

**THE DYNAMIC OF ABELIAN SUBGROUP OF $Diff^r(\mathbb{K}^n)$,
FIXING A POINT, ($\mathbb{K} = \mathbb{R}$ OR \mathbb{C})**

YAHYA N'DAO AND ADLENE AYADI

ABSTRACT. In this paper, we study the action of any abelian subgroup G of $Diff^r(\mathbb{K}^n)$, $r \geq 1$ on \mathbb{K}^n , ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}). We prove that there exist a decreasing finite sequence F_0, F_1, \dots, F_q of invariant closed sets such that $\Omega_j = F_j \setminus F_{j-1}$ is an open subset (for the relative topology) of F_j in which every orbit is isomorphic to any orbits meeting its closure. Moreover, if G has a dense orbit in \mathbb{K}^n then every orbit of Ω_q is dense in \mathbb{K}^n .

1. Introduction

Denote by $Diff^r(\mathbb{K}^n)$, $r \geq 1$, ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) the group of all C^r -diffeomorphisms of \mathbb{K}^n . Let G be an abelian subgroup of $Diff^r(\mathbb{K}^n)$, $r \geq 1$ such that $Fix(G) \neq \emptyset$, where $Fix(G) = \{x \in \mathbb{K}^n : f(x) = x, \forall f \in G\}$ be the global fixed point set of G . There is a natural action $G \times \mathbb{K}^n \rightarrow \mathbb{K}^n$. $(f, x) \mapsto f(x)$. For a point $x \in \mathbb{K}^n$, denote by $G(x) = \{f(x), f \in G\} \subset \mathbb{K}^n$ the orbit of G through x . A subset $E \subset \mathbb{K}^n$ is called G -invariant if $f(E) \subset E$ for any $f \in G$; that is E is a union of orbits. Denote by \overline{E} (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E .

A subset E of \mathbb{K}^n is called a *minimal set* of G if E is closed in \mathbb{K}^n , non empty, G -invariant and has no proper subset with these properties. It is equivalent to say that E is a G -invariant set such that every orbit contained in E is dense in it. If Ω is a G -invariant set in \mathbb{K}^n , we say that E is a *minimal set in Ω* if it is a minimal set of the restriction $G|_{\Omega}$ of G to Ω . An orbit $O \subset \Omega$ is called *minimal in Ω* if $\overline{O} \cap \Omega$ is a minimal set in Ω . This means that for every $x \in \overline{O} \cap \Omega$ we have $\overline{O} \cap \Omega = \overline{G(x)} \cap \Omega$. For example, a closed orbit in Ω is minimal in Ω . In particular, every point in $Fix(G)$ is minimal in \mathbb{K}^n .

In [2], S.Frimo proved that if G is an abelian subgroup of $Diff^1(\mathbb{R}^2)$ generated by any family of commuting diffeomorphisms of the plane which are C^1 -close to the identity and having a bounded orbit then $Fix(G) \neq \emptyset$. In [7], J. Franks, M. Handel and K. Parwani proved that if G is a finitely generated abelian subgroup of $Diff_+^1(\mathbb{R}^2)$ and if there is a compact G -invariant set $C \subset \mathbb{R}^2$, then $Fix(G)$ is non-empty.

This paper can be viewed as a generalization of the results given by S.Chihi in [5] for abelian linear group. We study here, the action of an abelian subgroup G of $Diff(\mathbb{K}^n)$.

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Our principal results can be stated as follows:

We generalize the result given in [1] for abelian subgroup of $GL(n, \mathbb{K})$:

For a subset $E \subset \mathbb{K}^n$, denote by $\text{vect}(E)$ the vector subspace of \mathbb{K}^n generated by all elements of E . Set $\mathcal{A}(G)$ be the algebra generated by G . In general, $\mathcal{A}(G)$ is not commutative. For a fixed vector $x \in \mathbb{K}^n \setminus \{0\}$, denote by:

- $\Phi_x : \mathcal{A}(G) \longrightarrow \Phi_x(\mathcal{A}(G)) \subset \mathbb{K}^n$ the linear map given by $\Phi_x(f) = f(x)$.
- $E(x) = \Phi_x(\mathcal{A}(G))$.
- $r(x) = \dim(E(x))$.

