense open subset for the C^2 topology has its holonomy equal to H_0 . Indeed, the first prolongation $\mathbf{I}^{(1)}$ of the ideal \mathbf{I} satisfies also Cartan’s criterion; this enables to show that any 2-jet of metric, integrable at the order 1 and such that $\{R(X, Y); X, Y \in T_0\mathcal{M}\} \subset \mathfrak{h}_0$, is the 2-jet of a metric with holonomy included in H_0 . The reasoning is presented, in the case $H = G_2$, in Proposition 3 p. 556 of [10]. It may be adapted here, as indicated in [12] §2.5 p. 126. So as, among such 2-jets, those satisfying $\{R(X, Y); X, Y \in T_0\mathcal{M}\} = \mathfrak{h}_0$ are generic, we get the result.

2 A way to understand n: parametrising metrics making parallel a nilpotent structure

2.1 Nilpotent endomorphisms: facts, notation

We sum up standard facts, in a presentation of our own. This makes Part 2 more self-contained and enables to introduce some coherent notation used all along afterwards. Let N be in $\text{End}(\mathbb{R}^d)$, nilpotent of index n .

We set $F^{a,b} = \text{Im } N^a \cap \ker N^b$, for $a \in \llbracket 1, n - 1 \rrbracket$ and $b \in \llbracket 1, n - a \rrbracket$. Though we do not use the $F^{a,b}$ explicitly here, we had them throughout in mind. They are ordered by inclusion as shown in Table 4 p. 18.

2.1 Notation (i) The invariant factors of N are:

$$\underbrace{(X, \dots, X)}_{d_1 \text{ times}}, \underbrace{(X^2, \dots, X^2)}_{d_2 \text{ times}}, \dots, \underbrace{(X^n, \dots, X^n)}_{d_n \text{ times}} = ((X^a)_{k=1}^{d_a})_{a=1}^n,$$

for some n -tuple $(d_a)_{a=1}^n$. We call here the $(d_a)_a$ the *characteristic dimensions* of N , see **(ii)** for a justification. We set $D_a = \sum_{k=1}^a d_k$, this notation will also be useful; $D_0 := 0$.

(ii) We denote by π the projection $\mathbb{R}^d \rightarrow \mathbb{R}^d / \text{Im } N$. Then for each $a \in \llbracket 1, n \rrbracket$, $d_a = \dim(\pi(\ker N^a) / \pi(\ker N^{a-1}))$. For any (a, b) , $F^{a,b} / (F^{a,b-1} + F^{a+1,b})$ is canonically isomorphic, through N^a , to $\pi(\ker N^{b-a}) / \pi(\ker N^{b-a-1})$.

2.2 Remark/Definition Let us denote by $\mathbb{R}[\nu]$ the real algebra generated by ν satisfying the unique relation $\nu^n = 0$. In other words, $\mathbb{R}[\nu] = \mathbb{R}[X]/(X^n) \simeq \mathbb{R}[N]$. Setting $\nu V := N(V)$ for $V \in \mathbb{R}^d$ turns \mathbb{R}^d into an $\mathbb{R}[\nu]$ -module. As such, it is isomorphic to $\prod_{a=1}^n (\nu^{n-a}\mathbb{R}[\nu])^{d_a}$ *i.e.* d_1 factors on which ν acts trivially, d_2 factors on which ν is 2-step nilpotent *etc.* Notice that this isomorphism is not canonical, even up to an automorphism of each of the factors. We set $D := D_n = \sum_{a=1}^n d_a$. We define a D -tuple of vectors $\beta = (X_i)_{i=1}^D$ to be an *adapted spanning family* of \mathbb{R}^d as an $\mathbb{R}[\nu]$ -module if each $(X_i)_{i=1+D_a}^{D_{a+1}}$, pushed on the quotient, is a basis of $\pi(\ker N^{a+1})/\pi(\ker N^a)$, see Notation 2.1. In other terms, β spans \mathbb{R}^d as an $\mathbb{R}[\nu]$ -module and the only relation the X_i satisfy is $\nu^a X_i = 0$ for $D_{a-1} < i \leq D_a$; or: each $(X_i)_{i=1+D_{a-1}}^{D_a}$ spans the factor $(\nu^{n-a}\mathbb{R}[\nu])^{d_a}$ as an $\mathbb{R}[\nu]$ -module. It is a basis if and only if the $\mathbb{R}[\nu]$ -module \mathbb{R}^d admits bases *i.e.* it is free *i.e.* $d_a = 0$ for $a < n$.

We denote by $n(i)$ the nilpotence index of N on each submodule $\langle X_i \rangle$; so $n(i) = a$ for $D_{a-1} < i \leq D_a$. We denote by $(X_i, (Y_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$ the basis $(X_i, (N^a X_i)_{a=1}^{n(i)-1})_{i=1}^D$ of \mathbb{R}^d as an \mathbb{R} -vector space.

Now let g be a symmetric bilinear form on \mathbb{R}^d such that $g(N\cdot, \cdot) = g(\cdot, N\cdot)$.

2.3 Remark/Definition For each $a \in \llbracket 1, n \rrbracket$, the symmetric bilinear form $g(\cdot, N^{a-1}\cdot)$ is well defined on the quotient $\pi(\ker N^a)/\pi(\ker N^{a-1}) \simeq \mathbb{R}^{d_a}$. Indeed, if $X \in \ker N^a$ and $Y \in \text{Im } N$, so $Y = NZ$, then:

$$g(X, N^{a-1}Y) = g(X, N^a Z) = g(N^a X, Z) = 0.$$

We denote by (r_a, s_a) its signature; $r_a + s_a \leq d_a$. It is standard that the couple (N, g) is characterised up to conjugation by this family of dimensions and signatures, called here its *characteristic signatures*. See *e.g.* the elementary exposition [21]. The form g is non degenerate if and only if each $g(\cdot, N^{a-1}\cdot)$ is and then, $[\cdot]$ being the floor function:

$$\text{sign}(g) = \left(\sum_{a=1}^n \left[\frac{a}{2} \right] d_a + \sum_{a \text{ odd}} r_a, \sum_{a=1}^n \left[\frac{a}{2} \right] d_a + \sum_{a \text{ odd}} s_a \right).$$

Assuming, by Proposition 2.4, the “ $\mathbb{R}[\nu]$ -module” viewpoint, Propositions 2.5 and 2.6 follow.

2.4 Proposition/Definition Set $E = (\mathbb{R}^d, N) \simeq \prod_{a=1}^n (\nu^{n-a}\mathbb{R}[\nu])^{d_a}$. On E , the $\mathbb{R}[\nu]$ -bilinear forms h are in bijection with the real forms g satisfying $g(N\cdot, \cdot) = g(\cdot, N\cdot)$, through:

$$h = \sum_{a=1}^n \nu^{a-1} g(\cdot, N^{n-a}\cdot).$$

We call such an h the $\mathbb{R}[\nu]$ -bilinear form associated with g , and g the real form associated with h . Be careful that g is not the real part of h , but the coefficient of the highest power ν^{n-1} of ν in h .

2.5 Proposition/Definition Let h be an $\mathbb{R}[\nu]$ -bilinear form on \mathbb{R}^d as in 2.4. If $\beta = (X_i)_{i=1}^D$ is an adapted spanning family (see 2.2) of \mathbb{R}^d , $\text{Mat}_\beta(h) = \sum_{a=0}^{n-1} \nu^a H_a \in \text{M}_D(\mathbb{R}[\nu])$ where:

– $H_a = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{H}^a \end{pmatrix}$, the upper left null square block, of size D_{n-1-a} , corresponding to $\text{span}_{\mathbb{R}[\nu]} \{X_i; N^{n-1-a} X_i = 0\}$,

– the upper left block \check{H}_0^a of \check{H}^a of size d_{n-a} , corresponding to $\text{span}_{\mathbb{R}[\nu]} \{X_i; N^{n-1-a}X_i \neq N^{n-a}X_i = 0\}$ is of signature (r_{n-a}, s_{n-a}) introduced in 2.3, hence of rank $r_{n-a} + s_{n-a}$.

We call the $(r_a, s_a)_{a=1}^D$ the signatures of h . So if $S \oplus \text{Im } N = \mathbb{R}^d$, (r_a, s_a) is the signature of the (well defined) form h_{n-a} on the quotient $(S \cap \ker N^a)/(S \cap \ker N^{a-1})$.

Proof. For the first point: $\nu^a h(X, \cdot) = 0$ as soon as $X \in \ker N^a$. q.e.d.

2.6 Proposition In Proposition 2.5, choosing an adequate β , we may take, for all a , \check{H}^a null except $\check{H}_0^a = I_{d_a, r_a, s_a} := \text{diag}(I_{r_a}, -I_{s_a}, 0_{d_a - r_a - s_a}) \in M_{d_a}(\mathbb{R}[\nu])$. So with such a β :

$$\begin{aligned} \text{Mat}_\beta(h) &= \text{diag} \left(\nu^{n-a} I_{d_a, r_a, s_a} \right)_{a=1}^n \\ &= \begin{pmatrix} \nu^{n-1} I_{d_1, r_1, s_1} & & \\ & \ddots & \\ & & \nu^0 I_{d_n, r_n, s_n} \end{pmatrix} \in M_D(\mathbb{R}[\nu]). \end{aligned}$$

Each block $\nu^{n-a} I_{d_a, r_a, s_a}$ corresponds to the factor $(\nu^{n-a} \mathbb{R}[\nu])^{d_a}$ i.e. to:

$$\text{span}_{\mathbb{R}[\nu]} \{(X_i)_{D_{a-1} < i \leq D_a}\} = \text{span}_{\mathbb{R}[\nu]} \{X_i; N^{a-1}X_i \neq N^aX_i = 0\} \subset \mathbb{R}^d.$$

The $\mathbb{R}[\nu]$ -conjugation class of h is given by the signatures of h , and h is non degenerate if and only if $r_a + s_a = d_a$ for each a .

Finally, take N a nilpotent structure on \mathcal{M} .

2.7 Notation The distributions $\text{Im } N^a$ and $\ker N^a$ are integrable, we denote their respective integral foliations by \mathcal{I}^a and \mathcal{K}^a and set $\mathcal{I} := \mathcal{I}^1$. **From now on, \mathcal{U} is an open neighbourhood of m on which those foliations are trivial.** We still denote by π the projection $\mathcal{U} \rightarrow \mathcal{U}/\mathcal{I}$.

Saying that N is integrable is saying that there are coordinates $(x_i, (y_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$ of \mathcal{U} such that at each point, the basis

$$\left(X_i, (Y_{i,a})_{a=1}^{n(i)-1} \right)_{i=1}^D := \left(\frac{\partial}{\partial x_i}, \left(\frac{\partial}{\partial y_{i,a}} \right)_{a=1}^{n(i)-1} \right)_{i=1}^D$$

is of the type given in Remark 2.2.

2.2 Preliminary : introducing a special class of functions

We introduce here some material which is a bit more general than our strict subject. Just afterwards, back to our germs of pseudo-Riemannian metrics with a parallel field of nilpotent endomorphisms, it will simplify a lot the statements and the proofs and above all make them natural.

We still denote $\mathbb{R}[X]/(X^n)$ by $\mathbb{R}[\nu]$. We now mimic the definition of a holomorphic function f from a manifold \mathcal{M} with a complex structure J , to $\mathbb{C} = \mathbb{R}[i]$. The latter is such that $df \circ J = i df$. Here \mathcal{M} is endowed with a “nilpotent structure”: an integrable field of endomorphisms such that $N^{n-1} \neq N^n = 0$. This leads to the following definition.

2.8 Definition If $f : (\mathcal{M}, N) \rightarrow \mathbb{R}[\nu]$, is differentiable, we call it here nilomorphic (for the nilpotent structure N) if $df \circ N = \nu df$.

2.9 Notation If η is a function or more generally a tensor with values in $\mathbb{R}[\nu]$, we denote by $\eta_a \in \mathbb{R}$ its coefficient of degree a in its expansion in powers of ν , so that: $\eta = \sum_{a=0}^{n-1} \eta_a \nu^a$.

2.10 Example The simplest example of such functions are “nilomorphic coordinates”, built once again similarly as complex coordinates $z_j := x_j + iy_j$ on a complex manifold. Take the N -integral coordinates $(x_i, (y_{i,a})_a)_i$ introduced on \mathcal{U} in Notation 2.7 and set:

$$z_i := x_i + \nu y_{i,1} + \nu^2 y_{i,2} + \dots + \nu^{n(i)-1} y_{i,n(i)-1} \in \mathbb{R}[\nu].$$

Then each $\nu^{n-n(i)} z_i$ is nilomorphic. Indeed, take X_i any coordinate vector transverse to $\text{Im } N$ and $a \in \mathbb{N}$. Then $(N^a X_j) \cdot (\nu^{n-n(i)} z_i) = 0$ if $i \neq j$ and $(N^a X_i) \cdot (\nu^{n-n(i)} z_i) = \nu^{n-n(i)} \nu^a$ (it is immediate if $N^a X_i \neq 0$; besides $N^a X_i = 0$ if and only if $a \geq n(i)$ so that both sides of the equality vanish simultaneously). In particular $(N^a X_i) \cdot z_i = \nu^a (X_i \cdot z_i)$.

2.11 Definition/Notation We now call the z_i of Example 2.10 themselves “nilomorphic coordinates” even if only the $\nu^{n-n(i)} z_i$ are nilomorphic functions. The reason appears in Remark 2.24. We also introduce a notation that will much alleviate the use of nilomorphic coordinates:

$$(\nu y_i) := \sum_{a=1}^{n(i)-1} \nu^a y_{i,a}, \text{ so that } z_i = x_i + (\nu y_i), \text{ and: } (\nu y) := ((\nu y_i)_{i=1}^D).$$

2.12 Remark Definition 2.8 may be stated for functions with value in any $\mathbb{R}[\nu]$ -module.

2.13 Remark A system of holomorphic coordinates provides an isomorphism between a small neighbourhood of any point m of (\mathcal{M}, J) onto a small neighbourhood of the origin in \mathbb{C}^K . So does a system of nilomorphic $(\nu^{n-n(i)} z_i)_i$, from a small neighbourhood of any point m of (\mathcal{M}, N) onto a small neighbourhood of the origin in some $\mathbb{R}[\nu]$ -module \mathbb{M} . Notice the small following difference: \mathbb{M} is not free *i.e.* $\mathbb{M} \not\cong (\mathbb{R}[\nu])^K$ in general, but $\mathbb{M} \simeq \prod_a (\nu^a \mathbb{R}[\nu])^{K(a)}$. See the previous section. This is linked to the $\nu^{n-n(i)}$ factoring the coordinates z_i .

2.14 Reminder A tensor θ on a foliated manifold $(\mathcal{M}, \mathcal{F})$ is said to be *basic* for \mathcal{F} , or \mathcal{F} -basic, if it is everywhere, locally, the pull back by $p : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{F}$ of some tensor $\bar{\theta}$ of \mathcal{M}/\mathcal{F} .

We will need also the following auxiliary definition.

2.15 Definition Let \mathbb{M} be an $\mathbb{R}[\nu]$ -module. A function $\check{f} : (\mathcal{U}/\mathcal{I}) \rightarrow \mathbb{M}$ is said to be adapted (to N) if for each $a \in \llbracket 0, n-1 \rrbracket$, $\nu^a \check{f}$ is $\pi(\mathcal{K}^a)$ -basic *i.e.* constant along the leaves of $\pi(\mathcal{K}^a)$. If $\mathbb{M} = \mathbb{R}[\nu]$, this means that each coefficient \check{f}_a is $\pi(\mathcal{K}^{n-1-a})$ -basic.

Similarly, a (multi)linear form $\check{\eta}$ defined on \mathcal{U}/\mathcal{I} with values in \mathbb{M} is called adapted if each $\nu^a \check{\eta}$ is $\check{\mathcal{K}}^a$ -basic. If $\mathbb{M} = \mathbb{R}[\nu]$, this means that each coefficient $\check{\eta}_a$ is $\check{\mathcal{K}}^{n-1-a}$ -basic.

Here is the main property of nilomorphic functions we will use.

2.16 Proposition Let \mathbb{M} be an $\mathbb{R}[\nu]$ -module and $f \in C^{n-1}(\mathcal{U}, \mathbb{M})$. Then f is nilomorphic for N if and only if, in any nilomorphic coordinates system $(z_i)_{i=1}^D = (x_i + (\nu y_i))_{i=1}^D$, it reads:

$$f = \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} (\nu y)^\alpha,$$

where \check{f} is some adapted function (see Def. 2.15) of the coordinates $(x_i)_{i=1}^D$ and where, classically:

$$\alpha \text{ is a multi-index } (\alpha_i)_{i=1}^D, \quad |\alpha| := \sum_{i=1}^D \alpha_i, \quad \alpha! := \prod_{i=1}^D \alpha_i!,$$

$$\frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} := \left(\frac{\partial^{|\alpha_1|}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_D}}{\partial x_D^{\alpha_D}} \right) \check{f}, \quad \text{and } (\nu y)^\alpha := \prod_{i=1}^D (\nu y_i)^{\alpha_i}.$$

2.17 Remark Proposition 2.16 is very similar to the fact that a function $(\mathcal{M}, J) \rightarrow \mathbb{C}$ is holomorphic if and only if it is equal to a power series in the neighbourhood \mathcal{U} of any point. In the complex case, we may consider that the coordinates x_i and y_i parametrise the integral leaves of $\text{Im } J$ (complicated manner to mean the whole \mathcal{U}), and that a single point m is a manifold transverse to this leaf. Then f is holomorphic if and only if it reads, in any holomorphic coordinates system:

$$f = \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial^{|\alpha|} f}{\partial z^\alpha} \right)_{|m} z^\alpha.$$

In the formula of Propostion 2.16 appear:

- instead of the z^α , the $(\nu y)^\alpha$, which are the powers of the coordinates parametrising the integral leaves \mathcal{I} of $\text{Im } N$ (those are not the whole \mathcal{U}),
- instead of the value and derivatives of f at the single point m (a “transversal” to \mathcal{U}), the values and derivatives of \check{f} , which is f along the level $\mathcal{T} := \{(\nu y) = 0\} = \{\forall i, (\nu y_i) = 0\}$, a transversal to the leaves of \mathcal{I} .

So in the complex case, you choose the value and derivatives of a holomorphic function at some point (ensuring a convergence condition), the rest of the function is given by a power series. In the “nilomorphic” case, you choose the value of f along some transversal \mathcal{T} to \mathcal{I} (ensuring the “adaptation” condition 2.15), the rest of the function is given by a power series. Notice finally that, as $\nu^n = 0$, this series in powers of (νy) is not only an analytic function, but even a *polynomial* one, of degree $n - 1$. So f is polynomial along the leaves of \mathcal{I} ; in particular, this means that this notion makes sense, in any N -nilomorphic coordinates system; see Example 2.26 for an explanatory point of view. Transversely to those leaves however, f may be only of class C^{n-1} .

2.18 Remark One of the interests of the development formula of Proposition 2.16 is that it holds even if the invariant factors of N have different degrees *i.e.* the $\mathbb{R}[\nu]$ -module $(T\mathcal{M}, N)$ is not free. This will enable to build metrics making N parallel in such cases, *e.g.* as in Example 2.39.

Proof of Proposition 2.16. (i) The “if” part. Take f of the form given in the proposition, X_i any coordinate vector transverse to $\text{Im } N$ and $a \in \mathbb{N}^*$. Let us check that $(N^a X_i).f = \nu^a (X_i.f)$. If $\alpha = (\alpha_i)_{i=1}^D$, $\alpha \pm 1_i$ stands for $(\alpha_1, \dots, \alpha_{i-1}, \alpha_i \pm 1, \alpha_{i+1}, \dots, \alpha_D)$.

$$\nu^a X_i. \left(\frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} (\nu y)^\alpha \right) = \nu^a \frac{\partial}{\partial x_i} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} (\nu y)^\alpha = \nu^a \frac{1}{\alpha!} \frac{\partial^{|\alpha+1|} \check{f}}{\partial x^{\alpha+1_i}} (\nu y)^\alpha.$$

As \check{f} is adapted (Def. 2.15), $\nu^{n(i)} \check{f}$ is constant along the leaves of $\mathcal{K}^{n(i)}$. So, as $X_i \in \ker N^{n(i)}$,

and setting $\alpha' := \alpha - 1_i$:

$$\begin{aligned} (N^a X_i) \cdot \left(\frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} (\nu y)^\alpha \right) &= \chi_{\{a < n(i)\}} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} \nu^a \alpha_i (\nu y)^{\alpha-1_i} \\ &= \chi_{\{a < n(i)\}} \nu^a \frac{1}{\alpha!} \frac{\partial^{|\alpha'|+1} \check{f}}{\partial x^{\alpha'+1_i}} (\nu y)^{\alpha'} \end{aligned}$$

As $(N^a X_i) \cdot \left(\frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}}{\partial x^\alpha} (\nu y)^\alpha \right) = 0$ if $\alpha_i = 0$, we get the following equality, which concludes:

$$(N^a X_i) \cdot f = \chi_{\{a < n(i)\}} \nu^a \sum_{\alpha'} \frac{1}{\alpha'!} \frac{\partial^{|\alpha'|+1} \check{f}}{\partial x^{\alpha'+1_i}} (\nu y)^{\alpha'} = \nu^a X_i \cdot f.$$

(ii) The “only if” part. If f is nilomorphic, then its restriction \check{f} to the level $\mathcal{T} = \{\forall i, (\nu y_i) = 0\}$, as a function of the x_i , is adapted: if $N^a X = 0$, $X \cdot (\nu^a \check{f}) = (N^a X) \cdot \check{f} = 0$. Therefore, denoting by \check{f}' the restriction of f to \mathcal{T} , viewed as a function of the x_i , the function f' defined as $f' := \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{f}'}{\partial x^\alpha} (\nu y)^\alpha$ is nilomorphic by **(i)**.

