

**ON THE DYNAMIC OF
HOLOMORPHIC DIFFEOMORPHISMS GROUPS
FIXING A COMMUNE POINT, ON \mathbb{C}^n**

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ABSTRACT. In this paper, we study the action on \mathbb{C}^n of any group G of holomorphic diffeomorphisms (automorphisms) of \mathbb{C}^n fixing 0. Suppose that there is $x \in \mathbb{C}^n$, having an orbit which generates \mathbb{C}^n and also $\tilde{E}(x) = \mathbb{C}^n$, where $\tilde{E}(x)$ is the vector space generated by $L_G = \{D_0 f x, f \in G\}$. We give an important condition so that an orbit $G(x)$ is isomorphic (by linear map) to the orbit $L_G(x)$ of the linear group L_G . More if G is abelian, we prove the existence of a G -invariant open set U , dense in \mathbb{C}^n , in which every orbit O is relatively minimal (i.e. $\overline{O} \cap U$ is a closed non empty, G -invariant set and has no proper subset with these properties). Moreover, if G has a dense orbit in \mathbb{C}^n then every orbit of U is dense in \mathbb{C}^n .

1. Introduction

By definition, diffeomorphisms of \mathbb{C}^n which are holomorphic are called automorphisms. Denote by $Aut(\mathbb{C}^n)$ the group of all automorphisms of \mathbb{C}^n , it is very large and complicated for $n \geq 2$. One attempt to study it, is to study its finite subgroups or more general its subgroups which are isomorphic to compact Lie groups. This paper is among the few who studied the topological dynamic of abelian infinite subgroups of $Aut(\mathbb{C}^n)$. Let G be a subgroup of $Aut(\mathbb{C}^n)$ fixing 0. We consider the action $G \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ given by $(f, x) \mapsto f(x)$ (i.e the image of x by f). For a point $x \in \mathbb{C}^n$, denote by $G(x) = \{f(x), f \in G\} \subset \mathbb{C}^n$ the orbit of G through x . A subset $E \subset \mathbb{C}^n$ is called G -invariant if $f(E) \subset E$ for any $f \in G$; that is E is a union of orbits. Denote by \overline{E} (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E .

A subset E of \mathbb{C}^n is called a *minimal set* of G if E is closed in \mathbb{C}^n , non empty, G -invariant and has no proper subset with these properties. It is equivalent to say that E is a G -invariant set such that every orbit contained in E is dense in it. If Ω is a G -invariant set in \mathbb{C}^n , we say that E is a *minimal set in Ω* if it is a minimal set of the restriction $G|_{\Omega}$ of G to Ω . An orbit $O \subset \Omega$ is called *relatively minimal in Ω* if $\overline{O} \cap \Omega$ is a minimal set in Ω . This means that for every $x \in \overline{O} \cap \Omega$ we have $\overline{O} \cap \Omega = \overline{G(x)} \cap \Omega$. For example, a closed orbit in Ω is relatively minimal in Ω . In particular, every point in $Fix(G)$ is relatively minimal in \mathbb{C}^n .

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In [2], the authors give a comprehensive introduction to holomorphic dynamics, that is the dynamics induced by the iteration of various analytic maps in complex number spaces. In [3] the first named author studied the subgroup $Aut_1(\mathbb{C}^n)$ consisting of automorphisms with Jaccobi determinant 1 ("volume-preserving automorphisms") and in [4], the authors proved that for $n \geq 2$ shears generate a dense subgroup of $Aut_1(\mathbb{C}^n)$, but this subgroup, at least $n = 2$, is not the entire group $Aut_1(\mathbb{C}^n)$.

For a complex hyperbolic manifold M , denote by $Aut(M)$ be the group of holomorphic automorphisms of M . Proper actions by holomorphic transformations are found in abundance. A fundamental result due to Kaup (see [5]) states that every closed subgroup of $Aut(M)$ that preserves a continuous distance on M acts properly on M . Thus, Lie groups acting properly and effectively on M by holomorphic transformations are precisely those closed subgroups of $Aut(M)$ that preserve continuous distances on M . In particular, if M is a Kobayashi-hyperbolic manifold, then $Aut(M)$ is a Lie group acting properly on M (see also [6]). Recent results concerning the dynamics of holomorphic diffeomorphisms of compact complex surfaces are described in [7], [8] and [9], that require a nice interplay between algebraic geometry, complex analysis, and dynamical systems.

In [10] and [11], the author proved that if f and g are two commuting holomorphic maps of the unit disc Δ , then they must share a fixed point (in the sense of non-tangential limit if this point is on the boundary). This result has been generalized to domains in several variables by the author, proving that a generic characteristic of commuting holomorphic mappings is that of sharing a fixed point in the closure of the domain.

In this paper, we give an important condition so that an orbit $G(x)$ is isomorphic (by linear map) to the orbit $L_G(x)$ of the linear group L_G . The abelian case can be viewed as a generalization of the results given in [12]. We use a construction analogous to that given by S.Chihi in [16] for abelian linear group.

