



HAL
open science

Emergence of a non trivial fluctuating phase in the XY model on regular networks

Sarah de Nigris, Xavier Leoncini

► **To cite this version:**

Sarah de Nigris, Xavier Leoncini. Emergence of a non trivial fluctuating phase in the XY model on regular networks. 2012. hal-00721437v1

HAL Id: hal-00721437

<https://hal.science/hal-00721437v1>

Preprint submitted on 27 Jul 2012 (v1), last revised 16 Jan 2013 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Emergence of a non trivial fluctuating phase in the XY model on regular networks

SARAH DE NIGRIS and XAVIER LEONCINI

Centre de Physique Théorique , CNRS - Aix-Marseille Université, Luminy, Case 907, F-13288 Marseille cedex 9, France

PACS 05.20.-y – Classical statistical mechanics
PACS 05.45.-a – Nonlinear dynamics and chaos

Abstract – We study the XY model on diluted networks. Considering the regular one-dimensional lattice topology, we focus on the influence of the dilution parameter $2 \geq \gamma \geq 1$. We find that for $\gamma < 1.5$, the system does not exhibit a phase transition, while for $\gamma > 1.5$ a second order transition of the magnetisation arises and displays identical properties as the mean field (HMF) regime. Hence $\gamma_c = 1.5$ appears to be a critical value, for which, in an identified energy range, the magnetisation shows important fluctuations. We resort to analytical calculations of the magnetisation in the low temperatures approximation regime and we show that our analytics breaks down below the threshold of γ_c while it gives the correct value above, confirming the critical value $\gamma_c = 1.5$.

In recent years systems with long-range interactions have attracted increasing attention and have been widely studied, proving to have a far richer phenomenology than the models with short-range potentials. For the latter, the rise of equilibrium, in the microcanonical ensemble, is only governed by the conserved momenta of the dynamics and this unique stationary state does not depend on the initial particle distribution [1]. Moreover the essential property of *additivity* allows to construct the canonical ensemble from the microcanonical, the two approaches resulting equivalent in the thermodynamic limit. This straightforward picture complexifies when dealing with systems interacting via a long-range unscreened potential: in first instance, the additivity property is no longer present and this loss leads to the necessity of a separate treatment of the two ensembles [2–4]. Even more interestingly, those systems keep track of the initial configuration which actually determines the stationary state: for a particular set of initial conditions, long-lasting quasistationary states (QSSs) arise whose duration diverges with the system size, implying ergodicity breaking [5–7]. Furthermore, recently, an oscillating metastable state has been observed [8], enriching the already various scenario of long-range systems. In this Letter, we address the issue of investigating the transition from short-range to long-range regime from a quite different point of view than previous works. Instead of focusing on dynamical constraints, we chose as control parameter a *topological* con-

dition, which is imposing the connectivity per interacting unit. We used the paradigmatic 1D-XY model for rotors and we will show that we can identify two limit regimes: a short-ranged one for low connectivity while, in the limit of high connectivity, the system shows global coherence via a second order phase transition. The main result of the paper is, however, the emergence of a peculiar new state in between in which the order parameter is affected by important fluctuations. Furthermore, we will show analytically that this state stems from the special *topological* condition on the connectivity we imposed.

In general the XY model describes a system of N pairwise interacting units. At each unit i is assigned a real number θ_i , which we refer to as the *spin* i . In the following, we will consider the XY model from the point of view of classical Hamiltonian dynamical systems by adding a kinetic energy term to the XY Hamiltonian. The total Hamiltonian H takes the form:

$$H = \sum_{i=1}^N \frac{p_i^2}{2} + \frac{J}{2k} \sum_{i,j=1}^N \epsilon_{i,j} (1 - \cos(\theta_i - \theta_j)). \quad (1)$$

Because of the periodicity of the cosine function in the Eq.(1), the phase space for θ_i is restricted to the interval $[0, 2\pi[$. We associate to each spin i a canonical momentum p_i whose coupled dynamics with the $\{\theta_i\}$ will be given by

the set of Hamilton equations:

$$\dot{\theta}_i = p_i, \quad \dot{p}_i = -\frac{J}{k} \sum_{i,j=1}^N \epsilon_{i,j} (\cos \theta_j \sin \theta_i - \sin \theta_j \cos \theta_i). \quad (2)$$