Claim. A nilomorphic function $g : \mathcal{U} \rightarrow \mathbb{M}$ null on \mathcal{T} is null.

Applying the claim to $g = f - f'$ gives $f = f'$, so that f is of the wanted form. Now we have to prove the claim. For any i and $a > 0$, as g is nilomorphic, $(N^a X_i) \cdot g = \nu^a (X_i \cdot g)$. In the quotient $\mathbb{M}/\nu\mathbb{M}$, this reads $(N^a X_i) \cdot [g]_{\mathbb{M}/\nu\mathbb{M}} = 0$ so $[g]_{\mathbb{M}/\nu\mathbb{M}} \equiv 0$. This gives rise to an induction: $X_i \cdot g \in \nu\mathbb{M}$ so in $\mathbb{M}/\nu^2\mathbb{M}$, $(N^a X_i) \cdot [g]_{\mathbb{M}/\nu^2\mathbb{M}} = 0$, hence $[g]_{\mathbb{M}/\nu^2\mathbb{M}} \equiv 0$. By induction we get $[g]_{\mathbb{M}/\nu^b\mathbb{M}} \equiv 0$ for all b and finally $g = 0$. q.e.d.

The tangent spaces $(T_m \mathcal{M}, N)$ are $\mathbb{R}[\nu]$ -modules, through: $\nu \cdot X := NX$. Seeing the action of N as that of a scalar leads naturally to introduce the “nilomorphic” version of the tensors: same theory, $\mathbb{R}[\nu]$ -linearity replacing \mathbb{R} -linearity. Let us take local nilomorphic coordinates $z_i = x_i + (\nu y_i)$ and, after Notation 2.7, $X_i := \frac{\partial}{\partial x_i}$, $Y_{i,a} := \frac{\partial}{\partial y_{i,a}}$. Then $(X_i)_{i=1}^D$ is an adapted spanning family of each $T_m \mathcal{M}$, see Definition 2.2.

2.19 Remark The adapted function \check{f} in Proposition 2.16 depends in general on the choice of the transversal $\mathcal{T} = \{(\nu y) = 0\}$. We do not study this dependence here. The interested reader may look at the link with the expansion of functions in jet bundles given in Example 2.26 to understand a meaning of it. Yet notice that \check{f}_0 is canonical *i.e.* does not depend on the choice of \mathcal{T} . More generally, the value of \check{f}_a along each leaf of $\pi(\mathcal{K}^{n-a})$ does not depend on it either, up to an additive constant.

2.20 Definition/Proposition A vector field V on (\mathcal{M}, N) is called nilomorphic if $\mathcal{L}_V N = 0$. Equivalently: in nilomorphic coordinates $z_i = (x_i + (\nu y_i))_i$, $V = \sum_i v_i X_i$ with nilomorphic functions v_i .

Proof. Any vector field reads $V := \sum_i v_i X_i$ with $v_i : \mathcal{M} \rightarrow \mathbb{R}[\nu]$. Now $\mathcal{L}_V N = 0$ if and only if, for any a and j , $[V, N^{a+1} X_j] = N[V, N^a X_j]$. The $(N^a X_i)_{a,i}$ commute, so $[V, N^{a+1} X_j] = -\sum_i (\mathcal{L}_{N^{a+1} X_j} v_i) X_i$ and $N[V, N^a X_j] = -\sum_i (\mathcal{L}_{N^a X_j} v_i) N X_i = \sum_i (\mathcal{L}_{N^a X_j} v_i) \nu \cdot X_i$. Hence $\mathcal{L}_V N = 0 \Leftrightarrow \forall i, j, a, \mathcal{L}_{N^{a+1} X_j} v_i = \nu \mathcal{L}_{N^a X_j} v_i$, the result. q.e.d.

2.21 Definition/Proposition Let η be some (multi)linear form on (\mathcal{M}, N) , with values in $\mathbb{R}[\nu]$. We say here that η is nilomorphic if:

(i) at each point m , it is $\mathbb{R}[\nu]$ -(multi)linear i.e., if $(V_i)_{i=1}^k \in T_m\mathcal{M}$:

$$\forall (a_i)_{i=1}^k \in \mathbb{N}^k, \eta(N^{a_1}V_1, \dots, N^{a_k}V_k) = \nu^{\sum_i a_i} \eta(V_1, \dots, V_k),$$

(ii) $\mathcal{L}_{NV}\eta = \nu\mathcal{L}_V\eta$ for all nilomorphic vector field V .

If (i) is verified, then (ii) means that in nilomorphic coordinates $z_i = x_i + (\nu y_i)$, the coefficients $\eta(X_{i_1}, \dots, X_{i_k})$ of η are nilomorphic functions (left to the reader).

2.22 Remark Point (i) above implies that $\nu^a\eta(V_1, \dots, V_k) = 0$, i.e. $\nu^{n-a}|\eta(V_1, \dots, V_k)$, as soon as some V_i is in $\ker N^a$. So in particular, setting $\eta = \sum_{a=0}^{n-1} \nu^a \eta_a$ with real η_a , at each point m , each η_a is the pull back of a (multi)linear application $(T_m\mathcal{M}/\ker N^{n-1-a})^k \rightarrow \mathbb{R}[\nu]$.

Applying proposition 2.16 to the coefficients of any nilomorphic (multi)linear form η gives:

2.23 Proposition Let η be an $\mathbb{R}[\nu]$ -(multi)linear form on \mathcal{U} . Then η is nilomorphic for N if and only if, introducing $(z_i)_{i=1}^D$ as in 2.11 it reads:

$$\eta = \sum_{(i_1, \dots, i_k)} \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{\eta}_{i_1, \dots, i_k}}{\partial x^\alpha} (\nu y)^\alpha dz_{i_1} \otimes \dots \otimes dz_{i_k}$$

where the $\check{\eta}_{i_1, \dots, i_k}$ are adapted functions of $(x_i)_{i=1}^D$ with value in $\nu^{n-\min_{i=1}^k n(i)} \mathbb{R}[\nu]$.

2.24 Remark So, in the coordinates, the ‘‘elementary’’ nilomorphic multilinear forms are the $\nu^{n-\min(n(i_1), \dots, n(i_k))} dz_{i_1} \otimes \dots \otimes dz_{i_k}$. If $k > 1$, the z_i are needed to write them, not only the $\nu^{n-n(i)} z_i$; thus we chose in 2.11 to define the former as the ‘‘nilomorphic coordinates’’. Using them:

$$\eta = \sum_{(i_1, \dots, i_k)} \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{\eta}_{i_1, \dots, i_k}}{\partial x^\alpha} (\nu y)^\alpha (\nu^{n-\min(n(i_1), \dots, n(i_k))} dz_{i_1} \otimes \dots \otimes dz_{i_k})$$

but here each $\check{\eta}_{i_1, \dots, i_k}$, valued in $\mathbb{R}[\nu]/(\nu^{\min_{i=1}^k n(i)})$, must be such that $\nu^{n-\min_i n(i)} \check{\eta}_{i_1, \dots, i_k}$ is adapted. So the expression of Prop. 2.23 is simpler.

The following result, which is now immediate, characterises the nilomorphic forms in terms of real ones.

2.25 Definition/Proposition Let $\theta \in \Gamma(\otimes^k T^*\mathcal{M})$ be a real k -linear form on (\mathcal{M}, N) . We call it pre-nilomorphic if:

- (i) for any $(V_j)_{j=1}^k$, the $\theta((V_j)_{j=1}^{i-1}, NV_i, (V_j)_{j=i+1}^k)$ are equal to each other, for all i ,
- (ii) $\mathcal{L}_{NV}\theta = \mathcal{L}_V\theta(N \cdot, \cdot, \dots, \cdot)$ for all nilomorphic vector field V .

Then the following $\mathbb{R}[\nu]$ -valued k -linear form is nilomorphic:

$$\Theta := \sum_{a=0}^{n-1} \nu^a \theta(N^{n-1-a} \cdot, \cdot, \dots, \cdot).$$

We call it the nilomorphic form associated with θ . In this sense, any nilomorphic k -linear form $\Theta = \sum_{a=0}^{n-1} \Theta_a \nu^a$, with real Θ_a , is associated with its coefficient Θ_{n-1} — which is, necessarily, pre-nilomorphic.

The example and comments below are unnecessary for the following. They are given as they are natural, and give another point of view on nilomorphic functions — making a link with another work [3, 4].

2.26 Example Natural manifolds with a nilpotent structure are the jet bundles $J^n\mathcal{W}$ over some differentiable manifold \mathcal{W} . The fibre at some point $m \in \mathcal{W}$ is $\{f :]-\varepsilon, \varepsilon[\rightarrow \mathcal{W} ; \varepsilon > 0 \text{ and } f(0) = m\} / \sim$, where $f \sim g$ if in some neighbourhood of 0, then in all of them, $\|f(t) - g(t)\| = o(t^n)$ when $t \rightarrow 0$. So $J^0\mathcal{W} = \mathcal{W}$ and $J^1\mathcal{W} = T\mathcal{W}$. With each local chart $\varphi = (x_i)_{i=1}^d : \mathcal{O} \rightarrow \mathbb{R}^d$ on some open set \mathcal{O} of \mathcal{W} is functorially associated a natural chart $\tilde{\varphi} = ((x_{i,a})_{a=0}^n)_{i=1}^d : J^n\mathcal{O} \rightarrow \mathbb{R}^d$, defined by:

$$\tilde{\varphi}([f]) = ((x_{i,a})_{a=0}^n)_{i=1}^d =: (\bar{x}_a)_{a=0}^n \text{ if } f(t) = \bar{x}_0 + t\bar{x}_1 + \dots + t^n\bar{x}_n + o(t^n).$$

A change of chart θ on \mathcal{W} induces a change of chart $\tilde{\theta}$ given by the action of the successive differentials of θ up to order n , on the \bar{x}_a . The projections

$$\mathcal{W} \leftarrow J^1\mathcal{W} \leftarrow J^2\mathcal{W} \leftarrow \dots \leftarrow J^n\mathcal{W} \leftarrow \dots$$

endow each $J^n\mathcal{W}$ with a flag of foliations $\mathcal{K}^1 \subset \dots \subset \mathcal{K}^n$: in any chart of the type $\tilde{\varphi}$, \mathcal{K}^a is given by the levels of $(\bar{x}_b)_{b=0}^{n-a}$. Now $\mathbb{R}[X]/(X^{n+1})$ acts naturally on $TJ^n\mathcal{W}$, through the endomorphism N defined as follows. If the path $(f + sg)_{s \in \mathbb{R}}$ represents, in some chart and at $s = 0$, some tangent vector v to $J^n\mathcal{W}$ at the point $[f]$, $N(v) := (f + s\hat{g})_{s \in \mathbb{R}}$ with $\hat{g}(t) = tg(t)$. This definition is consistent, all this is classical. At $v = (\bar{v}_a)_{a=0}^n \in T_{[f]}J^n\mathcal{W}$, in some chart of the type $\tilde{\varphi}$, N reads:

$$N((\bar{v}_a)_{a=0}^n) = (0, (\bar{v}_a)_{a=0}^{n-1}).$$

In this chart, $\text{Mat}(N)$ is constant, block-Jordan, so N is integrable; its invariant factors are the d -tuple $(X^{n+1}, \dots, X^{n+1})$. So $(J^n\mathcal{W}, N)$ is a manifold with a nilpotent structure. Each \mathcal{K}^a is the integral foliation of $\ker N^a = \text{Im } N^{n+1-a}$. Moreover, the submanifold \mathcal{T}_0 of the jets of constant functions is a privileged transversal to $\mathcal{I} = \mathcal{K}^n$. *Conversely, any manifold \mathcal{M} with a nilpotent structure N with invariant factors of the same degree $n+1$, and endowed with some fixed transversal \mathcal{T} to \mathcal{I} , is locally modelled on the n^{th} jet bundle of the (local) quotient \mathcal{M}/\mathcal{I} .* Explicitly, any nilomorphic coordinate system of \mathcal{M} is exactly given by some transversal \mathcal{T} to \mathcal{I} and some chart φ of \mathcal{M}/\mathcal{I} . If \mathcal{T} is fixed on \mathcal{M} , the correspondence Ψ , given on the left in nilomorphic coordinates induced by some chart φ of \mathcal{M}/\mathcal{I} , and on the right in the chart $\tilde{\varphi}$:

$$\begin{aligned} (\mathcal{M}, N) & \rightarrow J^n(\mathcal{M}/\mathcal{I}) \\ \Psi : (x_i + (\nu y_i))_{i=1}^D & = (x_i + \sum_{a=1}^n \nu^a y_{i,a})_{i=1}^D \mapsto \left[(x_i + \sum_{a=1}^n t^a y_{i,a})_{i=1}^D \right] \end{aligned}$$

is independent of the choice of φ — because of Proposition 2.16. This way, $(\mathcal{M}, \text{Diff}(\mathcal{M}, N, \mathcal{T}))$ identifies with $J^n(\mathcal{M}/\mathcal{I})$ with the standard action of $\text{Diff}(\mathcal{M}/\mathcal{I})$ on it. In this sense, a manifold with a nilpotent structure with invariant factors $(X^{n+1}, \dots, X^{n+1})$ is, locally, a jet bundle of order n where you “forgot” what the submanifold of constant jets is.

Introducing functions $\tilde{u} : J^n\mathcal{W} \rightarrow \mathbb{R}[\nu] = \mathbb{R}[X]/(X^{n+1}) \simeq J_0^n\mathbb{R}$ is therefore natural; they are (N, ν) -nilomorphic if and only if they represent, modulo some constant, the n -jet of functions $u : \mathcal{W} \rightarrow \mathbb{R}$. That is to say, \tilde{u} is nilomorphic if and only if there is some u and some f_0 in $C^\infty(\mathcal{M}, \mathbb{R})$ such that, for all $[f] \in J^n\mathcal{W}$:

$$\tilde{u}([f]) = [u \circ f] + [f_0], \text{ where } [\cdot] \text{ stands for “}n\text{-jet of”}.$$

This condition is independent of the choice of a privileged transversal \mathcal{T} , which amounts to a change of $[f_0]$. So through Ψ , it works on any (\mathcal{M}, N) with the invariant factors of N all of the same degree.

Note: the case $n = 1$ is elementary. The fibre of $J^1\mathcal{W} = T\mathcal{W} \xrightarrow{\pi} \mathcal{W}$ is a vector space, so for any $(m, v) \in T\mathcal{W}$, $T_{(m,v)}T\mathcal{W}$ identifies with $T_m\mathcal{W}$. So $N(V) := d\pi|_{(m,v)}(V)$ may be viewed as an element of $T_{(m,v)}T\mathcal{W}$. By construction, this $N \in \text{End}(TJ^1\mathcal{W})$ is 2-step nilpotent. We let the reader check it is the same as the N built above.

Slight adaptation for the case where N has any invariant factors. Let $\widehat{\mathcal{K}}$ be the integral flag $\pi(\mathcal{K}^1) \subset \dots \subset \pi(\mathcal{K}^n) \subsetneq \pi(\mathcal{M})$ of the $\pi(\ker N^a)$ on the local quotient $\pi(\mathcal{M}) = \mathcal{M}/\mathcal{I}$. We define a space $J^{\widehat{\mathcal{K}}}(\mathcal{M}/\mathcal{I})$ of “ $\widehat{\mathcal{K}}$ -jets” of functions from \mathbb{R} to $\pi(\mathcal{M})$ by: $f \sim g$ if, for each a , f and g have the same jet in $J^a(\pi(\mathcal{M})/\pi(\mathcal{K}^a))$. In coordinates adapted to $\widehat{\mathcal{K}}$, a $\widehat{\mathcal{K}}$ -jet $[f]$ is the data of the coordinates of f up to the order a , as soon as they are transverse to \mathcal{K}^a . To define Ψ as above we use, on the left side, coordinates adapted to $\widehat{\mathcal{K}}$ and factor those spanning each $\mathcal{K}^{a+1} \setminus \mathcal{K}^a$ by ν^{n-a} *i.e.* we use the $\nu^{n-n(i)}z_i$ introduced in Example 2.10. Then a function $\tilde{u} : \mathcal{M} \rightarrow \mathbb{R}[\nu]$ is nilomorphic if and only if it is, locally, the $\widehat{\mathcal{K}}$ -jet of some function $u : \pi(\mathcal{M}) \rightarrow \mathbb{R}$, plus some other fixed $\widehat{\mathcal{K}}$ -jet $[f_0]$.

2.27 Remark To build a Differential Calculus he calls “simplicial” on general topological spaces, see *e.g.* [3], W. Bertram studied those jet bundles (I thank him for the references). With A. Souvay [4], he used truncated polynomial rings $\mathbb{K}[X]/(X^n)$ with \mathbb{K} any topological ring. The obtained formulas are equivalent, in the case where the $\mathbb{R}[\nu]$ -module is free *i.e.* all the Jordan blocks of N have the same size, to some of this section, notably to Prop. 2.16. It is not immediately explicit in the statements of [4], but observe the “radial expansion” in Th. 2.8, and its consequences *e.g.* the expansion of a $\mathbb{K}[X]/(X^n)$ -valued function on the bottom of p. 14 in the proof of Th. 3.6, or that given in the proof of Th. 2.11. Besides, as non free $\mathbb{K}[X]/(X^n)$ -modules are direct sums of free ones, Theorem 4.5 of [4] generalises the principle of these expansions.

2.28 Remark Proposition 2.16 has a noticeable consequence: the fact, for a function $\mathcal{M} \rightarrow \mathbb{R}$, to be *polynomial along the leaves of \mathcal{I}* , in the form that appears in its statement, makes sense, regardless of the chosen N -integral local coordinates. So, as the datum of a complex structure on a manifold \mathcal{M} induces a real analytic structure on it, that of a nilpotent structure induces some “polynomial structure” along the leaves of \mathcal{I} . If $N^2 = 0$, this structure is of degree one *i.e.* is a flat affine structure. Let us see it directly, without Prop. 2.16. Take any $U = NV \in T_m\mathcal{I}$. There is a unique way, modulo $\ker N$, to extend V in a basic vector field along this leaf \mathcal{I}_m of \mathcal{I} . This induces a canonical way to extend U along \mathcal{I}_m *i.e.* \mathcal{I}_m is endowed with a flat affine structure, preserved by $\text{Diff}(\mathcal{M}, N)$. This amounts to a flat affine connection ∇ on $T\mathcal{I} = J^1\mathcal{I}$.

In the case where all the Jordan blocks of N have the same size n , using the point of view developed in Example 2.26, we see that the $\text{Diff}(\mathcal{M}, N)$ -invariant structure on the leaves of \mathcal{I} , providing their “polynomial structure” is a flat connection ∇ on its bundle $J^{n-1}\mathcal{I}$.

2.3 The germs of metrics making parallel some self adjoint nilpotent endomorphism

Here appears the link between what precedes and our subject:

2.29 Proposition *Let (\mathcal{M}, g) be a pseudo-Riemannian manifold admitting a parallel field $N \in \Gamma(\text{End}(T\mathcal{M}))$ of self adjoint endomorphisms, nilpotent of index n . Then N is integrable*

and g is pre-nilomorphic for N , in the sense of 2.25. After 2.4, g is the real metric associated with the nilomorphic metric $h := \sum_{a=0}^{n-1} \nu^a g(\cdot, N^{n-1-a} \cdot)$.

Conversely, suppose that $h = \sum_{a=0}^{n-1} \nu^a h_a$, with h_a real, is some non degenerate symmetric $\mathbb{R}[\nu]$ -bilinear form on a manifold \mathcal{M} with a nilpotent structure N . Set $g := h_{n-1}$. Then (\mathcal{M}, g) is pseudo-Riemannian, N is self adjoint and parallel on it — and, according to Definition 2.4, h is the nilomorphic metric associated with g .

We need the following lemma, to our knowledge (through [6]) first proven by [26], then independently by [22]; [29] is a short recent proof.

