

THE ACTION OF ABELIAN C^1 -DIFFEOMORPHISMS GROUP FIXING A POINT, ON \mathbb{C}^n

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ABSTRACT. In this paper, we study the action of any abelian subgroup G of $Diff^1(\mathbb{C}^n)$ on \mathbb{C}^n . Suppose that $0 \in Fix(G)$ and $dim(vect(L_G)) = n$, where $vect(L_G)$ is the vector space generated by $L_G = \{Df_0, f \in G\}$. We prove the existence of a decreasing finite sequence F_0, F_1, \dots, F_q of invariant closed sets such that $\Omega_j = F_j \setminus F_{j-1}$ is an open subset (for the relative topology) of F_j in which every orbit is minimal. Moreover, if G has a dense orbit in \mathbb{C}^n then every orbit of Ω_q is dense in \mathbb{C}^n .

1. Introduction

Denote by $Diff^1(\mathbb{C}^n)$ the group of all C^1 -diffeomorphisms of \mathbb{C}^n . Let G be an abelian subgroup of $Diff^1(\mathbb{C}^n)$, such that $0 \in Fix(G)$ and $dim(vect(L_G)) = n$, where $vect(L_G)$ is the vector space generated by $L_G = \{Df(0), f \in G\}$ and $Fix(G) = \{x \in \mathbb{C}^n : f(x) = x, \forall f \in G\}$ be the global fixed point set of G . There is a natural action $G \times \mathbb{C}^n \rightarrow \mathbb{C}^n$. $(f, x) \mapsto f(x)$. For a point $x \in \mathbb{C}^n$, denote by $G(x) = \{f(x), f \in G\} \subset \mathbb{C}^n$ the orbit of G through x . A subset $E \subset \mathbb{C}^n$ is called G -invariant if $f(E) \subset E$ for any $f \in G$; that is E is a union of orbits. Denote by \overline{E} (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E .

A subset E of \mathbb{C}^n is called a *minimal set* of G if E is closed in \mathbb{C}^n , non empty, G -invariant and has no proper subset with these properties. It is equivalent to say that E is a G -invariant set such that every orbit contained in E is dense in it. If Ω is a G -invariant set in \mathbb{C}^n , we say that E is a *minimal set in Ω* if it is a minimal set of the restriction $G|_{\Omega}$ of G to Ω . An orbit $O \subset \Omega$ is called *minimal in Ω* if $\overline{O} \cap \Omega$ is a minimal set in Ω . This means that for every $x \in \overline{O} \cap \Omega$ we have $\overline{O} \cap \Omega = \overline{G(x)} \cap \Omega$. For example, a closed orbit in Ω is minimal in Ω . In particular, every point in $Fix(G)$ is minimal in \mathbb{C}^n .

Many authors have studied the existence of commune fixed point of any abelian group of diffeomorphisms. In [5], S.Frimo proved that if G is an abelian subgroup of $Diff^1(\mathbb{R}^2)$ generated by any family of commuting diffeomorphisms of the plane which are C^1 -close to the identity and having a bounded orbit then $Fix(G) \neq \emptyset$. In [7], J. Franks, M. Handel and K. Parwani proved that if G is a finitely generated abelian subgroup of $Diff^1_+(\mathbb{R}^2)$ and if there is a compact G -invariant set $C \subset \mathbb{R}^2$, then $Fix(G)$ is non-empty.

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This paper can be viewed as a generalization of the results given in [3] whose generalize the structure's theorem given in [1] and it uses a construction analogous to that given by S.Chihi in [10] for abelian linear group.

Our principal results can be stated as follows:

Denote by $L_G = \{Df_0, f \in\}$ and $\text{vect}(L_G)$ be the vector space generated by L_G . We generalize the result given in [1] for abelian subgroup of $GL(n, \mathbb{C})$:

Theorem 1.1 (Structure's Theorem1). *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$, such that $0 \in \text{Fix}(G)$ and $\dim(\text{vect}(L_G)) = n$. Then there exist a finite increasing sequence of G -invariant closed subsets of \mathbb{C}^n , $F_0 \subset \dots \subset F_q$, $1 \leq q \leq n$, with the following properties:*

- (i) $F_0 = \emptyset$, $F_q = \mathbb{C}^n \setminus \{0\}$ and $\mathbb{C}^n = \bigcup_{j=1}^q \Omega_j$, where $\Omega_j = F_j \setminus F_{j-1}$.
- (ii) Every orbit of Ω_j is minimal in it.

Under the hypothesis of Theorem 1.1, denote by $U = \Omega_q = \mathbb{C}^n \setminus F_{q-1}$. We have the following corollaries.

Corollary 1.2. *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$, such that $0 \in \text{Fix}(G)$ and $\dim(\text{vect}(L_G)) = n$. If G has a dense orbit then every orbit in U is dense in \mathbb{C}^n .*

Corollary 1.3. *If G has a locally dense orbit O in \mathbb{C}^n and C is a connected component of U meeting O then every orbit meeting C is dense in it.*

For every $x \in \mathbb{C}^n$, denote by:

- $L_G(x) = \{Ax, A \in L_G\}$ the orbit of x defined by the natural action of the linear group L_G on \mathbb{C}^n .
- $\tilde{E}(x)$ be the vector space generated by $L_G(x)$.

Another version of the Structure's theorem is given by:

Theorem 1.4 (Structure's Theorem2). *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$, such that $0 \in \text{Fix}(G)$. If there exists $x \in \mathbb{C}^n$ with $\tilde{E}(x) = \mathbb{C}^n$, then there exist a finite increasing sequence of G -invariant closed subsets of \mathbb{C}^n , $F_0 \subset \dots \subset F_q$, $1 \leq q \leq n$, with the following properties:*

- (i) $F_0 = \emptyset$, $F_q = \mathbb{C}^n \setminus \{0\}$ and $\mathbb{C}^n = \bigcup_{j=1}^q \Omega_j$, where $\Omega_j = F_j \setminus F_{j-1}$.
- (ii) Every orbit of Ω_j is minimal in it.

In the second Theorem, we prove that the condition $\dim(\text{vect}(L_G)) = n$ can be replaced by $\tilde{E}(x) = \mathbb{C}^n$ for some $x \in \mathbb{C}^n$ and so the Corollaries 1.2 and 1.3 remains valid.

This paper is organized as follows: In Section 2, we give some results for abelian linear group. The Section 3 is devoted to prove the main results. In the section 4, we give two examples for $n = 2$.

2. Some results for abelian linear group

Denote by $M_n(\mathbb{C})$ the set of complex square matrices of order $n \geq 1$, and $GL(n, \mathbb{C})$ the group of the invertible matrices of $M_n(\mathbb{C})$.

- $\mathbb{T}_n(\mathbb{C})$ the set of all lower-triangular matrices over \mathbb{C} , of order n and with only one eigenvalue.
- $\mathbb{T}_n^*(\mathbb{C}) = \mathbb{T}_n(\mathbb{C}) \cap GL(n, \mathbb{C})$ (i.e. the subset of matrix of $\mathbb{T}_n(\mathbb{C})$ having a non zero eigenvalue), it is a subgroup of $GL(n, \mathbb{C})$.
- $\mathbb{D}_n(\mathbb{C})$ the set of diagonal matrix of $M_n(\mathbb{C})$.
- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.