The coupling constant J in Eqs.(1) and (2) is chosen positive in order to obtain a ferromagnetic behaviour and in the following it will be set at 1 without loss of generality. We encode the information about the links connecting the units in the *adjacency matrix* $\epsilon_{i,j}$:

$$\epsilon_{i,j} = \begin{cases} 1 & \text{if } i,j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}. \quad (3)$$

By construction, the adjacency matrix is a symmetric matrix with null trace. In Eq. (1) the normalisation constant k ensures the extensivity of the energy, according to the Kac prescription, and it corresponds to the number of links per spin, called the *degree*:

$$k \equiv \frac{1}{N} \sum_{i>j} \epsilon_{i,j} = \frac{2^{2-\gamma}(N-1)^\gamma}{N}. \quad (4)$$

In Eq.(4) the *dilution* γ , $\gamma \in [1,2]$ is introduced as the parameter of control to shift continuously from the short-range to the long-range regime [9]. It is straightforward to see that to the case $\gamma = 1$ corresponds to the linear chain with only nearest neighbours coupling and, on the other hand, $\gamma = 2$ corresponds to the full coupling of all the spins. In this latter case the Hamiltonian in Eq.(1) reduces to the *HMF* model [10]. We construct this way a lattice in which each spin is connected to $k/2$ neighbours on each side and the width of this neighbourhood is imposed by our choice of the dilution. To investigate the macroscopic behaviour of the system, we define the magnetisation $\mathbf{M} = (m_x, m_y)$, where $m_x = N^{-1} \sum_i \cos(\theta_i)$ and $m_y = N^{-1} \sum_i \sin(\theta_i)$. The modulus $M = |\mathbf{M}|$ indicates the degree of coherence of the spin angular distribution: the incoherent state will have $M = 0$, while finite values are naturally associated to more coherent states. Having set the structure of the lattice via the dilution, we performed simulations in the microcanonical ensemble and we studied the evolution of the total equilibrium magnetisation \bar{M} where the bar denotes the time average (we assume ergodicity). The system possesses two constants of motion preserved by the dynamics: the energy $H = E$ and the total angular momentum $P = \sum_i p_i$ which are set by the initial conditions. We chose to start the system with a Gaussian distribution for both for the spins and the momenta, we also impose $P = 0$. The numerical integration of Eqs. (2) is performed using a symplectic integrator [11], which ensures the conservation of the momenta E and P (which were monitored) and the symplectic structure. The thermodynamic quantities are calculated by averaging over time.

We first concentrated on low dilution values, $\gamma < 1.5$. For this regime of dilution, the system doesn't show a

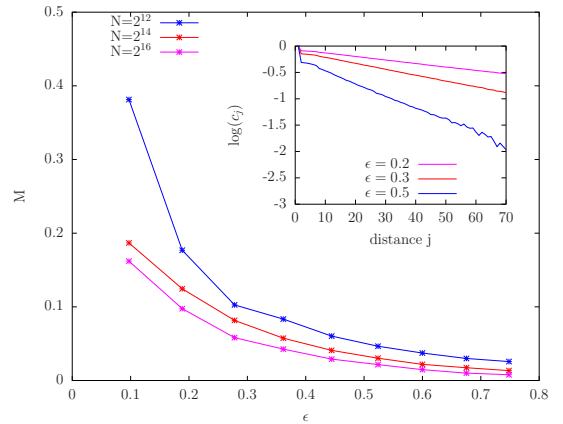


Fig. 1: (colour online) Equilibrium magnetisation for $\gamma = 1.25$ and different sizes; (inset) Correlation function c_j for $\gamma = 1.25$ and $N = 2^{14}$.