2.30 Lemma *A field of nilpotent endomorphisms N on \mathbb{R}^d with constant invariant factors is integrable if and only if its Nijenhuis tensor \mathcal{N}_N vanishes and the distributions $\ker N^k$ are involutive for all k .*

Proof of the proposition. D is torsion free and $DN = 0$, so $\mathcal{N}_N = 0$, and the distributions $\ker N^a$ are involutive: by Lemma 2.30, N is integrable. Hence g satisfies Definition 2.25. Indeed, (i) is the fact that N is self adjoint. For (ii) take V any nilomorphic vector field and check that $(\mathcal{L}_{NV}g)(A, B) = (\mathcal{L}_Vg^1)(A, B)$, where $g^1 := g(\cdot, N \cdot)$, for any A, B . By Prop. 2.20 we may suppose, without loss of generality, that the field V , and some fields extending A and B , are coordinate vector fields of some integral coordinate system for N , so that V, A, B, NV, NA and NB commute. Then we must check: $(NV).(g(A, B)) = V.(g(A, NB))$.

$$\begin{aligned}
& (NV).(g(A, B)) \\
&= g(D_{NV}A, B) + g(A, D_{NV}B) \\
&= g(D_A(NV), B) + g(A, D_B(NV)) \quad \text{as } [V, NA] = [V, NB] = 0, \\
&= g(ND_AV, B) + g(A, ND_BV) \quad \text{as } DN = 0, \\
&= g(D_AV, NB) + g(A, ND_BV) \quad \text{as } N^* = N, \\
&= g(D_VA, NB) + g(A, ND_VB) \quad \text{as } [V, A] = [V, B] = 0, \\
&= g(D_VA, NB) + g(A, D_V(NB)) \quad \text{as } DN = 0, \\
&= V.(g(A, NB)).
\end{aligned}$$

For the converse part, first g is non degenerate: if $g(V, \cdot) = 0$ then for any a , $h_a(V, \cdot) = h_{n-1}(V, N^{n-1-a} \cdot) = 0$ so $V = 0$. Then $N^* = N$ is immediate. To ensure $DN = 0$, it is sufficient to prove that, for any N -integral coordinate vector fields X_i, X_j, X_k and any a, b, c in \mathbb{N} , $g(D_{N^a X_i} N^b X_j, N^c X_k) = g(N^b D_{N^a X_i} X_j, N^c X_k)$.

$$\begin{aligned}
& 2g(D_{N^a X_i} N^b X_j, N^c X_k) \\
&= (N^a X_i).(g(N^b X_j, N^c X_k)) + (N^b X_j).(g(N^a X_i, N^c X_k)) \\
&\quad - (N^c X_k).(g(N^a X_i, N^b X_j)) \\
&= X_i.(g(X_j, N^{a+b+c} X_k)) + X_j.(g(X_i, N^{a+b+c} X_k)) \\
&\quad - X_k.(g(X_i, N^{a+b+c} X_j)) \\
&\quad \text{as } h \text{ is nilomorphic, so } g \text{ pre-nilomorphic, see 2.25,} \\
&= 2g(D_{X_i} X_j, N^{a+b+c} X_k),
\end{aligned}$$

which gives in particular the wanted equality.

q.e.d.

We are done: combining Propositions 2.29 and 2.23 provides exactly Theorem 2.31. In the statement, if needed, see Definitions 2.21, 2.4, 2.11, 2.15 and 2.6 for “nilomorphic”, “associated with”, “nilomorphic coordinates”, “adapted function” and “characteristic signatures”.

2.31 Theorem *Let N be a nilpotent structure of nilpotence index n on \mathcal{M} . Then N is self-adjoint and parallel for a pseudo-Riemannian metric g if and only if g is the real metric associated with a metric h nilomorphic for N .*

In nilomorphic local coordinates $(z_i)_{i=1}^D := ((x_i + (\nu y_i)))_{i=1}^D$, h is an $\mathbb{R}[\nu]$ -valued, $\mathbb{R}[\nu]$ -bilinear metric of the form:

$$\begin{aligned} h &= \sum_{i,j=1}^D h_{i,j} dz_i \otimes dz_j \quad \text{with nilomorphic functions } h_{i,j} = h_{j,i} \\ &\quad \text{applying in } \nu^{n-\max(n(i),n(j))} \mathbb{R}[\nu] \\ &\quad \text{and } (h_{i,j})_{i,j=1}^D \text{ non degenerate,} \\ &= \sum_{i,j=1}^D \sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} \check{h}_{i,j}}{\partial x^\alpha} (\nu y)^\alpha dz_i \otimes dz_j \\ &\quad \text{with } \alpha \text{ a multi-index } (\alpha_i)_{i=1}^D \text{ and } \check{h}_{i,j} = \check{h}_{j,i} \\ &\quad \text{adapted functions of the } (x_i)_i \text{ giving} \\ &\quad \text{the properties of } (h_{i,j})_{i,j=1}^D \text{ above.} \end{aligned}$$

The characteristic signatures $(r_a, s_a)_{a=1}^n$ of (N, g) are those of h .

In the theorem, recall that $h = \sum_{a=0}^{n-1} \nu^a g(\cdot, N^{n-1-a} \cdot)$, as stated in Proposition 2.29; in particular, g is the coefficient of ν^{n-1} . See also the important Remark 2.36, and a matricial formulation in Remark 2.40.

2.32 Remark Set $\check{h} = \sum_{i,j=1}^D \check{h}_{i,j} dz_i \otimes dz_j$. Then $\check{h} = \sum_{a=0}^{n-1} \check{h}_a \nu^a$ where each \check{h}_a is the value of $g(\cdot, N^{n-1-a} \cdot)$ along the transversal $\{(\nu y) = 0\}$ to \mathcal{I} . This gives more explicitly the link between g and h .

Using Notation 2.33, Corollary 2.34 translates Theorem 2.31 into purely real terms.

2.33 Notation (i) If $((\alpha_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$ is a multi-index designed so that: $y^\alpha := \prod_{i,a} y_{i,a}^{\alpha_{i,a}}$,

then x^α and \mathcal{L}_{X^α} denote:

$$x^\alpha := \prod_{i,a} x_i^{\alpha_{i,a}} = \prod_i x_i^{\sum_a \alpha_{i,a}} \quad \text{and} \quad \mathcal{L}_{X^\alpha} := \frac{\partial^{|\alpha|}}{\partial x^\alpha} := \prod_{i,a} \frac{\partial^{\alpha_{i,a}}}{\partial x_i^{\alpha_{i,a}}}.$$

(ii) Using point **(i)**, if $(x_i, (y_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$ are N -integral coordinates, if η is a (multi)-linear form on \mathcal{U}/\mathcal{I} (hence, depending only on the coordinates x_i) and if $b \in \llbracket 0, n-1 \rrbracket$, we set:

$$\eta^{(b)} := \sum_{\substack{\alpha \text{ such that} \\ \sum_{i,a} a \alpha_{i,a} = b}} \frac{1}{\alpha!} (\mathcal{L}_{X^\alpha} \eta) y^\alpha.$$

2.34 Corollary *In the framework of Theorem 2.31, α being a multi-index $((\alpha_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$, g is a metric defined by (on the right side, we make use of Notation 2.33 **(ii)**):*

$$\begin{aligned} g(X_i, N^c X_j) &= \sum_{b=c}^{n-1} \sum_{\substack{\alpha \text{ such that} \\ \sum_{i,a} a \alpha_{i,a} = n-1-b}} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial x^\alpha} (\tilde{B}^{b-c}(X_i, X_j)) y^\alpha \\ &= \sum_{b=c}^{n-1} (\tilde{B}^{b-c})^{(n-1-b)}, \end{aligned}$$

where each B^a , for $a \in \llbracket 0, n-1 \rrbracket$, is a $\pi(\mathcal{K}^{n-1-a})$ -basic symmetric bilinear form on $\pi(\mathcal{U})$, non degenerate along $\pi(\mathcal{K}^{n-a})/\pi(\mathcal{K}^{n-1-a})$, and \tilde{B}^* denotes π^*B^* . The characteristic signatures $(r_a, s_a)_{a=1}^n$ of the couple (N, g) are the signatures of the B^{n-a} restricted to $\pi(\ker N^{n-1-a})$.

Proof. By definition, the form h of Theorem 2.31 is equal to $\sum_{a=0}^{n-1} \nu^a g(\cdot, N^{n-1-a} \cdot)$. So $g(X_i, N^c X_j)$ is the coefficient of ν^{n-1-c} in h . Defining the \tilde{B}^a by $\check{h} = \sum_{a=0}^{n-1} \nu^a \tilde{B}^a$, we let the reader expand h as given in Theorem 2.31 in powers of ν , proving the Corollary. q.e.d.

2.35 Remark B^a is the matrix of $g(\cdot, N^{n-1-a} \cdot) = h_a$ on the transversal $\{(\nu y) = 0\}$ to \mathcal{I} .

2.36 Important Remark By Lemma 2.30, any parallel field of endomorphisms is integrable, so Theorem 2.31 and Corollary 2.34 give a parametrisation of the set of pseudo-Riemannian metrics on \mathcal{U} , with a holonomy representation preserving some fixed (arbitrary) self adjoint nilpotent endomorphism N . The parameters are the $(B^a)_{a=0}^{n-1}$, up to an action of $C^\infty(\mathbb{R}^{d-\text{rk } N}, \mathbb{R}^{\text{rk } N})$. Indeed, the B^a are chosen freely, and characterise g once the level $\{(\nu y) = 0\}$, *i.e.* some section of $\pi : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{I}$, is chosen. As $\dim \mathcal{I} = \text{rk } N$, $C^\infty(\mathbb{R}^{d-\text{rk } N}, \mathbb{R}^{\text{rk } N})$ acts simply transitively on those sections. We give here only an idea of this action, in Remark 2.43. Some part of the B^a is invariant under it, see Remark 2.37.

2.37 Remark The adapted bilinear form \check{h} in Theorem 2.31 depends on the chosen transversal $\mathcal{T} = \{(\nu y) = 0\}$; see a similar remark for nilomorphic functions in 2.19. Yet the restriction of each $\check{h}_a = g(\cdot, N^{n-1-a} \cdot)$ to the leaves of \mathcal{K}^{n-a} , — encoded below by the matrix \check{H}_a^0 in Remark 2.40 and by the matrix B_0^a in Example 2.39 —, is canonical *i.e.* does not depend on the choice of \mathcal{T} . Indeed for each a , $g(\cdot, N^a \cdot)$ does not pass on the quotient $\pi(\mathcal{U}/\mathcal{K}^a)$, but its restriction to the leaves of $\pi(\mathcal{K}^{a+1})$ does, see Remark 2.3. This invariant shall be noticed.

Therefore, it is natural to choose coordinates $(x_i; n(i) = a + 1) = (x_i)_{i=1+D_a}^{D_{a+1}}$ satisfying some property with respect to it, if they exist: *e. g.* if $h_{n-1-a} = g(\cdot, N^a \cdot)$ is flat along each leaf of $\pi(\mathcal{K}^{a+1})$, coordinates such that its matrix B_0^{n-1-a} is $I_{r_{a+1}, s_{a+1}}$. In particular if $d_{a+1} = 1$ we can take $B_0^{n-1-a} \equiv \pm 1$, after the corresponding characteristic signature.

As the \check{h}_a , except for their restriction to $\ker N^{n-a}$, depend on \mathcal{T} , may \mathcal{T} be chosen such that they satisfy some specific property ? The answer is given in Remarks 2.41 and 2.43.

The complicated expression of Corollary 2.34 is simpler in some cases. We present two of them. Case **(A)**: all invariant factors of N have the same degree *i.e.* $\text{Im } N^p = \ker N^{n-p}$ for $p \leq n$ *i.e.* N is conjugated to:

$$\left(\begin{array}{cccc} 0 & I_{d_n} & & \\ & \ddots & \ddots & \\ & & 0 & I_{d_n} \\ & & & 0 \end{array} \right), \quad \begin{array}{l} \text{with } n \text{ null blocks on the diagonal,} \\ d_n \text{ being the } n^{\text{th}} \text{ characteristic} \\ \text{dimension of } N, \text{ the other ones being null.} \end{array}$$

In other terms, the $\mathbb{R}[\nu]$ -module $E = (\mathbb{R}^d, N)$ is free: $E \simeq \mathbb{R}[\nu]^{d_n}$. Case **(B)**: the nilpotence index of N is small, namely we took $N^3 = 0$.

2.38 Example (A) After Corollary 2.34, a metric g makes N self adjoint and parallel if and only if, in N -integral coordinates giving N the block form displayed just above, its matrix reads:

$$\text{Mat}(g) = \begin{pmatrix} 0 & \cdots & 0 & G^0 \\ \vdots & \ddots & \ddots & G^1 \\ 0 & \ddots & \ddots & \vdots \\ G^0 & G^1 & \cdots & G^{n-1} \end{pmatrix},$$

with the following G^a . For each a , we denote also by B^a the matrix of the form B^a introduced in Corollary 2.34. Using Notation 2.33 (ii):

$$G^a = \sum_{b=0}^{n-1-a} (B^{a-b})^{(b)}.$$

The B^a are symmetric matrices, function of the coordinates x_i , with B^0 non degenerate. For each a , B^a represents $g(\cdot, N^{n-1-a} \cdot)$ along $\{(\nu y) = 0\}$. The couple (N, g) has only one characteristic signature, namely $(r_n, s_n) = \text{sign}(B^0)$. So $G^0 = B^0$, $G^1 = B^1 + \sum_i \left(\frac{\partial B^0}{\partial x_i}\right) y_{i,1}$ etc.; as an example, let us expand G^3 :

$$\begin{aligned} G^3 &= B^3 + B^{2(1)} + B^{1(2)} + B^{0(3)} \\ &= B^3 + \sum_i \left(\frac{\partial B^2}{\partial x_i}\right) y_{i,1} + \frac{1}{2!} \sum_{i,j} \left(\frac{\partial^2 B^1}{\partial x_i \partial x_j}\right) y_{i,1} y_{j,1} + \sum_i \left(\frac{\partial B^1}{\partial x_i}\right) y_{i,2} \\ &\quad + \frac{1}{3!} \sum_{i,j,k} \left(\frac{\partial^3 B^0}{\partial x_i \partial x_j \partial x_k}\right) y_{i,1} y_{j,1} y_{k,1} + \sum_{i,j} \left(\frac{\partial^2 B^0}{\partial x_i \partial x_j}\right) y_{i,2} y_{j,1} \\ &\quad \quad \quad + \sum_i \left(\frac{\partial B^0}{\partial x_i}\right) y_{i,3}. \end{aligned}$$

If $n = 2$, $\text{Mat}(g)$ is an affine function of the $y_{i,1}$:

$$\text{Mat}(g) = \begin{pmatrix} 0 & B^0 \\ B^0 & B^1 + \sum_i \frac{\partial B^0}{\partial x_i} y_{i,1} \end{pmatrix}.$$

To be totally explicit, on \mathbb{R}^4 with coordinates (x, x', y, y') , this means:

$$\text{Mat}(g) = \begin{pmatrix} 0 & B^0(x, x') \\ B^0(x, x') & B^1(x, x') + \frac{\partial B^0}{\partial x} y + \frac{\partial B^0}{\partial x'} y' \end{pmatrix}$$

with B^0 and B^1 symmetric 2-2 matrices, B^0 everywhere nondegenerate.

2.39 Example (B) Recall that, after Notation 2.1:

$$\begin{aligned} d_1 &= \dim(\pi(\ker N)), \quad d_2 = \dim(\pi(\ker N^2)/\pi(\ker N)) \\ \text{and } d_3 &= \dim(\pi(\ker N^3)/\pi(\ker N^2)) = \dim(\pi(T\mathcal{M})/\pi(\ker N^2)), \end{aligned}$$

Order the coordinates as: $((y_{i,2})_{i=d_1+d_2+1}^{d_1+d_2+d_3}, (y_{i,1})_{i=d_1+1}^{d_1+d_2+d_3}, (x_i)_{i=1}^{d_1+d_2+d_3})$.

$$\text{Then, if } N^3 = 0 \neq N^2, \text{Mat}(N) = \begin{pmatrix} 0 & 0 & I_{d_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{d_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{d_3} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

with columns and lines of respective sizes $d_3, d_2, d_3, d_1, d_2, d_3$. Columns 1, 2–3, 4–6 correspond respectively to the $y_{i,2}, y_{i,1}, x_i$. Then a metric g makes N self adjoint and parallel if and only if its matrix reads:

$$\text{Mat}(g) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & G^0 \\ 0 & 0 & 0 & 0 & \begin{pmatrix} G^1 \\ \end{pmatrix} \\ 0 & 0 & G^0 & 0 & \begin{pmatrix} G^1 \\ \end{pmatrix} \\ 0 & 0 & 0 & \begin{pmatrix} G^1 \\ \end{pmatrix} & \begin{pmatrix} G^2 \\ \end{pmatrix} \\ 0 & \begin{pmatrix} G^1 \\ \end{pmatrix} & \begin{pmatrix} G^1 \\ \end{pmatrix} & \begin{pmatrix} G^2 \\ \end{pmatrix} \\ G^0 & \begin{pmatrix} G^1 \\ \end{pmatrix} & \begin{pmatrix} G^1 \\ \end{pmatrix} & \begin{pmatrix} G^2 \\ \end{pmatrix} \end{pmatrix}$$

with, using again Notation 2.33, (ii), $G^0 = B^0$ and:

$$G^1 = B^1 + \begin{pmatrix} 0 & 0 \\ 0 & B^{0(1)} \end{pmatrix}, \quad G^2 = B^2 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \begin{pmatrix} B^{1(1)} \\ \end{pmatrix} \\ 0 & \begin{pmatrix} B^{1(1)} \\ \end{pmatrix} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B^{0(2)} \end{pmatrix},$$

where the B^a are symmetric matrices: B^0 depending on the $(x_i)_{i>d_1+d_2}$, B^1 on the $(x_i)_{i>d_1}$ and B^2 on all the $(x_i)_{i=1}^{d_1+d_2+d_3}$. We recall that:

$$B^{a(1)} = \sum_i \left(\frac{\partial B^a}{\partial x_i} \right) y_{i,1}, \quad B^{a(2)} = \frac{1}{2!} \sum_{i,j} \left(\frac{\partial^2 B^a}{\partial x_i \partial x_j} \right) y_{i,1} y_{j,1} + \sum_i \left(\frac{\partial B^a}{\partial x_i} \right) y_{i,2}.$$

Ensuring the non degeneracy of g is ensuring the non degeneracy condition stated in Corollary 2.34, *i.e.* here, B^0 is non degenerate and:

$$B^1 = \begin{pmatrix} B_0^1 & * \\ * & * \end{pmatrix} \text{ and } B^2 = \begin{pmatrix} B_0^2 & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \quad \text{with } B_0^1 \text{ and } B_0^2 \text{ non degenerate.}$$

The characteristic signatures of (N, g) , *cf.* Remark 2.3, are:

$$\begin{aligned} (\text{sign}(B_0^2), \text{sign}(B_0^1), \text{sign}(B_0^0)) &= ((r_1, s_1), (r_2, s_2), (r_3, s_3)), \\ \text{so: } \text{sign}(g) &= (d_2 + d_3 + r_1 + r_3, d_2 + d_3 + s_1 + s_3). \end{aligned}$$

If $N^2 = 0$, relabelling the G^a and B^a we find, setting $B^1 = \begin{pmatrix} B_0^1 & B^{1'} \\ {}_t B^{1'} & B^{1''} \end{pmatrix}$:

$$\begin{aligned} \text{Mat}(N) &= \begin{pmatrix} 0 & 0 & I_{d_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and} \\ \text{Mat}(g) &= \begin{pmatrix} 0 & 0 & G^0 \\ 0 & \begin{pmatrix} G^1 \\ \end{pmatrix} \\ G^0 & \begin{pmatrix} G^1 \\ \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & B^0 \\ 0 & B_0^1 & B^{1'} \\ B^0 & {}_t B^{1'} & B^{1''} + B^{0(1)} \end{pmatrix}, \end{aligned}$$

with B^0 and B_0^1 non degenerate, B^0 depending on the $(x_i)_{i>d_1}$ and B^1 on all the $(x_i)_{i=1}^{d_1+d_2}$. We recall that $B^{0(1)} = \sum_i \left(\frac{\partial B^0}{\partial x_i} \right) y_{i,1}$. In case $d_1 = 0$ we re-find the end of Example A above.

2.40 Remark For any nilpotence index n , ordering the coordinates as $((y_{i,n-1})_i, \dots, (y_{i,1})_i, (x_i)_i)$, one may build similarly $\text{Mat}(g)$. The principle is the same as when $N^3 = 0$ and no new phenomenon appears, but the matrix becomes rapidly very cumbersome. So it seems that the use of real coordinates, forms and matrices, if it may be avoided, should be, and

replaced by the use of $\mathbb{R}[\nu]$ -linear ones like in Theorem 2.31, just as complex expressions replace real ones in complex geometry.