Let $r \in \mathbb{N}^*$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ such that $\sum_{i=1}^r n_i = n$. Denote by:

- $\mathcal{K}_{\eta,r}(\mathbb{C}) = \{M = \text{diag}(T_1, \dots, T_r) \in M_n(\mathbb{C}) : T_k \in \mathbb{T}_{n_k}(\mathbb{C}), k = 1, \dots, r\}$.
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) = \mathcal{K}_{\eta,r}(\mathbb{C}) \cap GL(n, \mathbb{C})$, it is a subgroup of $GL(n, \mathbb{C})$.
- v^T the transpose of a vector $v \in \mathbb{C}^n$.
- $\mathcal{E}_n = (e_1, \dots, e_n)$ the standard basis of \mathbb{C}^n .
- I_n the identity matrix on \mathbb{C}^n .

Denote by:

- $u_0 = [e_{1,1}, \dots, e_{r,1}]^T \in \mathbb{C}^n$, where $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$, $1 \leq k \leq r$.
- $e^{(k)} = [0_{\mathbb{C}^{n_1}}, \dots, 0_{\mathbb{C}^{n_{k-1}}}, e_{k,1}^T, 0_{\mathbb{C}^{n_{k+1}}}, \dots, 0_{\mathbb{C}^{n_r}}]^T$, $1 \leq k \leq r$.

For a vector $v \in \mathbb{C}^n$, we write $v = \Re(v) + i\Im(v)$, where $\Re(v)$ and $\Im(v) \in \mathbb{R}^n$.

In [2], the authors proved the following

Proposition 2.1. ([2], Proposition 6.1.) *Let L be an abelian subgroup of $GL(n, \mathbb{C})$, then there exists $P \in GL(n, \mathbb{C})$ such that $\tilde{L} = P^{-1}LP$ is a subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$, for some $1 \leq r \leq n$ and $\eta \in \mathbb{N}_0^r$.*

For such matrix P define $v_0 = Pu_0$.

Let L be an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$. denote by:

- $(e_{k,1}, \dots, e_{k,n_k})$ be the canonical basis of \mathbb{C}^{n_k} , for every $k = 1, \dots, r$.
- $L_k = \{A_k, A = \text{diag}(A_1, \dots, A_r) \in L\}$.
- If $L \subset \mathbb{T}_n(\mathbb{C})$ denote by $F_L = \text{vect}\{(A - \mu_A I_n)e_i, A \in L, 1 \leq i \leq n-1\}$.
- In general, if $L \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$, denote by $F_{L_k} = \text{vect}\{(A_k - \mu_{A_k} I_{n_k})e_{k,i}, A_k \in L_k, 1 \leq i \leq n_k - 1\}$.

Lemma 2.2. ([2], Theorem 5.2) *Let L be an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$. Assume that $\dim(F_{L_k}) = n_k - 1$, $k = 1, \dots, r$. Then there exists a linear injective map $\rho : \mathbb{C}^n \rightarrow \mathcal{K}_{\eta,r}(\mathbb{C})$ such that:*

- i) $L \subset \rho(\mathbb{C}^n)$.
- ii) For every $v \in \mathbb{C}^n$, $\rho(v)u_0 = v$.

Lemma 2.3. *Let \tilde{L} be an abelian linear subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ with $\eta = (n_1, \dots, n_r)$. If $\dim(\text{vect}(\tilde{L})) = n$, then $\dim(F_{\tilde{L}_k}) = n_k - 1$ for every $k = 1, \dots, r$.*

Proof. Since $\dim(\text{vect}(\tilde{L})) = n$ then $\dim(\text{vect}(\tilde{L}_k)) = n_k$, $k = 1, \dots, r$. To show the Lemma for each \tilde{L}_k , we suppose that \tilde{L} is an abelian subgroup of $\mathbb{T}_n^*(\mathbb{C})$. We proceed by induction on the dimension $n \geq 1$:

- For $n = 1$, $\tilde{L} \subset \mathbb{C}^*$ and $F_1 = \{0\}$, so $\dim(F_1) = n - 1 = 0$.
- Suppose the property is true until dimension $n-1$ and let \tilde{L} be an abelian subgroup of $\mathbb{T}_n^*(\mathbb{C})$. Every $A \in \tilde{L}$ is written in the form:

$$A = \begin{bmatrix} A_1 & 0 \\ T_A & \mu_A \end{bmatrix}, \quad \text{with } A_1 \in \mathbb{T}_{n-1}^*(\mathbb{C}), T_A \in M_{1,n-1}(\mathbb{C})$$

and μ_A be the only eigenvalue of A . Consider the set $\tilde{L}^1 = \{A^1, A \in \tilde{L}\}$. It is clear that \tilde{L}^1 is an abelian subgroup of $\mathbb{T}_{n-1}^*(\mathbb{C})$. We can verify that $\dim(\text{vect}(\tilde{L}^1)) = n-1$, since $\dim(\text{vect}(\tilde{L})) = n$. By applying the induction property on \tilde{L}^1 , we obtain $\dim(F_{\tilde{L}^1}) = n-2$, where $F_{\tilde{L}^1} = \text{vect}((A^1 - \mu_A I_{n-1})e_i^1, A \in \tilde{L}, k = 1, \dots, n-2\}$ and $(e_1^1, \dots, e_{n-1}^1)$ be the canonical basis of \mathbb{C}^{n-1} . Now, if $\dim(F_{\tilde{L}}) < n-1$, then $\dim(F_{\tilde{L}}) = n-2$. Let $(v_1^1, \dots, v_{n-2}^1)$ be a basis of $F_{\tilde{L}^1}$ and denote by $v_k = (v_k^1, 0)$ for each k . We can assume that for any matrix $A \in \tilde{L}$, $T_A = 0$ leaving to change the canonical basis of \mathbb{C}^n by the basis $(v_1, \dots, v_{n-2}, e_{n-1}, e_n)$, because $F_{\tilde{L}^1}$ is \tilde{L}^1 -invariant. Let $\theta : \mathbb{T}_{n-1}(\mathbb{C}) \rightarrow \mathbb{T}_n(\mathbb{C})$ given by $\theta(A_1) = \begin{bmatrix} A_1 & 0 \\ 0 & \mu_{A_1} \end{bmatrix}$ is linear injective map such that $\tilde{L} = \theta(\tilde{L}^1)$, so $\text{vect}(\tilde{L}) = \theta(\text{vect}(\tilde{L}^1))$. It follows that $\dim(\text{vect}(\tilde{L})) = n-1 \neq n$, a contradiction. \square

Lemma 2.4. *Let L be an abelian linear subgroup of $GL(n, \mathbb{C})$. Then $\dim(\text{vect}(L)) = n$ if and only if $\dim(\text{vect}(L(u_0))) = n$.*

Proof. Firstly, by Proposition 2.1, suppose that $L \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$ and so $v_0 = u_0$. Since $\dim(\text{vect}(L)) = n$ then $\dim(\text{vect}(L_k)) = n_k$, $k = 1, \dots, r$.

Suppose that $\dim(\text{vect}(L)) = n$, then by Lemma 2.3, $\dim(F_{L_k}) = n_k - 1$, $k = 1, \dots, r$. By Lemma 2.2, there exists a linear injective map $\rho : \mathbb{C}^n \rightarrow \mathcal{K}_{\eta,r}(\mathbb{C})$ such that:

- i) $L \subset \rho(\mathbb{C}^n)$.
- ii) For every $v \in \mathbb{C}^n$, $\rho(v)u_0 = v$.