Let $a \in \text{Fix}(G)$, denote by $G' = T_{-a} \circ G \circ T_a$, where T_a is the translation by a . So $0 \in \text{Fix}(G')$. Define then:

- $L'(x) = \{Df_0(x), f \in G'\}$ and $\tilde{E}'(x) = \text{vect}(L'(x))$ the vector space generated by $L'(x)$.
- $r_x = r'_{x-a} = \dim(\tilde{E}'(x-a))$, for every $x \in \mathbb{K}^n$.

Theorem 1.1 (Structure's Theorem). *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $\text{Fix}(G) \neq \emptyset$. Then there exist a finite increasing sequence of G -invariant closed subsets of \mathbb{K}^n , $F_0 \subset \dots \subset F_q$, $1 \leq q \leq n$, with the following properties:*

- (i) $F_0 = \emptyset$, $F_q = \mathbb{K}^n \setminus \{0\}$ and $\mathbb{K}^n = \bigcup_{j=1}^q \Omega_j$, where $\Omega_j = F_j \setminus F_{j-1}$.
- (ii) For every $x \in \Omega_j$ and for every $y \in \overline{G(x)} \cap \Omega_j$ there exists a linear isomorphism Φ_y^x satisfying $\Phi_y^x(x) = y$ and $\Phi_y^x(G(x)) = G(y)$, for every $1 \leq j \leq q$.
- (iii) If $r(x) = r_x$ for some $x \in \mathbb{K}^n$, then for every $y \in \overline{G(x)} \cap \Omega_j$ we have $\overline{G(y)} \cap \Omega_j = \overline{G(x)} \cap \Omega_j$ for every $j \in \{1, \dots, q\}$.

Under the hypothesis of Theorem 1.1, denote by $U = \Omega_q = \mathbb{K}^n \setminus F_{q-1}$. We have the following corollaries.

Corollary 1.2. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $\text{Fix}(G) \neq \emptyset$. If G has a dense orbit then every orbit in U is dense in \mathbb{K}^n .*

Corollary 1.3. *If G has a locally dense orbit O in \mathbb{K}^n and C is a connected component of U meeting O then every orbit meeting C is dense in it.*

This paper is organized as follows: In Section 2, we introduce some properties of G -invariant vector space. The Section 3 is devoted to prove the main results. In the section 4, we give an example for $n = 2$.

2. G -invariant vector space

- $F(x)$ be a supplement to $\text{Ker}(\Phi_x)$ containing the identity map $\text{id}_{E(x)}$ of $E(x)$. (i.e. $F(x) \oplus \text{Ker}(\Phi_x) = \mathcal{A}(G)$). Denote by $U_x = \{y \in \mathbb{K}^n, r(y) = r(x)\}$.

Lemma 2.1. *Suppose that $0 \in \text{Fix}(G)$ and let $x \in U_j$. Then for every $y \in U_x$. Then there exists a vector space F of $\mathcal{A}(G)$ such that $\mathcal{A}(G) = F \oplus \text{Ker}(\Phi_x) = F \oplus \text{Ker}(\Phi_y)$. (i.e. $F_x = F_y = F$).*

Proof. Let $y \in U_x$. Then $r(x) = r(y) = j$, so $\text{cod}(Ker(\Phi_x)) = \text{cod}(Ker(\Phi_y)) = j \geq 1$. There are two cases:

- If $Ker(\Phi_y) = Ker(\Phi_x)$, then any supplement F of $Ker(\Phi_x)$ in $\mathcal{A}(G)$ is a supplement of $Ker(\Phi_y)$ in $\mathcal{A}(G)$.
- If $Ker(\Phi_y) \neq Ker(\Phi_x)$, we take $f_1 \in Ker(\Phi_x) \setminus Ker(\Phi_y)$ and $g_1 \in Ker(\Phi_y) \setminus Ker(\Phi_x)$, so $h_1 = f_1 + g_1 \notin Ker(\Phi_x) \cup Ker(\Phi_y)$. Denote by $H_1^x = \mathbb{K}h_1 \oplus Ker(\Phi_x)$ and $H_1^y = \mathbb{K}h_1 \oplus Ker(\Phi_y)$. We establish two cases:
 - If $H_1^x = H_1^y$, then any supplement F of H_1^x in $\mathcal{A}(G)$ is a supplement of H_1^y in $\mathcal{A}(G)$.
 - If $H_1^x \neq H_1^y$, we take $f_2 \in H_1^x \setminus H_1^y$ and $g_2 \in H_1^y \setminus H_1^x$, so $h_2 = f_2 + g_2 \notin H_1^x \cup H_1^y$. Denote by $H_2^x = \mathbb{K}h_2 \oplus H_1^x$ and $H_2^y = \mathbb{K}h_2 \oplus H_1^y$. We repeat the same processes until $j-1$ times and we obtain:
 - If $H_{j-1}^x = H_{j-1}^y$, then any supplement F of H_{j-1}^x in $\mathcal{A}(G)$ is a supplement of H_{j-1}^y in $\mathcal{A}(G)$.
 - If $H_{j-1}^x \neq H_{j-1}^y$, we take $f_j \in H_{j-1}^x \setminus H_{j-1}^y$ and $g_j \in H_{j-1}^y \setminus H_{j-1}^x$, so $h_j = f_j + g_j \notin H_{j-1}^x \cup H_{j-1}^y$. Denote by $H_j^x = \mathbb{K}h_j \oplus H_{j-1}^x$ and $H_j^y = \mathbb{K}h_j \oplus H_{j-1}^y$. We obtain then $H_j^x = H_j^y = \mathcal{A}(G)$, hence the proof is completed by taking $F = \text{vect}(h_1, \dots, h_j)$. \square

Lemma 2.2. *If $y \in \overline{G(x)} \cap U_x$, then:*

- (i) Φ_x and Φ_y are two isomorphisms from F_x to $E(x)$.
- (ii) $\Phi_y^x = \Phi_x \circ \Phi_y^{-1} : E(x) \rightarrow E(x)$ is an isomorphism satisfying $\Phi_y^x(x) = y$ and $\Phi_y^x(G(x)) = G(y)$.

Proof. (i) It is easy to show that Φ_x (resp. Φ_y) is an isomorphism from F_x (resp. F_y) to $E(x)$ (resp. $E(y)$). The proof results from Lemma 2.1, since $F_x = F_y$ and $E(x) = E(y)$.

(ii) By (i), Φ_y^x is an isomorphism. Now,

$$\begin{aligned} \Phi_y^x(x) &= \Phi_y \circ \Phi_x^{-1}(x) \\ &= \Phi_y(\text{id}_{E(x)}) = y. \end{aligned}$$

Let $f \in G$, so $f = f_1 + f_2$, $f_1 \in F_x$ and $f_2 \in Ker(\Phi_x)$. Then by (ii) we obtain:

$$\begin{aligned} \Phi_y^x(f(x)) &= \Phi_y \circ \Phi_x^{-1}(f_1(x)) \\ &= \Phi_y(f_1) \\ &= f_1(y) = f(y). \quad (\text{since } F_x = F_y) \end{aligned}$$

It follows that $\Phi_y^x(G(x)) = G(y)$. The proof is completed. \square

3. Proof of main results

throughout this section and the following sections, we assume that $0 \in \text{Fix}(G)$ leaving to replace G by $T_a \circ G \circ T_{-a}$, for some $a \in \text{Fix}(G)$.