The matrix of h given in Theorem 2.31 reads as follows. Take $(z_i)_{i=1}^D$ nilomorphic coordinates of \mathcal{U} , ordered by increasing values of $n(i)$. So, $(z_i)_{i=1}^{D_a} = (z_i; n(i) \leq a)$ parametrise the leaves of \mathcal{K}^a . Using Prop. 2.5,

$$\text{Mat}(h) = \sum_{\alpha} \frac{1}{\alpha!} \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} \check{H} \right) (\nu y)^\alpha$$

with \check{H} a sum of symmetric matrices $\nu^a \begin{pmatrix} 0 & 0 \\ 0 & \check{H}^a \end{pmatrix} \in \nu^a \text{M}_D(\mathbb{R})$ satisfying, for each $a \in \llbracket 0, n-1 \rrbracket$:

(i) (*Adaptation condition, see 2.15*) the block \check{H}^a , corresponding to the coordinates $(z_i; n(i) \geq n-a)$, depends only on the $(x_i; n(i) \geq n-a)$,

(ii) (*Non degeneracy condition, see 2.5*) cutting the $(z_i; n(i) \geq n-a)$ into $((z_i; n(i) = n-a), (z_i; n(i) > n-a))$, \check{H}^a splits into $\begin{pmatrix} \check{H}_0^a & * \\ * & * \end{pmatrix}$ with \check{H}_0^a non degenerate.

Finally, notice that $\check{H}^a = \nu^a B^a$, with $B^a = \text{Mat}(g(\cdot, N^{n-1-a} \cdot))$ along $\{(\nu y) = 0\}$, as appearing in Corollary 2.34.

2.41 Remark Another choice of “preferred” coordinates $(x_i)_{i=D_{a+1}}^{D_{a+1}} = (x_i; n(i) = a+1)$ for some a could be to try to get $B^a = \begin{pmatrix} B_0^a & 0 \\ 0 & B^{a''} \end{pmatrix}$, that is to say to choose them so that the $\{\frac{\partial}{\partial x_i}; n(i) > a+1\}$ be B^a -orthogonal to $\mathcal{K}^{a+1} \cap \{(\nu y) = 0\}$. It is impossible, as the orthogonal distribution to $\mathcal{K}^{n-a} \cap \{(\nu y) = 0\}$ is not integrable in general, even by seeking an “adequate” transversal $\mathcal{T} = \{(\nu y) = 0\}$ to \mathcal{I} . See Remark 2.43 for a proof. Notice here that on the contrary, this is possible for *alternate* nilomorphic bilinear forms, for which we can even achieve $B^a = \begin{pmatrix} B_0^a & 0 \\ 0 & 0 \end{pmatrix}$, see Lemma 3.12 and Rem. 3.13.

It works however in one case. If $\sum_{c=1}^b d_{n-a+c} = 1$ for some $b \geq c+1$ (all d_{n-a+c} null except one), the B^a -orthogonal to $\mathcal{K}^{n-a} \cap \mathcal{T}$ is 1-dimensional inside of the leaves of $\mathcal{K}^{n-a+b} \cap \mathcal{T}$, so may be integrated, within these leaves. This puts a null line under B_0^a and a null column on its right. Moreover, [8] provides a transversal \mathcal{T} such that the unique coordinate vector $\frac{\partial}{\partial x_i}$ transverse to \mathcal{K}^{n-a} in $\mathcal{K}^{a+b} \cap \mathcal{T}$ has a constant B^a -square, *e.g.* is isotropic. This puts a constant coefficient $B^{a''}$ below B_0^a on the right of it. See Example 2.42. If moreover $d_{n-a} = 1$, you must choose between achieving all this, and achieving $B_0^a \equiv \pm 1$ as in 2.37.

2.42 Example With Remarks 2.40 and 2.41, we treat the Lorentzian case. The indecomposable germs of Lorentzian metric making a non trivial self adjoint endomorphism N parallel are those with holonomy algebra included in:

$$\left\{ \begin{pmatrix} 0 & L & 0 \\ 0 & A & -{}^t L \\ 0 & 0 & 0 \end{pmatrix}; A \in \mathfrak{so}(n-2), L \in \mathbb{R}^{n-2} \right\},$$

in a basis where $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is parallel. Writing g on the form given at the end of 2.39 and setting $B^0 = 1$ by Remark 2.37 and $B^{1'} = 0$ by Remark

2.41 we get:

$$\text{Mat}(g) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & B_0^1 & 0 \\ 1 & 0 & b^{1''} \end{pmatrix},$$

with B_0^1 and $b^{1''}$ independent of the first coordinate. Again by 2.41, one can get $b^{1''}$ constant (zero or not), getting a classical form for g .

2.43 Remark The adapted bilinear form \check{h} appearing in Theorem 2.31 depends on the choice of $\mathcal{T} = \{(\nu y) = 0\}$. Let us illustrate here how this dependence works through the case $N^2 = 0$. In N -adapted coordinates:

$$((y_i)_{i=1}^{d_2}, (x_i)_{i=1}^{d_1}, (x_i)_{i=d_1+1}^{d_2}), \text{ we get: } \text{Mat}(N) = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\text{Mat}(g)$ is as given at the end of Example 2.39. Then B^0 is the matrix of $h_0 = g(\cdot, N \cdot)$, which is well defined on \mathcal{M}/\mathcal{K} , so does not depend on \mathcal{T} ; B^1 is the matrix of $\check{h}_1 = g|_{T\mathcal{T}}$. It depends on \mathcal{T} , which is the image of a section σ of $\pi : \mathcal{M} \rightarrow \mathcal{M}/\mathcal{I}$. Changing \mathcal{T} amounts to add to σ a vector field U defined along \mathcal{T} , and tangent to \mathcal{I} . Indeed, \mathcal{I} is endowed with a flat affine connection ∇ , see Remark 2.28, so identifies with its tangent space. In turn, this field U is equal to NV with V a section of $\pi(T\mathcal{M})/\pi(\ker N)$ defined on $\pi(\mathcal{M}) = \mathcal{M}/\mathcal{I}$. If $\sum_{i=d_1+1}^{d_2} v_i \frac{\partial}{\partial x_i}$ represents V , so that $U = \sum_{i=1}^{d_2} v_i \frac{\partial}{\partial y_i}$, it follows from the expression of $\text{Mat}(g)$ that B^1 becomes B_V^1 given by:

$$\begin{aligned} B_V^1\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) &= B^1\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) + V \cdot \left(B^0\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)\right) \\ &\quad + \sum_{i=1}^D \frac{\partial v_i}{\partial x_j} B^0\left(V, \frac{\partial}{\partial x_k}\right) + \frac{\partial v_i}{\partial x_k} B^0\left(\frac{\partial}{\partial x_j}, V\right) \end{aligned}$$

i.e. $B_V^1 = B^1 + \mathcal{L}_V B^0$. The Lie derivative $\mathcal{L}_V B^0$ is well defined, even if V is defined modulo $\ker N$, as $\ker N = \ker B^0$. So \check{h}_1 is defined, through the choice of \mathcal{T} , up to addition of an infinitesimal deformation of $h_0 = g(\cdot, N \cdot)$ by a diffeomorphism.

So, the orbit $\{B_V^1, V \in \Gamma(T\pi(\mathcal{M}))\}$ of the matrix B^1 when \mathcal{T} varies does not contain in general a matrix such that $B^{1'} = 0$ and/or $B^{1''} = 0$, with the notation of Example 2.39. Getting $B^{1'} = 0$ is impossible in general if $\dim(\mathcal{K} \cap \mathcal{T})$ is greater than 1. Indeed for all $i > d_1$ the 1-form $(\mathcal{L}_V B^0)(X_i, \cdot)$ is closed along the leaves of $\mathcal{K} \cap \mathcal{T}$. So $d(\iota_{X_i} B^1)$ is canonical, independent of \mathcal{T} . This proves the impossibility announced in Remark 2.41. Getting $B^{1''} = 0$ is also impossible in general: see below.

To simplify, we suppose now that $d_1 = 0$ *i.e.* $\ker N = \text{Im } N$; now h_0 is a non degenerate metric on $\pi(\mathcal{M}) = \mathcal{M}/\mathcal{I}$. After the infinitesimal part of Ebin's slice theorem (see *e.g.* [5] 4.2–5 for a short explanation):

$$\text{Sym}^2 T^* \mathcal{M} = \{\mathcal{L}_V h_0; V \in \Gamma(T\pi(\mathcal{M}))\} \oplus \ker \delta_{h_0}$$

with δ_{h_0} the divergence operator with respect to h_0 . I thank S. Gallot who let me think to this. So there is a local privileged choice of \mathcal{T} , making \check{h}_1 divergence free; this divergence free \check{h}_1 is canonical; besides it is given by $\frac{D(D-1)}{2}$ functions of D variables. The choice of \mathcal{T} , depending on D functions of D variables, acts “with a slice” on the value of \check{h}_1 , and each orbit of this action consists of forms depending also on D functions of D variables. Finally, up to diffeomorphism, g depends on $D(D-1)$ functions of D variables: $2\frac{D(D+1)}{2}$ for the couple (B^0, B^1) , minus the choice of \mathcal{T} and of a chart of \mathcal{M}/\mathcal{I} *i.e.* twice D functions of D variables.

2.44 Important Remark [The resulting algebra $\text{End}(T\mathcal{M})^{\mathfrak{h}}$ — Indecomposability of the obtained metrics] The form of the resulting algebra $\text{End}(T\mathcal{M})^{\mathfrak{h}}$ for metrics built in Theorem 2.31 is given by Corollary 3.5 in the next section. This corollary also discusses, together with Remark 3.6, if the metrics of Theorem 2.31 are decomposable.

2.45 Remark In the next section, Corollary 3.5 shows that in cases **(1)**–**(1 \mathbb{C})**, the holonomy group of a metric making N parallel acts trivially on $\text{Im } N^{n_2}$, which is non zero as soon as $(n =)n_1 > n_2$. So one may take N -adapted coordinates such that the $(Y_{1,a})_{a=n_2}^{n-1} := (N^a \frac{\partial}{\partial x_1})_{a=n_2}^{n-1}$ span $\text{Im } N^{n_2}$ and are parallel. Here is a direct way to see it:

– Take N -adapted coordinates such that $N^{n-1} \frac{\partial}{\partial x_1} \neq 0$; the metric $g_{n-1} := g(\cdot, N^{n-1} \cdot)$ is well-defined on $\mathcal{M}/\mathcal{K}^{n-1}$, which is 1-dimensional, so you may modify the coordinate x_1 so that $g(\frac{\partial}{\partial x_1}, N^{n-1} \frac{\partial}{\partial x_1}) \equiv \pm 1$ *i.e.* $B^0 = \pm 1$. If $n - n_2 = 1$ we are done.

– If $n - n_2 > 1$, then $\pi(\mathcal{M}/\mathcal{K}^{n-2}) = \pi(\mathcal{M}/\mathcal{K}^{n-1})$ is also 1-dimensional so the matrix B^1 is also a scalar. Modifying the level $\mathcal{T} = \{(\nu y) = 0\}$ as in Rem. 2.43 to get: $B_V^1 = B^1 + \mathcal{L}_V B^0$ with V a vector field on $\mathcal{M}/\mathcal{K}^{n-1}$, enables to modify arbitrarily B^1 . An adequate V gives $B_V^1 \equiv 0$. The new level $\{(\nu y) = 0\}$ is obtained by addition of the field $N^{n-1}V$ *i.e.*, setting $V = v \cdot \pi(\frac{\partial}{\partial x_1})$, by replacing the coordinate $y_{1,n-1}$ by $y_{1,n-1} - v$.

– We modify then similarly the coordinates $y_{1,n-a}$, for $a \in \llbracket 1, n - n_2 - 1 \rrbracket$, by induction on a : at each step, we choose $V = v \cdot \pi(\frac{\partial}{\partial x_1})$ such that $B^a + \mathcal{L}_V B^0 \equiv 0$ and replace $y_{1,n-a}$ by $y_{1,n-a} - v$. We let the reader check that this makes the scalars B^1, \dots, B^{n-n_2-1} null.

Because of the form of the metric given by Corollary 2.34, no coefficient of the metric g depends on the coordinates $(y_a)_{a=n_2}^{n-1}$. At m , we may take the $(N^a \frac{\partial}{\partial x_1})_{a=n-1}^0$ such that, on the subspace they span, $\text{Mat}(N) = N_n$ and $\text{Mat}(g) = \pm K_n$, see Notation 3.18 **(a)**. The \pm sign is given by the signature of the metric g_{n-1} introduced just above in this remark.

2.46 Example We end with an example of a pseudo-Riemannian metric where a parallel N naturally arises. I thank L. Bérard Bergery for having told me that this example works with non locally symmetric metrics. Example 2.26 showed that jet bundles carry naturally a nilpotent structure; in particular, tangent bundles carry a 2-step nilpotent structure N , see the *Note* at its end. Now if g is a (pseudo-)Riemannian metric on \mathcal{M} of dimension d , $T\mathcal{M}$ carries a natural metric \hat{g} , of signature (d, d) , making N self adjoint and parallel. In $TT\mathcal{M}$, we call “vertical” the tangent space \mathbf{V} to the fibre, and \mathbf{H} its “horizontal” complement given by the Levi-Civita connection D of g . If $X \in T_{(m,V)}T\mathcal{M}$, as the fibre of $T\mathcal{M}$ is a vector space, X is naturally identified with a vector of $T_m\mathcal{M}$, that we denote by X' . We set $\pi : T\mathcal{M} \rightarrow \mathcal{M}$. We define \hat{g} by:

$$\mathbf{V} \text{ and } \mathbf{H} \text{ are } \hat{g}\text{-totally isotropic and, if } (X, Y) \in \mathbf{V} \times \mathbf{H}, \hat{g}(X, Y) = g(X', d\pi(Y)).$$

Notice that N is \hat{g} -self adjoint. Moreover $\hat{D}N = 0$, with \hat{D} the Levi-Civita connection of \hat{g} .

Proof. We must show that, if X, Y, Z are vector fields tangent to $T\mathcal{M}$, $\hat{g}(\hat{D}_X NY, Z) = \hat{g}(\hat{D}_X Y, NZ)$. Let us show it at an arbitrary point $m \in T\mathcal{M}$. Around m , we build a frame field β on $T\mathcal{M}$ as follows. We take normal coordinate vector fields $(U_i)_{i=1}^d$ at $\pi(m)$ on \mathcal{M} . We lift them horizontally on $T\mathcal{M}$, getting a frame field $\beta_{\mathbf{H}}$ of \mathbf{H} . Besides we denote by $\beta_{\mathbf{V}}$ the frame field of \mathbf{V} such that $\beta'_{\mathbf{V}} = (U_i)_{i=1}^d$. In other words, $\beta_{\mathbf{V}} = N \cdot \beta_{\mathbf{H}}$. We set $\beta := (\beta_{\mathbf{V}}, \beta_{\mathbf{H}})$. In the following, X, Y, Z denote vectors of β . Notice that, by construction, $\beta_{\mathbf{V}}$ is constant along each fibre of $T\mathcal{M}$, so $[X, Y] = 0$ if $X, Y \in \beta_{\mathbf{V}}$; besides $[X, Y] \in \mathbf{V}$ if $X, Y \in \beta_{\mathbf{H}}$, as they are lifts of commuting fields on \mathcal{M} . Finally, by definition of \mathbf{H} , if $X \in \mathbf{H}$

and $Y \in \mathbf{V}$, $[X, Y] \in \mathbf{V}$ and $[X, Y]' = D_{d\pi(X)}Y'$. As the $(U_i)_{i=1}^d$ are normal coordinate vectors at $\pi(m)$, for all (i, j, k) , $U_i \cdot (g(U_j, U_k))$ is null at $\pi(m)$; combined with the definition of \widehat{g} , it gives that for any X, Y, Z in β , $X \cdot \widehat{g}(Y, Z)$ is also null at m . Then, if $X, Y, Z \in \beta$:

$$\begin{aligned} 2\widehat{g}(\widehat{D}_X NY, Z) &= X \cdot \widehat{g}(NY, Z) + NY \cdot \widehat{g}(NY, Z) - Z \cdot \widehat{g}(X, NY) \\ &\quad - \widehat{g}(X, [NY, Z]) - \widehat{g}(NY, [X, Z]) + \widehat{g}(Z, [X, NY]) \\ &\stackrel{(\text{at } m)}{=} -\widehat{g}(X, [NY, Z]) - \widehat{g}(NY, [X, Z]) + \widehat{g}(Z, [X, NY]). \end{aligned}$$

If $Y \in \beta_{\mathbf{V}}$, $NY = 0$ so this vanishes. If $Z \in \beta_{\mathbf{V}}$ (or if $X \in \beta_{\mathbf{V}}$, left to the reader) then:

- as $NY \in \beta_{\mathbf{V}}$, $[NY, Z] = 0$ so $\widehat{g}(X, [NY, Z]) = 0$,
- $[X, Z] \in \mathbf{V}$ so $\widehat{g}(NY, [X, Z]) \in \widehat{g}(\mathbf{V}, \mathbf{V}) = \{0\}$,
- as $NY \in \beta_{\mathbf{V}}$, $[X, NY] \in \mathbf{V}$ so $\widehat{g}(Z, [X, NY]) \in \widehat{g}(\mathbf{V}, \mathbf{V}) = \{0\}$.

So $(X \in \beta_{\mathbf{V}}$ or $Y \in \beta_{\mathbf{V}}$ or $Z \in \beta_{\mathbf{V}}) \Rightarrow g|_m(\widehat{D}_X NY, Z) = 0$. If $X, Y, Z \in \beta_{\mathbf{H}}$, at m , $\widehat{g}(\widehat{D}_X NY, Z)$ is equal to:

$$\frac{1}{2} (g(d\pi(X), D_{d\pi(Z)} d\pi(Y)) - 0 + g(d\pi(Z), D_{d\pi(X)} d\pi(Y))) = 0$$

as the $(U_i)_i$ are normal at m . Now similarly, at m , $2\widehat{g}(\widehat{D}_X Y, NZ) = -\widehat{g}(X, [Y, NZ]) - \widehat{g}(Y, [X, NZ]) + \widehat{g}(NZ, [X, Y])$, which also vanishes. So at any m , $\widehat{D}_X NY = N\widehat{D}_X Y$ *i.e.* $\widehat{D}N = 0$, Q.E.D.

We could build a similar \widehat{g} on any jet bundle $J^n \mathcal{M}$, making the N of Ex. 2.26 parallel.

3 Metrics such that $\text{End}(T\mathcal{M})^{\mathfrak{h}}$ has an arbitrary semi-simple part and a non trivial radical

In section 1, §1.4, we built metrics such that the semi-simple part \mathfrak{s} of $\text{End}(T\mathcal{M})^{\mathfrak{h}}$ is in each of the eight possibilities listed by Theorem 1.10. Then we built in section 2 metrics admitting an arbitrary self adjoint nilpotent structure as a parallel endomorphism; in particular, this makes in general the radical \mathfrak{n} of $\text{End}(T\mathcal{M})^{\mathfrak{h}}$ non trivial. Mixing here both arguments, we build metrics with \mathfrak{s} arbitrary in the list of Theorem 1.10, and whose holonomy commutes moreover with some arbitrary self adjoint nilpotent endomorphism N . We considered only the case where N is in the commutant of \mathfrak{s} . It is natural: it means that N is a complex endomorphism if $\mathfrak{s} = \langle J \rangle$ induces a complex structure *etc.*, see Remark 3.4. Besides by Proposition 1.8 and as we suppose N self adjoint, the non commuting case is strongly constrained.

This builds metrics whose holonomy group is the commutant of N in each of the holonomy groups given in Remark 1.15.

The principle is roughly the following. We repeat the constructions recalled in §1.4, on a manifold \mathcal{M} with a nilpotent structure N , replacing everywhere real differentiable or complex holomorphic functions by nilomorphic $\mathbb{R}[\nu]$ -valued, or nilomorphic+holomorphic $\mathbb{C}[\nu]$ -valued ones. This gives Theorem 3.2. To state it, if J (or L) and N are commuting (para)complex and nilpotent structures, we need integral coordinates for both J (or L) and N . They exist, this is Lemma 3.1, proven on p. 39.

3.1 Lemma (*small enhancement of Lemma 2.30*) *Suppose that \mathbb{R}^{2d} is endowed with a (para)complex structure J (or L) and a nilpotent structure N commuting with it. Coordinates are called here adapted to J (or L) if they make $\text{Mat}(J)$ (or $\text{Mat}(L)$) constant J_1 - (or*

L_1 -) block diagonal. Then there is a coordinate system simultaneously adapted to J (or L) and integral for N if and only if $\mathcal{N}_N = 0$ and the $\ker N^k$ are involutive for all k . This holds also with complex coordinates if all those endomorphisms are \mathbb{C} -linear on \mathbb{C}^{2d} .