Denote by $L_G = \{D_0f, f \in G\}$, it is a subgroup of $GL(n, \mathbb{C})$ (see Lemma 2.2) and $vect(L_G)$ be the vector space generated by L_G . For every $u \in \mathbb{C}^n$, denote by:

- $L_G(u) = \{Au, A \in L_G\}$ the orbit of u defined by the natural action of the linear group L_G on \mathbb{C}^n .
- $\tilde{E}(u)$ be the vector space generated by $L_G(u)$.
- $\mathcal{A}(G)$ be the vector space generated by G , we see that $\mathcal{A}(G)$ is not an algebra (i.e. it is not stable by composition).
- $E(x) = \{f(x), f \in \mathcal{A}(G)\}$.
- For a fixed point $x \in \mathbb{C}^n$, we define the linear map $\Phi_x : \mathcal{A}(G) \rightarrow \mathbb{C}^n$ given by $\Phi_x(f) = f(x), f \in \mathcal{A}(G)$.
- The linear map $\varphi : \mathcal{A}(G) \rightarrow M_n(\mathbb{C})$ the linear map given by $\varphi(f) = D_0f$, for every $f \in \mathcal{A}(G)$.

The group G is called *dominant* if there is a point $x \in \mathbb{C}^n$ such that each set $G(x)$ and $L_G(x)$ generates \mathbb{C}^n and $ker(\varphi) = Ker(\Phi_x)$ (i.e. G is dominant if $\tilde{E}(x) = E(x) = \mathbb{C}^n$ for some $x \in \mathbb{C}^n$ and $ker(\varphi) = Ker(\Phi_x)$).

We generalize the result given in [12] for abelian subgroup of $GL(n, \mathbb{C})$ in the following Theorem:

Theorem 1.1. *Let G be a dominant subgroup of $Aut(\mathbb{C}^n)$, with $0 \in Fix(G)$. Let $x \in \mathbb{C}^n$ such that $Ker(\Phi_x) = Ker(\varphi)$ then $G(x)$ is isomorphic (by a linear map) to $L_G(x)$.*

If G is abelian, we can better describe its dynamics as follow:

Theorem 1.2. *Let G be an abelian dominant subgroup of $Aut(\mathbb{C}^n)$, with $0 \in Fix(G)$. Then there exist a G -invariant open set U , dense in \mathbb{C}^n satisfying: For every $y \in U$ such that $Ker(\Phi_y) = Ker(\varphi)$ then $G(y)$ is relatively minimal in U .*

We have the following corollaries.

Corollary 1.3. *Let G be an abelian dominant subgroup of $Aut(\mathbb{C}^n)$, with $0 \in Fix(G)$. If G has a dense orbit then every orbit in U is dense in \mathbb{C}^n .*

The proofs in this paper will use the results of the action of the linear Abelian groups on \mathbb{C}^n , that will be applied to the linear group $\{D_0f, f \in G\}$. The complex analysis is not used here. This paper is organized as follows: In Section 2, we give some results for abelian linear group. The Section 3 is devoted to prove the main results. In the section 4, we give two examples for $n = 2$.

2. Proof of Theorem 1.1

Let G be a dominant subgroup of $Aut(\mathbb{C}^n)$ of \mathbb{C}^n fixing 0. Denote by:

- $C^1(\mathbb{C}^n, \mathbb{C}^n)$ the vector space of all C^1 -differentiable maps of \mathbb{C}^n , it is well known that $C^1(\mathbb{C}^n, \mathbb{C}^n)$ is an algebra.
- $\mathcal{A}(G)$ be the vector space generated by G . (i.e. The smaller vector subspace of $C^1(\mathbb{C}^n, \mathbb{C}^n)$ containing G and stable by compositions).
- For a fixed point $x \in \mathbb{C}^n$, we define the linear map $\Phi_x : \mathcal{A}(G) \rightarrow \mathbb{C}^n$ given by $\Phi_x(f) = f(x)$, $f \in \mathcal{A}(G)$.
- $E(x) = \Phi_x(\mathcal{A}(G))$.

Lemma 2.1. *Let G be an abelian subgroup of $Aut(\mathbb{C}^n)$, fixing 0. Then $g(0) = 0$ for every $g \in \mathcal{A}(G)$.*

Proof. Let $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$ with $f_k \in G$ and $\alpha_k \in \mathbb{C}$, so

$$g(0) = \sum_{k=1}^p \alpha_k f_k(0) = 0.$$

□

Denote by $\varphi : \mathcal{A}(G) \rightarrow M_n(\mathbb{C})$ the linear map given by $\varphi(f) = D_0f$, for every $f \in \mathcal{A}(G)$. One observes that $L_G = \varphi(G)$.

Lemma 2.2. *Let G be a subgroup of $Aut(\mathbb{C}^n)$, fixing 0. Then L_G is a subgroup of $GL(n, \mathbb{C})$. Moreover, if G is abelian so is L_G .*

Proof. Since $L_G = \varphi(G)$, so it suffices to show that φ is a morphism. Let $f, g \in G$, so $\varphi(f \circ g) = D(f \circ g)(0) = Df(g(0)).Dg(0)$. By Lemma 2.1, $g(0) = 0$, so $\varphi(f \circ g) = D(f)(0).Dg(0) = \varphi(f).\varphi(g)$. The proof is completed. \square

2.1. Hamel basis and norm. [14] The main of this section is to justify the existence of a basis of every vector space. This result is trivial in the finite case, is in fact rather surprising when one thinks of infinite dimensional vector spaces, and the definition of a basis. Recall that a *Hamel basis* or simply a basis of a vector space E is a linearly independent set \mathcal{B} (every finite subset of \mathcal{B} is linearly independent) such that for each nonzero $x \in E$ there are $a_1, \dots, a_k \in \mathcal{B}$ and nonzero scalars $\alpha_1, \dots, \alpha_k$ (all uniquely determined) such that $x = \sum_{i=1}^k \alpha_i a_i$. The following theorem is equivalent to the axiom of choice family of axioms and theorems. In [14], C.D.Aliprantis and K.C.Border proved, in the following theorem, that Zorn's lemma implies that every vector space has a basis.

