

# THE ACTION OF ABELIAN $C^1$ -DIFFEOMORPHISMS GROUP FIXING A POINT, ON $\mathbb{C}^n$

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ABSTRACT. In this paper, we study the action of any abelian subgroup  $G$  of  $Diff^1(\mathbb{C}^n)$  on  $\mathbb{C}^n$  fixing 0. Suppose that there is  $x \in \mathbb{C}^n$ , having an orbit which generates  $\mathbb{C}^n$  and also  $\tilde{E}(x) = \mathbb{C}^n$ , where  $\tilde{E}(x)$  is the vector space generated by  $\{Df_0x, f \in G\}$ . We prove the existence of a  $G$ -invariant open set  $U$ , dense in  $\mathbb{C}^n$ , in which every orbit is minimal. Moreover, if  $G$  has a dense orbit in  $\mathbb{C}^n$  then every orbit of  $U$  is dense in  $\mathbb{C}^n$ .

## 1. Introduction

Denote by  $Diff^1(\mathbb{C}^n)$  the group of all  $C^1$ -diffeomorphisms of  $\mathbb{C}^n$ . Let  $G$  be an abelian subgroup of  $Diff^1(\mathbb{C}^n)$  fixing 0. There is a natural action  $G \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ .  $(f, x) \mapsto f(x)$ . For a point  $x \in \mathbb{C}^n$ , denote by  $G(x) = \{f(x), f \in G\} \subset \mathbb{C}^n$  the orbit of  $G$  through  $x$ . A subset  $E \subset \mathbb{C}^n$  is called  $G$ -invariant if  $f(E) \subset E$  for any  $f \in G$ ; that is  $E$  is a union of orbits. Denote by  $\overline{E}$  (resp.  $\overset{\circ}{E}$ ) the closure (resp. interior) of  $E$ .

A subset  $E$  of  $\mathbb{C}^n$  is called a *minimal set* of  $G$  if  $E$  is closed in  $\mathbb{C}^n$ , non empty,  $G$ -invariant and has no proper subset with these properties. It is equivalent to say that  $E$  is a  $G$ -invariant set such that every orbit contained in  $E$  is dense in it. If  $\Omega$  is a  $G$ -invariant set in  $\mathbb{C}^n$ , we say that  $E$  is a *minimal set in  $\Omega$*  if it is a minimal set of the restriction  $G|_{\Omega}$  of  $G$  to  $\Omega$ . An orbit  $O \subset \Omega$  is called *minimal in  $\Omega$*  if  $\overline{O} \cap \Omega$  is a minimal set in  $\Omega$ . This means that for every  $x \in \overline{O} \cap \Omega$  we have  $\overline{O} \cap \Omega = \overline{G(x)} \cap \Omega$ . For example, a closed orbit in  $\Omega$  is minimal in  $\Omega$ . In particular, every point in  $Fix(G)$  is minimal in  $\mathbb{C}^n$ .

Many authors have studied the existence of commune fixed point of any abelian group of diffeomorphisms. In [5], S.Frimo proved that if  $G$  is an abelian subgroup of  $Diff^1(\mathbb{R}^2)$  generated by any family of commuting diffeomorphisms of the plane which are  $C^1$ -close to the identity and having a bounded orbit then  $Fix(G) \neq \emptyset$ . In [7], J. Franks, M. Handel and K. Parwani proved that if  $G$  is a finitely generated abelian subgroup of  $Diff_+^1(\mathbb{R}^2)$  and if there is a compact  $G$ -invariant set  $C \subset \mathbb{R}^2$ , then  $Fix(G)$  is non-empty.

This paper can be viewed as a generalization of the results given in [1] and [3]. We use a construction analogous to that given by S.Chihi in [10] for abelian linear group.

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Denote by  $L_G = \{Df_0, f \in G\}$ , it is an abelian subgroup of  $GL(n, \mathbb{C})$  (see Lemma 3.3) and  $\text{vect}(L_G)$  be the vector space generated by  $L_G$ . For every  $u \in \mathbb{C}^n$ , denote by:

- $L_G(u) = \{Au, A \in L_G\}$  the orbit of  $u$  defined by the natural action of the linear group  $L_G$  on  $\mathbb{C}^n$ .
- $\tilde{E}(u)$  be the vector space generated by  $L_G(u)$ .
- $\mathcal{A}(G)$  be the algebra generated by  $G$ .
- $E(x) = \{f(x), f \in \mathcal{A}(G)\}$ .

The group  $G$  is called *dominant* if there is a point  $x \in \mathbb{C}^n$  such that each set  $G(x)$  and  $L_G(x)$  generates  $\mathbb{C}^n$  (i.e.  $G$  is dominant if  $\tilde{E}(x) = E(x) = \mathbb{C}^n$  for some  $x \in \mathbb{C}^n$ ).

We generalize the result given in [1] for abelian subgroup of  $GL(n, \mathbb{C})$  in the following Theorem:

**Theorem 1.1.** *Let  $G$  be an abelian dominant subgroup of  $\text{Diff}^1(\mathbb{C}^n)$ , such that  $0 \in \text{Fix}(G)$ . Then there exist a  $G$ -invariant connected open set  $U$ , dense in  $\mathbb{C}^n$  in which every orbit is minimal.*

We have the following corollaries.

**Corollary 1.2.** *Let  $G$  be an abelian dominant subgroup of  $\text{Diff}^1(\mathbb{C}^n)$ , such that  $0 \in \text{Fix}(G)$ . If  $G$  has a dense orbit then every orbit in  $U$  is dense in  $\mathbb{C}^n$ .*

This paper is organized as follows: In Section 2, we give some results for abelian linear group. The Section 3 is devoted to prove the main results. In the section 4, we give two examples for  $n = 2$ .

## 2. Some results for abelian linear group

Let  $M_n(\mathbb{C})$  be the set of complex square matrices of order  $n \geq 1$ , and let  $GL(n, \mathbb{C})$  be the group of the invertible matrices of  $M_n(\mathbb{C})$ . Denote by

- $\mathbb{T}_n(\mathbb{C})$  the set of all lower-triangular matrices over  $\mathbb{C}$ , of order  $n$  and with only one eigenvalue.
- $\mathbb{T}_n^*(\mathbb{C}) = \mathbb{T}_n(\mathbb{C}) \cap GL(n, \mathbb{C})$  (i.e. the subset of matrix of  $\mathbb{T}_n(\mathbb{C})$  having a non zero eigenvalue), it is a subgroup of  $GL(n, \mathbb{C})$ .
- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$  and  $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$ .