Lemma 3.1. *Let G be an abelian subgroup of $Diff^r(\mathbb{K}^n)$, $r \geq 1$ and $x \in \mathbb{K}^n$. Then $E(x)$ is G -invariant.*

Proof. Suppose that $E(x)$ is generated by $f_1(x), \dots, f_p(x)$, with $f_k \in \mathcal{A}(G)$, $k = 1, \dots, p$. Let $y = \sum_{k=1}^p \alpha_k f_k(x) \in E(x)$ and $f \in G$, then $y = g(x)$, with $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$. Therefore $f(y) = f \circ g(x) = \Phi_x(f \circ g) \in E(x)$, since $f \circ g \in \mathcal{A}(G)$. \square

Lemma 3.2. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$. Then $g(0) = 0$ for every $g \in \mathcal{A}(G)$.*

Proof. Let $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$ with $f_k \in G$, $\alpha_k \in \mathbb{K}$, so $g(0) = \sum_{k=1}^p \alpha_k f_k(0) = 0$. Now, let $f_1, \dots, f_m, g_1, \dots, g_q \in \mathcal{A}(G)$ such that $f_k(0) = g_j(0) = 0$, $1 \leq k \leq m$, $1 \leq j \leq q$, so for every $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m \in \mathbb{K}$ we have

$$\begin{aligned} \left(\sum_{j=1}^q \beta_j g_j \right) \circ \left(\sum_{k=1}^m \alpha_k f_k \right) (0) &= \sum_{j=1}^q \beta_j g_j \left(\sum_{k=1}^m \alpha_k f_k(0) \right) \\ &= \sum_{j=1}^q \beta_j g_j(0) \\ &= 0 \end{aligned}$$

Since $\mathcal{A}(G)$ is the algebra generated by G , so it is stable by composition and by linear combinations, hence we obtain the results. \square

Lemma 3.3. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$. Then $L = \varphi(G)$ is an abelian subgroup of $GL(n, \mathbb{K})$.*

Proof. Let $f, g \in G$, so $\varphi(f \circ g) = D(f \circ g)(0) = Df(g(0)).Dg(0)$. By Lemma 3.2, $g(0) = 0$, so $\varphi(f \circ g) = D(f)(0).Dg(0) = \varphi(f).\varphi(g)$. The proof is completed. \square

Proposition 3.4. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$. Then for every $0 \leq t \leq r_G$, U_t is a G -invariant open subset of \mathbb{K}^n .*

To prove Proposition 3.4, we need the following notations. For a fixed point $x \in \mathbb{K}^n$, denote by:

$$- \tilde{U}_t = \{y \in \mathbb{K}^n \mid \tilde{r}_y \geq t\} \text{ for every } 0 \leq t \leq \tilde{r}_G.$$

Proof of Proposition 3.4. By Lemma 3.1, $E(x)$ is G -invariant and remark that the rank $r(y)$ is constant on any orbit $G(y)$, $y \in E(x)$. So U_t is G -invariant for every $0 \leq t \leq r_G$. Let's show that U_t is an open set: Let $y \in U_t$ and $r = r_y$, so $r \geq t$. Then there exist $f_1, \dots, f_r \in F(y)$ such that the r vectors $f_1(y), \dots, f_r(y)$ are linearly independent in $E(y)$. For all $z \in \mathbb{K}^n$, we consider the Gram's determinant

$$\Delta(z) = \det(\langle f_i(z) \mid f_j(z) \rangle)_{1 \leq i, j \leq r}$$

of the vectors $f_1(z), \dots, f_r(z)$ where $\langle \cdot | \cdot \rangle$ denotes the scalar product in \mathbb{K}^n . It is well known that these vectors are independent if and only if $\Delta(z) \neq 0$, in particular $\Delta(y) \neq 0$. Let

$$V_y = \{z \in \mathbb{K}^n, \Delta(z) \neq 0\}$$

The set V_y is open in \mathbb{K}^n , because the map $z \mapsto \Delta(z)$ is continuous. Now $\Delta(y) \neq 0$, and so $y \in V_y \subset U_t$. The proof is completed. \square