Theorem 2.3. ([14], *Theorem 1.8*) *Every nontrivial vector space has a Hamel basis.*

As a consequence, we found the important following results:

Theorem 2.4. *Every nontrivial vector space has a norm called Hamel norm.*

Proof. Let E be a nontrivial vector space over \mathbb{C} . By Theorem 2.3, E has a Hamel basis called $\mathcal{B} = (a_i)_{i \in I}$, for any set I of indices (not necessary countable). In this basis, every vector $x \in E$ has the form $x = \sum_{i \in I_x} \alpha_i a_i$, where $\alpha_i \in \mathbb{C}$ and $I_x \subset I$ with finite cardinal. The family $(\alpha_i)_{i \in I}$ with $\alpha_i = 0$ for every $i \in I \setminus I_x$, is called the coordinate of x . Now, define $\|x\| = \sum_{i \in I_x} |\alpha_i|$. It is easy to verify that $\|\cdot\|$ defines a norm on E by using the coordinate in the Hamel basis. We say that $\|\cdot\|$ is the Hamel norm associated to the Hamel basis \mathcal{B} . \square

Remark that any vector for the Hamel basis is with norm 1.

2.2. Linear map and isomorphism. A subset $E \subset \mathbb{C}^n$ is called G -invariant if $f(E) \subset E$ for any $f \in G$; that is E is a union of orbits. For a fixed vector $x \in \mathbb{R}^n \setminus \{0\}$, denote by:

- F_x is an algebraic supplement of $\text{Ker}(\Phi_x)$ in $\mathcal{A}(G)$. It is easy to show that $p_x := \dim(F_x) = \dim(E(x)) \leq n$ since $E(x) = \Phi_x(\mathcal{A}(G))$.
- $\mathcal{C}_x = (a_1, \dots, a_{p_x})$ is a basis of F_x .
- $\mathcal{B}_x = (b_i)_{i \in I}$ is a Hamel basis (Theorem 2.3) of $\text{Ker}(\Phi_x)$.
- $\mathcal{E}_x = (\mathcal{C}_x, \mathcal{B}_x)$ is a Hamel basis of $\mathcal{A}(G)$. By Theorem 2.4, $\mathcal{A}(G)$ is provided with the Hamel norm associated to the basis \mathcal{E}_x .

Lemma 2.5. *The linear map $\Phi_x : \mathcal{A}(G) \longrightarrow E(x)$ is continuous. In particular, $\text{Ker}(\Phi_x)$ is a closed subspace of $\mathcal{A}(G)$.*

Proof. Since Φ_x is linear and $\mathcal{A}(G)$ is a normed vector space (Lemma 2.4), we will verify the continuity of Φ_x on 0. Let $f \in \mathcal{A}(G)$ and write $f = f_1 + f_2$ with $f_1 \in F_x$ and $f_2 \in Ker(\Phi_x)$. Set $(\alpha_i)_{1 \leq i \leq p_x}$ and $(\beta_i)_{i \in I}$ be respectively the coordinates of f_1 and f_2 in \mathcal{C}_x and \mathcal{B}_x . Write $f = \sum_{i=1}^{p_x} \alpha_i a_i + \sum_{i \in I_2} \beta_i b_i$ where $I_2 \subset I$ with finite cardinal.

We have $\|f\| = \sum_{i=1}^{p_x} |\alpha_i| + \sum_{i \in I_2} |\beta_i|$ and $b_i(x) = 0$ for all $i \in I_2$. Therefore

$$\begin{aligned} \|\Phi_x(f)\| = \|f(x)\| &= \left\| \sum_{i=1}^{p_x} \alpha_i a_i(x) + \sum_{i \in I_2} \beta_i b_i(x) \right\| \\ &\leq \sum_{i=1}^{p_x} |\alpha_i| \|a_i(x)\| \\ &\leq \|f\| \sum_{i=1}^{p_x} \|a_i(x)\| \end{aligned}$$

Since $\sum_{i=1}^{p_x} \|a_i(x)\|$ is constant relative to f , then Φ_x is continuous. \square

Lemma 2.6. *Suppose that $\dim(\text{vect}(L_G)) = n$. Then the linear map $\varphi : \mathcal{A}(G) \rightarrow \text{vect}(L_G)$ is continuous. In particular, $Ker(\varphi)$ is a closed subspace of $\mathcal{A}(G)$.*

Proof. Since φ is linear and $\mathcal{A}(G)$ is a normed vector space (Lemma 2.4), we will verify the continuity of φ on 0. Firstly, see that $\text{cod}(Ker(\varphi)) = n$ is finite since $\dim(\text{vect}(L_G)) = n$. let F be an algebraic supplement to $Ker(\varphi)$ in $\mathcal{A}(G)$, $\mathcal{C}' = (a'_1, \dots, a'_n)$ and $\mathcal{B}' = (b'_i)_{i \in J}$ are respectively the Hamel basis of F and $Ker(\varphi)$ (Lemma 2.3). Let $f \in \mathcal{A}(G)$ and write $f = f_1 + f_2$ with $f_1 \in F$ and $f_2 \in Ker(\varphi)$. Set $(\alpha_i)_{1 \leq i \leq n}$ and $(\beta_i)_{i \in J}$ be respectively the coordinates of f_1 and f_2 in \mathcal{C}' and \mathcal{B}' . Write $f = \sum_{i=1}^n \alpha_i a'_i + \sum_{i \in I_2} \beta_i b'_i$ where $I_2 \subset J$ with finite cardinal. We have