phase transition of the order parameter, this could have been inferred since for very low dilutions, the system is more or less identical to just short range interactions system and in that case the Mermin-Wagner theorem disproves the existence of long-range order in a 1-D system. Still finite size effects are at play, and results are displayed in Fig. 1: the magnetisation vanishes with the system size, so that in the thermodynamic limit we expect the residual magnetisation to be zero. Nevertheless, quasi-long-range order could still arise at finite temperatures like in the 2D short-ranged *XY*-model which displays the Berezinskij-Kosterlitz-Thouless phase transition [12, 13]. This particular phase transition is characterized by the change in behaviour of the correlation function, which decays as a power law at low temperatures and exponentially in the high temperature phase. Hence

to test the eventual presence of a Kosterlitz-Thouless transition, we monitored the correlation function:

$$c(j) = \frac{1}{N} \sum_{i=1}^N \cos(\theta_i - \theta_{i+j[N]}). \quad (5)$$

At equilibrium, the correlation decays exponentially (See insert in Fig. 1) at any temperature in the considered physical range, confirming the absence of the aforementioned phase transition. For those values of γ , we can conclude that the number of links is still too low to entail a change in the 1-D behaviour and it is interesting to notice that even a configuration with quite a large neighborhood per spin like $\gamma = 1.4$ still corresponds to short range interactions.

Symmetrically, the other important range to consider is $\gamma > 1.5$, when we approach the full coupling of the spins. As shown in Fig. 2a, the mean field transition of the order parameter is recovered in this dilution regime: it is worth stressing here that we recover the mean field result even for γ significantly lower than 2, e.g. for $\gamma = 1.6$, imply-

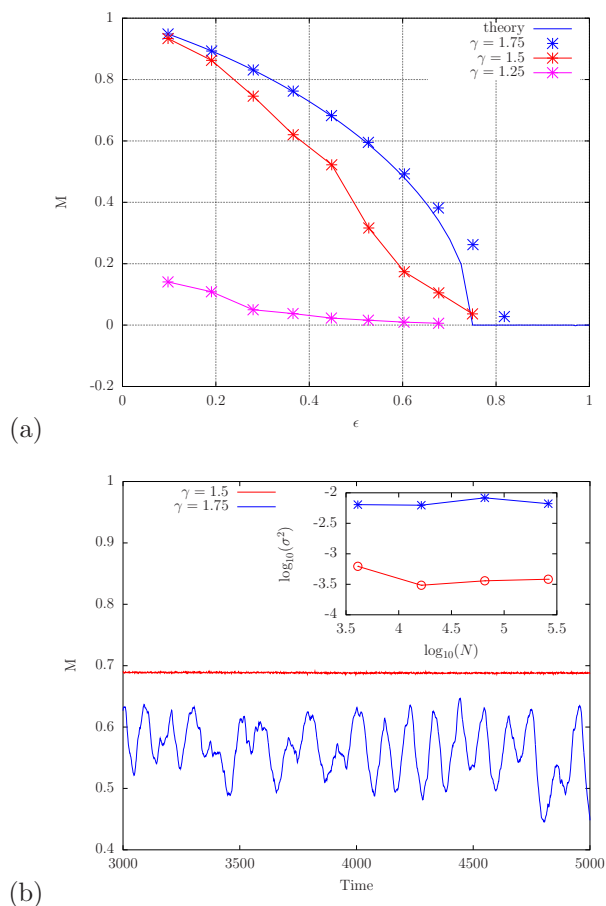


Fig. 2: (colour online) (a) Equilibrium magnetisation for $N = 2^{16}$ and different γ . (b) Time series for the order parameter with $N = 2^{18}$ and $\epsilon = 0.6$; (inset) Scaling of the magnetisation variance $\langle \sigma^2 \rangle$ for $\gamma = 1.5$, $\epsilon = 0.60$ (stars) and $\epsilon = 0.74$ (dots).

ing that global coherence is still reachable with a weaker condition than the full coupling. Naturally, in Fig. 2a, a shift exists between the simulations, performed at finite size, and the theoretical curve which is the one obtained for the mean field in the thermodynamic limit. Nevertheless this interval shrinks with the increasing size and, in that sense, it would be interesting to further investigate the finite size scaling of the critical energy. In both cases, $\gamma < 1.5$ and $\gamma > 1.5$, the variance of the magnetisation $\sigma^2 = \overline{(M - \bar{M})^2}$ vanishes linearly with the system size, ensuring the reaching of equilibrium in our simulations.