Let  $r \in \mathbb{N}^*$  and  $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$  such that  $\sum_{i=1}^r n_i = n$ . Denote by:

- $\mathcal{K}_{\eta, r}(\mathbb{C}) = \{M = \text{diag}(T_1, \dots, T_r) \in M_n(\mathbb{C}) : T_k \in \mathbb{T}_{n_k}(\mathbb{C}), k = 1, \dots, r\}$ .
- $\mathcal{K}_{\eta, r}^*(\mathbb{C}) = \mathcal{K}_{\eta, r}(\mathbb{C}) \cap GL(n, \mathbb{C})$ , it is a subgroup of  $GL(n, \mathbb{C})$ .
- $v^T$  the transpose of a vector  $v \in \mathbb{C}^n$ .
- $\mathcal{E}_n = (e_1, \dots, e_n)$  the standard basis of  $\mathbb{C}^n$ .
- $I_n$  the identity matrix on  $\mathbb{C}^n$ .
- $u_0 = [e_{1,1}, \dots, e_{r,1}]^T \in \mathbb{C}^n$ , where  $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$ ,  $1 \leq k \leq r$ .

For any subset  $E$  of  $\mathbb{C}^n$  (resp.  $M_n(\mathbb{C})$ ), denote by  $\text{vect}(E)$  the vector space generated by  $E$ .

In [2], the authors proved the following Proposition:

**Proposition 2.1.** ([2], Proposition 6.1.) *Let  $L$  be an abelian subgroup of  $GL(n, \mathbb{C})$ , then there exists  $P \in GL(n, \mathbb{C})$  such that  $\tilde{L} = P^{-1}LP$  is a subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ , for some  $1 \leq r \leq n$  and  $\eta \in \mathbb{N}_0^n$ .*

For such matrix  $P$  define  $v_0 = Pu_0$ . Let  $L$  be an abelian subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ . denote by:

-  $V = \prod_{k=1}^r \mathbb{C}^* \times \mathbb{C}^{n_k-1}$ . One has  $\mathbb{C}^n \setminus V = \bigcup_{k=1}^r H_k$ , where

$$H_k = \{u = [u_1, \dots, u_r]^T, \quad u_k \in \{0\} \times \mathbb{C}^{n_k-1}, \quad u_j \in \mathbb{C}^{n_j}, \quad j \neq k\}.$$

See that each  $H_k$  is a  $L$ -invariant vector space of dimension  $n - 1$ .

**Lemma 2.2.** ([10], Proposition 3.1) *Let  $L$  be an abelian subgroup of  $GL(n, \mathbb{C})$  and  $u \in \mathbb{C}^n$ . Then for every  $v \in \text{vect}(L(u))$  there exist  $B \in \text{vect}(L)$  such that  $Bu = v$ .*

**Proposition 2.3.** *Let  $\tilde{L}$  be an abelian linear subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$  with  $\eta = (n_1, \dots, n_r)$ . Then the following assertions are equivalent:*

- (i)  $\tilde{L}$  is dominant.
- (ii) For every  $u \in V$ , we have  $\text{vect}(\tilde{L}(u)) = \mathbb{C}^n$ . In particular,  $\text{vect}(\tilde{L}(u_0)) = \mathbb{C}^n$ .

*Proof.* Suppose that  $\tilde{L}$  is dominant, then there is  $u \in \mathbb{C}^n$  such that  $\text{vect}(\tilde{L}(v)) = \mathbb{C}^n$ . Remark that  $u \in V$ , since  $\mathbb{C}^n \setminus V$  is a union of  $r$   $\tilde{L}$ -invariant vector spaces with dimensions  $n - 1$  and let  $v \in V$ . By applying lemma 2.2 on  $\tilde{L}$ , there exist  $B \in \text{vect}(\tilde{L})$  such that  $Bu = v$ . As  $\mathcal{K}_{\eta, r}(\mathbb{C})$  is a vector space then  $\text{Vect}(L) \subset \mathcal{K}_{\eta, r}(\mathbb{C})$ . Write  $u = [u_1, \dots, u_r]^T$ ,  $v = [v_1, \dots, v_r]^T$  with  $u_k = [x_{k,1}, \dots, x_{k, n_k}]^T$ ,  $v_k = [y_{k,1}, \dots, y_{k, n_k}]^T \in \mathbb{C}^* \times \mathbb{C}^{n_k-1}$  and  $B = \text{diag}(B_1, \dots, B_r)$  with

$$B_k = \begin{bmatrix} \mu_{B_k} & & & 0 \\ a_{2,1}^{(k)} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{n_k,1}^{(k)} & \dots & a_{n_k, n_k-1}^{(k)} & \mu_{B_k} \end{bmatrix}, \quad 1 \leq k \leq r$$

then  $\mu_{B_k} x_{k,1} = y_{k,1}$ , so  $\mu_{B_k} = \frac{y_{k,1}}{x_{k,1}} \neq 0$ , hence  $B \in GL(n, \mathbb{C})$ . Then  $B(\tilde{L}(u)) = \tilde{L}(v)$ . We conclude that  $\text{vect}(\tilde{L}(v)) = \mathbb{C}^n$ . The converse is obvious.  $\square$

Denote by:

-  $\Psi_x : \text{vect}(L_G) \longrightarrow \tilde{E}(x) \subset \mathbb{C}^n$  the linear map given by  $\Psi_x(A) = Ax$ .

**Lemma 2.4.** *Let  $G$  be an abelian subgroup of  $\text{Diff}^1(\mathbb{C})$  such that  $0 \in \text{Fix}(G)$ . Then  $\tilde{E}(x)$  is  $L_G$ -invariant and the linear map  $\Psi_x : \text{vect}((L_G)_{|\tilde{E}(x)}) \longrightarrow \tilde{E}(x)$  is an isomorphism, where  $(L_G)_{|\tilde{E}(x)}$  is the restriction of  $L_G$  on  $\tilde{E}(x)$ .*

*Proof.* By construction,  $\tilde{E}(x)$  is  $L_G$ -invariant and  $\Psi_x$  is linear and surjective. Let  $A \in \text{Ker}(\Psi_x)$  and  $y \in \text{vect}(L_G(x))$ . Then there is  $B \in \text{vect}((L_G)_{/\tilde{E}(x)})$  such that  $y = Bx$ . Now,  $Ay = ABx = BAx = 0$ , so  $A = 0$ . Hence  $\Psi_x$  is injective, so it is an isomorphism.  $\square$

**Corollary 2.5.** *Let  $G$  be an abelian dominant subgroup of  $\text{Diff}^1(\mathbb{C})$  such that  $0 \in \text{Fix}(G)$ . Then  $\dim(\text{vect}(L_G)) = n$ .*

*Proof.* Since  $G$  is dominant, then there is  $x \in \mathbb{C}^n$  such that  $\tilde{E}(x) = \mathbb{C}^n$ . Then by lemma 2.4,  $\Psi_x : \text{vect}(L_G) \rightarrow \mathbb{C}^n$  is an isomorphism, so  $\dim(\text{vect}(L_G)) = n$ .  $\square$

Denote by:

-  $\tilde{\Omega}_n = \{x \in \mathbb{C}^n, \dim(\text{vect}(\tilde{L}(x))) = n\}$ .