$\|f\| = \sum_{i=1}^n |\alpha_i| + \sum_{i \in I_2} |\beta_i|$ and $\varphi(b'_i) = Db'_i(0) = 0$ for all $i \in I_2$. Therefore

$$\begin{aligned} \|\varphi(f)\| = \|Df(0)\| &= \left\| \sum_{i=1}^n \alpha_i Da'_i(0) + \sum_{i \in I_2} \beta_i Db'_i(0) \right\| \\ &\leq \sum_{i=1}^n |\alpha_i| \|Da'_i(0)\| \\ &\leq \|f\| \sum_{i=1}^n \|Da'_i(0)\| \end{aligned}$$

Since $\sum_{i=1}^n \|Da'_i(0)\|$ is constant relative to f , then φ is continuous. \square

Lemma 2.7. ([15], 3.5) *Let E be a topological vector space, and let M be a closed subspace of finite codimension. Then $E = M \oplus N$ is a topological sum, for every algebraic complementary subspace N of M .*

Corollary 2.8. *The algebraic sum $\mathcal{A}(G) = F_x \oplus Ker(\Phi_x)$ is topological. In particular, $F_x \oplus Ker(\Phi_x)$ and $\mathcal{A}(G)$ are topological isomorphic by the map: $(f_1, f_2) \mapsto f_1 + f_2$.*

Proof. By the Theorem 2.4, $\mathcal{A}(G)$ is a normed vector space so it is a topological vector space. By Lemma 2.5, Φ_x is continuous and so its kernel is closed vector space with finite codimension. The results follows directly by applying the Lemma 2.7 for $E = \mathcal{A}(G)$ and $M = Ker(\Phi_x)$. \square

By Corollary 2.8, we can identify $\mathcal{A}(G)$ with $F_x \oplus Ker(\Phi_x)$, so every $f \in \mathcal{A}(G)$ is denoted by $f = (f_1, f_2) = f_1 + f_2$ with $f_1 \in F_x$ and $f_2 \in Ker(\Phi_x)$.

Lemma 2.9. *Let H and K be two closed vector subspaces of $\mathcal{A}(G)$ such that $cod(H) = cod(K) = j \geq 1$. Let $\psi \in \mathcal{A}(G) \setminus (H \cup K)$. Then there exists a vector space F of $\mathcal{A}(G)$, containing ψ and satisfying $\mathcal{A}(G) = F \oplus H = F \oplus K$.*

Proof. There are two cases;

Case 1: If $j = 1$, then we take $F = \mathbb{C}\psi$.

Case 2: Suppose that $j \geq 2$. Denote by $H' = H \oplus \mathbb{C}\psi$ and $K' = K \oplus \mathbb{C}\psi$. Suppose that $H' \neq K'$ (otherwise, it is easy to take a comment supplement) and let $f_1 \in H' \setminus K'$ and $g_1 \in K' \setminus H'$, so $h_1 = f_1 + g_1 \notin H' \cup K'$. Denote by $H_1 = \mathbb{C}h_1 \oplus H'$ and $K_1 = \mathbb{C}h_1 \oplus K'$. We establish two cases:

- If $H_1 = K_1$, then any supplement F of H_1 in $\mathcal{A}(G)$ is a supplement of K_1 in $\mathcal{A}(G)$, the proof follows then.

- If $H_1 \neq K_1$, we take $f_2 \in H_1 \setminus K_1$ and $g_2 \in K_1 \setminus H_1$, so $h_2 = f_2 + g_2 \notin H_1 \cup K_1$. Denote by $H_2 = \mathbb{C}h_2 \oplus H_1$ and $K_2 = \mathbb{C}h_2 \oplus K_1$.

We repeat the same processes until $j - 2$ times and we obtain:

- If $H_{j-2} = K_{j-2}$, then any supplement F of H_{j-2} in $\mathcal{A}(G)$ is a supplement of K_{j-2} in $\mathcal{A}(G)$.

- If $H_{j-2} \neq K_{j-2}$, we take $f_{j-1} \in H_{j-2} \setminus K_{j-2}$ and $g_{j-1} \in K_{j-2} \setminus H_{j-2}$, so $h_{j-1} = f_{j-1} + g_{j-1} \notin H_{j-2} \cup K_{j-2}$. Denote by $H_{j-1} = \mathbb{C}h_{j-1} \oplus H_{j-2}$ and $K_{j-1} = \mathbb{C}h_{j-1} \oplus K_{j-2}$. We obtain then $H_{j-1} = K_{j-1} = \mathcal{A}(G)$. Hence the proof is completed by taking $F = vect(\psi, h_1, \dots, h_{j-1})$. \square

Denote by $\tilde{r}_x = dim(\tilde{E}(x))$.