The transition between the 1-D behaviour and the mean field phase appears to be critical for $\gamma_c = 1.5$: for low energies $0.45 \leq \epsilon \leq 0.75$ the magnetisation is affected by important fluctuations and it is not clear if it does not reach an equilibrium state (Fig. 2b) on the time scales considered, or if these fluctuations are the mere reflection of the critical behaviour and will persist forever. Moreover the correlation function in Eq. (5) does not prove helpful in characterizing this peculiar state: it acquires the exponential behaviour only for densities of energy above $\epsilon = 0.7$,

while in the interesting interval of energies it is heavily affected by the fluctuations and it is impossible to properly determine its behaviour. We observed these effects on several sizes from $N = 2^{12}$ up to $N = 2^{18}$ and, when considering the scaling of σ^2 with the size (reported in the inset in Fig. 2b), it is evident that the variance is not affected by the increasing system size.

We argue that at $\gamma = 1.5$ the number of links is at its lower value to allow the arising of long range order and to shed light on the mechanism underneath, we derive an analytical form for the magnetisation which shows that the critical factor is embedded in the adjacency matrix, via its spectrum. As first hypothesis, we restrict our analysis to the low temperature regime, hence assuming that the difference $\theta_i - \theta_j \forall i, j$ is small. We therefore obtain a simple quadratic Hamiltonian: $H = \sum_i \frac{p_i^2}{2} + \frac{J}{4k} \sum_{i,j} \epsilon_{ij} (\theta_i - \theta_j)^2$. This assumption is justified by the simulations, as previously discussed; to proceed further, we consider a representation for the $\{\theta_i, p_i\}$ as a sum of random phased waves [14]:

$$\begin{aligned} \theta_i &= \sum_l \alpha_l(t) \cos\left(\frac{2\pi li}{N} + \phi_l\right), \\ p_i &= \sum_l \hat{\alpha}_l(t) \cos\left(\frac{2\pi li}{N} + \phi_l\right), \end{aligned} \quad (6)$$

where ϕ_l are randomly distributed phases on the circle. Since we make the hypothesis that the time dependence is totally encoded in the amplitudes α_l , the momenta are simply related to the angles via the first Hamilton equation $\dot{\theta}_i = p_i$. The basic idea behind this reasoning is that, at equilibrium, the momenta are Gaussian distributed variables, justifying the representation in Eqs. (6). We also observe that it consists in a linear changing of variable since we use N modes for our representation. If we now consider different sets of phases $\{\phi_l\}_m$ labeled by m , we have that each one of them corresponds to a phase space trajectory and, hence, it is possible to replace the ensemble average with the average on the random phases [14]. Consequently, injecting Eqs. (6) in the linearised Hamiltonian and averaging on the random phases, we obtain:

$$\frac{\langle H \rangle}{N} = \frac{1}{2} \sum_{l=1}^N \dot{\alpha}_l^2 + \alpha_l^2 (1 - \lambda_l), \quad (7)$$

where

$$\lambda_l = \frac{2}{k} \sum_{m=1}^{k/2} \cos\left(\frac{2\pi ml}{N}\right) \quad (8)$$

are the eigenvalues of the adjacency matrix. Using the second Hamilton equation $\frac{d}{dt} \left(\frac{\partial \langle H \rangle}{\partial \dot{\alpha}_l} \right) = - \frac{\partial \langle H \rangle}{\partial \alpha_l}$, we can thus derive from Eq. (7) a dispersion relation for the waves amplitudes that embeds two levels of information: at the microscopical level, the structure of the links, via the adjacency matrix spectrum and, from a more macroscopical point of view, Eq. (7) results from averaging on the random phases which, as explained, accounts for the ensemble averaging. Imposing the equipartition of energy at equilibrium for the obtained collection of harmonic oscillators, gives an additional relation between the frequencies ω_l and