3.2 Theorem (Corollary and generalisation of Theorem 2.31) *Let $H_{\mathfrak{s}}$ be the generic holonomy group corresponding to an algebra \mathfrak{s} in any of the eight cases of Theorem 1.10, and N any self adjoint nilpotent endomorphism in the commutant of \mathfrak{s} i.e. the bicommutant of $H_{\mathfrak{s}}$. We denote by $\mathcal{G}_{H_{\mathfrak{s}}^N}$ the set of germs of metrics whose holonomy group H is included in the commutant $H_{\mathfrak{s}}^N$ of N in $H_{\mathfrak{s}}$. If $g \in \mathcal{G}_{H_{\mathfrak{s}}^N}$, then N extends as a parallel endomorphism, in particular as a nilpotent structure. Besides $\text{End}(T\mathcal{M})^{\mathfrak{h}} \supset \langle \mathfrak{s} \cup \{N\} \rangle$.*

(a) For \mathfrak{s} in each case of Theorem 1.10, $g \in \mathcal{G}_{H_{\mathfrak{s}}^N}$ if and only if:

– In case (1^C), g is the real metric associated with the real part of a non degenerate, $\mathbb{C}[\nu]$ -valued, $\mathbb{C}[\nu]$ -bilinear metric h , which is both holomorphic and nilomorphic on $(\mathcal{M}, \underline{J}, N)$.

– In case (2), and in N -adapted coordinates $(x_i, (y_{i,a})_{a=1}^{n(i)-1})_{i=1}^D$ such that $J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$ for any odd i , given by Lemma 3.1, g is the real metric associated with a non degenerate nilomorphic metric h given by:

$$\begin{aligned} h\left(\frac{\partial}{\partial w_i}, \frac{\partial}{\partial \bar{w}_j}\right) &= \frac{\partial^2 u}{\partial w_i \partial \bar{w}_j}, \quad \text{with } u \text{ an } \mathbb{R}[\nu]\text{-valued nilomorphic function,} \\ &\quad \text{and } \frac{\partial}{\partial w_{(j+1)/2}} := \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial x_{j+1}} \in \mathbb{T}^{1,0} \mathbb{R}^d \\ &\quad \text{for all odd } j, \\ &= \sum_{\alpha} \frac{\partial^{|\alpha|}}{\partial x_{\alpha}} \left(\frac{\partial^2 \check{u}}{\partial w_i \partial \bar{w}_j} \right) (\nu y)^{\alpha} \quad \text{with } \check{u} \text{ some } \mathbb{R}[\nu]\text{-valued} \\ &\quad \text{adapted function of} \\ &\quad \text{the coordinates } (x_i)_i. \end{aligned}$$

– In case (2'), and in N -adapted coordinates such that $L \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$ for any odd i , given by Lemma 3.1, g is the real metric associated with a non degenerate nilomorphic metric h given by:

$$\begin{aligned} h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{j+1}}\right) &= \frac{\partial^2 u}{\partial x_i \partial x_{j+1}} \quad \text{for } i, j \text{ odd, with } u \text{ some } \mathbb{R}[\nu]\text{-valued} \\ &\quad \text{nilomorphic function,} \\ &= \sum_{\alpha} \frac{\partial^{|\alpha|}}{\partial x_{\alpha}} \left(\frac{\partial^2 \check{u}}{\partial x_i \partial x_{j+1}} \right) (\nu y)^{\alpha} \quad \text{with } \check{u} \text{ some } \mathbb{R}[\nu]\text{-valued} \\ &\quad \text{adapted function of} \\ &\quad \text{the coordinates } (x_i)_i. \end{aligned}$$

– In case (2^C), and in complex N -adapted coordinates such that $J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$, or $L \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$, for i odd, g is the real part of the complex metric associated with a non degenerate complex nilomorphic metric h given by the complexifications, which are equivalent, of any of the two formulas given in the previous cases. The potential u is then a $\mathbb{C}[\nu]$ -valued holomorphic and nilomorphic function.

– In cases (3), (3') and (3^C), and in the real analytic category, it is the solution of the exterior differential system given in §1.4, formulated in nilomorphic coordinates and with $\mathbb{R}[\nu]$ - or $\mathbb{C}[\nu]$ -valued nilomorphic functions instead of real or complex ones. In particular, in cases (3) and (3'), the elements of $\mathcal{G}_{H_{\mathfrak{s}}^N}$, considered up to diffeomorphism, are parametrised by $b \frac{D}{2}$ real analytic functions of $\frac{D}{2} + 1$ variables, where $b := \min\{a \in \llbracket 1, n \rrbracket; d_a \neq 0\}$, with D

the number of invariant factors of N and d_a the number of repetitions of X^a among them. In case $(\mathbf{3}^{\mathbb{C}})$, they are by $b\frac{D}{4}$ holomorphic functions of $\frac{D}{4} + 1$ complex variables.

(b) In a dense open subset for the C^2 topology in $\mathcal{G}_{H_s^N}$ (inside of the real analytic category in cases $(\mathbf{3})$ – $(\mathbf{3}')$ – $(\mathbf{3}^{\mathbb{C}})$), the holonomy group H of the metric is exactly H_s^N . So those commutants are holonomy groups.

3.3 Remark For the meaning of D and the d_a , see also Notation 2.1.

3.4 Remark When does a nilpotent endomorphism N commute with \mathfrak{s} ? Exactly when $\text{Id} + N$ does. Now by definition, the automorphisms commuting with \mathfrak{s} are those preserving the G -structure defined by the commutant of \mathfrak{s} . In cases $(\mathbf{1}^{\mathbb{C}})$ and $(\mathbf{2})$, they are exactly the J - or \underline{J} -complex automorphisms, and in case $(\mathbf{2}')$, the paracomplex ones *i.e.* $\left\{ \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \right.$ with $U, V \in \text{GL}_{d/2}(\mathbb{R})$. In case $(\mathbf{3})$, they are the (J_1, J_2, J_3) -quaternionic automorphisms. In case $(\mathbf{3}')$, if $\text{Mat}(L) = I_{d/2, d/2}$ and $\text{Mat}(J) = J_{d/2}$, they are $\left\{ \begin{pmatrix} U & 0 \\ 0 & -J U J \end{pmatrix} \right.$ with $U \in \text{GL}_{d/2}(\mathbb{R})$. Cases $(\mathbf{2}^{\mathbb{C}})$ and $(\mathbf{3}^{\mathbb{C}})$ are the complexification of cases $(\mathbf{2})$ – $(\mathbf{2}')$ and $(\mathbf{3})$ – $(\mathbf{3}')$.

Now, in the different cases of Theorem 3.2, Corollary 3.5 gives the form of $\mathfrak{e} := \text{End}(T\mathcal{M})^{\mathfrak{h}}$, and notably that of its radical \mathfrak{n} , asked for in the introduction. Here I thank the referee for his noticing a mistake in my first version. The commutant $\mathfrak{c} := \text{End}(T\mathcal{M})^{(H_s^N)}$ of H_s^N contains \mathfrak{s} and the bicommutant $N^{\text{cc}} := \text{End}(T\mathcal{M})^{(\text{End}(T\mathcal{M})^N)}$ of N (classically equal to $\mathbb{R}[N]$). In fact $\mathfrak{c} = \langle \mathfrak{s} \cup \{N\} \rangle$, except a bit surprisingly in some sub-cases of $(\mathbf{1})$ and $(\mathbf{1}^{\mathbb{C}})$. This could be expected: look at the lowest possible dimension in the case $N = 0$. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathfrak{o}(g)^{\mathfrak{s}} = \mathfrak{o}_1(\mathbb{K}) = \{0\}$ is trivial in cases $(\mathbf{1})$ – $(\mathbf{1}^{\mathbb{C}})$, so $\text{End}_{\mathbb{K}}(T\mathcal{M}) = \mathfrak{c} \supseteq \mathfrak{o}(g)$, whereas in the other cases $\mathfrak{o}(g)^{\mathfrak{s}} \neq \{0\}$, *e.g.* in case $(\mathbf{2})$, $\text{End}(T\mathcal{M}) \supseteq \mathfrak{c} = \mathfrak{u}_1 = \langle J \rangle$.

Corollary 3.5 gives the details. It follows from a sequence of results in linear algebra, gathered at the end of this section, pp. 45 *sq.* To know the general form of the matrices at stake in it, or related to algebras involved in this article — commutants *etc.* —, notably in the delicate cases $(\mathbf{1})$ and $(\mathbf{1}^{\mathbb{C}})$, see Lemma 3.20. Replace in it the real coefficients by complex ones to get cases $(\mathbf{1}^{\mathbb{C}})$ – $(\mathbf{2}^{\mathbb{C}})$ – $(\mathbf{3}^{\mathbb{C}})$.

3.5 Corollary Suppose that H is the holonomy group of a metric g and that $H \subset H_s^N$ (see Theorem 3.2). Then immediately, $\text{End}(T\mathcal{M})^H \supset \text{End}(T\mathcal{M})^{H_s^N}$, with equality if $H = H_s^N$ (which holds generically). Now:

(a) If \mathfrak{s} is not in case $(\mathbf{1})$ or $(\mathbf{1}^{\mathbb{C}})$ of Theorem 1.10 *i.e.* $\mathfrak{s} \neq \mathbb{R} \cdot \text{Id}$ and $\mathfrak{s} \neq \langle \underline{J} \rangle$ with \underline{J} a self adjoint complex structure then:

$$\text{End}(T\mathcal{M})^{H_s^N} = \langle \mathfrak{s} \cup \{N\} \rangle = \mathfrak{s} \oplus (N),$$

so if $H = H_s^N$, (\mathcal{M}, g) is indecomposable. The sum $\mathfrak{s} \oplus (N)$ is the decomposition of $\langle \mathfrak{s} \cup \{N\} \rangle$ into a semi-simple part and its radical; (N) is the radical spanned by N in $\text{End}(T\mathcal{M})^{H_s^N}$.

(b) If \mathfrak{s} is in case $(\mathbf{1})$ or $(\mathbf{1}^{\mathbb{C}})$ of Theorem 1.10, set $n = n_1 \geq n_2 \geq \dots \geq n_D$ the sizes of the Jordan blocks of N — viewed as a \underline{J} -complex endomorphism in case $(\mathbf{1}^{\mathbb{C}})$. By convention, we set $n_i = 0$ for $i > D$.

(b1) If $n_1 > 2n_2$, locally, $(\mathcal{M}, g) \simeq (\mathcal{M}', g') \times (\mathbb{R}^{n_1-2n_2}, g_{\text{flat}})$ is decomposable. The sizes of the Jordan blocks of $N' := N|_{T\mathcal{M}'}$ are $(2n_2, n_2, \dots, n_D)$. So the situation on (\mathcal{M}', g') is described by **(b2)**.

(b2) Else, set $\mathbb{K} := \mathbb{R}$ or $\mathbb{K} := \mathbb{R}[\underline{J}] \simeq \mathbb{C}$, so that $\mathbb{K} \simeq \mathfrak{s}$. Set $\pi' : T\mathcal{M} \rightarrow T\mathcal{M}/\ker N^{n_2}$, $\pi'' : T\mathcal{M} \rightarrow T\mathcal{M}/\ker N^{n_3}$ and $g_{n_3} := g(\cdot, N^{n_3} \cdot)$ [or its complexification in case $\mathbb{K} = \mathbb{C}$], defined on $\pi''(T\mathcal{M})$. Then:

$$\begin{aligned} \text{End}(T\mathcal{M})^{H_{\mathfrak{s}}^N} &= (\mathbb{K}[N] + (N^{n_2})_* \pi'^* \text{End}_{\mathbb{K}}(T\mathcal{M}/\ker N^{n_2})) \\ &\quad \oplus (N^{n_3})_* \pi''^* \mathfrak{o}_{\mathbb{K}}(g_{n_3})^N \end{aligned}$$

where $\varphi_* \psi^* A := \{\varphi \circ f \circ \psi, f \in A\}$. So (\mathcal{M}, g) is indecomposable if $H = H_{\mathfrak{s}}^N$. A semi-simple part of $\text{End}(T\mathcal{M})^{H_{\mathfrak{s}}^N}$ is $\mathfrak{s} = \mathbb{K} \cdot \text{Id} \subset \mathbb{K}[N]$ and its radical is $(N\mathbb{K}[N] + (N^{n_2})_* \pi'^* \text{End}_{\mathbb{K}}(T\mathcal{M}/\ker N^{n_2})) \oplus (N^{n_3})_* \pi''^* \mathfrak{o}_{\mathbb{K}}(g_{n_3})^N$.

If $n_1 = n_2$ and $n_3 = 0$, $\pi' = 0$ and $\pi'' = \text{Id}$ so: $\text{End}(T\mathcal{M})^{H_{\mathfrak{s}}^N} = \mathbb{K}[N] \oplus \mathfrak{o}_{\mathbb{K}}(g)^N$. In this case, $\mathfrak{o}_{\mathbb{K}}(g)^N$ contains J or L a skew adjoint (para)complex structure commuting with N , so semi-simple part \mathfrak{s}' of $\text{End}(T\mathcal{M})^{H_{\mathfrak{s}}^N}$ is not $\mathfrak{s} \simeq \mathbb{K} \text{Id}$, but $\mathfrak{s}' = \mathbb{K} \cdot J$ or $\mathfrak{s}' = \mathbb{K} \cdot L$, $\text{End}(T\mathcal{M})^{H_{\mathfrak{s}}^N} = \mathfrak{s}' \oplus (N)$ and we are respectively in case **(2)** or **(2')** of Table 1 if $\mathbb{K} = \mathbb{R}$, and in case **(2^C)** if $\mathbb{K} = \mathbb{C}$.

Moreover, the signature of the flat metric g_{flat} appearing in the first point is $(\frac{d''}{2}, \frac{d''}{2})$ or $(\frac{d'' \pm 1}{2}, \frac{d'' \mp 1}{2})$ with $d'' := n_1 - 2n_2$, according to the cases given in Proposition 3.22.

Proof. Apply Proposition 3.22, complexified for cases **(1^C)**–**(2^C)**–**(3^C)**. For the last, exceptional case, see the second point of Remark 3.9. q.e.d.

3.6 Important Remark Corollary 3.5 shows that in Theorem 3.2, generic metrics are indecomposable, except in cases **(1)**–**(1^C)** when $n_1 > 2n_2$.

3.7 Remark In Corollary 3.5 **(b2)**, the situation is like in **(a)** i.e. $\text{End}(T\mathcal{M})^{H_{\mathfrak{s}}^N} = \mathbb{K}[N] = \mathbb{K} \text{Id} \oplus (N)$ if and only if $\pi' = \pi'' = 0$ i.e. $n_1 = n_2 = n_3$ i.e. N has at least three Jordan blocks of maximal size.

3.8 Remark The cases where $T\mathcal{M}/\ker N^{n_2} \neq \{0\}$ i.e. $(n =)n_1 > n_2$ (considering, in case **(1^C)**, the complex Jordan blocks) are exactly those where the commutation with N (or with $\{N, \underline{J}\}$ in case **(1^C)**) forces the holonomy group to act trivially on a non null subspace, namely $\text{Im } N^{n_2}$.

3.9 Remark Some very elementary cases turn out to be “exceptional” in the classification of Corollary 3.5.

– If N consists of only one Jordan block, so of order $n = d > 0$, then immediately $\mathfrak{h} \subset \mathfrak{o}(g)^N = \{0\}$, do the calculation or use Lemma 3.20 **(b)** and **(c)**. So the metric is flat and $\text{End}(T\mathcal{M})^{\mathfrak{h}} = \text{End}(T\mathcal{M})$. This is case **(1)** with $n_2 = 0$ so $2n_1 > n_2$, and $\text{End}_{\mathbb{R}}(T\mathcal{M}/\ker N^{n_2}) = \text{End}(T\mathcal{M})$.

– If N consists of exactly two Jordan blocks, of the same size $n = \frac{d}{2} > 0$, then a skew adjoint complex structure J , if $\text{sign}(g_{n-1}) \in \{(2, 0), (0, 2)\}$, or paracomplex structure L , if $\text{sign}(g_{n-1}) = (1, 1)$, commuting with N , is also parallel, and we are in case **(2)** or **(2')**. Then N , as a (para)complex endomorphism, has one single Jordan block and $\text{End}(T\mathcal{M})^{\mathfrak{h}} = \mathfrak{s}\mathbb{R}[N]$

with $\mathfrak{s} = \langle J \rangle$ or $\mathfrak{s} = \langle L \rangle$. Apply case (1) of Corollary 3.5 with $n_1 = n_2 > 0$ and $n_3 = 0$; then $g_{n_3} = g$ and $\text{End}(T\mathcal{M})^{\mathfrak{h}} = \mathbb{R}[N] \oplus \mathfrak{o}(g)^N$.

– This does not go on: if N consists of $k > 3$ Jordan blocks of the same size $n = \frac{d}{k} > 0$, $\text{End}(T\mathcal{M})^{\mathfrak{h}} = \mathbb{K}[N]$ is the bicommutant of N .

– All this appears in the case $N = 0$. For g generic, $\text{End}(T\mathcal{M})^{\mathfrak{h}} = \text{End}(T\mathcal{M})^{\mathfrak{o}(g)}$. If $d \geq 2$, $\mathfrak{o}(g)$ is abelian and $\text{End}(T\mathcal{M})^{\mathfrak{h}}$ is the (Euclidian or Lorentzian) conformal group. If $d > 2$, the commutant of $\mathfrak{o}(g)$ is trivial, $\text{End}(T\mathcal{M})^{\mathfrak{h}} = \mathbb{R} \cdot \text{Id}$. You may see this difference appear in case (1), $(n_1, n_2, n_3) = (1, 1, 0)$ if $d = 2$ and $(n_1, n_2, n_3) = (1, 1, 1)$ if $d > 2$.

Proof of Lemma 3.1. We do it in the real case. The “only if” is immediate. The converse is immediate in the case of J : repeat the proof of [29] with the field \mathbb{C} replacing \mathbb{R} : $(\mathbb{R}^{2d}, J) \simeq \mathbb{C}^d$, so work with complex coordinates. For the case of L , or of J for an alternative proof, we have to check that the proof works with L -adapted coordinates at each step (on p. 610 of [29]). The author builds the coordinates by induction on the nilpotence index n of N . For $n = 1$ the result is empty hence true. If it holds for index $n - 1$ and if $N^{n-1} \neq N^n = 0$, as $\ker N$ is L -invariant we may find coordinates $((x_i)_i, (y_i)_i)$ that are:

- L -adapted *i.e.*: $L \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$ and $L \frac{\partial}{\partial y_i} = \frac{\partial}{\partial y_{i+1}}$ for any odd i ,
- such that the $(y_i)_i$ parametrise the leaves of \mathcal{K} .

Then the induction assumption applies on $\mathbb{R}^{2d}/\mathcal{K}$, providing coordinates $(x_i)_i$ of the wished type on $\mathbb{R}^{2d}/\mathcal{K}$. As the fields $(N \frac{\partial}{\partial x_i})_i$ commute with each other, [29] extends these coordinates to the whole \mathbb{R}^{2d} , obtaining N -adapted coordinates $((x_i)_i, (\bar{y}_i)_i)$. Here, we need moreover to check that they are also L -adapted *i.e.* (i) $L \left(\frac{\partial}{\partial \bar{x}_i} \right) = \frac{\partial}{\partial \bar{x}_{i+1}}$ and (ii) $L \left(\frac{\partial}{\partial \bar{y}_i} \right) = \frac{\partial}{\partial \bar{y}_{i+1}}$ for i odd. We follow [29]: the $\frac{\partial}{\partial \bar{x}_i}$ are equal to some $N \left(\frac{\partial}{\partial x_j} \right)$ if they are in $\text{Im } N$, else they are equal to $\frac{\partial}{\partial x_i}$. As $LN = NL$, this gives (i). The $\frac{\partial}{\partial \bar{y}_i}$ are equal to some $N \left(\frac{\partial}{\partial x_k} \right)$ if they are in $\text{Im } N$, else they are chosen freely. As $LN = NL$, this gives (ii). q.e.d.

3.10 Remark (*This will be used in Part 4*) The key properties used in the proof are that J , or L , has a constant matrix in the basis $(\frac{\partial}{\partial x_i})_i$, and that it commutes with N . So the same proof, and result, hold for any integrable field of endomorphism playing the role of J or L .

To show the theorem we need the following remarks.