Lemma 2.10. *If $r(x) = \tilde{r}_x$, then there exists a commune vector space F_x supplement to $Ker(\Phi_x)$ and to $ker(\varphi)$ in $\mathcal{A}(G)$. (i.e $F_x \oplus Ker(\Phi_x) = F_x \oplus Ker(\varphi) = \mathcal{A}(G)$).*

Proof. By Lemmas 2.5 and 2.6, the maps Φ_x and φ are continuous, so $Ker(\Phi_x)$ and $Ker(\varphi)$ are closed. Since $cod(Ker(\Phi_x)) = cod(Ker(\varphi)) = n$, then by lemma 2.9, there exists a commune supplement F_x to $Ker(\Phi_x)$ and to $ker(\Phi_x)$ in $\mathcal{A}(G)$ containing id (identity map of \mathbb{C}^n), because $id \in \mathcal{A}(G) \setminus (Ker(\Phi_x) \cup Ker(\varphi))$. \square

Lemma 2.11. *The linear map $\Phi_x : F_x \longrightarrow E(x)$ given by $\Phi_x(f) = f(x)$ is an isomorphism.*

Proof. Here, F_x is considered as a supplement to $Ker(\Phi_x)$ in $\mathcal{A}(G)$. The proof follows directly from the fact that Φ_x is linear surjective and $dim(F_x) = dim(E(x)) = n$. \square

Lemma 2.12. *If $r(x) = \tilde{r}_x = n$, then the restriction $\varphi_1 : F_x \longrightarrow vect(L_G)$ of φ from F_x unto $vect(L_G)$ is an isomorphism.*

Proof. Here, F_x is considered as a supplement to $Ker(\varphi)$ in $\mathcal{A}(G)$. The proof follows directly from the fact that φ_1 is linear surjective and $dim(F_x) = dim(vect(L_G)) = n$ (Corollary 3.5). \square

Lemma 2.13. *If $r(x) = \tilde{r}_x = n$ and $ker(\varphi) = Ker(\Phi_x)$, then the map $\varphi_x := \tilde{\Phi}_x \circ \varphi_1 \circ \Phi_x^{-1} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ defined by $\varphi_x(f(x)) = D_0f(x)$, $f \in F_x$, is an isomorphism and satisfying:*

- (i) $\varphi_x(G(x)) = L_G(x)$.
- (ii) Let $y \in \mathbb{C}^n$ and $z = \varphi_x(y)$ then $\varphi_x(G(y)) = \overline{L_G(z)}$.
- (iii) for every $y \in \overline{G(x)}$ we have $z = \varphi_x(y) \in \overline{L_G(x)}$.

Proof. By Lemma 2.10, we can assume that F_x is a commune supplement to $Ker(\Phi_x)$ and to $Ker(\varphi)$ in $\mathcal{A}(G)$. Since $r(x) = \tilde{r}_x = n$, then $E(x) = \tilde{E}(x) = \mathbb{C}^n$.

(i) φ_x is well defined; Indeed, let $f \in \mathcal{A}(G)$ and write $f = g' + h$ with $g' \in Ker(\Phi_x)$ and $h \in Ker(\varphi)$, so $f(x) = g'(x)$. Then $\varphi_x(f(x)) = \tilde{\Phi}_x \circ \varphi_1 \circ \Phi_x^{-1}(g'(x)) = \tilde{\Phi}_x \circ \varphi_1(g') = \tilde{\Phi}_x(D_0g') = D_0g'(x)$. Since $Ker(\Phi_x) = Ker(\varphi)$ then $D_0h = 0$, so $D_0f = D_0g'$. It follows that $\varphi_x(f(x)) = D_0f(x)$.

φ_x is an isomorphisms, since $\varphi_x = \tilde{\Phi}_x \circ \varphi_1 \circ \Phi_x^{-1}$ and by Lemmas 2.12, 3.4 and 2.11, it is composed by isomorphisms.

Let $y \in \varphi_x(G(x))$, there exists $f \in G$ such that $y = \varphi_x(f(x))$, then $\varphi_x(y) = \varphi_x(f(x)) = D_0f(x) \in L_G(x)$. For the converse, let $y \in L_G(x)$, there exists $f \in G$ such that $y = D_0f(x)$. Write $f = g + h$ with $g \in F_x$ and $h \in Ker(\varphi)$. Therefore $y = D_0g(x) + D_0h(x) = D_0g(x) = \varphi_x(g(x))$. Since $Ker(\Phi_x) = Ker(\varphi)$, then $h \in Ker(\Phi_x)$, so $h(x) = 0$. As $f(x) = g(x) + h(x) = g(x)$, it follows that $\varphi_x(f(x)) = y$. Hence, $y \in \varphi_x(G(x))$.

(ii) Let $y \in \mathbb{C}^n$ and $z = \varphi_x(y)$. By Lemma 2.11, $\Phi_x(F_x) = E(x) = \mathbb{C}^n$, then there exists $g \in F_x$ such that $y = g(x)$, so $z = D_0g(x)$. Let $f \in G$ then $\varphi_x(f(y)) = D_0(f \circ g)(x) = D_0f \circ D_0g(x) = D_0f(z)$. Hence, $\varphi_x(f(y)) \in L_G(z)$.

For the converse, let $a \in L_G(z)$, there exists $f \in G$ such that $a = D_0f(z)$. Then $\varphi_x(f(y)) = \varphi_x(f \circ g(x)) = D_0(f \circ g)(x) = D_0f \circ D_0g(x) = D_0f(z) = a$. Hence, $a \in \varphi_x(G(y))$.