Let $M^{(1)}, \dots, M^{(n)} \in L$ be the generators of $\text{vect}(L)$. By (i), there exist $v_1, \dots, v_n \in \mathbb{C}^n$ such that $\rho(v_i) = M^{(i)}$ for each i . By (ii), $v_i = \rho(v_i)u_0 = M^{(i)}u_0 \in L(u_0)$. It follows that v_1, \dots, v_n generate $\text{vect}(L(u_0))$. Since ρ is injective and $M^{(1)}, \dots, M^{(n)}$ are free, then (v_1, \dots, v_n) is a basis of $\text{vect}(L(u_0))$ and so $\text{vect}(L(u_0)) = \mathbb{C}^n$.

Conversely, suppose that $\dim(\text{vect}(L_G(u_0))) = n$, so $\dim(\text{vect}(L_k(e_{k,1}))) = n_k$, where $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$ and $L_k := (L_G)_k$ for every $k = 1, \dots, r$. It follows that $\dim(F_{L_k}) = n_k - 1$. We repeat the same proof, as in above, by using the Lemma 2.2. \square

Denote by:

- $\Psi_x : \text{vect}(L_G) \longrightarrow \tilde{E}(x) \subset \mathbb{C}^n$ the linear map given by $\Psi_x(A) = Ax$.

Lemma 2.5. *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C})$ such that $0 \in \text{Fix}(G)$ and $\dim(\text{vect}(L_G)) = n$. Then $\tilde{E}(x)$ is L_G -invariant and the linear map $\Psi_x : \text{vect}((L_G)_{/\tilde{E}(x)}) \longrightarrow \tilde{E}(x)$ is an isomorphism., where $(L_G)_{/\tilde{E}(x)}$ is the restriction of L_G on $\tilde{E}(x)$.*

Proof. By construction, $\tilde{E}(x)$ is L_G -invariant and Ψ_x is linear and surjective. Let $A \in \text{Ker}(\Psi_x)$ and $y \in \text{vect}(L_G(x))$. Then there is $B \in \text{vect}((L_G)_{/\tilde{E}(x)})$ such that $y = Bx$. Now, $Ay = ABx = BAx = 0$, so $A = 0$. \square

Let \tilde{L} be an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$. Denote by:

- $\tilde{\Omega}_n = \{x \in \mathbb{C}^n, \dim(\text{vect}(\tilde{L}(x))) = n\}$.
- $\tilde{V} = \prod_{k=1}^r \mathbb{C}^* \times \mathbb{C}^{n_k-1}$.

Lemma 2.6. *For every $x \in \tilde{V}$ there is $P \in \mathcal{K}_{\eta,r}^*(\mathbb{C})$ such that $x = Pu_0$.*

Proof. Write $x = (x_1, \dots, x_r)$ with $x_k = (x_{k,1}, \dots, x_{k,n_k})$ for each k . Let $P =$

$$\text{diag}(P_1, \dots, P_r) \text{ such that } P_k = \begin{bmatrix} x_{k,1} & & & & \\ & x_{k,2} & \ddots & & 0 \\ & \vdots & 0 & \ddots & \\ & \vdots & \vdots & \ddots & \ddots \\ x_{k,n_k} & 0 & \dots & 0 & x_{k,1} \end{bmatrix} \in \mathbb{T}_{n_k}^*(\mathbb{C}), \text{ since}$$

$x \in \tilde{V}$, so $x_{k,1} \neq 0$ for all k . It follows that $x = Pu_0$, since $P_k e_{k,1} = x_k$. \square

Lemma 2.7. *Let \tilde{L} be an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ such that $\dim(\text{vect}(\tilde{L})) = n$. Then $\tilde{\Omega}_n = \tilde{V}$.*

Proof. Firstly, remark that $u_0 \in \tilde{\Omega}_n$ since $\dim(\text{vect}(\tilde{L})) = n$ and by Lemma 2.4, $\dim(\text{vect}(\tilde{L}(u_0))) = n$. Let $x \in \tilde{V}$. By Lemma 2.6, there is $P \in \mathcal{K}_{\eta,r}^*(\mathbb{C})$ such that $x = Pu_0$. Denote by $L = P^{-1}\tilde{L}P$, so $L \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$ and $\dim(\text{vect}(L)) = n$, hence by applying Lemma 2.4 on L , we have $\dim(\text{vect}(L(u_0))) = n$. Now, $\tilde{L}(x) = PLP^{-1}(x) = PL(u_0)$, then $\text{vect}(\tilde{L}(x)) = P(\text{vect}(L(u_0)))$, so $\dim(\text{vect}(\tilde{L}(x))) = n$, hence $x \in \tilde{\Omega}_n$.

Conversely, let $x \in \tilde{\Omega}_n$, so $\dim(\text{vect}(\tilde{L}(x))) = n$. Since $\mathbb{C}^n \setminus \tilde{V} = \bigcup_{k=1}^r \tilde{H}_k$, where

$$\tilde{H}_k = \{y = (y_1, \dots, y_r), y_k \in \{0\} \times \mathbb{C}^{n_k-1}, y_i \in \mathbb{C}^{n_i}, \text{ for every } i \neq k\}.$$

By the form of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$, we can verify, that each \tilde{H}_k is \tilde{L} -invariant, so $x \notin H_k$ because $\dim(H_k) = n - 1$. It follows that $x \in \tilde{V}$. \square

3. Proof of main results

Lemma 3.1. *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$ and $x \in \mathbb{C}^n$. Then $E(x)$ is G -invariant.*

Proof. Suppose that $E(x)$ is generated by $f_1(x), \dots, f_p(x)$, with $f_k \in \mathcal{A}(G)$, $k = 1, \dots, p$. Let $y = \sum_{k=1}^p \alpha_k f_k(x) \in E(x)$ and $f \in G$, then $y = g(x)$, with $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$. Therefore $f(y) = f \circ g(x) = \Phi_x(f \circ g) \in E(x)$, since $f \circ g \in \mathcal{A}(G)$. \square

Lemma 3.2. *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$. Then $g(0) = 0$ for every $g \in \mathcal{A}(G)$.*

Proof. Let $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$ with $f_k \in G$, $\alpha_k \in \mathbb{C}$, so $g(0) = \sum_{k=1}^p \alpha_k f_k(0) = 0$. Now, let $f_1, \dots, f_m, g_1, \dots, g_q \in \mathcal{A}(G)$ such that $f_k(0) = g_j(0) = 0$, $1 \leq k \leq m$, $1 \leq j \leq q$, so for every $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_m \in \mathbb{C}$ we have

$$\begin{aligned} \left(\sum_{j=1}^q \beta_j g_j \right) \circ \left(\sum_{k=1}^m \alpha_k f_k \right) (0) &= \sum_{j=1}^q \beta_j g_j \left(\sum_{k=1}^m \alpha_k f_k(0) \right) \\ &= \sum_{j=1}^q \beta_j g_j(0) \\ &= 0 \end{aligned}$$

Since $\mathcal{A}(G)$ is the algebra generated by G , so it is stable by composition and by linear combinations, hence we obtain the results. \square

Denote by $\varphi : \mathcal{A}(G) \rightarrow M_n(\mathbb{C})$ the linear map given by $\varphi(f) = Df_0$, for every $f \in \mathcal{A}(G)$. Then $L_G = \varphi(G)$.

