

**THE DYNAMIC OF ABELIAN SUBGROUP OF  $Diff^r(\mathbb{K}^n)$ ,  
FIXING A POINT, ( $\mathbb{K} = \mathbb{R}$  OR  $\mathbb{C}$ )**

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ABSTRACT. In this paper, we study the action of any abelian subgroup  $G$  of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  on  $\mathbb{K}^n$ , ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ). We prove that there exist a decreasing finite sequence  $F_0, F_1, \dots, F_q$  of invariant closed sets such that  $\Omega_j = F_j \setminus F_{j-1}$  is an open subset (for the relative topology) of  $F_j$  in which every orbit is minimal in it. Moreover, if  $G$  has a dense orbit in  $\mathbb{K}^n$  then every orbit of  $\Omega_q$  is dense in  $\mathbb{K}^n$ .

### 1. Introduction

Denote by  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$ , ( $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ ) the group of all  $C^r$ -diffeomorphisms of  $\mathbb{K}^n$ . Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  such that  $Fix(G) \neq \emptyset$ , where  $Fix(G) = \{x \in \mathbb{K}^n : f(x) = x, \forall f \in G\}$  be the global fixed point set of  $G$ . There is a natural action  $G \times \mathbb{K}^n \rightarrow \mathbb{K}^n$ .  $(f, x) \mapsto f(x)$ . For a point  $x \in \mathbb{K}^n$ , denote by  $G(x) = \{f(x), f \in G\} \subset \mathbb{K}^n$  the orbit of  $G$  through  $x$ . A subset  $E \subset \mathbb{K}^n$  is called  $G$ -invariant if  $f(E) \subset E$  for any  $f \in G$ ; that is  $E$  is a union of orbits. Denote by  $\overline{E}$  (resp.  $\overset{\circ}{E}$ ) the closure (resp. interior) of  $E$ .

A subset  $E$  of  $\mathbb{K}^n$  is called a *minimal set* of  $G$  if  $E$  is closed in  $\mathbb{K}^n$ , non empty,  $G$ -invariant and has no proper subset with these properties. It is equivalent to say that  $E$  is a  $G$ -invariant set such that every orbit contained in  $E$  is dense in it. If  $\Omega$  is a  $G$ -invariant set in  $\mathbb{K}^n$ , we say that  $E$  is a *minimal set in  $\Omega$*  if it is a minimal set of the restriction  $G|_{\Omega}$  of  $G$  to  $\Omega$ . An orbit  $O \subset \Omega$  is called *minimal in  $\Omega$*  if  $\overline{O} \cap \Omega$  is a minimal set in  $\Omega$ . This means that for every  $x \in \overline{O} \cap \Omega$  we have  $\overline{O} \cap \Omega = \overline{G(x)} \cap \Omega$ . For example, a closed orbit in  $\Omega$  is minimal in  $\Omega$ . In particular, every point in  $Fix(G)$  is minimal in  $\mathbb{K}^n$ , where  $Fix(G) = \{x \in \mathbb{K}^n, f(x) = x, \forall f \in G\}$  is the set of all fixed point by  $G$ .

In [2], S.Frimo proved that if  $G$  is an abelian subgroup of  $Diff^1(\mathbb{R}^2)$  generated by any family of commuting diffeomorphisms of the plane which are  $C^1$ -close to the identity and having a bounded orbit then  $Fix(G) \neq \emptyset$ . In [7], J. Franks, M. Handel and K. Parwani proved that if  $G$  is a finitely generated abelian subgroup of  $Diff_+^1(\mathbb{R}^2)$  and if there is a compact  $G$ -invariant set  $C \subset \mathbb{R}^2$ , then  $Fix(G)$  is non-empty.

This paper can be viewed as a generalization of the results given in [1] for abelian linear group. We study here, the action of an abelian subgroup  $G$  of  $Diff(\mathbb{K}^n)$ .

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Our principal results can be stated as follows:

We generalize the result given in [1] for abelian subgroup of  $GL(n, \mathbb{K})$ :

**Theorem 1.1** (Structure's Theorem). *Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  such that  $Fix(G) \neq \emptyset$ . Then there exist a finite increasing sequence of  $G$ -invariant closed subsets of  $\mathbb{K}^n$ ,  $F_0 \subset \dots \subset F_q$ ,  $1 \leq q \leq n$ , with the following properties:*

- (i)  $F_0 = \emptyset$ ,  $F_q = \mathbb{K}^n \setminus \{0\}$  and  $\mathbb{K}^n = \bigcup_{j=1}^q \Omega_j$ , where  $\Omega_j = F_j \setminus F_{j-1}$ .
- (ii) There exists  $p \leq q$  such that  $\Omega_j$  is dense in  $F_j$  for every  $1 \leq j \leq p$ .
- (iii) Every orbit in  $\Omega_j$  is minimal in  $\Omega_j$ ,  $1 \leq j \leq q$ .