(iii) Since $y \in \mathbb{C}^n$, there exists $g \in F_x$ such that $y = g(x)$, so $z = D_0g(x)$. By continuity of φ_x and by (i), we have $z \in \varphi_x(\overline{G(x)}) \subset \overline{\varphi_x(G(x))} = \overline{L_G(x)}$. \square

3. The abelian case

3.1. Some results for abelian linear group. Let $M_n(\mathbb{C})$ be the set of complex square matrices of order $n \geq 1$, and let $GL(n, \mathbb{C})$ be the group of the invertible matrices of $M_n(\mathbb{C})$. Denote by

- $\mathbb{T}_n(\mathbb{C})$ the set of all lower-triangular matrices over \mathbb{C} , of order n and with only one eigenvalue.
- $\mathbb{T}_n^*(\mathbb{C}) = \mathbb{T}_n(\mathbb{C}) \cap GL(n, \mathbb{C})$ (*i.e.* the subset of matrix of $\mathbb{T}_n(\mathbb{C})$ having a non zero eigenvalue), it is a subgroup of $GL(n, \mathbb{C})$.
- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.

Let $r \in \mathbb{N}^*$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ such that $\sum_{i=1}^r n_i = n$. Denote by:

- $\mathcal{K}_{\eta, r}(\mathbb{C}) = \{M = \text{diag}(T_1, \dots, T_r) \in M_n(\mathbb{C}) : T_k \in \mathbb{T}_{n_k}(\mathbb{C}), k = 1, \dots, r\}$.
 - $\mathcal{K}_{\eta, r}^*(\mathbb{C}) = \mathcal{K}_{\eta, r}(\mathbb{C}) \cap GL(n, \mathbb{C})$, it is a subgroup of $GL(n, \mathbb{C})$.
 - v^T the transpose of a vector $v \in \mathbb{C}^n$.
 - $\mathcal{E}_n = (e_1, \dots, e_n)$ the standard basis of \mathbb{C}^n .
 - I_n the identity matrix on \mathbb{C}^n .
 - $u_0 = [e_{1,1}, \dots, e_{r,1}]^T \in \mathbb{C}^n$, where $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$, $1 \leq k \leq r$.
- For any subset E of \mathbb{C}^n (resp. $M_n(\mathbb{C})$), denote by $\text{vect}(E)$ the vector space generated by E .

In [13], the authors proved the following Proposition:

Proposition 3.1. ([13], Proposition 6.1.) *Let L be an abelian subgroup of $GL(n, \mathbb{C})$, then there exists $P \in GL(n, \mathbb{C})$ such that $\tilde{L} = P^{-1}LP$ is a subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$, for some $1 \leq r \leq n$ and $\eta \in \mathbb{N}_0^r$.*

For such matrix P define $v_0 = Pu_0$. Let L be an abelian subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$. denote by:

- $V = \prod_{k=1}^r \mathbb{C}^* \times \mathbb{C}^{n_k-1}$. One has $\mathbb{C}^n \setminus V = \bigcup_{k=1}^r H_k$, where

$$H_k = \{u = [u_1, \dots, u_r]^T, u_k \in \{0\} \times \mathbb{C}^{n_k-1}, u_j \in \mathbb{C}^{n_j}, j \neq k\}.$$

See that each H_k is a L -invariant vector space of dimension $n - 1$.

Lemma 3.2. ([16], Proposition 3.1) *Let L be an abelian subgroup of $GL(n, \mathbb{C})$ and $u \in \mathbb{C}^n$. Then for every $v \in \text{vect}(L(u))$ there exist $B \in \text{vect}(L)$ such that $Bu = v$.*

For an abelian subgroup L of $L(n, \mathbb{C})$, it is called dominant if $\text{vect}(L(x)) = \mathbb{C}^n$ for some $x \in \mathbb{C}^n$, where $L(x) = \{Ax, A \in G\}$.

Proposition 3.3. *Let \tilde{L} be an abelian linear subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ with $\eta = (n_1, \dots, n_r)$. Then the following assertions are equivalent:*

- (i) \tilde{L} is dominant.
- (ii) For every $u \in V$, we have $\text{vect}(\tilde{L}(u)) = \mathbb{C}^n$. In particular, $\text{vect}(\tilde{L}(u_0)) = \mathbb{C}^n$.

Proof. Suppose that \tilde{L} is dominant, then there is $u \in \mathbb{C}^n$ such that $vect(\tilde{L}(u)) = \mathbb{C}^n$. Remark that $u \in V$, since $\mathbb{C}^n \setminus V$ is a union of r \tilde{L} -invariant vector spaces with dimensions $n - 1$ and let $v \in V$. By applying lemma 3.2 on \tilde{L} , there exist $B \in vect(\tilde{L})$ such that $Bu = v$. As $\mathcal{K}_{\eta,r}(\mathbb{C})$ is a vector space then $Vect(L) \subset \mathcal{K}_{\eta,r}(\mathbb{C})$. Write $u = [u_1, \dots, u_r]^T$, $v = [v_1, \dots, v_r]^T$ with $u_k = [x_{k,1}, \dots, x_{k,n_k}]^T$, $v_k = [y_{k,1}, \dots, y_{k,n_k}]^T \in \mathbb{C}^* \times \mathbb{C}^{n_k-1}$ and $B = diag(B_1, \dots, B_r)$ with

$$B_k = \begin{bmatrix} \mu_{B_k} & & & 0 \\ a_{2,1}^{(k)} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{n_k,1}^{(k)} & \dots & a_{n_k,n_k-1}^{(k)} & \mu_{B_k} \end{bmatrix}, \quad 1 \leq k \leq r$$

then $\mu_{B_k} x_{k,1} = y_{k,1}$, so $\mu_{B_k} = \frac{y_{k,1}}{x_{k,1}} \neq 0$, hence $B \in GL(n, \mathbb{C})$. Then $B(\tilde{L}(u)) = \tilde{L}(v)$. We conclude that $vect(\tilde{L}(v)) = \mathbb{C}^n$. The converse is obvious. \square

Denote by:

- $\Psi_x : vect(L_G) \longrightarrow \tilde{E}(x) \subset \mathbb{C}^n$ the linear map given by $\Psi_x(A) = Ax$.