Lemma 3.3. *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$, such that $0 \in \text{Fix}(G)$. Then L_G is an abelian subgroup of $GL(n, \mathbb{C})$.*

Proof. Let $f, g \in G$, so $\varphi(f \circ g) = D(f \circ g)(0) = Df(g(0)).Dg(0)$. By Lemma 3.2, $g(0) = 0$, so $\varphi(f \circ g) = D(f)(0).Dg(0) = \varphi(f).\varphi(g)$. The proof is completed. \square

Denote by:

- $U_t = \{x \in \mathbb{C}^n, r(x) \geq t\}$, for every $t \in \mathbb{N}$.
- $r_G = \max\{r(x), x \in \mathbb{C}^n\}$.

Proposition 3.4. *Let G be an abelian subgroup of $\text{Diff}^1(\mathbb{C}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$. Then for every $0 \leq t \leq r_G$, U_t is a G -invariant open subset of \mathbb{C}^n .*

Proof of Proposition 3.4. In the first, remark that the rank $r(y)$ is constant on any orbit $G(y)$, $y \in E(x)$. So U_t is G -invariant for every $0 \leq t \leq r_G$. Let's show that U_t is an open set: Let $y \in U_t$ and $r = r_y$, so $r \geq t$. Then there exist $f_1, \dots, f_r \in F(y)$ such that the r vectors $f_1(y), \dots, f_r(y)$ are linearly independent in $E(y)$. For all $z \in \mathbb{C}^n$, we consider the Gram's determinant

$$\Delta(z) = \det(\langle f_i(z) | f_j(z) \rangle)_{1 \leq i, j \leq r}$$

of the vectors $f_1(z), \dots, f_r(z)$ where $\langle \cdot | \cdot \rangle$ denotes the scalar product in \mathbb{C}^n . It is well known that these vectors are independent if and only if $\Delta(z) \neq 0$, in particular $\Delta(y) \neq 0$. Let

$$V_y = \{z \in \mathbb{C}^n, \Delta(z) \neq 0\}$$

The set V_y is open in \mathbb{C}^n , because the map $z \mapsto \Delta(z)$ is continuous. Now $\Delta(y) \neq 0$, and so $y \in V_y \subset U_t$. The proof is completed. \square

3.1. Hamel basis and norm. The main of this section is to justify the existence of a basis of every vector space. This result is trivial in the finite case, is in fact rather surprising when one thinks of infinite dimensional vector spaces, and the definition of a basis. Recall that a *Hamel basis* or simply a basis of a vector space E is a linearly independent set \mathcal{B} (every finite subset of \mathcal{B} is linearly independent) such that for each nonzero $x \in E$ there are $a_1, \dots, a_k \in \mathcal{B}$ and nonzero scalars $\alpha_1, \dots, \alpha_k$ (all uniquely determined) such that $x = \sum_{i=1}^k \alpha_i a_i$. The following theorem is equivalent to the axiom of choice family of axioms and theorems. In [4], C.D.Aliprantis and K.C.Border proved, in the following theorem, that Zorn's lemma implies that every vector space has a basis.

Theorem 3.5. ([4], *Theorem 1.8*) *Every nontrivial vector space has a Hamel basis.*

As a consequence, we found the important following results:

Theorem 3.6. *Every nontrivial vector space has a norm called Hamel norm.*

Proof. Let E be a nontrivial vector space over \mathbb{R} . By Theorem 3.5, E has a Hamel basis called $\mathcal{B} = (a_i)_{i \in I}$, for any set I of indices (not necessary countable). In this basis, every vector $x \in E$ has the form $x = \sum_{i \in I_x} \alpha_i a_i$, where $\alpha_i \in \mathbb{K}$ and $I_x \subset I$ with finite cardinal. The family $(\alpha_i)_{i \in I}$ with $\alpha_i = 0$ for every $i \in I \setminus I_x$, is called the coordinate of x . Now, define $\|x\| = \sum_{i \in I_x} |\alpha_i|$. It is easy to verify that $\|\cdot\|$ defines a norm on E by the coordinate in the Hamel basis. \square

Remark that any vector for the Hamel basis is with norm 1.

3.2. Linear map and isomorphism. A subset $E \subset \mathbb{R}^n$ is called G -invariant if $f(E) \subset E$ for any $f \in G$; that is E is a union of orbits. Set $\mathcal{A}(G)$ be the algebra generated by G . For a fixed vector $x \in \mathbb{R}^n \setminus \{0\}$, denote by:

- $\Phi_x : \mathcal{A}(G) \longrightarrow \Phi_x(\mathcal{A}(G)) \subset \mathbb{R}^n$ the linear map given by $\Phi_x(f) = f(x)$.
- $E(x) = \Phi_x(\mathcal{A}(G))$.
- F_x is an algebraic supplement of $\text{Ker}(\Phi_x)$ in $\mathcal{A}(G)$. It is easy to show that $p_x := \dim(F_x) = \dim(\Phi_x(\mathcal{A}(G))) \leq n$.
- $\mathcal{C}_x = (a_1, \dots, a_{p_x})$ is a basis of F_x .
- $\mathcal{B}_x = (b_i)_{i \in I}$ is a Hamel basis (Theorem 3.5) of $\text{Ker}(\Phi_x)$.
- $\mathcal{E}_x = (\mathcal{C}_x, \mathcal{B}_x)$ is a Hamel basis of $\mathcal{A}(G)$. By theorem 3.6, $\mathcal{A}(G)$ is provided with the Hamel norm defined in the Theorem 3.6 associated to the basis \mathcal{E}_x .

Lemma 3.7. *The linear map $\Phi_x : \mathcal{A}(G) \longrightarrow E(x)$ is continuous. In particular, $\text{Ker}(\Phi_x)$ is a closed subspace of $\mathcal{A}(G)$.*

Proof. Since Φ_x is linear and $\mathcal{A}(G)$ is a normed vector space (Lemma 3.6), we will verify the continuity of Φ_x on 0. Let $f \in \mathcal{A}(G)$ and write $f = f_1 + f_2$ with $f_1 \in F_x$ and $f_2 \in \text{Ker}(\Phi_x)$. Set $(\alpha_i)_{1 \leq i \leq p_x}$ and $(\beta_i)_{i \in I}$ be respectively the coordinates of f_1 and f_2 in \mathcal{C}_x and \mathcal{B}_x . Write $f = \sum_{i=1}^{p_x} \alpha_i a_i + \sum_{i \in I_2} \beta_i b_i$ where $I_2 \subset I$ with finite cardinal.