Let  $O$  be an orbit of  $G$ . Denote by  $cl(O) = \{x \in \mathbb{K}^n, \overline{G(x)} = \overline{O}\}$ . The orbit  $O$  is said to be at *level 0* if  $O$  is minimal in  $\mathbb{K}^n$ . Inductively, we say that  $O$  is at *level  $p$* ,  $p \geq 1$  if every orbit  $\gamma \subset \overline{O} \setminus cl(O)$  is at level  $< p$  with at least one orbit at level  $k$  for every  $k < p$ . The upper bound of levels of orbits of  $G$  is called the *height* of  $G$  and denoted by  $ht(G)$ . For example, if  $ht(G)$  is finite, say  $p$  this means that  $p$  is the supremum of  $k \in \mathbb{N}$  such that there exist orbits  $\gamma_0, \gamma_1, \dots, \gamma_k$  of  $G$  with  $\overline{\gamma_0} \subset \overline{\gamma_1} \subset \dots \subset \overline{\gamma_k}$  and  $\overline{\gamma_i} \neq \overline{\gamma_j}$ , for every  $0 \leq i, j \leq k$ ,  $i \neq j$ .

**Corollary 1.2.**  $ht(G) \leq q \leq n$ .

**Corollary 1.3.**  $G$  admits a minimal set in  $\mathbb{K}^n \setminus Fix(G)$ .

Under the hypothesis of Theorem 1.1, denote by  $U = \Omega_q = \mathbb{K}^n \setminus F_{q-1}$ . We have the following corollaries.

**Corollary 1.4.** *Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  such that  $Fix(G) \neq \emptyset$ . If  $G$  has a dense orbit then every orbit in  $U$  is dense in  $\mathbb{K}^n$ .*

**Corollary 1.5.** *If  $G$  has a locally dense orbit  $O$  in  $\mathbb{K}^n$  and  $C$  is a connected component of  $U$  meeting  $O$  then every orbit meeting  $C$  is dense in it.*

For  $n = 1$ , we obtain the following result.

**Corollary 1.6.** *Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K})$ ,  $r \geq 1$  such that  $Fix(G) \neq \emptyset$ . If  $G$  has a dense orbit then  $Fix(G) = \{x_0\}$  is trivial and the orbit of every  $x \in \mathbb{K} \setminus \{x_0\}$ , is dense in  $\mathbb{K}$ .*

This paper is organized as follows: In Section 2, we introduce some properties of  $G$ -invariant vector space. In Section 3, we prove the Structure's Theorem (Theorem 1.1) and the Corollary 1.4. The section 4, is devoted to study particular cases, we prove the Corollary 1.6 and we give an example for  $n = 2$ .

## 2. $G$ -invariant vector space

For a subset  $E \subset \mathbb{K}^n$ , denote by  $vect(E)$  the vector subspace of  $\mathbb{K}^n$  generated by all elements of  $E$ . Set  $\mathcal{A}(G)$  be the algebra generated by  $G$ .

For every  $x \in \mathbb{K}^n$ , We consider the vector space  $E(x) = vect(G(x))$  generated by the orbit  $G(x)$ .

**Lemma 2.1.** *Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  and  $x \in \mathbb{K}^n$ . Then  $E(x)$  is  $G$ -invariant.*

*Proof.* Suppose that  $E(x)$  is generated by  $f_1(x), \dots, f_p(x)$ , with  $f_k \in G$ ,  $k = 1, \dots, p$ . Let  $y = \sum_{k=1}^p \alpha_k f_k(x) \in E(x)$  and  $f \in G$ , then  $y = g(x)$ , with  $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$ . Therefore  $f(y) = f \circ g(x) = \Phi_x(f \circ g) \in E(x)$ , since  $f \circ g \in \mathcal{A}(G)$ .  $\square$

For a fixed vector  $x \in \mathbb{K}^n$ , denote by:  
-  $F(x) = \mathcal{A}(G|_{E(x)})$  the algebra generated by the restriction  $G|_{E(x)}$  of  $G$  to  $E(x)$ .  
We will show that  $F(x)$  has a finite dimension (Proposition 2.3).