Lemma 3.4. *Let G be an abelian subgroup of $Aut(\mathbb{C}^n)$ such that $0 \in Fix(G)$. Then $\tilde{E}(x)$ is L_G -invariant and the linear map $\Psi_x : vect((L_G)_{|\tilde{E}(x)}) \longrightarrow \tilde{E}(x)$ is an isomorphism, where $(L_G)_{|\tilde{E}(x)}$ is the restriction of L_G on $\tilde{E}(x)$.*

Proof. By construction, $\tilde{E}(x)$ is L_G -invariant and Ψ_x is linear and surjective. Let $A \in Ker(\Psi_x)$ and $y \in \tilde{E}(x)$. Then there is $B \in vect((L_G)_{|\tilde{E}(x)})$ such that $y = Bx$. Now, $Ay = ABx = BAx = 0$, so $A = 0$. Hence Ψ_x is injective, so it is an isomorphism. \square

Corollary 3.5. *Let G be an abelian dominant subgroup of $Aut(\mathbb{C}^n)$ such that $0 \in Fix(G)$. Then $dim(vect(L_G)) = n$.*

Proof. Since G is dominant, then there is $x \in \mathbb{C}^n$ such that $\tilde{E}(x) = \mathbb{C}^n$. Then by lemma 3.4, $\Psi_x : vect(L_G) \longrightarrow \mathbb{C}^n$ is an isomorphism, so $dim(vect(L_G)) = n$. \square

Denote by:

- $\tilde{\Omega}_n = \{x \in \mathbb{C}^n, dim(vect(\tilde{L}(x))) = n\}$.

Lemma 3.6. *Let \tilde{L} be a dominant abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$. Then $\tilde{\Omega}_n = V$.*

Proof. Since \tilde{L} is dominant, then by Proposition 3.3, for every $u \in V$, $vect(\tilde{L}(u)) = \mathbb{C}^n$, hence $V \subset \tilde{\Omega}_n$. For the converse, let $u \in \tilde{\Omega}_n$, then $dim(vect(\tilde{L}(u))) = n$. It follows that $u \in V$ because $\mathbb{C}^n \setminus V$ is a union of r \tilde{L} -invariant vector spaces of dimension $n - 1$. This completes the proof. \square

Denote by:

- $\tilde{\Omega}_k = \{y \in \mathbb{C}^n, \tilde{r}_y \geq k\}$, for every $0 \leq k \leq n$. By applying Lemma 3.11 given in [16] to the abelian linear group L_G , we found the following result:

Lemma 3.7. ([16], Lemma 3.11) $\tilde{\Omega}_k$ is a L_G -invariant dense open subset of \mathbb{C}^n .

Lemma 3.8. ([16], Theorem 3.10) Let $x \in \tilde{\Omega}_n$ then for every $y \in \overline{L_G(x)} \cap \tilde{\Omega}_n$ we have $\overline{L_G(y)} \cap \tilde{\Omega}_n = \overline{L_G(x)} \cap \tilde{\Omega}_n$.

Denote by:

- $\Omega_n = \{y \in \mathbb{C}^n, \dim(E(y)) = n\}$.

Lemma 3.9. (Under the above notations) Suppose that $r(x) = \tilde{r}_x = n$ and $\ker(\varphi) = \text{Ker}(\Phi_x)$. We have $\varphi_x(\Omega_n) = \tilde{\Omega}_n$.

Proof. Let $y \in \Omega_n$ and $z = \varphi_x(y)$. By Lemma 2.13,(ii), $\varphi_x(G(y)) = L_G(z)$. Since φ_x is linear, then $\varphi_x(\text{vect}(G(y))) = \tilde{E}(z) = \mathbb{C}^n$, since $\tilde{r}_z = n$, where $\text{vect}(G(y))$ is the vector space generated by $G(y)$. As $\text{vect}(G(y)) \subset E(y)$, then $\varphi_x(E(y)) = \mathbb{C}^n$, so $r(y) = n$. It follows that $z \in \tilde{\Omega}_n$. For the converse we use the same proof for φ_x^{-1} . \square

3.2. Proof of Theorem 1.2.

Denote by:

- $r(x) = \dim(E(x))$.
- $U_k = \{x \in \mathbb{C}^n, r(x) \geq k\}$, for every $k \in \mathbb{N}$.
- $r_G = \max\{r(x), x \in \mathbb{C}^n\}$.

Proposition 3.10. Let G be an abelian subgroup of $\text{Aut}(\mathbb{C}^n)$, such that $0 \in \text{Fix}(G)$. Then for every $0 \leq k \leq r_G$, U_k is a G -invariant open subset of \mathbb{C}^n .