We have $\|f\| = \sum_{i=1}^{p_x} |\alpha_i| + \sum_{i \in I_2} |\beta_i|$ and $b_i(x) = 0$ for all $i \in I_2$. Therefore

$$\begin{aligned} \|\Phi_x(f)\| = \|f(x)\| &= \left\| \sum_{i=1}^{p_x} \alpha_i a_i(x) + \sum_{i \in I_2} \beta_i b_i(x) \right\| \\ &\leq \sum_{i=1}^{p_x} |\alpha_i| \|a_i(x)\| \\ &\leq \|f\| \sum_{i=1}^{p_x} \|a_i(x)\| \end{aligned}$$

Since $\sum_{i=1}^{p_x} \|a_i(x)\|$ is constant relative to f , then Φ_x is continuous. \square

Lemma 3.8. *Suppose that $\dim(\text{vect}(L_G)) = n$. The linear map $\varphi : \mathcal{A}(G) \longrightarrow \text{vect}(L_G)$ is continuous. In particular, $\text{Ker}(\varphi)$ is a closed subspace of $\mathcal{A}(G)$.*

Proof. Since φ is linear and $\mathcal{A}(G)$ is a normed vector space (Lemma 3.6), we will verify the continuity of φ on 0. Firstly, see that $\text{cod}(\text{Ker}(\varphi)) = n$ is finite since $\dim(\text{vect}(L_G)) = n$. Let F be an algebraic supplement to $\text{Ker}(\varphi)$ in \mathcal{A} , $\mathcal{C}' = (a'_1, \dots, a'_n)$ and $\mathcal{B}' = (b'_i)_{i \in J}$ are respectively the Hamel basis of F and $\text{Ker}(\varphi)$ (Lemma 3.5). Let $f \in \mathcal{A}(G)$ and write $f = f_1 + f_2$ with $f_1 \in F$ and $f_2 \in \text{Ker}(\varphi)$. Set $(\alpha_i)_{1 \leq i \leq n}$ and $(\beta_i)_{i \in J}$ be respectively the coordinates of f_1 and f_2 in \mathcal{C}' and \mathcal{B}' . Write $f = \sum_{i=1}^n \alpha_i a'_i + \sum_{i \in I_2} \beta_i b'_i$ where $I_2 \subset J$ with finite cardinal. We have

$\|f\| = \sum_{i=1}^n |\alpha_i| + \sum_{i \in I_2} |\beta_i|$ and $\varphi(b_i) = Db_i(0) = 0$ for all $i \in I_2$. Therefore

$$\begin{aligned} \|\varphi(f)\| &= \|Df(0)\| = \left\| \sum_{i=1}^n \alpha_i Da_i(0) + \sum_{i \in I_2} \beta_i Db_i(0) \right\| \\ &\leq \sum_{i=1}^n |\alpha_i| \|Da_i(0)\| \\ &\leq \|f\| \sum_{i=1}^n \|Da_i(0)\| \end{aligned}$$

Since $\sum_{i=1}^n \|Da_i(0)\|$ is constant relative to f , then φ is continuous. \square

Lemma 3.9. ([6], 3.5) *Let E be a topological vector space over \mathbb{R} , and let M be a closed subspace of finite codimension. Then $E = M \oplus N$ is a topological sum, for every algebraic complementary subspace N of M .*

Corollary 3.10. *The algebraic sum $\mathcal{A}(G) = F_x \oplus Ker(\Phi_x)$ is topological. In particular, $F_x \oplus Ker(\Phi_x)$ and $\mathcal{A}(G)$ are topological isomorphic by the map: $(f_1, f_2) \mapsto f_1 + f_2$.*

Proof. By the Theorem 3.6, $\mathcal{A}(G)$ is a normed vector space so it is a topological vector space. By Lemma 3.7, Φ_x is continuous and so its kernel is closed vector space with finite codimension. The results follows directly by applying the Lemma 3.9 for $E = \mathcal{A}(G)$ and $M = Ker(\Phi_x)$. \square

By Corollary 3.10, we can identify $\mathcal{A}(G)$ with $F_x \oplus Ker(\Phi_x)$, so every $f \in \mathcal{A}(G)$ is denoted by $f = (f_1, f_2) = f_1 + f_2$ with $f_1 \in F_x$ and $f_2 \in Ker(\Phi_x)$.

Recall that $\tilde{E}(x) = \psi_x(\mathcal{A}(G))$.

Lemma 3.11. *Let H and K be two closed vector subspaces of $\mathcal{A}(G)$ such that $cod(H) = cod(K) = j \geq 1$. Let $\psi \in \mathcal{A}(G) \setminus (H \cup K)$. Then there exists a vector space F of $\mathcal{A}(G)$, containing ψ and satisfying $\mathcal{A}(G) = F \oplus H = F \oplus K$.*

Proof. There are two cases;

Case 1: If $j = 1$, then we take $F = \mathbb{K}\psi$.

Case 2: Suppose that $j \geq 2$. Denote by $H' = H \oplus \mathbb{K}\psi$ and $K' = K \oplus \mathbb{K}\psi$. Suppose that $H' \neq K'$ (otherwise, it is easy to take a comment supplement) and let $f_1 \in H' \setminus K'$ and $g_1 \in K' \setminus H'$, so $h_1 = f_1 + g_1 \notin H' \cup K'$. Denote by $H_1 = \mathbb{K}h_1 \oplus H'$ and $K_1 = \mathbb{K}h_1 \oplus K'$. We establish two cases:

- If $H_1 = K_1$, then any supplement F of H_1 in $\mathcal{A}(G)$ is a supplement of K_1 in $\mathcal{A}(G)$, the proof follows then.

- If $H_1 \neq K_1$, we take $f_2 \in H_1 \setminus K_1$ and $g_2 \in K_1 \setminus H_1$, so $h_2 = f_2 + g_2 \notin H_1 \cup K_1$. Denote by $H_2 = \mathbb{K}h_2 \oplus H_1$ and $K_2 = \mathbb{K}h_2 \oplus K_1$.

We repeat the same processes until $j - 2$ times and we obtain:

- If $H_{j-1} = K_{j-1}$, then any supplement F of H_{j-2} in $\mathcal{A}(G)$ is a supplement of K_{j-2} in $\mathcal{A}(G)$.
- If $H_{j-2} \neq K_{j-2}$, we take $f_j \in H_{j-2} \setminus K_{j-2}$ and $g_j \in H_{j-2} \setminus K_{j-2}$, so $h_j = f_j + g_j \notin H_{j-2} \cup K_{j-2}$. Denote by $H_{j-1} = \mathbb{K}h_{j-1} \oplus H_{j-2}$ and $K_{j-1} = \mathbb{K}h_{j-1} \oplus K_{j-2}$. We obtain then $H_{j-1} = K_{j-1} = \mathcal{A}(G)$, hence the proof is completed by taking $F = \text{vect}(\psi, h_1, \dots, h_j)$. \square

Denote by:

- $\tilde{E}(x) = \text{vect}(L_G(x))$.
- $\tilde{r}_x = \dim(\tilde{E}(x))$. Since $\text{vect}(L_G(x)) = n$ then by Lemma 2.4, $\dim(\tilde{E}(v_0)) = n$, so $\tilde{r}_G := \sup_{x \in \mathbb{C}^n} \{\tilde{r}_x\} = n$.