3.11 Remark (natural matricial form for an $\mathbb{R}[\nu]$ -bilinear alternate 2-form) Using Notation 2.1 and 2.2, we adapt 2.3, 2.4 and 2.5 for ω an $\mathbb{R}[\nu]$ -bilinear *alternate* 2-form on $E = (\mathbb{R}^d, N)$.

$$\omega = \sum_{a=0}^{n-1} \nu^a \omega_a \quad \text{with } \omega_{n-1} \text{ such that } \omega_{n-1}(\cdot, N \cdot) = \omega_{n-1}(N \cdot, \cdot) \\ \text{and for any } a, \omega_a = \omega_{n-1}(\cdot, N^{n-1-a} \cdot).$$

For $a \in \llbracket 1, n \rrbracket$ we denote by $r_a \in 2\mathbb{N}$ the rank of $\omega_{n-a} := \omega_{n-1}(\cdot, N^{a-1} \cdot)$ defined on $\ker N^a / \ker N^{a-1}$. It is standard that the couple (N, ω_{n-1}) on \mathbb{R}^d is characterised up to conjugation by the $(r_a)_{a=1}^n$, called here the *ranks of ω* . See *e.g.* [21] like in 2.3.

If $\beta = (X_i)_{i=1}^D$ is an adapted spanning family (see 2.2) of E , $\text{Mat}_\beta(\omega) = \sum_{a=0}^{n-1} \nu^a \Omega_a \in \text{M}_D(\mathbb{R}[\nu])$ where:

(i) $\Omega_a = \begin{pmatrix} 0 & 0 \\ 0 & \check{\Omega}^a \end{pmatrix}$, the upper left null square block, of size D_{n-1-a} , corresponding to $\text{span}_{\mathbb{R}[\nu]} \{X_i; N^{n-1-a}X_i = 0\}$,

(ii) the upper left block $\check{\Omega}_0^a$ of $\check{\Omega}^a$ of size d_{n-a} , corresponding to $\text{span}_{\mathbb{R}[\nu]} \{X_i; N^{n-1-a}X_i \neq N^{n-a}X_i = 0\}$ is of rank r_{n-a} . So if $S \oplus \text{Im } N = E$, r_a is the rank of the (well defined) form ω_{n-a} on the quotient $(S \cap \ker N^a)/(S \cap \ker N^{a-1})$.

For $r \leq \delta$ and r even, we denote by $J_{\delta,r/2}$ the matrix $\text{diag}(J_{r/2}, 0) \in M_\delta(\mathbb{R}[\nu])$. There are adapted spanning families $\beta = (X_i)_{i=1}^D$ of E such that $\check{\Omega}_a$ is null except $\check{\Omega}_0^a = J_{d_a, r_a}$, for all a i.e.:

$$\begin{aligned} \text{Mat}_\beta(\omega) &= \text{diag}(\nu^{n-a} J_{d_a, r_a/2})_{a=1}^n \\ &= \begin{pmatrix} \nu^{n-1} J_{d_1, r_1/2} & & \\ & \ddots & \\ & & \nu^0 J_{d_n, r_n/2} \end{pmatrix} \in M_D(\mathbb{R}[\nu]). \end{aligned}$$

Each block $\nu^{n-a} J_{d_a, r_a/2}$ corresponds to the factor $(\nu^{n-a} \mathbb{R}[\nu])^{d_a} = \text{span}((X_i)_{D_{a-1} < i \leq D_a})$ of E . The form ω is non degenerate if and only if each ω_a is i.e. $r_a = d_a$ for all a .

3.12 Lemma *The Poincaré and Darboux lemmas admit a natural “nilomorphic” version. For example, if B is some ball in (\mathbb{R}^d, N) with N nilpotent, in constant Jordan form:*

(a) *If $\lambda \in \Lambda_{\mathbb{R}[\nu]}^k(B)$ with $p > 0$ is a closed nilomorphic k -form on B then there is a nilomorphic $\alpha \in \Lambda_{\mathbb{R}[\nu]}^{k-1}(B)$ such that $\lambda = d\alpha$.*

(b) *If $\omega \in \Lambda_{\mathbb{R}[\nu]}^2(B)$ is closed and has constant ranks, there exist nilomorphic coordinates on B in which $\text{Mat}(\omega)$ has the (constant) form given at the end of Remark 3.11.*

Proof. (a) Classically, if λ is a closed real form and X a vector field on B whose flow $(\varphi^t)_{t \in [0, +\infty[}$ is a retraction of B on a point, then:

$$\alpha := \int_0^\infty \iota_X (\varphi^{t*} \lambda) dt$$

fits. In our case, use the retraction $(e^{-t} \text{Id}_B)_{t \in [0, +\infty[}$ generated by $X = -\text{Id}_{\mathbb{R}^d}$; φ^t and X being nilomorphic, so is the obtained integral form α .

(b) As ω is closed and has constant ranks, its kernel integrates in some foliation \mathcal{F} and N acts on the quotient B/\mathcal{F} . So we may suppose that ω is non degenerate. Then we use Moser’s path method. We set ω_0 the constant nondegenerate 2-form on \mathbb{R}^d , given in Remark 3.11. To simplify, we suppose that $\omega_t := \omega_0 + t(\omega - \omega_0)$ never degenerates. Else, iterate the method along an adequate piecewise affine path from ω_0 to ω . We want to build a nilomorphic homotopy $(\varphi^t)_{t \in [0, 1]}$ such that:

$$\varphi_t^* \omega_t = \omega_0. \tag{*}$$

Once this is done, $\varphi_1^* \omega = \omega_0$ and $\varphi_1^* N = N$ as we want. Let X_t be the field such that $X_t(\varphi^t(p)) = \frac{d}{dt} \varphi^t(p)$, then (*) amounts to:

$$\varphi^{t*} \left(\mathcal{L}_{X_t} \omega_t + \frac{d}{dt} \omega_t \right) = 0$$

i.e., as $d\omega_t = 0$, $d(\iota_{X_t}\omega_t) + \omega = 0$. By **(a)**, there is a *nilomorphic* 1-form λ on B with $\omega = d\lambda$. Then it is sufficient to find X_t with $\iota_{X_t}\omega_t + \lambda = 0$. As ω_t is nondegenerate, this defines indeed X_t ; as ω_t and λ are nilomorphic, so is X_t . q.e.d.

3.13 Remark An infinitesimal deformation of nilomorphic Darboux coordinates for a non degenerate ω is a nilomorphic field X such that $0 = \mathcal{L}_X\omega = d(\iota_X\omega)$. So classically, X is the symplectic gradient of some (nilomorphic) potential f . By Prop. 2.23, f is given by its “adapted” restriction $f = \sum_a f_a \nu^a$ to $\mathcal{T} = \{(\nu y) = 0\}$ *i.e.* by one real function (f_{n-1}) of $D - D_0 = D$ variables, one (f_{n-2}) of $D - D_1$ variables *etc.*, and one (f_0) of $D - D_{n-1}$ variables. Set $b := \min\{a \in \llbracket 1, n \rrbracket; d_a \neq 0\}$. Then $0 = D_0 = \dots = D_{b-1}$. So f_{n-1} up to f_{n-b} depend, each, on $D - D_0 = D$ variables — strictly more variables than any other f_c . So the Darboux coordinates depend on b functions of D variables.

In passing, we add the following. After Rem. 2.3, on each leaf of the (quotient) foliation $\pi(\mathcal{K}^{a+1})/\pi(\mathcal{K}^a)$, the (real) symplectic form $\omega(\cdot, N^a \cdot)$ is well defined. Choosing nilomorphic Darboux coordinates means in particular choosing Darboux coordinates for them, but also choosing a transversal $\mathcal{T} = \{(\nu y) = 0\}$ to \mathcal{I} such that for all a :

- the orthogonal distribution to $\mathcal{T} \cap \mathcal{K}^a$ with respect to $\omega(\cdot, N^{a-1} \cdot)$ is integrable,
- its intersection with \mathcal{T} is totally isotropic.

In fact, it amounts exactly to both these choices.

Proof of Theorem 3.2. The fact that N extends as a nilpotent structure was given by Lemma 2.30 in §2.3. To prove **(a)** and **(b)**, we begin with cases **(3)** and **(3')** — case **(3^C)** is only their complexification.

Part (a) We follow the line of §1.4. The triple (J, U, N) is given and we look for a quadruple (g, J, U, N) . This is equivalent to a quadruple $(J, N, \tilde{\omega}_0, \tilde{\omega})$ where:

$$\tilde{\omega}_0 := \sum_{a=0}^{n-1} \omega_0(\cdot, N^{n-1-a} \cdot) \nu^a$$

is the (J) -complex nilomorphic symplectic 2-form associated with the complex symplectic 2-form $\omega_0 := g(\cdot, U \cdot) + ig(\cdot, JU \cdot)$, and where:

$$\tilde{\omega} := \sum_{a=0}^{n-1} \omega(\cdot, N^{n-1-a} \cdot) \nu^a$$

is the symplectic nilomorphic (1,1)-form associated with $\omega := \omega_0(\cdot, U \cdot)$. As $UN = NU$, $\tilde{\omega} = \tilde{\omega}_0(\cdot, U \cdot)$. By Lemma 3.12 **(b)**, we may use local coordinates $(x_j, (y_{j,a})_a)_{j=1}^D$ adapted to N and such that for all odd j , $J \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_{j+1}}$. As usual, we set $z_j := x_j + (\nu y_j) \in \mathbb{R}[\nu]$ for all j and introduce the “complex and nilomorphic” local coordinates $(w_j)_{j=1}^{D/2}$ by:

$$w_{(j+1)/2} := z_j + iz_{j+1} \in \mathbb{C}[\nu].$$

Through those coordinates, we consider that we are in some ball \mathcal{B} of $(\mathbb{C}^{d/2}, N)$ with N a complex endomorphism field. The matrix Ω_0 of ω_0 as a $\mathbb{C}[\nu]$ -bilinear form is as at the end of Remark 3.11, with non degenerate $J_{d_a, r_a/2} = J_{d_a}$. In the complexification $T^{\mathbb{C}}\mathcal{B} = T\mathcal{B} \otimes \mathbb{C}$ of the tangent bundle, we introduce also, for j odd, the vector fields $\frac{\partial}{\partial w_{(j+1)/2}} := \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial x_{j+1}}$

and $\frac{\partial}{\partial \bar{w}_{(j+1)/2}} := \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial x_{j+1}}$. We introduce the matrix $V = (v_{i,j})_{i,j=1}^{D/2} \in M_{D/2}(\mathbb{C}[\nu])$ of the ν -linear, J -antilinear morphism U by:

$$U \left(\frac{\partial}{\partial w_i} \right) = \sum_{j=1}^{D/2} v_{i,j} \frac{\partial}{\partial \bar{w}_j} \quad \text{i.e., if } v_{i,j} = \sum_{a=0}^{n-1} v_{i,j,a} \nu^a, \text{ then:}$$

$$\forall b \in \llbracket 0, n-1 \rrbracket, U \left(N^b \frac{\partial}{\partial w_i} \right) = \sum_{j,a} v_{i,j,a} N^{b+a} \frac{\partial}{\partial \bar{w}_j}.$$

Notice that, as $N^a \frac{\partial}{\partial \bar{w}_j} = 0$ if $a \geq n(j)$, $\deg_\nu v_{i,j} < n(j)$ for all i, j . In other terms, in the block decomposition of V corresponding to the flag of the $\pi(\ker N^a)$, i.e. to the blocks of $\text{Mat}(\omega)$ given in Remark 3.11, the ν -degree of the a^{th} line of blocks is strictly less than a .

We keep following §1.4. Let dw be the column $(dw_j)_{j=1}^{D/2}$, then:

$$\omega_0 = {}^t dw \wedge \Omega_0 \wedge dw \quad \text{and, setting } H := -\Omega_0 V, \quad \omega = \frac{i}{2} {}^t dw \wedge H \wedge d\bar{w}.$$

Here, notice that $\Phi : V \mapsto -\Omega_0 V = H$ is injective. Indeed, it associates with the matrix V of U , the matrix H of the form $\omega_0(\cdot, U \cdot)$, and ω_0 is non degenerate. An alternative, computational argument is the fact that, as $\deg_\nu v_{i,j} < n(j)$ for all coefficient $v_{i,j}$ of V , no product $\nu^a \nu^b$ with $a + b \geq n$ appears when computing $\Omega_0 V$. This shows also the following. Introduce:

$$\Omega_0^{-1} := -\text{diag}(\nu^{-n+1} J_{d_1/2}, \nu^{-n+2} J_{d_2/2}, \dots, \nu^{-1} J_{d_{n-1}/2}, J_{d_n/2}),$$

ν^{-a} standing for the application $\sum_{b \geq a} f_b \nu^b \mapsto \sum_{b \geq a} f_b \nu^{b-a}$, from $\nu^a \mathbb{C}[\nu]$ to $\mathbb{C}[\nu]$ i.e. for the left inverse of the multiplication by ν^a . Then $H \mapsto -\Omega_0^{-1} H$ is well-defined on the space of the matrices of $\mathbb{R}[\nu]$ -bilinear forms, as they satisfy property **(i)** of Remark 3.11. More precisely, Φ is a bijection:

$$\begin{aligned} \Phi : \{V = (v_{i,j}) \in M_D(\mathbb{R}[\nu]); \deg_\nu v_{i,j} < n(j) \text{ for all } i, j, \\ {}^t \bar{V} \Omega_0 = -\Omega_0 V \text{ and } V \bar{V} = \varepsilon I\} \rightarrow \\ \tilde{\mathcal{H}}_\varepsilon := \{H \in M_D(\mathbb{R}[\nu]); H \text{ satisfy property (i) of 3.11,} \\ {}^t \bar{H} = H \text{ and } H \Omega_0^{-1} \bar{H} = \varepsilon \Omega_0\}. \end{aligned}$$

Notice that in the set above, H is necessarily non degenerate, so satisfies property **(ii)** of 3.11 with $r_a = d_a$ for all a . Now that this adaptation to the nilomorphic case is done, we may perform Cartan's test. In fact, it works just like in §1.4. Indeed:

– We look for an N -stable integral manifold of the exterior differential equation **I** : ${}^t dw \wedge dH \wedge d\bar{w} = 0$ i.e. for a nilomorphic function $H : (\mathbb{C}^{d/2}, N) \rightarrow \tilde{\mathcal{H}}_\varepsilon$ around the origin, whose graph is an integral manifold of **I**. By Proposition 2.16, it amounts to find an *adapted* function $H : \mathcal{T} \rightarrow \tilde{\mathcal{H}}_\varepsilon$ (see 2.15), with \mathcal{T} the transversal $\{(\nu y) = 0\}$ to the foliation \mathcal{I} . So in the following we work on \mathcal{T} , identified with $\mathbb{C}^{D/2}$ by the coordinates.

– Then, Cartan's test rests on the equation of the tangent space $\widetilde{W}_\varepsilon := T_{H(0)} \tilde{\mathcal{H}}_\varepsilon$, which we will see to be nearly the same as in §1.4. More precisely, along \mathcal{T} , the $M_D(\mathbb{C}[\nu])$ -valued function H we look for reads $H = \sum_{a=0}^{n-1} H_a \nu^a$, each H_a being the pull back of some complex valued matrix function on $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^{n-1-a})$. We will see that the coefficient of ν^a in **I** is an exterior differential equation involving only the function H_a , thus is an exterior differential equation defined on $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^{n-1-a})$. Each of those equations is like that of §1.4

To alleviate the formulas, we write them in the case $\varepsilon = -1$ and g positive definite. The other cases work alike. We introduce $\widehat{\Omega}_0 := \text{diag}(J_{d_1/2}, J_{d_2/2}, \dots, J_{d_n/2})$. As in §1.4, at the origin, we may take $V = \widehat{\Omega}_0$ *i.e.*, at the origin:

$$H = \widehat{I} := \text{diag}(\nu^{n-1}I_{d_1/2}, \nu^{n-2}I_{d_2/2}, \dots, \nu I_{d_{n-1}/2}, I_{d_n/2}),$$

$$\begin{aligned} \text{then: } W \in \widetilde{W}_\varepsilon &\Leftrightarrow W\widehat{\Omega}_0^{-1}\widehat{I} + \widehat{I}\widehat{\Omega}_0^{-1}W = 0 \\ &\Leftrightarrow W\widehat{\Omega}_0 + \widehat{\Omega}_0W = 0 \\ &\Leftrightarrow \sum_{a=0}^{n-1} \nu^a (W_a\widehat{\Omega}_0 + \widehat{\Omega}_0W_a) = 0. \end{aligned}$$

The coefficient of ν^a involves only W_a . So, as announced, the coefficient of ν^a in \mathbf{I} is an equation involving only H_a . This equation is stated on the $(D - D_{n-1-a})/2$ -dimensional quotient $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^{n-1-a})$ as we look for a \mathcal{K}^{n-1-a} -basic function H_a . (This is consistent with the fact that only the bottom right square of W_a , appearing in (i) of Remark 3.11 and corresponding to the quotient by $\ker N^{n-1-a}$, is non vanishing.) Now on this quotient, the vectors may be reordered so that $\widehat{\Omega}_0$ reads $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and $W_a\widehat{\Omega}_0 + \widehat{\Omega}_0W_a = 0$ is the same equation as that defining W_ε in §1.4. So Cartan's criterion is fulfilled and the solutions H_a depend on $(D - D_{n-1-a})/2$ of $(D - D_{n-1-a})/2 + 1$ variables. Finally, set $b := \min\{a \in \llbracket 1, n \rrbracket; d_a \neq 0\}$. Then $0 = D_0 = \dots = D_{b-1}$. So H_{n-1} up to H_{n-b} depend, each, on $(D - D_0)/2 = \frac{D}{2}$ functions of $(D - D_0)/2 + 1 = \frac{D}{2} + 1$ variables — strictly more variables than any other H_c . So the whole function H depends on $b\frac{D}{2}$ functions of $(D - D_0)/2 + 1 = \frac{D}{2} + 1$ variables. By Remark 3.13, the choice of the complex Darboux coordinates for ω amounts to that of b function of $D/2$ variables, so this does not interfere.

Note. The above technique may be used as a standard reasoning to adapt arguments about an exterior differential system to the nilomorphic framework.

Part (b) for cases (3)–(3')–(3^C). Following the note just above, (b) is given by the reasoning referred to in Remark 1.34, adapted to the nilomorphic case *i.e.* applied to adapted functions, or jets, defined on \mathcal{T} .

Part (a) for case (1^C). Use the first point of Reminder 1.24, and repeat the very proof of Theorem 2.31, with the field \mathbb{C} and holomorphic functions replacing \mathbb{R} and smooth functions.

Part (a) for cases (2) and (2'), hence (2^C), which is their complexification. We are now quicker. Repeat the classic proofs (see respectively *e.g.* [24] §11.2 and §8.3, and [2] §2) with nilomorphic coordinates and functions replacing real ones. In other words:

– Take a nilomorphic coordinate system which is also integral for J or L *i.e.* such that $J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$ or $L\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_{i+1}}$ for all odd i . This is given by Lemma 3.1.

– Set $\mathcal{T} := \{(\nu y) = 0\} = \{\forall i, (\nu y_i) = 0\}$, then apply these proofs along \mathcal{T} , to each of the (real) coefficients f_a , factor of ν^a , of the nilomorphic functions f that appear. As the latter are adapted along \mathcal{T} (see 2.15), this means applying the proofs on $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^{n-1-a})$ for each f_a . Then extend the value of all functions along the leaves of \mathcal{I} by the formula of Proposition 2.16. So, you get the announced potentials u .

Part (b) for cases (1), (2) and (2') — hence also for their complexifications (1^C) and (2^C). We adapt the standard arguments given in Proposition 1.27. First, we take nilomorphic coordinates being, at the origin in $\mathcal{T} = \{(\nu y) = 0\}$, tangent to normal coordinates *i.e.* such that the $X_i.g(X_j, X_k)$ vanish; in particular all $D_{X_i}X_j$ are null. As $DN = 0$, so are

the $D_{N^b X_i} N^a X_j$, and we still get, for any vectors A, B, U, V among the $N^a X_i$ at the origin, that $g(R(A, B)U, V)$ is equal to:

$$\frac{1}{2}(A.U.(g(B, V)) - B.U.(g(A, V)) - A.V.(g(B, U)) + B.V.(g(A, U))).$$

As N is also self adjoint, $R(N^a A, B) = R(A, N^a B)$ for all a . The $R(X_i, N^a X_j)$ are determined by their restriction on $T\mathcal{T}$ i.e. by the $R(X_i, N^a X_j)X_k$. We denote by \widehat{R} this restriction.