Proof. In the first, remark that the rank $r(y)$ is constant on any orbit $G(y)$, $y \in E(x)$. So U_k is G -invariant for every $0 \leq k \leq r_G$. Let's show that U_t is an open set: Let $y \in U_t$ and $r = r_y$, so $r \geq t$. Then there exist $f_1, \dots, f_r \in G$ such that the r vectors $f_1(y), \dots, f_r(y)$ are linearly independent in $E(y)$. For all $z \in \mathbb{C}^n$, we consider the Gram's determinant

$$\Delta(z) = \det(\langle f_i(z) | f_j(z) \rangle)_{1 \leq i, j \leq r}$$

of the vectors $f_1(z), \dots, f_r(z)$ where $\langle \cdot | \cdot \rangle$ denotes the scalar product in \mathbb{C}^n . It is well known that these vectors are independent if and only if $\Delta(z) \neq 0$, in particular $\Delta(y) \neq 0$. Let

$$V_y = \{z \in \mathbb{C}^n, \Delta(z) \neq 0\}$$

The set V_y is open in \mathbb{C}^n , because the map $z \mapsto \Delta(z)$ is continuous. Now $\Delta(y) \neq 0$, and so $y \in V_y \subset U_k$. The proof is completed. \square

Proof of Theorem 1.2. Since G is dominant, then there is $a \in \mathbb{C}^n$ such that $r(a) = \tilde{r}_a = n$ and $\ker(\varphi) = \ker(\Phi_a)$, so $\Omega_n \neq \emptyset$. Let $U = \Omega_n \cap \tilde{\Omega}_n$, since $\overline{\Omega_n} = \overline{\tilde{\Omega}_n} = \mathbb{C}^n$, then $\overline{U} = \mathbb{C}^n$. Let $x \in U$ such that $\ker(\varphi) = \ker(\Phi_x)$ and let $y \in \overline{G(x)} \cap U$ such that. By Lemma 2.13, there exists an isomorphism $\varphi_x : \mathbb{C}^n \rightarrow \mathbb{C}^n$ satisfying $\varphi_x(G(x)) = L_G(x)$ and $\varphi_x(G(y)) = L_G(z)$ with $z = \varphi_x(y)$. By Lemma 3.9, we have $\varphi_x(\Omega_n) = \tilde{\Omega}_n$, so $z \in \varphi_x(\overline{G(x)} \cap \Omega_n) = \overline{L_G(x)} \cap \tilde{\Omega}_n$. By Lemma 3.8, we have

$$\overline{L_G(z)} \cap \tilde{\Omega}_n = \overline{L_G(x)} \cap \tilde{\Omega}_n \quad (1).$$

Therefore by (1) we obtain

$$\begin{aligned} \overline{G(x)} \cap \Omega_n &= \varphi_x^{-1}(\overline{L_G(x)}) \cap \varphi_x^{-1}(\tilde{\Omega}_n) \\ &= \varphi_x^{-1}(\overline{L_G(x)} \cap \tilde{\Omega}_n) \\ &= \varphi_x^{-1}(\overline{L_G(z)} \cap \tilde{\Omega}_n) \\ &= \varphi_x^{-1}(\overline{L_G(z)}) \cap \varphi_x^{-1}(\tilde{\Omega}_n) \\ &= \overline{G(y)} \cap \Omega_n. \end{aligned}$$

We conclude that $\overline{G(x)} \cap U = \overline{G(y)} \cap U$. This completes the proof. \square

Proof of Corollary 1.3. Let O be a dense orbit in \mathbb{C}^n (i.e. $\overline{O} = \mathbb{C}^n$). Then for every $x \in O$, we have $r(x) = n$, so $O \subset U$ and $\overline{O} \cap U = U$. Since O is relatively minimal in U (Theorem 1.2), then for every orbit $L \subset U$, we have $\overline{L} \cap U = \overline{O} \cap U$. Therefore $\overline{L} = \overline{O} = \mathbb{C}^n$. \square

4. Particular cases

4.1. Linear case. Suppose that G is a dominant abelian subgroup of $GL(n, \mathbb{C})$. In this case, we have $G = L_G$ and $\mathcal{A}(G) = \text{vect}(G)$. It follows that $\mathcal{A}(G)$ has a finite dimension. Moreover, $G(x) = L_G(x)$, so the theorem1.1 becomes trivial. For the theorem1.2, we found the structure's theorem given in [12], but we give an other form of the G -invariant open set U as follow:

$$U := \{x \in \mathbb{C}^n, A \in \text{vect}(G) \text{ with } Ax = 0 \implies A = 0\}.$$

This means that

$$U := \{x \in \mathbb{C}^n, \ker(\Phi_x) = \{0\}\}.$$

4.2. Affine case. Suppose that G is a dominant abelian group of affine maps of \mathbb{C}^n fixing a commune point. Denote by $Fix(G)$ the set of all commune fixed points of all elements of G , so $Fix(G) \neq \emptyset$. In this case, every $f \in G$ has the form: $f(x) = Ax + b$ with $A \in GL(n, \mathbb{C})$ and $b \in \mathbb{C}^n$. Write $f := (A, b)$ and we have $L_G = \{A, (A, b) \in G\}$ (i.e. L_G is the group of linear part of all $f \in G$). In [1], (Proposition 3.1), the authors proved that G is \mathcal{T} -conjugate (conjugation by translation) to L_G , so there exists a translation T of \mathbb{C}^n such that $G(x) = T(L_G(T^{-1}x))$. But by theorem1.1, we give a linear isomorphism φ_x satisfying $G(x) = \varphi_x(L_G(x))$. Here, $\mathcal{A}(G) = \text{vect}(G)$. It follows that $\mathcal{A}(G)$ has a finite dimension. For the theorem1.2, we found the

structure's theorem given in [1], but we give an other form of the G -invariant open set U as follow:

$$U := \{x \in \mathbb{C}^n, (A, b) \in \text{vect}(G) \text{ with } Ax + b = 0 \implies A = 0\}.$$

This means that

$$U := \{x \in \mathbb{C}^n, \text{Ker}(\Phi_x) = \{0\}\}.$$

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