By Lemma 3.1, $E(x)$ is G -invariant, then we will restraint the construction of Φ_x to $E(x)$ as follow:

Set $f_1, \dots, f_r \in \mathcal{A}(G)$ such that $f_1(x), \dots, f_r(x)$ generate $E(x)$. Denote by $H_x = \text{vect}(f_1, \dots, f_r)$ the vector space generated by f_1, \dots, f_r and by $\psi_x = \Phi_x/H_x : H_x \rightarrow E(x)$ given by $\psi_x(f) = f(x)$.

Lemma 3.5. *If $r(x) = r_x$, then the map $\varphi_x : E(x) \rightarrow \tilde{E}(x)$ defined by $\varphi_x(f(x)) = Df_0(x)$, for every $f \in H$, is an isomorphism and satisfying:*

- (i) $\varphi_x(G(x)) = L(x)$.
- (ii) for every $y \in \overline{G(x)}$, $z = \varphi_x(y) \in \overline{L(x)}$ and $\varphi_x(G(y)) = L(z)$.

Proof. (i) By construction, φ_x is linear and surjective. The proof results since $r(x) = r_x = r$.

Let $f_1, \dots, f_r \in \mathcal{A}(G)$ such that $Df_1(0)(x), \dots, Df_r(0)(x)$ generate $\tilde{E}(x)$. Set $H = \text{vect}(f_1, \dots, f_r)$, it is easy to show that $E(x) = \Phi_x(H) = \text{vect}(f_1(x), \dots, f_r(x))$. Now, let $f = \sum_{k=1}^r \alpha_k f_k \in H$, so $\varphi_x(f(x)) = \sum_{k=1}^r \alpha_k Df_k(0)(x) = Df_0(x) \in L(x)$. It follows that $\varphi_x(G(x)) = L(x)$.

(ii) Since $y \in \overline{E(x)}$, there exists $g \in H$ such that $y = g(x)$, so $z = Dg_0(x)$. By continuity of φ_x and by (i), we have $z \in \varphi_x(\overline{G(x)}) \subset \overline{\varphi_x(G(x))} = \overline{L(x)}$. Let $f \in G$, we have $f = \sum_{k=1}^r \alpha_k f_k$, so $\varphi_x(f(y)) = \varphi_x(f \circ g(y)) = Df_0 \cdot Dg_0(x) = Df_0(z) \in L(z)$. It follows that $\varphi_x(G(y)) = L(z)$. \square

By applying Lemma 3.11 given in [5] to the abelian linear group \tilde{L} , we found the following result:

Lemma 3.6. ([5], Lemma 3.11) \tilde{U}_t is a \tilde{G} -invariant dense open subset of \mathbb{K}^n .

Consider the distinct values $\tilde{r}_0 = 0 < \tilde{r}_1 < \dots < \tilde{r}_p = \tilde{r}_G$ taken by the map $\tilde{r} : \mathbb{K}^n \rightarrow \mathbb{N}$ given by $x \mapsto r_x$, $x \in \mathbb{K}^n$ and let $\tilde{F}_j = \{x \in \mathbb{K}^n / r_x \leq \tilde{r}_j\}$, $j = 0, 1, \dots, p$. Evidently \tilde{F}_j ($0 \leq j \leq p$) is the complementary of the L -invariant open set $\tilde{U}_{\tilde{r}_{j-1}}$ (Lemma 3.6) and so the sequence $\tilde{F}_0, \dots, \tilde{F}_p$ is an increasing sequence of closed subsets of \mathbb{K}^n and $\tilde{\Omega}_j = \tilde{F}_j \setminus \tilde{F}_{j-1} = \tilde{F}_j \cap \tilde{U}_{\tilde{r}_{j-1}}$. By Lemma 3.3, L is an abelian subgroup of $GL(n, \mathbb{K})$. Then we will use the following result given in [5]:

We have used the same construction of the open sets $\tilde{\Omega}_j$ given by S.Chihi in the proof of Theorem 3.10.