For an abelian subgroup  $L$  of  $L(n, \mathbb{C})$ , it is called dominant if  $\text{vect}(L(x)) = \mathbb{C}^n$  for some  $x \in \mathbb{C}^n$ , where  $L(x) = \{Ax, A \in G\}$ .

**Lemma 2.6.** *Let  $\tilde{L}$  be a dominant abelian subgroup of  $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ . Then  $\tilde{\Omega}_n = V$ .*

*Proof.* Since  $\tilde{L}$  is dominant, then by Proposition 2.3, for every  $u \in V$ ,  $\text{vect}(\tilde{L}(u)) = \mathbb{C}^n$ , hence  $V \subset \tilde{\Omega}_n$ . For the converse, let  $u \in \tilde{\Omega}_n$ , then  $\dim(\text{vect}(\tilde{L}(u))) = n$ . It follows that  $u \in V$  because  $\mathbb{C}^n \setminus V$  is a union of  $r$   $\tilde{L}$ -invariant vector spaces of dimension  $n - 1$ . This completes the proof.  $\square$

### 3. Proof of main results

Let  $G$  be an abelian dominant subgroup of  $C^1$ -diffeomorphisms of  $\mathbb{C}^n$  fixing 0.

Denote by:

-  $C^1(\mathbb{C}^n, \mathbb{C}^n)$  the vector space of all  $C^1$ -differentiable maps of  $\mathbb{C}^n$ , it is well known that  $C^1(\mathbb{C}^n, \mathbb{C}^n)$  is an algebra.

-  $\mathcal{A}(G)$  be the algebra generated by  $G$ . (i.e. The smallest vector subspace of  $C^1(\mathbb{C}^n, \mathbb{C}^n)$  containing  $G$  and stable by compositions).

- For a fixed point  $x \in \mathbb{C}^n$ , we define the linear map  $\Phi_x : \mathcal{A}(G) \rightarrow \mathbb{C}^n$  given by  $\Phi_x(f) = f(x)$ ,  $f \in \mathcal{A}(G)$ .

-  $E(x) = \Phi_x(\mathcal{A}(G))$ .

**Lemma 3.1.** *Let  $G$  be an abelian subgroup of  $\text{Diff}^1(\mathbb{C}^n)$  and  $x \in \mathbb{C}^n$ . Then  $E(x)$  is  $G$ -invariant.*

*Proof.* Suppose that  $E(x)$  is generated by  $f_1(x), \dots, f_p(x)$ , with  $f_k \in \mathcal{A}(G)$ ,  $k = 1, \dots, p$ . Let  $y = \sum_{k=1}^p \alpha_k f_k(x) \in E(x)$  and  $f \in G$ , then  $y = g(x)$ , with  $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$ . Therefore  $f(y) = f \circ g(x) = \Phi_x(f \circ g) \in E(x)$ , since  $f \circ g \in \mathcal{A}(G)$ .  $\square$

**Lemma 3.2.** *Let  $G$  be an abelian subgroup of  $\text{Diff}^1(\mathbb{C}^n)$ , fixing 0. Then  $g(0) = 0$  for every  $g \in \mathcal{A}(G)$ .*

*Proof.* Let  $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$  with  $f_k \in G$ ,  $\alpha_k \in \mathbb{C}$ , so  $g(0) = \sum_{k=1}^p \alpha_k f_k(0) = 0$ . Now, let  $f_1, \dots, f_m, g_1, \dots, g_q \in \mathcal{A}(G)$  such that  $f_k(0) = g_j(0) = 0$ ,  $1 \leq k \leq m$ ,  $1 \leq j \leq q$ , so for every  $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m \in \mathbb{C}$  we have

$$\begin{aligned} \left( \sum_{j=1}^q \beta_j g_j \right) \circ \left( \sum_{k=1}^m \alpha_k f_k \right) (0) &= \sum_{j=1}^q \beta_j g_j \left( \sum_{k=1}^m \alpha_k f_k(0) \right) \\ &= \sum_{j=1}^q \beta_j g_j(0) \\ &= 0 \end{aligned}$$

Since  $\mathcal{A}(G)$  is the algebra generated by  $G$ , so it is stable by composition and by linear combinations, hence we obtain the results.  $\square$

Denote by  $\varphi : \mathcal{A}(G) \rightarrow M_n(\mathbb{C})$  the linear map given by  $\varphi(f) = Df_0$ , for every  $f \in \mathcal{A}(G)$ . One observes that  $L_G = \varphi(G)$ .

**Lemma 3.3.** *Let  $G$  be an abelian subgroup of  $Diff^1(\mathbb{C}^n)$ , fixing 0. Then  $L_G$  is an abelian subgroup of  $GL(n, \mathbb{C})$ .*

*Proof.* Let  $f, g \in G$ , so  $\varphi(f \circ g) = D(f \circ g)(0) = Df(g(0)).Dg(0)$ . By Lemma 3.2,  $g(0) = 0$ , so  $\varphi(f \circ g) = D(f)(0).Dg(0) = \varphi(f).\varphi(g)$ . The proof is completed.  $\square$

Denote by:

- $r(x) = \dim(E(x))$ .
- $U_k = \{x \in \mathbb{C}^n, r(x) \geq k\}$ , for every  $k \in \mathbb{N}$ .
- $r_G = \max\{r(x), x \in \mathbb{C}^n\}$ .