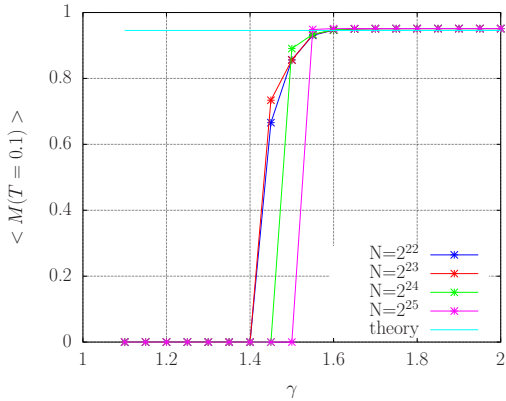


Fig. 3: (colour online) Approximated magnetisation $\langle M \rangle = \exp(-\frac{T}{2N} \sum_l \frac{1}{1-\lambda_l})$ for $T = 0.1$ versus the dilution parameter γ .

the amplitudes α_l : $\alpha_l^2 \omega_l^2 = 2T/N$. We evaluate now the magnetisation in the low temperature regime using the same approach: we inject the representation (6) and we average on the phases, obtaining [15]:

$$\langle \mathbf{M} \rangle = \prod_l J_0(\alpha_l) (\cos \theta_0, \sin \theta_0), \quad (9)$$

where θ_0 is the average of the $\{\theta_i\}$ which is a constant because of the conservation of the total momentum $P = 0$. The absolute value of the magnetisation $\langle M \rangle$ will hence be, from Eq. (9), the product over the l modes of the Bessel functions. To evaluate the logarithm of $\langle M \rangle$, we observe that, at equilibrium and in the limit of large system size, we expect to have small α_l^2 . We can thus approximate the Bessel functions in the limit of small amplitudes α_l which is, therefore, the low temperatures regime. This finally leads to:

$$\ln(\langle M \rangle) = - \sum_l \frac{\alpha_l^2}{4} = - \frac{T}{2N} \sum_l \frac{1}{1-\lambda_l}. \quad (10)$$

We calculated numerically Eq. (10) for increasing N and in Fig. 3 we show how it correctly grasps the behaviour for the magnetisation: in the low temperature regime, it retrieves the theoretical value for $\gamma > 1.5$ and it vanishes when $\gamma < 1.5$. Moreover, with the increasing size, the difference between the two regimes becomes sharper confirming the critical nature of $\gamma_c = 1.5$. The key for this peculiar effect at $\gamma = 1.5$ appears thus to be fully encoded in the spectrum of the adjacency matrix, which drives the system to the mean field regime or to the short range one according to the dilution parameter γ . Nevertheless, by a rapid inspection of Eq. (8), it appears not trivial to isolate the dependence of the eigenvalues on the dilution and on the size, each eigenvalue consisting in a sum of $k/2$ contributions. In Fig. 4, we show the behaviour of the spectrum for three representative values of γ : clearly the spectra qualitatively differ according to the dilution, but

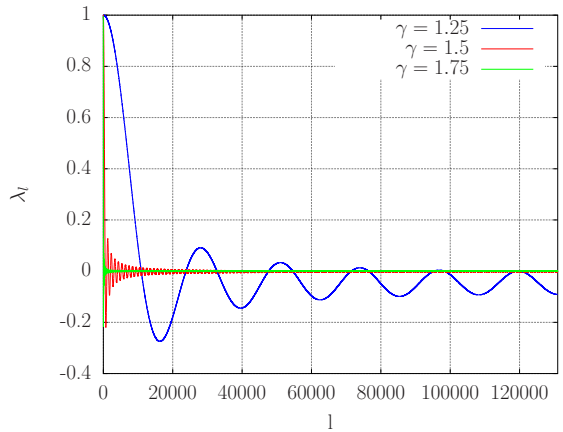


Fig. 4: (colour online) Spectra λ_l for $N = 2^{18}$ and different γ values.

how to quantify this difference is still object of a more refined analysis to precisely relate the spectrum properties to the magnetisation behaviour.