Lemma 3.7. ([5], Theorem 3.10) *Let $1 \leq j \leq \tilde{r}_G$ and $x \in \tilde{\Omega}_j$ then for every $y \in \overline{L(x)} \cap \tilde{\Omega}_j$ we have $\overline{L(y)} \cap \tilde{\Omega}_j = \overline{L(x)} \cap \tilde{\Omega}_j$.*

Proof of Structure's Theorem. By Proposition 3.4, U_t is a G -invariant open set of \mathbb{K}^n for every $0 \leq t \leq r_G$ and it is dense in \mathbb{K}^n for every $0 \leq t \leq \tilde{r}_G$. Consider the distinct values $r_0 = 0 < r_1 < \dots < r_q = r_G$ taken by the map $r : \mathbb{K}^n \rightarrow \mathbb{N}$ given by $x \mapsto r(x)$, $x \in \mathbb{K}^n$ and let $F_j = \{x \in \mathbb{K}^n / r(x) \leq r_j\}$, $j = 0, 1, \dots, q$. Evidently F_j ($0 \leq j \leq q$) is the complementary of the G -invariant open set $U_{r_{j-1}}$ and so the sequence F_0, \dots, F_q is an increasing sequence of closed subsets of \mathbb{K}^n and $\Omega_j = F_j \setminus F_{j-1} = F_j \cap U_{r_{j-1}}$. Yet, by Proposition 3.4, $U_{r_{j-1}}$ is a G -invariant open set of \mathbb{K}^n and it is dense in \mathbb{K}^n if $r_{j-1} \leq \tilde{r}_G$, so Ω_j is a dense open set of F_j if $r_{j-1} \leq \tilde{r}_G$, for the relative topology. Let $p \geq 0$ such that $r_j \leq \tilde{r}_G$ for every $0 \leq j \leq p$. This proves (i).

The proof of (ii): Let $x \in \Omega_j$ ($1 \leq j \leq q$) and $y \in \overline{G(x)} \cap \Omega_j$. We have $r(x) = r(y) = r_j$ since $x, y \in \Omega_j$. Then $E(x) = E(y)$, so $y \in \overline{G(x)} \cap U_x$. The proof follows from Lemma 2.2.(iii) since $\Phi_y^x(G(x)) = G(y)$.

The proof of (iii): Let $y \in \overline{G(x)} \cap \Omega_j$. By Lemma 3.5, there exists an isomorphism $\varphi_x : E(x) \rightarrow \tilde{E}(x)$ satisfying $\varphi_x(G(x)) = L(x)$ and $\varphi_x(G(y)) = L(z)$ with $z = \varphi_x(y)$. By Lemma 3.7, we have

$$\overline{L(y)} \cap \tilde{\Omega}_j = \overline{L(x)} \cap \tilde{\Omega}_j \quad (1).$$

By Lemma 3.5, we have φ_x is an isomorphism satisfying $\varphi_x(E(x)) = \tilde{E}_x$ and $\varphi_x(G(x)) = L(x)$. It is easily to show that $\varphi_x(\Omega_j) = \tilde{\Omega}_j$, therefore by (1) we obtain

$$\begin{aligned} \overline{G(x)} \cap \Omega_j &= \varphi_x^{-1}(\overline{L(x)}) \cap \varphi_x^{-1}(\tilde{\Omega}_j) \\ &= \varphi_x^{-1}(\overline{L(x)} \cap \tilde{\Omega}_j) \\ &= \varphi_x^{-1}(\overline{L(y)} \cap \tilde{\Omega}_j) \\ &= \overline{\varphi_x^{-1}(L(y))} \cap \Omega_j \\ &= \overline{G(y)} \cap \Omega_j. \end{aligned}$$

It follows that $\overline{G(x)} \cap \Omega_j = \overline{G(y)} \cap \Omega_j$. The proof is completed. \square

Remark 3.8. (i) See that $q \leq r_G$ and $p \leq \tilde{r}_G$.