**Proposition 3.4.** *Let  $G$  be an abelian subgroup of  $Diff^1(\mathbb{C}^n)$ , such that  $0 \in \text{Fix}(G)$ . Then for every  $0 \leq k \leq r_G$ ,  $U_k$  is a  $G$ -invariant open subset of  $\mathbb{C}^n$ .*

*Proof.* In the first, remark that the rank  $r(y)$  is constant on any orbit  $G(y)$ ,  $y \in E(x)$ . So  $U_k$  is  $G$ -invariant for every  $0 \leq k \leq r_G$ . Let's show that  $U_t$  is an open set: Let  $y \in U_t$  and  $r = r_y$ , so  $r \geq t$ . Then there exist  $f_1, \dots, f_r \in G$  such that the  $r$  vectors  $f_1(y), \dots, f_r(y)$  are linearly independent in  $E(y)$ . For all  $z \in \mathbb{C}^n$ , we consider the Gram's determinant

$$\Delta(z) = \det(\langle f_i(z) | f_j(z) \rangle)_{1 \leq i, j \leq r}$$

of the vectors  $f_1(z), \dots, f_r(z)$  where  $\langle \cdot | \cdot \rangle$  denotes the scalar product in  $\mathbb{C}^n$ . It is well known that these vectors are independent if and only if  $\Delta(z) \neq 0$ , in particular  $\Delta(y) \neq 0$ . Let

$$V_y = \{z \in \mathbb{C}^n, \Delta(z) \neq 0\}$$

The set  $V_y$  is open in  $\mathbb{C}^n$ , because the map  $z \mapsto \Delta(z)$  is continuous. Now  $\Delta(y) \neq 0$ , and so  $y \in V_y \subset U_k$ . The proof is completed.  $\square$

**3.1. Hamel basis and norm.** [4] The main of this section is to justify the existence of a basis of every vector space. This result is trivial in the finite case, is in fact rather surprising when one thinks of infinite dimensional vector spaces, and the definition of a basis. Recall that a *Hamel basis* or simply a basis of a vector space  $E$  is a linearly independent set  $\mathcal{B}$  (every finite subset of  $\mathcal{B}$  is linearly independent) such that for each nonzero  $x \in E$  there are  $a_1, \dots, a_k \in \mathcal{B}$  and nonzero scalars  $\alpha_1, \dots, \alpha_k$  (all uniquely determined) such that  $x = \sum_{i=1}^k \alpha_i a_i$ . The following theorem is equivalent to the axiom of choice family of axioms and theorems. In [4], C.D.Aliprantis and K.C.Border proved, in the following theorem, that Zorn's lemma implies that every vector space has a basis.

**Theorem 3.5.** ([4], *Theorem 1.8*) *Every nontrivial vector space has a Hamel basis.*

As a consequence, we found the important following results:

**Theorem 3.6.** *Every nontrivial vector space has a norm called Hamel norm.*

*Proof.* Let  $E$  be a nontrivial vector space over  $\mathbb{C}$ . By Theorem 3.5,  $E$  has a Hamel basis called  $\mathcal{B} = (a_i)_{i \in I}$ , for any set  $I$  of indices (not necessary countable). In this basis, every vector  $x \in E$  has the form  $x = \sum_{i \in I_x} \alpha_i a_i$ , where  $\alpha_i \in \mathbb{C}$  and  $I_x \subset I$  with finite cardinal. The family  $(\alpha_i)_{i \in I}$  with  $\alpha_i = 0$  for every  $i \in I \setminus I_x$ , is called the coordinate of  $x$ . Now, define  $\|x\| = \sum_{i \in I_x} |\alpha_i|$ . It is easy to verify that  $\|\cdot\|$  defines a norm on  $E$  by using the coordinate in the Hamel basis. We say that  $\|\cdot\|$  is the Hamel norm associated to the Hamel basis  $\mathcal{B}$ .  $\square$

Remark that any vector for the Hamel basis is with norm 1.

**3.2. Linear map and isomorphism.** A subset  $E \subset \mathbb{C}^n$  is called  $G$ -invariant if  $f(E) \subset E$  for any  $f \in G$ ; that is  $E$  is a union of orbits. For a fixed vector  $x \in \mathbb{R}^n \setminus \{0\}$ , denote by:

- $F_x$  is an algebraic supplement of  $\text{Ker}(\Phi_x)$  in  $\mathcal{A}(G)$ . It is easy to show that  $p_x := \dim(F_x) = \dim(E(x)) \leq n$  since  $E(x) = \Phi_x(\mathcal{A}(G))$ .
- $\mathcal{C}_x = (a_1, \dots, a_{p_x})$  is a basis of  $F_x$ .
- $\mathcal{B}_x = (b_i)_{i \in I}$  is a Hamel basis (Theorem 3.5) of  $\text{Ker}(\Phi_x)$ .
- $\mathcal{E}_x = (\mathcal{C}_x, \mathcal{B}_x)$  is a Hamel basis of  $\mathcal{A}(G)$ . By Theorem 3.6,  $\mathcal{A}(G)$  is provided with the Hamel norm associated to the basis  $\mathcal{E}_x$ .

**Lemma 3.7.** *The linear map  $\Phi_x : \mathcal{A}(G) \rightarrow E(x)$  is continuous. In particular,  $\text{Ker}(\Phi_x)$  is a closed subspace of  $\mathcal{A}(G)$ .*

*Proof.* Since  $\Phi_x$  is linear and  $\mathcal{A}(G)$  is a normed vector space (Lemma 3.6), we will verify the continuity of  $\Phi_x$  on 0. Let  $f \in \mathcal{A}(G)$  and write  $f = f_1 + f_2$  with  $f_1 \in F_x$  and  $f_2 \in \text{Ker}(\Phi_x)$ . Set  $(\alpha_i)_{1 \leq i \leq p_x}$  and  $(\beta_i)_{i \in I}$  be respectively the coordinates of  $f_1$  and  $f_2$  in  $\mathcal{C}_x$  and  $\mathcal{B}_x$ . Write  $f = \sum_{i=1}^{p_x} \alpha_i a_i + \sum_{i \in I_2} \beta_i b_i$  where  $I_2 \subset I$  with finite cardinal.