Lemma 3.12. *If $r(x) = \tilde{r}_x = n$, then there exists a commune vector space $F_x \subset \varphi^{-1}(L_G)$ supplement to $\text{Ker}(\Phi_x)$ and to $\text{ker}(\varphi)$ in $\mathcal{A}(G)$. (i.e $F_x \oplus \text{Ker}(\Phi_x) = F_x \oplus \text{Ker}(\varphi) = \mathcal{A}(G)$).*

Proof. Since L_G is abelian so $\text{vect}(L_G)$ is an abelian algebra and φ is a morphism of ring, then $\varphi^{-1}(L_G)$ is an algebra containing G . hence $\varphi^{-1}(L_G) = \mathcal{A}(G)$. Then any supplement of $\text{Ker}(\Phi_x)$ in $\mathcal{A}(G)$ is contained in $\varphi^{-1}(L_G)$. Now by Lemma 3.8, φ is continuous. By Lemmas 3.7 and 3.8 both $\text{Ker}(\Phi_x)$ and $\text{Ker}(\varphi)$ are closed and since $\text{cod}(\text{Ker}(\Phi_x)) = \text{cod}(\text{Ker}(\varphi))$, then by lemma 3.11, there exists a commune supplement F_x to $\text{Ker}(\Phi_x)$ and to $\text{ker}(\Phi_x)$ in $\mathcal{A}(G)$ containing I_d (because $I_d \in \mathcal{A}(G) \setminus \text{Ker}(\Phi_x) \cup \text{Ker}(\varphi)$). \square

Lemma 3.13. *The linear map $\Phi_x : F_x \longrightarrow E(x)$ given by $\Phi_x(f) = f(x)$ is an isomorphism.*

Proof. The proof follows directly from the fact that Φ_x is linear surjective and $\dim(F_x) = \dim(E(x)) = n$. \square

Lemma 3.14. *If $r(x) = \tilde{r}_x = n$, then the restriction $\varphi_1 : F_x \longrightarrow \text{vect}(L_G)$ of φ from F_x unto $\text{vect}(L_G)$ is an isomorphism.*

Proof. The proof follows directly from the fact that φ_1 is linear surjective and $\dim(F_x) = \dim(\text{vect}(L_G)) = n$. \square

Lemma 3.15. *If $r(x) = \tilde{r}_x = n$, then the map $\varphi_x := \tilde{\Phi}_x \circ \varphi_1 \circ \Phi_x^{-1} : E(x) \longrightarrow \tilde{E}(x)$ defined by $\varphi_x(f(x)) = Df_0(x)$, $f \in H$, is an isomorphism and satisfying:*

- (i) $\varphi_x(G(x)) = L_G(x)$.
- (ii) Let $y \in E(x)$ and $z = \varphi_x(y)$ then $\varphi_x(G(y)) = L_G(z)$.
- (iii) for every $y \in \overline{G(x)}$ we have $z = \varphi_x(y) \in \overline{L_G(x)}$.

Proof. (i) φ_x is an isomorphisms, since $\varphi_x = \tilde{\Phi}_x \circ \varphi_1 \circ \Phi_x^{-1}$ and by Lemmas 3.14, 2.5 and 3.13, it is composed by isomorphisms.

By Lemma 3.12, we can assume that F_x is a commune supplement to $Ker(\Phi_x)$ and to $Ker(\varphi)$ in $\mathcal{A}(G)$. Let $f_1, \dots, f_r \in \mathcal{A}(G)$ be the generator of F_x . So the restriction $(\Phi_x)_{/F_x}$ (resp. $(\psi_x)_{/F_x}$) of Φ_x (resp. ψ_x) on F_x are bijective from F_x to $E(x)$ (resp. $\tilde{E}(x)$). Then $f_1(x), \dots, f_r(x)$ generate $E(x)$ and $Df_1(0)(x), \dots, Df_r(0)(x)$ generate $\tilde{E}(x)$. Now, let $f = \sum_{k=1}^r \alpha_k f_k \in F_x$, so $\varphi_x(f(x)) = \sum_{k=1}^r \alpha_k Df_k(0)(x) = Df_0(x) \in L(x)$. It follows that $\varphi_x(G(x)) = L_G(x)$.

(ii) Let $y \in E(x)$ and $z = \varphi_x(y)$. Let $f \in G$ and write $f = \sum_{k=1}^r \alpha_k f_k$, where (f_1, \dots, f_r) be a basis of F_x . Then by Lemma 3.2, $g(0) = 0$ and so $\varphi_x(f(y)) = \varphi_x(f \circ g(x)) = Df_0.Dg_0(x) = Df_0(z) \in L_G(z)$. It follows that $\varphi_x(G(y)) \subset L_G(z)$. Conversely, let $a \in L_G(z)$, so $a = Df_0(z)$ for some $f \in G$. Since $z = Dg_0(x)$ and $g \in G$, so $a = D(f \circ g)_0(x) = \varphi_x(f \circ g)$. Hence $a \in \varphi_x(G(x))$.

(iii) Since $y \in E(x)$, there exists $g \in F_x$ such that $y = g(x)$, so $z = Dg_0(x)$. By continuity of φ_x and by (i), we have $z \in \varphi_x(\overline{G(x)}) \subset \varphi_x(G(x)) = L_G(x)$. \square

Denote by:

- $\tilde{U}_t = \{y \in \mathbb{C}^n, \tilde{r}_y \geq t\}$, for every $0 \leq t \leq n$.

By applying Lemma 3.11 given in [10] to the abelian linear group L_G , we found the following result:

Lemma 3.16. ([10], Lemma 3.11) \tilde{U}_t is a L_G -invariant dense open subset of \mathbb{C}^n .

Consider the distinct values $\tilde{r}_0 = 0 < \tilde{r}_1 < \dots < \tilde{r}_p = n$ taken by the map $\tilde{r} : \mathbb{C}^n \rightarrow \mathbb{N}$ given by $x \mapsto \tilde{r}_x$, $x \in \mathbb{C}^n$ and let $\tilde{F}_j = \{x \in \mathbb{C}^n / \tilde{r}_x \leq \tilde{r}_j\}$, $j = 0, 1, \dots, p$. Evidently \tilde{F}_j ($0 \leq j \leq p$) is the complementary of the L -invariant open set $\tilde{U}_{\tilde{r}_{j-1}}$ (Lemma 3.16) and so the sequence $\tilde{F}_0, \dots, \tilde{F}_p$ is a increasing sequence of closed subsets of \mathbb{C}^n and $\tilde{\Omega}_j = \tilde{F}_j \setminus \tilde{F}_{j-1} = \tilde{F}_j \cap \tilde{U}_{\tilde{r}_{j-1}}$. By Lemma 3.3, L is an abelian subgroup of $GL(n, \mathbb{C})$. We set the same construction of the open sets $\tilde{\Omega}_j$ which is given by S.Chihhi in the proof of Theorem 3.10, for using the following result:

medskip

Lemma 3.17. ([10], Theorem 3.10) Let $1 \leq j \leq p$ and $x \in \tilde{\Omega}_j$ then for every $y \in \overline{L_G(x)} \cap \tilde{\Omega}_j$ we have $\overline{L_G(y)} \cap \tilde{\Omega}_j = \overline{L_G(x)} \cap \tilde{\Omega}_j$.

Lemma 3.18. (Under the above notations) For every $x \in \Omega_j$, we have $\varphi_x(\Omega_j \cap E(x)) = \tilde{\Omega}_j \cap \tilde{E}(x)$ for every $1 \leq j \leq p$.