(ii) If $\tilde{r}_G = r_G$ then $p = q$ and so Ω_j is dense in F_j for every $j \leq q$. In general, $\tilde{r}_G < r_G$ (see example 4.1).

Proof of Corollary 1.2. Suppose that $\overline{G(x)} = \mathbb{K}^n$ and let $y \in U$. We have $y \in \overline{G(x)} \cap U$ and $E(x) = E(y) = \mathbb{K}^n$, so by Theorem 1.1.(iii), there exists an isomorphism Φ_x^y such that $G(y) = \Phi_x^y(G(x))$. Hence $\overline{G(y)} = \Phi_x^y(\overline{G(x)}) = \Phi_x^y(\mathbb{K}^n) = \mathbb{K}^n$. The proof is complete. \square

Proof of Corollary 1.3. Let $x \in \mathbb{K}^n$ such that $\overline{G(x)} \neq \emptyset$, so $r(x) = n$ and so $x \in U$. Denote by C the connected component of U meeting $G(x)$, say that $x \in C$, (otherwise we let $y \in G(x) \cap C$). Now, let's show that $B := (\overline{G(x)} \cap C) \setminus \overline{G(x)}$ is an open and closed subset of C : By construction, B is G -invariant and closed in C . Let $y \in B$, so $G(y) \subset \overline{G(x)} \cap U$. By Theorem 1.1.(ii), there exists an isomorphism $\Phi_y^x : E(x) \rightarrow E(y)$ satisfying $\Phi_y^x(x) = y$ and $\Phi_y^x(G(x)) = G(y)$. Then $\Phi_y^x(\overline{G(x)}) = \overline{G(y)}$. Thus $\overline{G(y)} \neq \emptyset$, so $y \in \overline{G(y)} \subset B$, hence B is an open and closed subset of C . It follows that $B = \emptyset$, since $x \in C \setminus B$. Then $\overline{G(x)} \cap C$ is a non empty open and closed subset of C , so $G(x)$ is dense in C . By the same proof, for every $z \in C \subset \overline{G(x)}$, we have $\overline{G(z)} \neq \emptyset$. The proof follows then from the above.

This completes the proof. \square

4. Example in the dimension $n=2$

Example 4.1. Let G be the group generated by $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the diffeomorphism given by $f(x, y) = \left(x - \frac{y-1}{2((y-1)^2+1)} - \frac{1}{4}, y - \frac{x+1}{2((x+1)^2+1)} + \frac{1}{4} \right)$ with $f(0, 0) = (0, 0)$. Then $r_G = 2$ and $\tilde{r}_G = 1$.

Proof. Let $u = (1, -1)$, $g = T_u \circ f \circ T_{-u}$ and $G' = T_u \circ f \circ T_{-u}$, where T_u denotes the translation by u . Then $g(x, y) = \left(x - \frac{y}{2(y^2+1)} - \frac{1}{4}, y - \frac{x}{2(x^2+1)} + \frac{1}{4} \right)$. We will show that g is a diffeomorphism, $r_{G'} = 2$ and $\tilde{r}_{G'} = 1$. Let $\varphi : x \mapsto \frac{-x}{2(x^2+1)} - \frac{1}{4}$ and $\psi : x \mapsto \varphi(x) + \frac{1}{2}$, be two differentiable map of \mathbb{R} satisfying $|\varphi'(x)| = |\psi'(x)| < \frac{1}{2}$ for every $x \in \mathbb{R}$, since $\varphi'(x) = \frac{x^2-1}{2(x^2+1)^2}$. We have $g(x, y) = (x + \varphi(y), y + \psi(x))$. Firstly, we will show that g is bijective: Let $(a, b), (x, y) \in \mathbb{R}^2$ then

$$g(x, y) = (a, b) \iff \begin{cases} x + \varphi(y) = a \\ y + \psi(x) = b \end{cases} \iff \begin{cases} x + \varphi(b - \psi(x)) = a \\ y = b - \psi(x) \end{cases}$$