We have  $\|f\| = \sum_{i=1}^{p_x} |\alpha_i| + \sum_{i \in I_2} |\beta_i|$  and  $b_i(x) = 0$  for all  $i \in I_2$ . Therefore

$$\begin{aligned} \|\Phi_x(f)\| = \|f(x)\| &= \left\| \sum_{i=1}^{p_x} \alpha_i a_i(x) + \sum_{i \in I_2} \beta_i b_i(x) \right\| \\ &\leq \sum_{i=1}^{p_x} |\alpha_i| \|a_i(x)\| \\ &\leq \|f\| \sum_{i=1}^{p_x} \|a_i(x)\| \end{aligned}$$

Since  $\sum_{i=1}^{p_x} \|a_i(x)\|$  is constant relative to  $f$ , then  $\Phi_x$  is continuous.  $\square$

**Lemma 3.8.** *Suppose that  $\dim(\text{vect}(L_G)) = n$ . Then the linear map  $\varphi : \mathcal{A}(G) \rightarrow \text{vect}(L_G)$  is continuous. In particular,  $\text{Ker}(\varphi)$  is a closed subspace of  $\mathcal{A}(G)$ .*

*Proof.* Since  $\varphi$  is linear and  $\mathcal{A}(G)$  is a normed vector space (Lemma 3.6), we will verify the continuity of  $\varphi$  on 0. Firstly, see that  $\text{cod}(\text{Ker}(\varphi)) = n$  is finite since  $\dim(\text{vect}(L_G)) = n$ . let  $F$  be an algebraic supplement to  $\text{Ker}(\varphi)$  in  $\mathcal{A}$ ,  $\mathcal{C}' = (a'_1, \dots, a'_n)$  and  $\mathcal{B}' = (b'_i)_{i \in J}$  are respectively the Hamel basis of  $F$  and  $\text{Ker}(\varphi)$  (Lemma 3.5). Let  $f \in \mathcal{A}(G)$  and write  $f = f_1 + f_2$  with  $f_1 \in F$  and  $f_2 \in \text{Ker}(\varphi)$ . Set  $(\alpha_i)_{1 \leq i \leq q}$  and  $(\beta_i)_{i \in J}$  be respectively the coordinates of  $f_1$  and  $f_2$  in  $\mathcal{C}'$  and  $\mathcal{B}'$ . Write  $f = \sum_{i=1}^n \alpha_i a'_i + \sum_{i \in I_2} \beta_i b'_i$  where  $I_2 \subset J$  with finite cardinal. We have

$\|f\| = \sum_{i=1}^n |\alpha_i| + \sum_{i \in I_2} |\beta_i|$  and  $\varphi(b'_i) = Db'_i(0) = 0$  for all  $i \in I_2$ . Therefore

$$\begin{aligned} \|\varphi(f)\| = \|Df(0)\| &= \left\| \sum_{i=1}^n \alpha_i Da'_i(0) + \sum_{i \in I_2} \beta_i Db'_i(0) \right\| \\ &\leq \sum_{i=1}^n |\alpha_i| \|Da'_i(0)\| \\ &\leq \|f\| \sum_{i=1}^n \|Da'_i(0)\| \end{aligned}$$

Since  $\sum_{i=1}^n \|Da'_i(0)\|$  is constant relative to  $f$ , then  $\varphi$  is continuous.  $\square$

**Lemma 3.9.** ([6], 3.5) *Let  $E$  be a topological vector space, and let  $M$  be a closed subspace of finite codimension. Then  $E = M \oplus N$  is a topological sum, for every algebraic complementary subspace  $N$  of  $M$ .*

**Corollary 3.10.** *The algebraic sum  $\mathcal{A}(G) = F_x \oplus \text{Ker}(\Phi_x)$  is topological. In particular,  $F_x \oplus \text{Ker}(\Phi_x)$  and  $\mathcal{A}(G)$  are topological isomorphic by the map:  $(f_1, f_2) \mapsto f_1 + f_2$ .*

*Proof.* By the Theorem 3.6,  $\mathcal{A}(G)$  is a normed vector space so it is a topological vector space. By Lemma 3.7,  $\Phi_x$  is continuous and so its kernel is closed vector space with finite codimension. The results follows directly by applying the Lemma 3.9 for  $E = \mathcal{A}(G)$  and  $M = Ker(\Phi_x)$ .  $\square$

By Corollary 3.10, we can identify  $\mathcal{A}(G)$  with  $F_x \oplus Ker(\Phi_x)$ , so every  $f \in \mathcal{A}(G)$  is denoted by  $f = (f_1, f_2) = f_1 + f_2$  with  $f_1 \in F_x$  and  $f_2 \in Ker(\Phi_x)$ .

**Lemma 3.11.** *Let  $H$  and  $K$  be two closed vector subspaces of  $\mathcal{A}(G)$  such that  $cod(H) = cod(K) = j \geq 1$ . Let  $\psi \in \mathcal{A}(G) \setminus (H \cup K)$ . Then there exists a vector space  $F$  of  $\mathcal{A}(G)$ , containing  $\psi$  and satisfying  $\mathcal{A}(G) = F \oplus H = F \oplus K$ .*

*Proof.* There are two cases;

*Case 1:* If  $j = 1$ , then we take  $F = \mathbb{C}\psi$ .

*Case 2:* Suppose that  $j \geq 2$ . Denote by  $H' = H \oplus \mathbb{C}\psi$  and  $K' = K \oplus \mathbb{C}\psi$ . Suppose that  $H' \neq K'$  (otherwise, it is easy to take a comment supplement) and let  $f_1 \in H' \setminus K'$  and  $g_1 \in K' \setminus H'$ , so  $h_1 = f_1 + g_1 \notin H' \cup K'$ . Denote by  $H_1 = \mathbb{C}h_1 \oplus H'$  and  $K_1 = \mathbb{C}h_1 \oplus K'$ . We establish two cases:

- If  $H_1 = K_1$ , then any supplement  $F$  of  $H_1$  in  $\mathcal{A}(G)$  is a supplement of  $K_1$  in  $\mathcal{A}(G)$ , the proof follows then.

- If  $H_1 \neq K_1$ , we take  $f_2 \in H_1 \setminus K_1$  and  $g_2 \in K_1 \setminus H_1$ , so  $h_2 = f_2 + g_2 \notin H_1 \cup K_1$ . Denote by  $H_2 = \mathbb{C}h_2 \oplus H_1$  and  $K_2 = \mathbb{C}h_2 \oplus K_1$ .

We repeat the same processes until  $j - 2$  times and we obtain:

- If  $H_{j-2} = K_{j-2}$ , then any supplement  $F$  of  $H_{j-2}$  in  $\mathcal{A}(G)$  is a supplement of  $K_{j-2}$  in  $\mathcal{A}(G)$ .

- If  $H_{j-2} \neq K_{j-2}$ , we take  $f_{j-1} \in H_{j-2} \setminus K_{j-2}$  and  $g_{j-1} \in K_{j-2} \setminus H_{j-2}$ , so  $h_{j-1} = f_{j-1} + g_{j-1} \notin H_{j-2} \cup K_{j-2}$ . Denote by  $H_{j-1} = \mathbb{C}h_{j-1} \oplus H_{j-2}$  and  $K_{j-1} = \mathbb{C}h_{j-1} \oplus K_{j-2}$ . We obtain then  $H_{j-1} = K_{j-1} = \mathcal{A}(G)$ . Hence the proof is completed by taking  $F = vect(\psi, h_1, \dots, h_{j-1})$ .  $\square$

Denote by  $\tilde{r}_x = dim(\tilde{E}(x))$ .