**Lemma 2.2.** *Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  such that  $0 \in Fix(G)$ . Then  $g(0) = 0$  for every  $g \in \mathcal{A}(G)$ .*

*Proof.* Let  $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$  with  $f_k \in G$ ,  $\alpha_k \in \mathbb{K}$ , so  $g(0) = \sum_{k=1}^p \alpha_k f_k(0) = 0$ .  
Now, let  $f_1, \dots, f_m, g_1, \dots, g_q \in \mathcal{A}(G)$  such that  $f_k(0) = g_j(0) = 0$ ,  $1 \leq k \leq m$ ,  $1 \leq j \leq q$ , so for every  $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_m \in \mathbb{K}$  we have

$$\begin{aligned} \left( \sum_{j=1}^q \beta_j g_j \right) \circ \left( \sum_{k=1}^m \alpha_k f_k \right) (0) &= \sum_{j=1}^q \beta_j g_j \left( \sum_{k=1}^m \alpha_k f_k(0) \right) \\ &= \sum_{j=1}^q \beta_j g_j(0) \\ &= 0 \end{aligned}$$

Since  $\mathcal{A}(G)$  is the algebra generated by  $G$ , so it is stable by composition and by linear combinations, hence we obtain the results.  $\square$

Denote by  $\Phi_x : F(x) \rightarrow \Phi_x(F(x)) \subset \mathbb{K}^n$  the linear map given by  $\Phi_x(f) = f(x)$ . So  $E(x) = \Phi_x(F(x))$ .

**Proposition 2.3.** *Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  such that  $0 \in Fix(G)$  and  $x \in \mathbb{K}^n$ . Then  $\Phi_x$  is a linear isomorphism. In particular,  $F(x)$  has a finite dimension.*

*Proof.* Let  $f \in \text{Ker}(\Phi_x)$ , so  $f(x) = 0$ . Let  $y \in E(x)$ , since  $\Phi_x : F(x) \rightarrow E(x)$  is surjective there exists  $g \in \mathcal{A}(G)$  such that  $y = g(x)$ . As  $G$  is abelian,  $\mathcal{A}(G)$  is also. Then  $g \circ f = f \circ g$ . Therefore  $f(y) = f(g(x)) = g(f(x)) = g(0)$ . By Lemma 2.2,  $g(0) = 0$ , so  $f(y) = 0$ . It follows that  $f = 0$ . Hence  $\text{Ker}(\Phi_x) = \{0\}$ . This completes the proof.  $\square$

Denote by:

- $DF(x)$  the group of all diffeomorphisms of  $F(x)$ .
- $\overline{G_{/E(x)}}$  the closure of the restriction  $G_{/E(x)}$  in  $F(x)$  for the relative topology of  $\mathcal{C}^r(\mathbb{K}^n, \mathbb{K}^n)$  (the vector space of all  $C^r$ -differentiable maps from  $\mathbb{K}^n$  to  $\mathbb{K}^n$ ).
- $\tilde{G} = \overline{G_{/E(x)}} \cap DF(x)$ . It is a closed subgroup of  $DF(x)$ .
- $\tilde{G}(x) = \Phi_x(\tilde{G})$ .

**Lemma 2.4.** *We have  $\overline{\tilde{G}(x)} = \overline{G(x)}$ .*

*Proof.* Recall that  $G(x) = \Phi_x(G_{/E(x)})$  and  $\tilde{G}(x) = \Phi_x(\tilde{G})$ . Then  $\overline{G(x)} \subset \overline{\tilde{G}(x)}$ . Let  $y \in \overline{\tilde{G}(x)}$ , so  $y = \lim_{m \rightarrow +\infty} f_m(x)$  for some sequence  $(f_m)_{m \in \mathbb{N}}$  in  $\tilde{G}$ . Therefore, for every  $m \in \mathbb{N}$ , there exists a sequence  $(f_{m,k})_{k \in \mathbb{N}}$  in  $G_{/E(x)}$  tending to  $f_m$ . As  $\Phi_x$  is continuous,  $\lim_{k \rightarrow +\infty} \Phi_x(f_{m,k}) = \Phi_x(f_m)$ , so  $\lim_{k \rightarrow +\infty} f_{m,k}(x) = f_m(x)$ . Thus for every  $\varepsilon > 0$ , there exists  $M > 0$  and for every  $m \geq M$ , there exists  $k_m > 0$ , such that for every  $k \geq k_m$ , we have  $\|f_m(x) - y\| < \frac{\varepsilon}{2}$  and  $\|f_{m,k}(x) - f_m(x)\| < \frac{\varepsilon}{2}$ . Then, for every  $m > M$ ,

$$\|f_{m,k_m}(x) - y\| \leq \|f_{m,k_m}(x) - f_m(x)\| + \|f_m(x) - y\| < \varepsilon.$$