Proof. Let $y \in \Omega_j \cap E(x)$ and $z = \varphi_x(y)$. By Lemma 3.15,(ii), $\varphi_x(G(y)) = L_G(z)$. Since φ_x is linear, then $\varphi_x(E(y)) = \tilde{E}(z)$, so $r(y) = \tilde{r}_z = j$. It follows that $z \in \tilde{\Omega}_j \cap \tilde{E}(x)$. For the converse we use the same proof for φ_x^{-1} . \square

Proof of Structure's Theorem 1. By Proposition 3.4, U_t is a G -invariant open set of \mathbb{C}^n for every $0 \leq t \leq r_G$ and it is dense in \mathbb{C}^n for every $0 \leq t \leq \tilde{r}_G$. Consider the distinct values $r_0 = 0 < r_1 < \dots < r_q = r_G$ taken by the map $r : \mathbb{C}^n \rightarrow \mathbb{N}$ given by $x \mapsto r(x)$, $x \in \mathbb{C}^n$ and let $F_j = \{x \in \mathbb{C}^n / r(x) \leq r_j\}$, $j = 0, 1, \dots, q$. Evidently F_j ($0 \leq j \leq q$) is the complementary of the G -invariant open set $U_{r_{j-1}}$ and so the sequence F_0, \dots, F_q is an increasing sequence of closed subsets of \mathbb{C}^n and $\Omega_j = F_j \setminus F_{j-1} = F_j \cap U_{r_{j-1}}$ for every $1 \leq j \leq q$. Yet, by Proposition 3.4, $U_{r_{j-1}}$ is a G -invariant open set of \mathbb{C}^n . This proves (i).

The proof of (ii): Let $x \in \Omega_j$ and $y \in \overline{G(x)} \cap \Omega_j$. By Lemma 3.15, there exists an isomorphism $\varphi_x : E(x) \rightarrow \tilde{E}(x)$ satisfying $\varphi_x(G(x)) = L(x)$ and $\varphi_x(G(y)) = L_G(z)$ with $z = \varphi_x(y)$. By Lemma 3.17, we have

$$\overline{L_G(z)} \cap \tilde{\Omega}_j = \overline{L_G(x)} \cap \tilde{\Omega}_j \quad (1).$$

By Lemma 3.15, we have φ_x is an isomorphism satisfying $\varphi_x(E(x)) = \tilde{E}_x$ and $\varphi_x(G(x)) = L_G(x)$. By Lemma 3.18, we have $\varphi_x(\Omega_j \cap E(x)) = \tilde{\Omega}_j \cap \tilde{E}(x)$, therefore by (1) we obtain

$$\begin{aligned} \overline{G(x)} \cap \Omega_j &= \overline{G(x)} \cap \Omega_j \cap E(x) \\ &= \varphi_x^{-1}(\overline{L_G(x)}) \cap \varphi_x^{-1}(\tilde{\Omega}_j \cap \tilde{E}(x)) \\ &= \overline{\varphi_x^{-1}(L_G(x))} \cap \Omega_j \cap E(x) \\ &= \overline{L_G(z)} \cap \Omega_j \cap E(x) \\ &= \overline{G(y)} \cap \Omega_j. \end{aligned}$$

This completes the proof. \square

Denote by $\tilde{r}_G = \max\{\tilde{r}_x, x \in \mathbb{C}^n\}$.

Proof of Corollary 1.2. Let O be a dense orbit in \mathbb{C}^n (i.e. $\overline{O} = \mathbb{C}^n$). Then for every $x \in O$, we have $r(x) = n$, so $O \subset U$ and $\overline{O} \cap U = U$. Since O is minimal in U (Theorem 1.1), then for every orbit $L \subset U$, we have $\overline{L} \cap U = \overline{O} \cap U$. Therefore $\overline{L} = \overline{O} = \mathbb{C}^n$. \square

Proof of Corollary 1.3. If O is a locally dense orbit in \mathbb{C}^n (i.e. $\overset{\circ}{\overline{O}} \neq \emptyset$) then $O \subset U$ (because, as above, for any $x \in O$, $r(x) = n$). Let C be a connected component of U meeting O . Then $\overline{O} \cap C$ is a nonempty closed subset in C . Lets show that $\overline{O} \cap C$ is open in C . Let $y \in \overline{O} \cap C$. Since O is minimal in U (Theorem 1.1) then $\overline{O} \cap U = \overline{G(y)} \cap U$. So, $\overset{\circ}{\overline{G(y)}} = \overset{\circ}{\overline{O}} \neq \emptyset$. Then $y \in \overset{\circ}{\overline{G(y)}} \cap C \subset \overset{\circ}{\overline{O}} \cap C$. Lets show that every orbit meeting C is dense in C : if O' is an orbit meeting C then $O' \subset U$. Since O is dense in C then $O' \subset \overline{O}$ and then $O' \cap C = O \cap C = C$. This completes the proof. \square

Proof of Structure's Theorem 2. Suppose that $\tilde{E}(x) = \mathbb{C}^n$, for some $x \in \mathbb{C}^n$. Firstly, by Proposition 2.1, we can assume that $L_G \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$ and so $v_0 = u_0$. Secondly, we verify that $x \in V := \prod_{k=1}^r \mathbb{C}^* \times \mathbb{C}^{n_k-1}$: We have $\mathbb{C}^n \setminus V = \bigcup_{k=1}^r H_k$, where

$$H_k = \{y = (y_1, \dots, y_r), \quad y_k \in \{0\} \times \mathbb{C}^{n_k-1}, \quad y_i \in \mathbb{C}^{n_i} \text{ for } i \neq k\}.$$

By construction of $\mathcal{K}_{\eta,r}(\mathbb{C})$, each H_k is L_G -invariant vector space with dimension $n-1$, so $x \in V$.

Thirdly, we can take $x = u_0$, leaving to replace L_G by $PL_G P^{-1}$ for any passage matrix P from the canonical basis to (x, e_2, \dots, e_n) .

Finally, the proof of Theorem 1.4 follows directly from Theorem 1.1 and Lemma 2.4. \square

3.3. Locally results. We can remove the condition $\dim(\text{vect}(L_G)) = n$ in the Theorem 1.1, we obtain a locally decomposition as follow:

Corollary 3.19. *Let G be an abelian subgroup of $Diff^1(\mathbb{C}^n)$, such that $0 \in \text{Fix}(G)$. Let $x \in \mathbb{C}^n$. If $r(x) = \tilde{r}_x$, there exist a finite increasing sequence of G -invariant closed subsets of $E(x)$, $F_0 \subset \dots \subset F_q$, $1 \leq q \leq n$, with the following properties:*

- (i) $F_0 = \emptyset$, $F_q = \mathbb{C}^n \setminus \{0\}$ and $\mathbb{C}^n = \bigcup_{j=1}^q \Omega_j$, where $\Omega_j = F_j \setminus F_{j-1}$.
- (ii) Every orbit of Ω_j is minimal in it.

Proof. By Lemma 3.1, $E(x)$ is G -invariant. Then the proof results directly by applying Theorem ?? on the restriction $G_{/E(x)}$ of G on $E(x)$. \square

3.4. Some results for the dimension $n = 1$. We can remove the condition $\dim(\text{vect}(L_G)) = 1$ in the Theorem 1.1, we obtain a global decomposition as follow:

Corollary 3.20. *Let G be an abelian subgroup of $Diff^1(\mathbb{C})$, such that $0 \in \text{Fix}(G)$. Then every orbit of $\mathbb{C} \setminus \{0\}$ is minimal in it.*

Proof. Since $id \in G$, then $id \in L_G$ and so $\dim(L_G) = 1$. The proof results from Theorem 1.1, for $F_0 = \emptyset$, $F_1 = \{0\}$ and $F_2 = \mathbb{C}$. \square

Remark 3.21. Remark that $\dim(L_G) \geq 1$, since $I_n \in L_G$, where I_n is the identity matrix with order n .