**Lemma 3.12.** *If  $r(x) = \tilde{r}_x$ , then there exists a commune vector space  $F_x \subset \varphi^{-1}(L_G)$  supplement to  $Ker(\Phi_x)$  and to  $ker(\varphi)$  in  $\mathcal{A}(G)$ . (i.e  $F_x \oplus Ker(\Phi_x) = F_x \oplus Ker(\varphi) = \mathcal{A}(G)$ ).*

*Proof.* Since  $L_G$  is abelian so  $vect(L_G)$  is an abelian algebra and  $\varphi$  is a morphism of algebra, then  $\varphi^{-1}(L_G)$  is an algebra containing  $G$ . hence  $\varphi^{-1}(L_G) = \mathcal{A}(G)$ . Therefore any supplement of  $Ker(\Phi_x)$  in  $\mathcal{A}(G)$  is contained in  $\varphi^{-1}(L_G)$ . Now by Lemma 3.8,  $\varphi$  is continuous. By Lemmas 3.7 and 3.8 both  $Ker(\Phi_x)$  and  $Ker(\varphi)$  are closed and since  $cod(Ker(\Phi_x)) = cod(Ker(\varphi)) = n$ , then by lemma 3.11, there exists a commune supplement  $F_x$  to  $Ker(\Phi_x)$  and to  $ker(\Phi_x)$  in  $\mathcal{A}(G)$  containing  $id$  (identity map of  $\mathbb{C}^n$ ), because  $id \in \mathcal{A}(G) \setminus Ker(\Phi_x) \cup Ker(\varphi)$ .  $\square$

**Lemma 3.13.** *Then the linear map  $\Phi_x : F_x \longrightarrow E(x)$  given by  $\Phi_x(f) = f(x)$  is an isomorphism.*

*Proof.* Here,  $F_x$  is considered as a supplement to  $Ker(\Phi_x)$  in  $\mathcal{A}(G)$ . The proof follows directly from the fact that  $\Phi_x$  is linear surjective and  $dim(F_x) = dim(E(x)) = n$ .  $\square$

**Lemma 3.14.** *If  $r(x) = \tilde{r}_x = n$ , then the restriction  $\varphi_1 : F_x \rightarrow vect(L_G)$  of  $\varphi$  from  $F_x$  unto  $vect(L_G)$  is an isomorphism.*

*Proof.* Here,  $F_x$  is considered as a supplement to  $Ker(\varphi)$  in  $\mathcal{A}(G)$ . The proof follows directly from the fact that  $\varphi_1$  is linear surjective and  $dim(F_x) = dim(vect(L_G)) = n$  (Corollary 2.5).  $\square$

**Lemma 3.15.** *If  $r(x) = \tilde{r}_x = n$ , then the map  $\varphi_x := \tilde{\Phi}_x \circ \varphi_1 \circ \Phi_x^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}^n$  defined by  $\varphi_x(f(x)) = Df_0(x)$ ,  $f \in F_x$ , is an isomorphism and satisfying:*

- (i)  $\varphi_x(G(x)) = L_G(x)$ .
- (ii) Let  $y \in \mathbb{C}^n$  and  $z = \varphi_x(y)$  then  $\varphi_x(G(y)) = L_G(z)$ .
- (iii) for every  $y \in \overline{G(x)}$  we have  $z = \varphi_x(y) \in \overline{L_G(x)}$ .

*Proof.* By Lemma 3.12, we can assume that  $F_x$  is a commune supplement to  $Ker(\Phi_x)$  and to  $Ker(\varphi)$  in  $\mathcal{A}(G)$ . Since  $r(x) = \tilde{r}_x = n$ , then  $E(x) = \tilde{E}(x) = \mathbb{C}^n$ .

(i)  $\varphi_x$  is an isomorphisms, since  $\varphi_x = \tilde{\Phi}_x \circ \varphi_1 \circ \Phi_x^{-1}$  and by Lemmas 3.14, 2.4 and 3.13, it is composed by isomorphisms.

Let  $f_1, \dots, f_r \in \mathcal{A}(G)$  be the generator of  $F_x$ . So the restriction  $(\Phi_x)_{/F_x}$  (resp.  $(\psi_x)_{/F_x}$ ) of  $\Phi_x$  (resp.  $\psi_x$ ) on  $F_x$  are bijective from  $F_x$  to  $\mathbb{C}^n$ . Then  $f_1(x), \dots, f_n(x)$  generate  $E(x)$  and  $Df_1(0)(x), \dots, Df_n(0)(x)$  generate  $\mathbb{C}^n$ . Now, let  $f = \sum_{k=1}^n \alpha_k f_k \in F_x$ , so  $\varphi_x(f(x)) = \sum_{k=1}^n \alpha_k Df_k(0)(x) = Df_0(x) \in L(x)$ . It follows that  $\varphi_x(G(x)) = L_G(x)$ .

(ii) Let  $y \in E(x)$  and  $z = \varphi_x(y)$ . Let  $f \in G$  and write  $f = \sum_{k=1}^n \alpha_k f_k$ , where  $(f_1, \dots, f_n)$  be a basis of  $F_x$ . Then by Lemma 3.2,  $g(0) = 0$  and so  $\varphi_x(f(y)) = \varphi_x(f \circ g(x)) = Df_0.Dg_0(x) = Df_0(z) \in L_G(z)$ . It follows that  $\varphi_x(G(y)) \subset L_G(z)$ . Conversely, let  $a \in L_G(z)$ , so  $a = Df_0(z)$  for some  $f \in G$ . Since  $z = Dg_0(x)$  and  $g \in G$ , so  $a = D(f \circ g)_0(x) = \varphi_x(f \circ g)$ . Hence  $a \in \varphi_x(G(x))$ .