Hence  $\lim_{m \rightarrow +\infty} f_{m,k_m}(x) = y$ , so  $y \in \overline{G(x)}$ . It follows that  $\overline{\tilde{G}(x)} \subset \overline{G(x)}$ . The proof is completed.  $\square$

### 3. Proof of main results

For a fixed point  $x \in \mathbb{K}^n$ , denote by:

- $p = \dim(F(x))$ .
- $\varphi_x : F(x) \rightarrow \varphi_x(F(x)) \subset M_p(\mathbb{K})$  the linear map given by  $\varphi_x(f) = Df(0)$ , where  $Df(0)$  is the differential of  $f$  on 0.

**Lemma 3.1.** *Let  $G$  be an abelian subgroup of  $\text{Diff}^r(\mathbb{K}^n)$ ,  $r \geq 1$  such that  $0 \in \text{Fix}(G)$ . Then  $\varphi_x$  is a homomorphism of groups. In particular,  $\varphi_x(DF(x))$  is an abelian subgroup of  $GL(p, \mathbb{K})$ .*

*Proof.* Let  $f, g \in DF(x)$ , so  $\varphi_x(f \circ g) = D(f \circ g)(0) = Df(g(0)).Dg(0)$ . By Lemma 2.2,  $g(0) = 0$ , so  $\varphi_x(f \circ g) = D(f)(0).Dg(0) = \varphi_x(f).\varphi_x(g)$ . The proof is completed.  $\square$

**Lemma 3.2.** *Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  such that  $0 \in Fix(G)$ . Let  $x, y \in \mathbb{K}^n$  such that  $E(x) = E(y)$ . Then  $y = f(x)$  for some  $f \in DF(x)$ . In particular,  $G(y) = f(G(x))$ .*

*Proof.* Since  $E(y) = E(x)$ , we have  $G_{/E(x)} = G_{/E(y)}$ , so  $F(x) = F(y)$ . Then there exist  $f, h \in F(x)$  such that  $f(x) = y$  and  $h(y) = x$ . So  $f \circ h(y) - y = 0$ . As  $f \circ h - id_{E(x)} \in F(y)$ , where  $id_{E(x)}$  is the identity of  $E(x)$ , we obtain  $\Phi_y(f \circ h - id_{E(x)}) = 0$ . By Proposition 2.3,  $\Phi_y : F(y) \rightarrow E(y)$  is an isomorphism, so  $f \circ h - id_{E(x)} = 0$ . Hence  $f \circ h = h \circ f = id_{E(x)}$ . It follows that  $f \in DF(x)$ . This completes the proof since  $f \circ g = g \circ f$  for every  $g \in G$ .  $\square$

For every  $x \in \mathbb{K}^n$ , denote by:

- $r(x) = \dim(E(x))$  is called *the rank* of  $x$  over  $G$ .
- $r_G = \max\{r(y), y \in \mathbb{K}^n\}$ .
- $\tilde{\Phi}_x : \varphi_x(F(x)) \rightarrow E(x)$  the linear map given by  $\tilde{\Phi}_x(A) = Ax$ ,  $A \in \varphi_x(F(x))$ .
- $\tilde{E}_x = \tilde{\Phi}_x \circ \varphi_x(F(x))$ .

**Lemma 3.3.** *Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  such that  $0 \in Fix(G)$ . Then the linear map  $\tilde{\Phi}_y : \varphi_y(F(y)) \rightarrow \tilde{E}(y)$  is an isomorphism.*

*Proof.* The proof results directly by applying the Proposition 2.3 to  $\varphi_y(F(y))$ , because all matrices in  $\varphi_y(F(y))$  fixed 0 and commute two by two and  $\varphi_y$  is surjective.  $\square$

Denote by:

- $U = \{y \in \mathbb{K}^n, r(y) = r_G\}$ .
- $U_t = \{y \in \mathbb{K}^n / r(y) \geq t\}$  for every  $0 < t \leq r_G$ .