4. Example in the dimension $n=2$

We give some examples of abelian subgroup of $Diff^1(\mathbb{C})$ which are near enough to the linear group. Recall that $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Example 4.1. Let Γ be the group abelian of all diffeomorphisms f of \mathbb{C}^2 defined by $f(x, y) = (x, y + P(x))$, $P \in \mathbb{C}[X]$ and G be a subgroup of Γ generated by $f : (x, y) \mapsto (x, y + P(x))$ and $g : (x, y) \mapsto (x, y + Q(x))$ with $P, Q \in \mathbb{C}[X]$, $Q(0) = P(0) = 0$ and $P'(0) \neq 0$. Then every orbit is minimal in the open set $\mathbb{C}^* \times \mathbb{C}$.

Proof. We can easily verify that G is abelian, $0 \in Fix(G)$. In the begin, we have $f^n \circ g^m(x, y) = (x, y + nP(x) + mQ(x))$ for every $n, m \in \mathbb{Z}$. Then every $h \in \mathcal{A}(G)$ has the form:

$$h(x, y) = (\alpha x, \beta y + R(x)) \quad \text{where } R \in \mathbb{C}[X] \quad (1)$$

We have $Df_0 = \begin{bmatrix} 1 & 0 \\ P'(0) & 1 \end{bmatrix}$. Then $L_G \subset \mathbb{T}_2^*(\mathbb{C})$. Here $u_0 = e_1 = (1, 0)$ then $\tilde{E}(u_0)$ is generated by e_1 and $Df_0(u_0) = (1, P'(0))$, so $dim(\tilde{E}(u_0)) = 2$, hence $\tilde{E}(e_1) = \mathbb{C}^2$. By Lemma 2.7, $\tilde{\Omega}_2 = \mathbb{C}^* \times \mathbb{C}$ and by Lemma 3.18, we have $\varphi_{e_1}(\Omega_2) = \tilde{\Omega}_2$, then

$$\Omega_2 = \varphi_{e_1}^{-1}(\mathbb{C}^* \times \mathbb{C}) \quad (2)$$

By Theorem 1.4, every orbit of Ω_2 is minimal in it. Now let's prove that $\varphi_{e_1}^{-1}(\mathbb{C}^* \times \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}$: Recall that $\varphi_{e_1}(x, y) = \tilde{\Phi}_{e_1} \circ \varphi \circ \Phi_{e_1}^{-1}(x, y)$ (see Lemma 3.15). Let $(x, y) \in \mathbb{C}^* \times \mathbb{C}$. Then there exists $h \in \mathcal{A}(G)$ such that $(x, y) = Dh_0 e_1$. Using (2) and by integration, h has the form $h : (s, t) \mapsto (xs, \beta t + ys + sR(s) + \gamma)$, with $R \in \mathbb{C}[X]$, then $h(e_1) = (x, y + R(1))$ and $Dh_0 e_1 = (x, y)$. Therefore

$$\begin{aligned} \varphi_{e_1}^{-1}(x, y) &= \Phi_{e_1} \circ \varphi^{-1} \circ \tilde{\Phi}_{e_1}^{-1}(x, y) \\ &= \Phi_{e_1} \circ \varphi^{-1}(Dh_0) \\ &= \Phi_{e_1}(h) \\ &= h(e_1) \\ &= (x, y + R(1) + \gamma) \in \mathbb{C}^* \times \mathbb{C} \end{aligned}$$

Conversely, Let $(a, b) \in \mathbb{C}^* \times \mathbb{C}$ then there is $h \in \mathcal{A}(G)$ such that $h(e_1) = (a, b)$. By (2), h has the form $h(s, t) = (as, \beta t + bs + sR(s) + \gamma)$, then $Dh_0(e_1) = (a, b + R(1) + R'(1))$. Therefore

$$\begin{aligned} \varphi_{e_1}(a, b) &= \tilde{\Phi}_{e_1} \circ \varphi \circ \Phi_{e_1}^{-1}(a, b) \\ &= \tilde{\Phi}_{e_1} \circ \varphi(h) \\ &= \tilde{\Phi}_{e_1}(Dh_0) \\ &= h(e_1) \\ &= (a, b + R(1) + R'(1)) \in \mathbb{C}^* \times \mathbb{C} \end{aligned}$$

It follows that $\varphi_{e_1}^{-1}(\mathbb{C}^* \times \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}$. □

Example 4.2. Let G be the abelian group generated by $f(x, y) = (x^q, y + x^k)$ and $g(x, y) = (x^{q'}, y + x^{k'})$ q and q' are odd integers, $k, k' \in \mathbb{N}$. Then every orbit of $\mathbb{C}^* \times \mathbb{C}$ is minimal in it.

Proof. We can easily verify that G is abelian, $0 \in Fix(G)$. In the begin, we have $f^n(x, y) = (x^{nq}, y + x^k + \sum_{i=1}^n x^{iqk})$ and so

$$f^n \circ g^m(x, y) = \left(x^{nqm q'}, y + x^k + \sum_{i=1}^n x^{iqk} + x^{nqk'} + \sum_{j=1}^m x^{nqj q' k'} \right)$$

for every $n, m \in \mathbb{Z}$. Then every $h \in \mathcal{A}(G)$ has the form:

$$h(x, y) = (\alpha x, \beta y + R(x)) \quad \text{where } R \in \mathbb{C}[X] \quad (1)$$

We have $Did_0 = id$ and $Df_0 = Dg_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $L_G \subset \mathbb{T}_1^*(\mathbb{C}) \oplus \mathbb{T}_1^*(\mathbb{C})$.

Here $u_0 = e_1 + e_2 = (1, 1)$ then $\tilde{E}(u_0)$ is generated by e_1 and $Df_0(u_0) = (0, 1)$, so $dim(\tilde{E}(u_0)) = 2$, hence $\tilde{E}(u_0) = \mathbb{C}^2$. By Lemma 2.7, $\tilde{\Omega}_2 = \mathbb{C}^* \times \mathbb{C}$ and by Lemma 3.18, we have $\varphi_{u_0}(\Omega_2) = \tilde{\Omega}_2$, then

$$\Omega_2 = \varphi_{u_0}^{-1}(\mathbb{C}^* \times \mathbb{C}) \quad (2)$$

By Theorem 1.4, every orbit of Ω_2 is minimal in it. As in the proof of example 4.1, we prove that $\varphi_{u_0}^{-1}(\mathbb{C}^* \times \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}$. \square

Question1 : Is there an abelian subgroup of $Diff^1(\mathbb{C}^n)$ fixing some point having no minimal set in $\mathbb{C}^n \setminus Fix(G)$?

Question2 : An abelian subgroup of $Diff^1(\mathbb{C}^n)$ fixing some point and satisfying $r(x) = \tilde{r}_x$ for every $x \in \mathbb{C}^n$, is-it conjugate to a linear group?

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