(iii) Since  $y \in \mathbb{C}^n$ , there exists  $g \in F_x$  such that  $y = g(x)$ , so  $z = Dg_0(x)$ . By continuity of  $\varphi_x$  and by (i), we have  $z \in \varphi_x(\overline{G(x)}) \subset \overline{\varphi_x(G(x))} = \overline{L_G(x)}$ .  $\square$

Denote by:

-  $\tilde{\Omega}_k = \{y \in \mathbb{C}^n, \tilde{r}_y \geq k\}$ , for every  $0 \leq k \leq n$ . By applying Lemma 3.11 given in [10] to the abelian linear group  $L_G$ , we found the following result:

**Lemma 3.16.** ([10], Lemma 3.11)  $\tilde{\Omega}_k$  is a  $L_G$ -invariant dense open subset of  $\mathbb{C}^n$ .

**Lemma 3.17.** ([10], Theorem 3.10) Let  $x \in \tilde{\Omega}_n$  then for every  $y \in \overline{L_G(x)} \cap \tilde{\Omega}_n$  we have  $\overline{L_G(y)} \cap \tilde{\Omega}_n = \overline{L_G(x)} \cap \tilde{\Omega}_n$ .

**Lemma 3.18.** *(Under the above notations) Suppose that  $r(x) = \tilde{r}_x = n$ . For every  $x \in \Omega_n$ , we have  $\varphi_x(\Omega_n) = \tilde{\Omega}_n$ .*

*Proof.* Let  $y \in \Omega_n$  and  $z = \varphi_x(y)$ . By Lemma 3.15,(ii),  $\varphi_x(G(y)) = L_G(z)$ . Since  $\varphi_x$  is linear, then  $\varphi_x(E(y)) = \tilde{E}(z)$ , so  $r(y) = \tilde{r}_z = n$ . It follows that  $z \in \tilde{\Omega}_n$ . For the converse we use the same proof for  $\varphi_x^{-1}$ .  $\square$

*Proof of Theorem 1.1.* Since  $G$  is dominant, then there is  $x \in \mathbb{C}^n$  such that  $r(x) = \tilde{r}_x = n$ . Let  $y \in \overline{G(x)} \cap \Omega_n$ . By Lemma 3.15, there exists an isomorphism  $\varphi_x : \mathbb{C}^n \rightarrow \mathbb{C}^n$  satisfying  $\varphi_x(G(x)) = L_G(x)$  and  $\varphi_x(G(y)) = L_G(z)$  with  $z = \varphi_x(y)$ . By Lemma 3.18, we have  $\varphi_x(\Omega_n) = \tilde{\Omega}_n$ , so  $z \in \varphi_x(\overline{G(x)} \cap \Omega_n) = \overline{L_G(x)} \cap \tilde{\Omega}_n$ . By Lemma 3.17, we have

$$\overline{L_G(z)} \cap \tilde{\Omega}_n = \overline{L_G(x)} \cap \tilde{\Omega}_n \quad (1).$$

Therefore by (1) we obtain

$$\begin{aligned} \overline{G(x)} \cap \Omega_n &= \varphi_x^{-1}(\overline{L_G(x)}) \cap \varphi_x^{-1}(\tilde{\Omega}_n) \\ &= \varphi_x^{-1}(\overline{L_G(x)} \cap \tilde{\Omega}_n) \\ &= \varphi_x^{-1}(\overline{L_G(z)} \cap \tilde{\Omega}_n) \\ &= \varphi_x^{-1}(\overline{L_G(z)}) \cap \varphi_x^{-1}(\tilde{\Omega}_n) \\ &= \overline{G(y)} \cap \Omega_n. \end{aligned}$$

On the other hand,  $U = \Omega_n = \varphi_x(\tilde{\Omega}_n)$  and by Lemma 2.6,  $\tilde{\Omega}_n = V$  which is connected and dense open set in  $\mathbb{C}^n$ , so is  $U$  since  $U = \varphi_x(V)$ . This completes the proof.  $\square$

**Remark 3.19.** As a consequence of the proof of Theorem 1.1,  $U = \varphi_x(V)$ , for a particular point  $x \in \mathbb{C}^n$  satisfying  $E(x) = \tilde{E}(x) = \mathbb{C}^n$ .

*Proof of Corollary 1.2.* Let  $O$  be a dense orbit in  $\mathbb{C}^n$  (i.e.  $\overline{O} = \mathbb{C}^n$ ). Then for every  $x \in O$ , we have  $r(x) = n$ , so  $O \subset U$  and  $\overline{O} \cap U = U$ . Since  $O$  is minimal in  $U$  (Theorem 1.1), then for every orbit  $L \subset U$ , we have  $\overline{L} \cap U = \overline{O} \cap U$ . Therefore  $\overline{L} = \overline{O} = \mathbb{C}^n$ .  $\square$

**3.3. The Theorem 1.1 in the dimension  $n = 1$ .** We can remove the condition  $G$  is dominant in the Theorem 1.1, we obtain a global decomposition as follow:

**Corollary 3.20.** *Let  $G$  be an abelian subgroup of  $\text{Diff}^1(\mathbb{C})$ , such that  $0 \in \text{Fix}(G)$ . Then every orbit of  $\mathbb{C} \setminus \{0\}$  is minimal in it.*

*Proof.* Since  $id \in G$ , then  $id \in L_G$  and so  $\dim(E(1)) = \dim(\tilde{E}(1)) = 1$ . On the other hand  $V = \mathbb{C}^*$ . The proof results directly from Theorem 1.1 and Remark 3.19, because  $U = \varphi_1(V) = \mathbb{C}^*$ .  $\square$

#### 4. Example in the dimension $n=2$

We give some examples of abelian subgroup of  $Diff^1(\mathbb{C})$  which are near enough to linear group. Recall that  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ .

**Example 4.1.** Let  $\Gamma$  be the group abelian of all diffeomorphisms  $f$  of  $\mathbb{C}^2$  defined by  $f(x, y) = (x, y + P(x))$ ,  $P \in \mathbb{C}[X]$  and  $G$  be a subgroup of  $\Gamma$  generated by  $f : (x, y) \mapsto (x, y + P(x))$  and  $g : (x, y) \mapsto (x, y + Q(x))$  with  $P, Q \in \mathbb{C}[X]$ ,  $Q(0) = P(0) = 0$ ,  $P(1) \neq 0$  and  $P'(0) \neq 0$ . Then every orbit is minimal in the open set  $\mathbb{C}^* \times \mathbb{C}$ .