Consider the map $\chi_b : x \mapsto x + \varphi(b - \psi(x))$, then χ_b is continuous, differentiable and $\chi_b'(x) = 1 - \psi'(x)\varphi'(b - \psi(x))$, so $\chi_b'(x) > 0$ because $|\psi'(x)\varphi'(b - \psi(x))| < 1$. Therefore, χ_b is strictly increasing. Moreover φ and ψ are $\frac{1}{2}$ Lipschitz: $|\varphi(t) - \varphi(0)| \leq \frac{|t|}{2}$ and $|\psi(t) - \psi(0)| \leq \frac{|t|}{2}$, then $|\varphi(t)| \leq \frac{|t|}{2} + |\varphi(0)|$ and $|\psi(t)| \leq \frac{|t|}{2} + |\psi(0)|$, so

$$\begin{aligned} |\varphi(b - \psi(x))| &\leq \frac{1}{2}|b - \psi(x)| - \frac{1}{4} \\ &\leq \frac{1}{2}|\psi(x)| + \frac{|b|}{2} - \frac{1}{4} \\ &\leq \frac{1}{4}|x| + \frac{|b|}{2} - \frac{1}{8} \end{aligned}$$

therefore,

$$\chi_b(x) \geq \left(1 - \frac{1}{4}\right)x - \left(\frac{|b|}{2} - \frac{1}{8}\right) \rightarrow +\infty, \text{ if } x \rightarrow +\infty$$

and

$$\chi_b(x) \leq (1 - \frac{1}{4})x + \left(\frac{|b|}{2} - \frac{1}{8}\right) \longrightarrow -\infty, \text{ if } x \longrightarrow -\infty$$

Thus χ_b realizes a bijection from \mathbb{R} unto \mathbb{R} . As a consequence:

$$g(x, y) = (a, b) \iff \begin{cases} x = \chi_b^{-1}(a) \\ y = b - \psi(\chi_b^{-1}(a)) \end{cases}$$

This implies that g is bijective.

Secondly, we will show that g is a local diffeomorphism: The Jacobien matrix of g on any point $(x, y) \in \mathbb{R}^2$ is given by

$$Dg_{(x,y)} = \begin{bmatrix} 1 & \psi'(x) \\ \varphi'(y) & 1 \end{bmatrix}.$$

Then $\det(Dg_{(x,y)}) = 1 - \varphi'(x)\psi'(y) \neq 0$ for every $(x, y) \in \mathbb{R}^2$. Since g is bijective, it follows that g is a diffeomorphism, so is f . Now $g(u) = u$ and $Dg_u = I_2$. As a consequence, we have $f(0, 0) = (0, 0)$ and $Df_{(0,0)} = Dg_u = I_2$. For $v = (1, 0)$, we have $f(v) = (1, \frac{1}{20})$. Since $(v, f(v))$ is a basis of \mathbb{R}^2 , so $E(v) = \mathbb{R}^2$, hence $r_G = 2$. On the other hand, $\tilde{G} = G$ and $L = \varphi_v(G) = \{I_2\}$, so $L(x, y) = \mathbb{R}(x, y)$, so $\dim(L(x, y)) = 1$, hence $\tilde{r}_G = 1$. \square

Question : An abelian subgroup of $Diffr(\mathbb{K}^n)$ fixing some point is-it conjugate to a linear group?

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YAHYA N'DAO, UNIVERSITY OF MONCTON, OF MATHEMATICS AND STATISTICS, CANADA
E-mail address: yahiandao@yahoo.fr

ADLENE AYADI, UNIVERSITY OF GAFSA, FACULTY OF SCIENCES, DEPARTMENT OF MATHEMATICS, GAFSA, TUNISIA.
E-mail address: adlenesoo@yahoo.com