In this Letter we first introduced our model for the interaction, the XY model and focused on the regular lattice topology in which we controlled the degree of each spin via the dilution parameter γ . We showed that three different regimes existed, a low dilution regime ($\gamma < 1.5$), where the long-range order is absent, a high dilution phase in which the global coherence and mean field behaviour is recovered ($\gamma > 1.5$) and a peculiar behaviour at the threshold of $\gamma = 1.5$. Interestingly, we show that the mean field transition does not necessitate the full coupling of the spins, like in the HMF model or in a random diluted network, and it still arises for a regular topology even for $\gamma = 1.6$, quite far hence from the extremal configuration of $\gamma = 2$. However we consider that the main result of our analysis is the evidence of a unsteady almost turbulent like state when $\gamma = 1.5$: the important fluctuations affecting the order parameter and the invariance of these effects on the system size in a whole interval of energies are in total contrast with what observed in the other regimes, where with the same initial conditions the convergence to equilibrium is rapid. We presented a analytical calculation for the magnetisation, based on the method in [14], which is able to catch the appropriate behaviour in the two limits discussed before. This result points out that $\gamma = 1.5$ is indeed the critical value for this passage from the 1-D topology to the mean field frame. Moreover it proves that the spectrum of the adjacency matrix, which carries the information on the links, is crucial to understand this shift. As anticipated before, this unstable state stems from *topological features* of the lattice, instead of from a particular choice of the initial conditions as in [16–18]. This intrinsic difference indicates the important role of this state that it is observed, at our knowledge, for the first time. We anticipate that the same kind of phenomenon can be observed with different topologies and probably lower dilutions, and

believe that if we should find an efficient way to modify the dilution parameters, these systems could prove to be useful on-off switches for a somewhat large temperature range.

* * *

The authors are grateful to W. Ettoumi for discussions. S.d.N. has been supported by DGA/DS/MRIS.

REFERENCES

- [1] G.GALLAVOTTI, *eprint chao-dyn*, **9403004** (1994) .
- [2] J.BARRE, D.MUKAMEL S., *Phys Rev Lett*, **87** (2001) 030601.
- [3] LEYVRAZ F. and S.RUFFO, *J.Phys A*, **35** (2002) 285.
- [4] TORCINI A. and ANTONI M., *Phys. Rev. E*, **59** (1999) 2746.
- [5] D.MUKAMEL, S.RUFFO N., *Phys Rev Lett*, **95** (2005) 240604.
- [6] CAMPA A., DAUXOIS T. and RUFFO S., *Phys. Rep.*, **480** (2009) 57.
- [7] P.H. CHAVANIS J. V. and BOUCHET F., *Eur. Phys. J. B*, **46** (2005) 61.
- [8] H.MORITA K., *Phys Rev Lett*, **96** (2006) 050602.
- [9] CIANI A., RUFFO S. and FANELLI D., *Long-range Interaction, Stochasticity and Fractional Dynamics- Dedication to George M. Zaslavsky (1935-2008)* (HEP and Springer) 2010.
- [10] ANTONI M. and RUFFO S., *Phys. Rev. E*, **52** (1995) 3261.
- [11] MCLACHLAN R. I. and ATELA P., *Nonlinearity*, **5** (1992) 541.
- [12] V.L.BEREZINSKIJ, *Sov. Phys. JETP*, **32** (1971) 494.
- [13] KOSTERLITZ J. M. and THOULESS D. J., *Journal of Physics C: Solid State Physics*, **6** (1973) 1181.
- [14] LEONCINI X. and VERGA A., *Phys. Rev. E*, **64** (2001) 066101.
- [15] LEONCINI X., VERGA A. and RUFFO S., *Phys. Rev. E*, **57** (1998) 6377.
- [16] ANTONIAZZI A., FANELLI D., RUFFO S. and YAMAGUCHI Y. Y., *Phys. Rev. Lett.*, **99** (2007) 040601.
- [17] A.ANTONIAZZI, F.CALIFANO, D.FANELLI and S.RUFFO, *Phys Rev Lett*, **98** (2007) 150602.
- [18] ANTONIAZZI A., FANELLI D., BARRE J., CHAVANIS P.-H., DAUXOIS T. and RUFFO S., *Phys. Rev. E*, **75** (2007) 011112.