*Proof.* We can easily verify that  $G$  is abelian,  $0 \in Fix(G)$ . In the begin, we have  $f^n \circ g^m(x, y) = (x, y + nP(x) + mQ(x))$  for every  $n, m \in \mathbb{Z}$ . Then every  $h \in \mathcal{A}(G)$  has the form:

$$h(x, y) = (\alpha x, \beta y + R(x)) \quad \text{where } R \in \mathbb{C}[X] \quad (1)$$

We have  $Df_0 = \begin{bmatrix} 1 & 0 \\ P'(0) & 1 \end{bmatrix}$ . Then  $L_G \subset \mathbb{T}_2^*(\mathbb{C})$ . Here  $u_0 = e_1 = (1, 0)$  then  $\tilde{E}(u_0)$  is generated by  $e_1$  and  $Df_0(u_0) = (1, P'(0))$ , so  $dim(\tilde{E}(u_0)) = 2$ , hence  $\tilde{E}(e_1) = \mathbb{C}^2$ . On the other hand,  $f(u_0) = [1, P(1)]^T$  and  $id(u_0) = [1, 0]^T$ , so  $dim(E(u_0)) = dim(\tilde{E}(u_0)) = 2$  since  $P(1) \neq 0$ , so  $G$  is dominant. By Lemma 2.6,  $\tilde{\Omega}_2 = \mathbb{C}^* \times \mathbb{C}$  and by Lemma 3.18, we have  $\varphi_{e_1}(\Omega_2) = \tilde{\Omega}_2$ , then

$$\Omega_2 = \varphi_{e_1}^{-1}(\mathbb{C}^* \times \mathbb{C}) \quad (2)$$

By Theorem 1.1, every orbit of  $\Omega_2$  is minimal in it. Now let's prove that  $\varphi_{e_1}^{-1}(\mathbb{C}^* \times \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}$ : Recall that  $\varphi_{e_1}(x, y) = \tilde{\Phi}_{e_1} \circ \varphi \circ \Phi_{e_1}^{-1}(x, y)$  (see Lemma 3.15). Let  $(x, y) \in \mathbb{C}^* \times \mathbb{C}$ . Then there exists  $h \in \mathcal{A}(G)$  such that  $(x, y) = Dh_0 e_1$ . Using (2) and by integration,  $h$  has the form  $h : (s, t) \mapsto (xs, \beta t + ys + sR(s) + \gamma)$ , with  $R \in \mathbb{C}[X]$ , then  $h(e_1) = (x, y + R(1))$  and  $Dh_0 e_1 = (x, y)$ . Therefore

$$\begin{aligned} \varphi_{e_1}^{-1}(x, y) &= \Phi_{e_1} \circ \varphi^{-1} \circ \tilde{\Phi}_{e_1}^{-1}(x, y) \\ &= \Phi_{e_1} \circ \varphi^{-1}(Dh_0) \\ &= \Phi_{e_1}(h) \\ &= h(e_1) \\ &= (x, y + R(1) + \gamma) \in \mathbb{C}^* \times \mathbb{C} \end{aligned}$$

Conversely, Let  $(a, b) \in \mathbb{C}^* \times \mathbb{C}$  then there is  $h \in \mathcal{A}(G)$  such that  $h(e_1) = (a, b)$ . By (2),  $h$  has the form  $h(s, t) = (as, \beta t + bs + sR(s) + \gamma)$ , then  $Dh_0(e_1) = (a, b + R(1) + R'(1))$ . Therefore

$$\begin{aligned} \varphi_{e_1}(a, b) &= \tilde{\Phi}_{e_1} \circ \varphi \circ \Phi_{e_1}^{-1}(a, b) \\ &= \tilde{\Phi}_{e_1} \circ \varphi(h) \\ &= \Phi_{e_1}(Dh_0) \\ &= h(e_1) \\ &= (a, b + R(1) + R'(1)) \in \mathbb{C}^* \times \mathbb{C} \end{aligned}$$

It follows that  $\varphi_{e_1}^{-1}(\mathbb{C}^* \times \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}$ . □

**Example 4.2.** Let  $G$  be the abelian group generated by  $f(x, y) = (x^q, y + x^k)$  and  $g(x, y) = (x^{q'}, y + x^{k'})$   $q$  and  $q'$  are odd integers,  $k, k' \in \mathbb{N}$ . Then every orbit of  $\mathbb{C}^* \times \mathbb{C}$  is minimal in it.

*Proof.* We can easily verify that  $G$  is abelian,  $0 \in \text{Fix}(G)$ . In the begin, we have  $f^n(x, y) = (x^{nq}, y + x^k + \sum_{i=1}^n x^{iqk})$  and so

$$f^n \circ g^m(x, y) = \left( x^{nqm q'}, y + x^k + \sum_{i=1}^n x^{iqk} + x^{nqk'} + \sum_{j=1}^m x^{nqj q' k'} \right)$$

for every  $n, m \in \mathbb{Z}$ . Then every  $h \in \mathcal{A}(G)$  has the form:

$$h(x, y) = (\alpha x, \beta y + R(x)) \quad \text{where } R \in \mathbb{C}[X] \quad (1)$$

We have  $Did_0 = id$  and  $Df_0 = Dg_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ . Then  $L_G \subset \mathbb{T}_1^*(\mathbb{C}) \oplus \mathbb{T}_1^*(\mathbb{C})$ .

Here  $u_0 = e_1 + e_2 = (1, 1)$  then  $\tilde{E}(u_0)$  is generated by  $e_1$  and  $Df_0(u_0) = (0, 1)$ , so  $\dim(\tilde{E}(u_0)) = 2$ , hence  $\tilde{E}(u_0) = \mathbb{C}^2$ . By Lemma 2.6,  $\tilde{\Omega}_2 = \mathbb{C}^* \times \mathbb{C}$  and by Lemma 3.18, we have  $\varphi_{u_0}(\Omega_2) = \tilde{\Omega}_2$ , then

$$\Omega_2 = \varphi_{u_0}^{-1}(\mathbb{C}^* \times \mathbb{C}) \quad (2)$$

On the other hand,  $f(u_0) = [1, 2]^T$  and  $id(u_0) = [1, 1]^T$ , so  $\dim(E(u_0)) = \dim(\tilde{E}(u_0)) = 2$ , so  $G$  is dominant. By Theorem 1.1, every orbit of  $\Omega_2$  is minimal in it. As in the proof of example 4.1, we prove that  $\varphi_{u_0}^{-1}(\mathbb{C}^* \times \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}$ .  $\square$

**Question1 :** Is there an abelian subgroup of  $\text{Diff}^1(\mathbb{C}^n)$  fixing some point having no minimal set in  $\mathbb{C}^n \setminus \text{Fix}(G)$ ?

**Question2 :** An abelian subgroup of  $\text{Diff}^1(\mathbb{C}^n)$  fixing some point and satisfying  $r(x) = \tilde{r}_x$  for every  $x \in \mathbb{C}^n$ , is-it conjugate to a linear group?

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