In case (1), for each a , $\widehat{R}(X_i, N^a X_j)$, which is defined at the origin on $T\mathcal{T}/\ker N^a$, is the alternate part of the bilinear form:

$$\beta_{i,j,a} : (U, V) \mapsto X_i.U.g(N^a X_j, V) - X_j.U.g(N^a X_i, V),$$

also defined on $T\mathcal{T}/\ker N^a$. For each a , the $\beta_{i,j,a}$ depend on the second derivatives at 0 of the coefficients of $g(\cdot, N^a \cdot)$, defined on $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^a)$. Those derivatives are free in normal coordinates. Indeed $g(\cdot, N^a \cdot)$ is the coefficient of ν^{n-1-a} of the nilomorphic metric h given in Theorem 2.31, hence is chosen freely. So, on a dense open subset of the 2-jets of metrics, the alternate parts of the $(\beta_{i,j,a})_{i,j=1}^D$ are linearly independent and hence span a $\frac{\delta(\delta-1)}{2}$ -dimensional space, with $\delta = \dim(\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^a)) = \#\{i; N^a X_i \neq 0\}$. So, the sum for all a of those dimensions is the number K of triples (i, j, a) with $i < j$ and $N^a X_j \neq 0$, and generically, the holonomy algebra is K -dimensional. Now an element γ of $\mathfrak{o}_d(\mathbb{R})^N$ is precisely given by the $g(\gamma(N^a X_j), X_i)$ for those triples (i, j, a) i.e. $\dim \mathfrak{o}_d(\mathbb{R})^N = K$. We are done.

The adaptation for case (2) is similar. The forms $\beta_{i,j,a}$ are:

$$\beta_{i,j,a} : (Z_k, \overline{Z}_l) \mapsto \frac{1}{2}(-\overline{Z}_j.Z_k.(g(Z_i, N^a \overline{Z}_l)) - Z_i.\overline{Z}_l.(g(\overline{Z}_j, N^a Z_k))),$$

defined for i, j in $\llbracket 1, \frac{D}{2} \rrbracket$ and $a \in \llbracket 0, n-1 \rrbracket$, see the proof of Proposition 1.27. Each $\beta_{i,j,a}$ is given by the derivatives of u_{n-1-a} , the coefficient of ν^{n-1-a} in the potential u given by part (a) of the theorem. This u_{n-1-a} is defined on $\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^a)$, so the $(\beta_{i,j,a})_{i,j}$ may be chosen freely and span a $(\frac{\delta}{2})^2$ -dimensional space, with $\delta = \dim(\mathcal{T}/(\mathcal{T} \cap \mathcal{K}^a)) = \#\{i; N^a X_i \neq 0\}$. The sum for all a of those dimensions is the number K' of triples (i, j, a) with $i < j \leq \frac{D}{2}$ and $N^a Z_j \neq 0$. So generically, the holonomy algebra is K' -dimensional. Now an element γ of $\mathfrak{u}_{d/2}^N$ is precisely given by the $g(\gamma(N^a Z_j), \overline{Z}_j)$ for those triples (i, j, a) i.e. $\dim \mathfrak{u}_{d/2}^N = K'$. We are done. Case (2') is entirely similar and left to the reader. q.e.d.

3.14 Remark [A simpler way to build metrics — but not the whole space of them — in all the cases of Theorem 3.2: mimic real analytic metrics, in the nilomorphic framework] Take g a metric on \mathbb{R}^D having a holonomy group H_0 of one of the eight types of Remark 1.15. Then $\text{End}(T\mathcal{M})^{h_0} = \mathfrak{s}$ is semi-simple, and of any of the eight types of Theorem 1.10. If g is *real analytic*, then around the origin, in coordinates $(x_i)_{i=1}^D$, each of its coefficients $g_{i,j}$ is a power series $\sum_{\alpha} g_{i,j,\alpha} x^\alpha$ where $\alpha = (\alpha_1, \dots, \alpha_D) \in \mathbb{N}^D$ is a multi-index. Tensorise the space \mathbb{R}^D by $\mathbb{R}[\nu] \simeq \mathbb{R}[X]/(X^n)$, obtaining $(E, N) := \mathbb{R}[\nu]^D$. Each x_i becomes $z_i := x_i + (\nu y_i)$ and the expansions of the $g_{i,j}$ become $h_{i,j} = \sum_{\alpha} g_{i,j,\alpha} z^\alpha$, which are nilomorphic by construction. We let the reader check that the real metric associated with this nilomorphic metric h has $H_0 \otimes \mathbb{R}[\nu] \simeq \widehat{H}_0^N$ as holonomy group, with \widehat{H}_0 the group of the type of H_0 in the table of Remark 1.15, acting on \mathbb{R}^{nD} instead of \mathbb{R}^D . By this means, as in Theorem 3.2, we build a metric with \widehat{H}_0^N as holonomy group, with N in the bicommutant of \widehat{H}_0 , if all the invariant factors of N have the same degree n . So this works like a complexification.

Yet this does not parametrise the set of all such metrics. Indeed the restriction of h to $\{(\nu y) = 0\}$, *i.e.* h applied to tangent vectors to $\{(\nu y) = 0\}$, has value in \mathbb{R} .

In fact, we may proceed similarly for a nilpotent endomorphism N of any similarity type. As it is a bit more cumbersome, we present it apart; we also let some checkings to the reader. Consider a metric g like above. For each i set, formally, $z_i := x_i + (\nu y)_i$ and choose some $n(i) \in \llbracket 1, n \rrbracket$. The $\nu^{n-n(i)} z_i$ parametrise an $\mathbb{R}[\nu]$ -module $(E, N) := \prod_{i=1}^D (\nu^{n-n(i)} \mathbb{R}[\nu])$, the original \mathbb{R}^D being identified with the level $\{(\nu y) = 0\}$. Denote by d the real dimension of E . Suppose additionally that:

- the $n(i)$ are chosen so that the induced flag $\ker N \subset \ker N^2 \subset \dots \subset \ker N^{n-1} \subset E$ is \mathfrak{s} -stable,

- the $g_{i,j,\alpha}$ are such that the $\mathbb{R}[\nu]$ -valued form:

$$h := \sum_{i,j=1}^D \sum_{\alpha} g_{i,j,\alpha} \nu^{n-n(\alpha)} z^\alpha dz_i \otimes dz_j \quad \text{with } n(\alpha) := \min_{\{i;\alpha_i \neq 0\}} \{n(i)\}$$

makes sense and is nilomorphic, *i.e.* that the value \check{h} of h along $\{(\nu y) = 0\}$ is adapted (see Definition 2.15). We let the reader check that it means: $g_{i,j,\alpha} = 0$ as soon as $n(i) < n(\alpha)$ or $n(j) < n(\alpha)$.

Denote by \mathbf{K} the trace on $\{(\nu y) = 0\}$ of the flag of the $\ker N^a$. Then, among the sequences $(g_{i,j,\alpha})_{i,j,\alpha}$ such that the holonomy group of the corresponding metric g is some H_0 in the list of Remark 1.15, these additional constraints induce metrics whose holonomy group is $H'_0 := \{\gamma \in H_0; \gamma(\mathbf{K}) = \mathbf{K}\}$. Moreover, the real metric associated with h has \widehat{H}_0^N as holonomy group, with \widehat{H}_0 the group of the same type as H_0 in the table of Remark 1.15, but acting on \mathbb{R}^d . This gives also the existence part of Theorem 3.2, but does not parametrise the whole set of metrics sharing this holonomy group, for the same reason as above: the value \check{h} of h along $\{(\nu y) = 0\}$ is such that each \check{h}_a is given by a matrix of the form $B^a = \begin{pmatrix} B_0^a & 0 \\ 0 & 0 \end{pmatrix}$, as given in Remark 2.41. This is not the case for a generic metric with this holonomy group.

3.15 Important Remark The note p. 43 points out that the technique used in the proof of Theorem 3.2 may be used as a standard way to generalise reasonings on germs of real functions to germs of nilomorphic ones. In particular, by this means, or as in Remark 3.14 above, for the analytic category, we might show similar statements as Theorem 3.2 for H any semi-simple classical pseudo-Riemannian holonomy group.

We finally state the sequence of results in linear algebra leading to Corollary 3.5. This provides the matrix of the elements of all algebras appearing here. To let clearly appear the reason the final Proposition 3.22 works, we give the statements as a sequence of steps; this makes the work to prove each step quite clear. So we let the proofs to the reader.

3.16 Notation The two cases are here labelled in reference to Theorem 1.10. We recall standard facts in point denoted by **(2)**, to introduce in **(2')** a “paracomplex” counterpart of them.

(i) If J is a g -skew adjoint morphism of \mathbb{R}^{2d} with $J^2 = -\text{Id}$, its commutant $U(g) = O(g)^J$ in $O(g)$ may be seen as a subgroup of $GL_d(\mathbb{C})$. More precisely, in a basis such that $J = \text{diag}(J_1, \dots, J_1)$, $U(g)$ is a group of real matrices consisting of square subblocks in:

$$M_2(\mathbb{R})^{J_1} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}; a, b \in \mathbb{R} \right\} \simeq \{a + ib; a, b \in \mathbb{R}\} = \mathbb{C},$$

so in such a basis, these matrices may be considered as complex.

(ii) Similarly, if L is a g -skew adjoint morphism of \mathbb{R}^{2d} with $L^2 = \text{Id}$, we denote here by $U_L(g) = O(g)^L$ its commutant in $O(g)$. It may be seen as a subgroup of $\text{GL}_d(\mathbb{R} \oplus \mathbb{R})$, so as a group of matrices with entries in $\mathbb{R} \oplus \mathbb{R}$. More precisely, in a basis such that $L = \text{diag}(I_{1,1}, \dots, I_{1,1})$, $U_L(g)$ is a group of real matrices consisting of square subblocks in:

$$M_2(\mathbb{R})^{I_{1,1}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a, b \in \mathbb{R} \right\} \simeq \{(a, b); a, b \in \mathbb{R}\} = \mathbb{R} \oplus \mathbb{R},$$

with the natural product $(a, b) \cdot (a', b') = (aa', bb')$ on the ring $\mathbb{R} \oplus \mathbb{R}$. These subblocks may be considered as elements of $M_{d/2}(\mathbb{R} \oplus \mathbb{R})$.

The inclusion $\mathbb{R} \cdot \text{Id} \subset M_2(\mathbb{R})^{J_1} \simeq \mathbb{C}$ sends \mathbb{R} onto $\{a + ib; b = 0\}$, stabilised by the involution $a + ib \mapsto \overline{a + ib} = a - ib$. Similarly $\mathbb{R} \cdot \text{Id} \subset M_2(\mathbb{R})^{I_{1,1}} \simeq \mathbb{R} \oplus \mathbb{R}$ sends \mathbb{R} onto $\{(a, b); a = b\}$, stabilised by the natural involution $(a, b) \mapsto (b, a)$, called here “ L -” or “para-complex” conjugation and denoted by $(a, b) \mapsto \overline{(a, b)}$. Notice this involution at line (2’) of Table 1 p. 8. To sum up, in terms of real 2-2 matrices or submatrices:

$$\text{in (i), } \overline{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ and in (ii), } \overline{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.$$

3.17 Notation The two cases are here labelled in reference to Theorem 1.10. We recall standard facts in point denoted by (3) about the group $\text{Sp}(p, q)$, to introduce in (3’) a “paraquaternionic” counterpart of them.

(i) If $\langle J, J', J'' \rangle$ is a g -skew adjoint quaternionic structure on \mathbb{R}^{4d} , i.e. $(J'')^2 = -\text{Id}$ and $JJ' = -J'J = J''$, its commutant $\text{Sp}(g) = O(g)^{\{J, J', J''\}}$ in $O(g)$ may be seen as a subgroup of $\text{GL}_d(\mathbb{H})$. More precisely, identifying (\mathbb{R}^{4d}, J) with $(\mathbb{C}^{2d}, iI_{1,1})$, in a basis such that $J' = -\text{diag}(J_1, \dots, J_1)$, $\text{Sp}(g)$ is a group of real matrices consisting of 4-4 square subblocks in $M_4(\mathbb{R})^{\{J, J'\}}$ identified with:

$$\left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}; \alpha, \beta \in \mathbb{C} \right\} \simeq \{a + ia' + jb + kb'; \alpha = a + ia', \beta = b + ib'\} = \mathbb{H},$$

so in such a basis, these matrices may be considered as quaternionic.

(ii) Similarly, if $\langle L, L', J \rangle$ is a g -skew adjoint paraquaternionic structure on \mathbb{R}^{4d} , i.e. $(L')^2 = \text{Id}$ and $LL' = -L'L = J$, we denote here by $\text{Sp}_{L, L'}(g)$ its commutant $O(g)^{\{L, L', J\}}$ in $O(g)$. It may be seen as a subgroup of $\text{GL}_d(M_2(\mathbb{R}))$. More precisely, in a basis such that $L = \text{diag}(I_{2,2}, \dots, I_{2,2})$ and $L' = \text{diag}(L'_2, \dots, L'_2)$ where $L'_2 = \begin{pmatrix} 0 & J_1 \\ -J_1 & 0 \end{pmatrix}$, $\text{Sp}_{L, L'}(g)$ is a group of real matrices consisting of square subblocks in:

$$M_4(\mathbb{R})^{\{I_{2,2}, L'_2\}} = \left\{ h_P := \begin{pmatrix} P & 0 \\ 0 & t\tilde{P} \end{pmatrix}; P \in M_2(\mathbb{R}) \right\} \simeq M_2(\mathbb{R}),$$

where \tilde{P} stands for the comatrix of P .

In (i), the inclusion $\mathbb{R} \cdot \text{Id} \subset M_4(\mathbb{R})^{\{J, J'\}} \simeq \mathbb{H}$ sends \mathbb{R} to $\{a + ia' + jb + kb'; a' = b = b' = 0\}$, stabilised by the quaternionic conjugation. The inclusion $\mathbb{R} \cdot \text{Id} \subset M_4(\mathbb{R})^{\{I_{2,2}, L'_2\}} \simeq M_2(\mathbb{R})$, sends \mathbb{R} to $\{h_{\lambda \text{Id}}\}$, stabilised by $h_P \mapsto h_{t\tilde{P}}$ called here “paraquaternionic conjugation” and denoted by $h_P \mapsto \overline{h_P}$. This involution appears at line (3’) of Table 1 p. 8.

3.18 Notation (a) For each $p \in \mathbb{N}$ and for $\delta \in \{1, 2, 4\}$ we introduce:

$$N_p^{(\delta)} := \begin{pmatrix} 0 & I_\delta & & \\ & \ddots & \ddots & \\ & & \ddots & I_\delta \\ & & & 0 \end{pmatrix} \text{ and: } K_p^{(\delta)} := \begin{pmatrix} & & & I_\delta \\ & & \ddots & \\ & & & \\ I_\delta & & & \end{pmatrix},$$

both in $M_{\delta p}(\mathbb{R})$. We will also denote $N_p^{(1)}$ by N_p and $K_p^{(1)}$ by K_p .

(b) If $M = (m_{i,j})_{i=1,j=1}^{p,q}$ is a matrix with p lines and q columns, we denote here by ${}^{\sharp}M$ its “transpose with respect to the anti diagonal” ${}^{\sharp}M = (m_{q-j,p-i})_{j=1,i=1}^q = K_q \cdot {}^tM \cdot K_p$.

3.19 Lemma/Notation Let g be a pseudo-Riemannian metric on \mathbb{R}^d , and N a g -self adjoint nilpotent endomorphism of \mathbb{R}^d , of nilpotence index n . In five different cases, labelled as in Theorem 1.10, let us introduce a Lie subgroup Q of $GL_d(\mathbb{R})$, its Lie algebra \mathfrak{q} , and an induced set of bases of \mathbb{R}^d , called here “privileged”, on which Q^N acts simply transitively. The $N_i, N_i^{(\delta)}, K_i, K_i^{(\delta)}$ are as in Notation 3.18 **(a)** and $\varepsilon_i \in \{-1, 1\}$.

(1) Here $Q := O(g)$. There are bases of \mathbb{R}^d in which:

$$\begin{cases} \text{Mat}(N) = \text{diag}(N_{n_1}, \dots, N_{n_D}) \text{ with } n = n_1 \geq \dots \geq n_D \\ \text{Mat}(g) = \text{diag}(\varepsilon_{n_1} K_{n_1}, \dots, \varepsilon_{n_D} K_{n_D}). \end{cases}$$

(2) Here $J \in \text{End}(\mathbb{R}^d)$ with $J^* = -J$ and $J^2 = -\text{Id}$; $Q := U(g) = O(g)^J$. There are bases of \mathbb{R}^d in which $\text{Mat}(J) = \text{diag}(J_1, \dots, J_1)$ and:

$$\begin{cases} \text{Mat}(N) = \text{diag}(N_{n_1}^{(2)}, \dots, N_{n_D}^{(2)}) \text{ with } n = n_1 \geq \dots \geq n_D \\ \text{Mat}(g) = \text{diag}(\varepsilon_{n_1} K_{n_1}^{(2)}, \dots, \varepsilon_{n_D} K_{n_D}^{(2)}). \end{cases}$$

(2') Here $L \in \text{End}(\mathbb{R}^d)$ with $L^* = -J$ and $L^2 = -\text{Id}$; $Q := U_L(g) = O(g)^L$, see Notation 3.16 **(ii)**. There are bases of \mathbb{R}^d in which $\text{Mat}(L) = \text{diag}(I_{1,1}, \dots, I_{1,1})$ and:

$$\begin{cases} \text{Mat}(N) = \text{diag}(N_{n_1}^{(2)}, \dots, N_{n_D}^{(2)}) \text{ with } n = n_1 \geq \dots \geq n_D \\ \text{Mat}(g) = \text{diag}(K_{2n_1}, \dots, K_{2n_D}). \end{cases}$$

(3) Here (J, J', J'') is a g -skew adjoint quaternionic structure on \mathbb{R}^d , see 3.17 **(i)**; $Q := \text{Sp}(g) = O(g)^{\{J, J'\}}$. There are bases of $\mathbb{R}^d \simeq \mathbb{C}^{d/2}$ in which $\text{Mat}(J) = i \text{diag}(I_{1,1}, \dots, I_{1,1})$, $\text{Mat}(J) = -\text{diag}(J_1, \dots, J_1)$ and:

$$\begin{cases} \text{Mat}(N) = \text{diag}(N_{n_1}^{(4)}, \dots, N_{n_D}^{(4)}) \text{ with } n = n_1 \geq \dots \geq n_D \\ \text{Mat}(g) = \text{diag}(\varepsilon_{n_1} K_{n_1}^{(4)}, \dots, \varepsilon_{n_D} K_{n_D}^{(4)}). \end{cases}$$

(3') Here (L, L', J) is a g -skew adjoint paraquaternionic structure on \mathbb{R}^d , see 3.17 **(ii)**; $Q = \text{Sp}_{L, L'}(g) = O(g)^{\{L, L'\}}$. There are bases of \mathbb{R}^d in which $\text{Mat}(L) = \text{diag}(I_{2,2}, \dots, I_{2,2})$, $\text{Mat}(L') = \text{diag}(L'_2, \dots, L'_2)$ with L'_2 as in 3.17 **(ii)** and:

$$\begin{cases} \text{Mat}(N) = \text{diag}(N_{n_1}^{(4)}, \dots, N_{n_D}^{(4)}) \text{ with } n = n_1 \geq \dots \geq n_D \\ \text{Mat}(g) = \text{diag}(K_{2n_1}^{(2)}, \dots, K_{2n_D}^{(2)}). \end{cases}$$

Reference for the proof. Privileged bases are provided by [21].

3.20 Lemma Take $\gamma \in \text{End}(\mathbb{R}^d)$. Using a privileged basis given by Lemma 3.19, we consider its matrix M as a matrix with coefficients in $\mathbb{A} := \mathbb{R}$, $\mathbb{A} := \mathbb{C}$, $\mathbb{A} := \mathbb{R} \oplus \mathbb{R}$, $\mathbb{A} := \mathbb{H}$ or $\mathbb{A} := M_2(\mathbb{R})$ in cases **(1)**, **(2)**, **(2')**, **(3)** or **(3')** respectively, see Notation 3.16 and 3.17. Consider $(M_{i,j})_{i,j=1}^D$ the block-decomposition of M corresponding to the Jordan blocks of N , and the Lie algebra \mathfrak{q} introduced in Lemma 3.19.

(a) $\gamma \in \mathfrak{q}$ if and only if, with the notation \sharp set in 3.18 (b):

- in case **(1)**, $M_{j,i} = -\varepsilon_i \varepsilon_j \sharp M_{i,j}$ for all $i, j \in \llbracket 1, D \rrbracket$,
- in cases **(2)** and **(3)**, $M_{j,i} = -\varepsilon_i \varepsilon_j \sharp \overline{M_{i,j}}$ for all $i, j \in \llbracket 1, D \rrbracket$, with the complex or quaternionic conjugation in cases **(2)** and **(3)** respectively,
- in case **(2')** and **(3')**, $M_{j,i} = -\sharp \overline{M_{i,j}}$ for all $i, j \in \llbracket 1, D \rrbracket$, with the paracomplex or paraquaternionic conjugation, in cases **(2')** and **(3')** respectively, introduced in Notation 3.16 **(2')** and 3.17 **(3')**.

(b) γ commutes with N , and additionally with J in case **(2)**, L in case **(2')**, $\{J, J', J''\}$ in case **(3)** or $\{L, L', J\}$ in case **(3')** if and only if, for all $i \leq j$:

$$(\star) \begin{cases} M_{i,j} = \begin{pmatrix} M'_{i,j} \\ 0_{n_i - n_j, n_j} \end{pmatrix}, M_{j,i} = \begin{pmatrix} 0_{n_j, n_i - n_j} & M'_{j,i} \end{pmatrix}, \text{ i.e. } M_{i,i} = M'_{i,i} \\ \text{where: } M'_{i,j} = \begin{pmatrix} m_{i,j}^1 & m_{i,j}^2 & \cdots & m_{i,j}^{n_j} \\ 0 & m_{i,j}^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & m_{i,j}^2 \\ 0 & \cdots & 0 & m_{i,j}^1 \end{pmatrix} \in M_{n_k}(\mathbb{A}). \end{cases}$$

(c) Hence, $\gamma \in \mathfrak{q}^N$ if and only if the $M_{i,j}$ are as above and:

- in case **(1)**, $M'_{j,i} = -\varepsilon_i \varepsilon_j M'_{i,j}$ for all $i, j \in \llbracket 1, D \rrbracket$,
- in cases **(2)** and **(3)**, $M'_{j,i} = -\varepsilon_i \varepsilon_j \overline{M'_{i,j}}$ for all $i, j \in \llbracket 1, D \rrbracket$,
- in cases **(2')** and **(3')**, $M'_{j,i} = -\overline{M'_{i,j}}$ for all $i, j \in \llbracket 1, D \rrbracket$, with the conjugation introduced in Notation 3.16 **(2')** and 3.17 **(3')**.

Indeed, the $M'_{i,j}$ being as in (\star) , $\sharp M'_{i,j} = M'_{i,j}$. So finally, for $i < j$, $M_{j,i}$ is given by $M_{i,j}$, which is freely chosen as in (\star) , and:

- in case **(1)**, the $M'_{i,i}$ are null,
- in cases **(2)**–**(3)**, the $M'_{i,i}$ are as in (\star) , with purely imaginary coefficients i.e. in $i\mathbb{R} \subset \mathbb{C}$ for case **(2)** and in $\text{span}\{i, j, k\} \subset \mathbb{H}$ for **(3)**.
- in cases **(2')**–**(3')**, the $M'_{i,i}$ are as in (\star) , with coefficients of the type $(a, -a) \in \mathbb{R} \oplus \mathbb{R}$ for **(2')**, and $\text{diag}(P, \sharp P)$ with $\text{tr } P = 0$ for **(3')**.

(d) Therefore, $\gamma \in \text{End}(\mathbb{R}^d)^{(\mathfrak{q}^N)}$ if and only if:

– in the other cases than **(1)**, all the $M_{i,j}$ are null except when $i = j$, where $M_{i,i}$ is as in (\star) .

– in case **(1)**, the situation is the same except for some similarity types of N . All the $M_{i,j}$ are null except when $i = j$, and except possibly $M_{1,2}$ and $M_{2,1}$. Remind that $n = n_1 \geq n_2 \geq \dots \geq n_D$ are the sizes of the Jordan blocks of N ; conventionally, we set $n_i = 0$ for $i > D$, for example $n_3 = 0$ if $D = 2$ i.e. N has two Jordan blocks. Then:

- All $M_{i,i}$ for $i \geq 2$ are as in (\star) , and $M_{1,1} = M'_{1,1} + M''_{1,1}$ where $M'_{1,1}$ is as in (\star) and:

$$M''_{1,1} = \begin{pmatrix} 0_{n_1-n_2, n_2} & * \\ 0_{n_2, n_2} & 0_{n_2, n_1-n_2} \end{pmatrix}, \text{ with } * \text{ arbitrary in } M_{n_1-n_2}(\mathbb{R}).$$

- $M_{1,2} = \begin{pmatrix} 0_{n_2-n_3, n_3} & M''_{1,2} \\ 0_{n_3, n_3} & 0_{n_3, n_2-n_3} \end{pmatrix}$ and $M_{2,1} = -\#M_{1,2}$, with:

$$M''_{1,2} = \begin{pmatrix} m^1 & m^2 & \dots & m^{n_2-n_3} \\ 0 & m^1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & m^2 \\ 0 & \dots & 0 & m^1 \end{pmatrix} \in M_{n_2-n_3}(\mathbb{R}).$$

Thus, $M''_{1,1} = 0$ i.e. $M_{1,1}$ is as in (\star) if and only if $n_1 = n_2$ i.e. $d_n > 1$ i.e. N has several Jordan blocks of maximal size, and $M_{1,2} = -\#M_{1,2} = 0$ if and only if $n_2 = n_3$. So (1) is like the other cases if and only if $n_1 = n_2 = n_3$.

Hint for the proof. In (d) , for cases (2) – $(2')$ and (3) – $(3')$, to get $M_{i,j} = 0$ if $i \neq j$, it is sufficient to involve two Jordan blocks. For case (1) , involving three blocks is necessary to get the result, and four or more give no additional constraint.

3.21 Remark The reason case (1) behaves differently is that $M_{i,i} = -\#M_{i,i} \Rightarrow M_{i,i} = 0$, whereas $M_{i,i} = -\#\overline{M_{i,i}} \not\Rightarrow M_{i,i} = 0$.

3.22 Proposition Take \mathfrak{s} any of the subalgebras of $\text{End}(\mathbb{R}^d)$ appearing in Theorem 1.10. With the assumptions of Lemma 3.20:

– In cases (2) , $(2')$, (3) and $(3')$, the action of $\mathfrak{o}(g)^{\mathfrak{s} \cup \{N\}}$ on (\mathbb{R}^d, g) is indecomposable and: $\text{End}(T\mathcal{M})^{(\mathfrak{o}(g)^{\mathfrak{s} \cup \{N\}})} = \mathfrak{s} \cdot \mathbb{R}[N]$.

– In case (1) , the action of $\mathfrak{o}(g)^{\mathfrak{s} \cup \{N\}} = \mathfrak{o}(g)^N$ on (\mathbb{R}^d, g) is indecomposable if and only if $2n_2 < n_1$ i.e. N has only one Jordan block of maximal size, the second largest size being more than twice smaller.

• If $2n_2 < n_1$, take F any supplement of $\ker N^{n_2}$ in $\text{Im } N^{n_2}$. By Lemma 3.20 (c), $\mathfrak{o}(g)^N$ acts trivially on $\text{Im } N^{n_2}$ thus on F ; besides $\ker N^{n_2} = \ker(g|_{\text{Im } N^{n_2}})$, so $g'' := g|_F$ is non degenerate. So $\mathfrak{o}(g)^N$ acts trivially on F and stabilises F^\perp . In particular, $\text{End}(F)^{(\mathfrak{o}(g)^N)} = \text{End}(F)$. Moreover, $d'' := \dim F = n_1 - 2n_2$ and the signature of g'' is:

- * $(\frac{d''}{2}, \frac{d''}{2})$ if d'' is even i.e. n is,
- * $(\frac{d''+1}{2}, \frac{d''-1}{2})$, respectively $(\frac{d''-1}{2}, \frac{d''+1}{2})$, if d'' is odd i.e. n is, and $g(\cdot, N^{n-1} \cdot)$ is positive, respectively negative on the line $\mathbb{R}^d / \ker N^{n-1}$.

On F^\perp , the operator $N' = N|_{F^\perp}$ is of nilpotence index $n' = 2n_2$ and has Jordan blocks of sizes $(2n_2, n_2, n_3, \dots, n_D)$. So the action of $\mathfrak{o}(g)^N$ on F^\perp is described by the second item below.

• If $2n_2 \geq n_1$, set $\pi' : \mathbb{R}^d \rightarrow \mathbb{R}^d / \ker N^{n_2}$ and $\pi'' : \mathbb{R}^d \rightarrow \mathbb{R}^d / \ker N^{n_3}$. The (pseudo-)euclidian product $g_{n_3} := g(\cdot, N^{n_3} \cdot)$ is well-defined and non degenerate on $\mathbb{R}^d / \ker N^{n_3}$, on which N acts. This defines the group $O(g_{n_3})$ and the commutant $O(g_{n_3})^N$ of N in it. Besides, we may see the N^{n_i} as morphisms $\mathbb{R}^d / \ker N^{n_i} \rightarrow \text{Im } N^{n_i}$. Then:

$$\text{End}(\mathbb{R}^d)^{(\mathfrak{o}(g)^N)} = \left(\mathbb{R}[N] + N_*^{n_2} \pi'^* \text{End}(\mathbb{R}^d / \ker N^{n_2}) \right) \oplus N_*^{n_3} \pi''^* \mathfrak{o}(g_{n_3})^N.$$

Hint for the proof. For the (in)decomposability, you see in Lemma 3.20 (c) that, in cases (2)–(2′) and (3)–(3′), no proper subspace of \mathbb{R}^d is stable by $\mathfrak{o}(g)^{\mathfrak{s} \cup \{N\}}$, except possibly isotropic lines in some very exceptional subcases of (2′). In case (1), the subspace on which $\mathfrak{o}(g)^N$ acts trivially is exactly $\{0\}$ if $n_1 = n_2$ and $\text{Im } N^{n_2}$ if $n_1 > n_2$.

4 A glimpse on the case where the holonomy group is the commutant of several algebraically independent nilpotent endomorphisms

We investigate here the simplest example where $\mathfrak{h} = \mathfrak{o}(g)^{\{N, N'\}}$ with (N, N') algebraically independent. This will show a phenomenon appearing when \mathfrak{n} is abelian, non principal, see Comment 4.3. We will see that the results of both Parts 1 and 2 are needed to describe it. Here is it. We parametrise the set \mathcal{G} of germs of metrics such that $\mathfrak{n} = (N, N')$ with N and N' self adjoint and $N^2 = N'^2 = NN' = N'N = 0$. To simplify a bit more, we assume $\text{Im } N = \ker N$ — our goal here is no kind of general theory. Besides, taking $\dim \mathcal{M} \geq 6$ ensures $n_1 = n_2 = n_3$ so we are not in the exceptional cases of Corollary 3.5.

4.1 Remark (i) Then, there is a $U \in \text{End}(T\mathcal{M}/\text{Im } N)$ such that $N' = NU$, which makes sense as the argument of N may be defined only modulo $\ker N$. Indeed, N gives an isomorphism $\theta_N : T\mathcal{M}/\ker N \rightarrow \ker N$. As $NN' = N'N$, N' gives also a morphism $\theta_{N'} : T\mathcal{M}/\ker N \rightarrow \ker N$, and is determined by it. Set $U := \theta_N^{-1} \circ \theta_{N'} \in \text{End}(T\mathcal{M}/\ker N)$.

(ii) Identify U and its matrix. In coordinates $(y_i, x_i)_{i=1}^{d/2}$ adapted to N :

$$\text{Mat}(N) = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \quad \text{and:} \quad \text{Mat}(N') = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}.$$

Then, using Lemma 3.1 complemented by Remark 3.10, we take coordinates such that $\text{Mat}(U)$ is also constant — and for instance in Jordan form. Explicitly, U is well defined on \mathcal{M}/\mathcal{I} (in other words it is \mathcal{I} -basic) and, if $\prod_{\alpha} P_{\alpha}^{n_{\alpha}}$ is the decomposition of its minimal polynomial in powers of irreducible polynomials, we may take coordinates $(x_i)_i$ which are:

- product coordinates for $\mathcal{M}/\mathcal{I} \simeq \prod_k \overline{\mathcal{M}}_k$, the integration of the decomposition $\bigoplus_{\alpha}^{\perp} \ker P_{\alpha}^{n_{\alpha}}$,
- on each factor, adapted to the nilpotent part of U on it and, if $\deg P_{\alpha} = 2$ i.e. if the semi-simple part of U induces a complex structure \underline{J}_{α} on it, also complex coordinates for it.

4.2 Proposition *With the $U \in \text{End}(T\mathcal{M}/\text{Im } N)$ introduced in Remark 4.1 (i), a metric g makes N and N' parallel if and only if:*

(a) *it is the real metric h_1 associated with a (ν, N) -nilomorphic metric $h = h_0 + \nu h_1$ with value in $\mathbb{R}[\nu] = \mathbb{R}[X]/(X^2)$,*

(b) *the bilinear form h_0 , defined on \mathcal{M}/\mathcal{I} , makes U parallel (recall that $h_0 = g(\cdot, N \cdot)$) so is nondegenerate, hence is a metric, on \mathcal{M}/\mathcal{I} .*

Then there exist coordinates $(x_i, y_i)_i$ simultaneously N - and N' -adapted, as given in Remark 4.1 (ii). In such coordinates, on each factor $\overline{\mathcal{M}}_k$, h_0 is itself of the form given by Theorem 2.31, for the nilpotent part N_k of U on $\overline{\mathcal{M}}_k$ (i.e. h_0 is the real metric associated with some (μ, N_k) -nilomorphic metric, with $\mathbb{R}[\mu] = \mathbb{R}[X]/(X^{n_k})$), and also complex Riemannian for \underline{J}_k if $\deg P_k = 2$ (then $\mathbb{R}[\mu]$ is replaced by $\mathbb{C}[\mu]$).

Proof. After Th. 2.31, g makes N parallel if and only if it satisfies **(a)**. It makes also N' parallel if and only if for any (i, j, k) , $g(D_{X_i}N'X_j, X_k) = g(D_{X_i}X_j, N'X_k)$, that is to say $g(D_{X_i}N'X_j, X_k) = g(D_{X_i}X_j, N'X_k)$, or $g(D_{X_i}UX_j, NX_k) = g(D_{X_i}X_j, N'X_k)$, as $DN = 0$. This means $h_0(D_{X_i}UX_j, X_k) = h_0(D_{X_i}X_j, UX_k)$ *i.e.* U is parallel for h_0 . q.e.d.

In real terms, following Example 2.38, and in the basis $((Y_i)_i, (X_i)_i)$, $\text{Mat}(g) = \begin{pmatrix} 0 & G^0 \\ G^0 & G^1 \end{pmatrix}$ where:

- G^0 , depending only on the $(x_i)_i$, is the matrix of a metric on \mathcal{M}/\mathcal{I} making U parallel,
- $G^1 = B^1 + \sum_i \left(\frac{\partial G^1}{\partial x_i}\right) y_{i,1}$, with B^1 the matrix of any bilinear symmetric form, depending only on the $(x_i)_i$.

So building a metric making N and N' parallel means taking a metric such that N is, and adding constraints on it by repeating the general story of this article, on the quotient \mathcal{M}/\mathcal{I} , endowed with the metric $g(\cdot, N\cdot)$ and of a parallel endomorphism U induced by N' .

4.3 Comment If (N, N') is a general pair of commuting nilpotent endomorphisms, their characteristic flags of subspaces $F^{a,b}$ introduced at the beginning of §2.1 induce, each alone and with each other, a lot of quotient spaces E_α on each of which the metric g and the pair (N, N') induce an endomorphism U_α and a metric g_α . Building a metric making N and N' parallel means in particular making each U_α parallel for g_α , so repeating Theorem 2.31 on E_α , applied to some coefficient of some nilomorphic metric, sometimes several times successively. In fact, the situation is more complicated, as a pair of commuting nilpotent endomorphisms is not something simple — *e.g.* think simply to the case treated in Proposition 4.2, without the assumption $\text{Im } N = \ker N$. Finally, if more than two endomorphisms N and N' are involved, it may appear quotients of the type of the E_α on which any of the situations of Theorem 1.10 appears, even if $\mathfrak{s} = \mathbb{R} \text{Id}$ on $T\mathcal{M}$ itself.

So the general situation, though not entirely new with respect to the case where \mathfrak{n} is principal, seems to be complicated. What the good next questions are it is still unclear. The case where \mathfrak{n} is not abelian may induce new phenomena, but is strongly constrained, when \mathfrak{n} consists of self adjoint elements, by Proposition 1.8.

5 Parallel endomorphisms and Ricci curvature

The Ricci form $\text{ric}(\cdot, J\cdot)$ has remarkable properties on Kähler manifolds. Let us determine the properties of the corresponding forms when g admits other parallel endomorphism fields than a Kähler structure.

5.1 Theorem *Suppose U is a parallel endomorphism field for a pseudo-Riemannian metric g ; (a, b) denote any two tangent vectors at some point.*

- (i) *If U is self adjoint:*
 - a) $\text{ric}(a, Ub) = \text{ric}(Ua, b) = \text{tr}(U(R(a, \cdot)b))$; U and $R(a, \cdot)b$ commute,
 - b) (standard result) if $U = \underline{J}$ is a complex structure, g is the real part of the \underline{J} -complex metric $g_{\mathbb{C}} := g(\cdot, \cdot) - ig(\cdot, \underline{J}\cdot)$, and the Ricci curvature of $g_{\mathbb{C}}$ is $\text{ric}_{\mathbb{C}} = \text{ric}(\cdot, \cdot) - i\text{ric}(\cdot, \underline{J}\cdot)$,
 - c) if $U = N \neq 0$ is nilpotent, ric is degenerate and $\text{Im } N \subset \ker \text{ric}$.
- (ii) *If U is skew adjoint:*
 - a) $\text{ric}(a, Ub) = -\text{ric}(Ua, b) = \frac{1}{2} \text{tr}(U \circ R(a, b))$,

b) if $U = N \neq 0$ is nilpotent, ric is degenerate and $\text{Im } N \subset \ker \text{ric}$,

c) if V is another skew symmetric parallel endomorphism with $VU = -UV$, and if U and V are invertible, then $\text{ric} = 0$. So (standard result) cases **(3)**, **(3')**, **(3^C)** of Theorem 1.10 are Ricci-flat.

Proof. Take U self adjoint, then the whole of **a)** follows from Remark 1.9. Point **b)** is standard and for **c)**, after **a)**, $\text{ric}(a, Nb) = \text{tr}(N(R(a, \cdot)b))$, and as N and $R(a, \cdot)b$ commute, their product is also nilpotent, so trace free. Now take U skew adjoint.

$$\begin{aligned} \text{ric}(a, Ub) &= \text{tr}(R(a, \cdot)Ub) \\ &= \text{tr}(U(R(a, \cdot)b)) \text{ as } U, \text{ being parallel, commutes with } R(a, \cdot), \\ &= \text{tr}(R(a, U\cdot)b) \text{ as } \text{tr}(UV) = \text{tr}(VU), \\ &= -\text{tr}(R(Ua, \cdot)b). \end{aligned}$$

For the last line, take any u, v, w : $g(R(Ua, u)v, w) = g(R(v, w)Ua, u) = g(UR(v, w)a, u) = -g(R(v, w)a, Uu) = -g(R(a, Uu)v, w)$. So finally, $\text{ric}(a, Ub) = -\text{ric}(Ua, b)$. Besides:

$$\begin{aligned} \text{ric}(Ua, b) &= \text{ric}(b, Ua) \\ &= \text{tr}(U(R(b, \cdot)a)) \\ &= -\text{tr}(U(R(\cdot, a)b)) - \text{tr}(U(R(a, b)\cdot)) \text{ by the Bianchi identity,} \\ &= \text{tr}(R(a, \cdot)Ub) - \text{tr}(U \circ R(a, b)) \text{ as } U \text{ commutes with} \\ &= \text{ric}(a, Ub) - \text{tr}(U \circ R(a, b)). \qquad R(a, \cdot) = -R(\cdot, a), \end{aligned}$$

As $\text{ric}(Ua, b) = -\text{ric}(a, Ub)$, we get **a)**. Point **b)** follows: $\text{ric}(a, Nb) = \frac{1}{2} \text{tr}(N \circ R(a, b)) = 0$ as $N \circ R(a, b) = R(a, b) \circ N$ is nilpotent. Point **c)** is only a way to re-find that $\text{ric} = 0$ in cases **(3)**, **(3')**, **(3^C)**, using **a)**. Indeed, if U and V are as announced, any b can be written $b = UVc$, and: $\text{ric}(a, b) = \text{ric}(a, UVc) = -\text{ric}(Ua, Vc) = -\frac{1}{2} \text{tr}(V \circ R(Ua, c)) = \frac{1}{2} \text{tr}(V \circ R(a, Uc)) = \text{ric}(a, VUc) = -\text{ric}(a, UVc) = -\text{ric}(a, b)$. q.e.d.

5.2 Corollary Let Ric be the endomorphism such that $\text{ric} = g(\cdot, \text{Ric} \cdot)$. If the metric is indecomposable (in a local Riemannian product) and such that ric is parallel, then Ric is either semi-simple or 2-step nilpotent.

Proof. As g is indecomposable, the minimal polynomial of Ric is of the form P^α with P irreducible, see *Claim 1* p. 10 in the proof of Theorem 1.10. So Ric is either invertible or nilpotent. Apply Theorem 5.1 **(i) c)** to the nilpotent part N_{Ric} of Ric : if Ric is invertible, $\ker \text{ric} = \{0\}$ so $N_{\text{Ric}} = 0$, else $\text{Ric}^2 = N_{\text{Ric}}^2 = 0$. We re-find here the result of [7]. q.e.d.

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