**Proposition 3.4.** *Let  $G$  be an abelian subgroup of  $Diff^r(\mathbb{K}^n)$ ,  $r \geq 1$  such that  $0 \in Fix(G)$ . Then for every  $0 \leq t \leq r_G$ ,  $U_t$  is a  $G$ -invariant open subset of  $\mathbb{K}^n$ . In particular, is  $U$ . Moreover, for every  $0 \leq t \leq \tilde{r}_G$ ,  $U_t$  is dense in  $\mathbb{K}^n$ .*

To prove Proposition 3.4, we need the following notations. For a fixed point  $x \in \mathbb{K}^n$ , denote by:

- $\tilde{L} = \varphi_x(\tilde{G})$ .
- $\tilde{L}(x) = \tilde{\Phi}_x \circ \varphi_x(\tilde{G})$ , so  $\tilde{E}_x$  is the vector space generated by  $\tilde{L}(x)$ .
- $\tilde{r}_y = \dim(\tilde{E}(y))$ .
- $\tilde{r}_G = \max\{\tilde{r}_y, y \in \mathbb{K}^n\}$ .
- $\tilde{U} = \{y \in \mathbb{K}^n : \dim(\tilde{E}(y)) = \tilde{r}_G\}$ .
- $\tilde{U}_t = \{y \in \mathbb{K}^n / \tilde{r}_y \geq t\}$  for every  $0 \leq t \leq \tilde{r}_G$ .

By applying Lemma 3.11 given in [5] to the abelian linear group  $\tilde{L}$ , we found the following result:

**Lemma 3.5.** ([5], Lemma 3.11)  $\tilde{U}_t$  is a  $\tilde{G}$ -invariant dense open subset of  $\mathbb{K}^n$ .

**Lemma 3.6.**  $\dim(E(y)) \geq \dim(\tilde{E}_y)$  for every  $y \in \mathbb{K}^n$ . In particular, for every  $0 \leq t \leq \tilde{r}_G$ , we have  $\tilde{U}_t \subset U_t$ .

*Proof.* Let  $y \in \mathbb{K}^n$ . Denote by  $r = \tilde{r}_y$  and let  $f_1, \dots, f_r \in F(y)$  such that  $Df_1(0)(y), \dots, Df_r(0)(y)$  generate  $\tilde{E}_y$ . By Lemma 3.3,  $\tilde{\Phi}_y : \varphi_y(F(y)) \rightarrow \tilde{E}_y$  is an isomorphism, then  $Df_1(0), \dots, Df_r(0)$  generate  $\varphi_y(F(y))$ . Let's show that  $f_1, \dots, f_r$  are linear independent in  $F(y)$ : If  $\alpha_1 f_1 + \dots + \alpha_r f_r = 0$  for some  $\alpha_1, \dots, \alpha_r \in \mathbb{K}$  then  $\alpha_1 Df_1(0) + \dots + \alpha_r Df_r(0) = 0$ , hence  $\alpha_1 = \dots = \alpha_r = 0$ . It follows that  $\dim(F(y)) \geq r$ . By Proposition 2.3,  $\Phi_y : F(y) \rightarrow E(y)$  is an isomorphism, so  $\dim(E(y)) \geq r$ . The proof is completed.  $\square$

*Proof of Proposition 3.4.* Remark that the rank is constant on the orbits, so  $U_t$  is  $G$ -invariant for every  $0 \leq t \leq r_G$ . Let's show that  $U_t$  is an open set: Let  $y \in U_t$  and  $r = r_y$ , so  $r \geq t$ . Then there exist  $f_1, \dots, f_r \in F(y)$  such that the  $r$  vectors  $f_1(y), \dots, f_r(y)$  are linearly independent in  $E(y)$ . For all  $z \in \mathbb{K}^n$ , we consider the Gram's determinant

$$\Delta(z) = \det(\langle f_i(z) | f_j(z) \rangle)_{1 \leq i, j \leq r}$$

of the vectors  $f_1(z), \dots, f_r(z)$  where  $\langle \cdot | \cdot \rangle$  denotes the scalar product in  $\mathbb{K}^n$ . It is well known that these vectors are independent if and only if  $\Delta(z) \neq 0$ , in particular  $\Delta(y) \neq 0$ . Let

$$V_y = \{z \in \mathbb{K}^n, \Delta(z) \neq 0\}$$

The set  $V_y$  is open in  $\mathbb{K}^n$ , because the map  $z \mapsto \Delta(z)$  is continuous. Now  $\Delta(y) \neq 0$ , and so  $y \in V_y \subset U_t$ . On the other hand, for every  $0 \leq t \leq \tilde{r}_G$ , we have  $\tilde{U}_t \subset U_t$  (Lemma 3.6) and by Lemma 3.5,  $\tilde{U}_t$  is dense in  $\mathbb{K}^n$ , so is  $U_t$ . The proof is completed.  $\square$

*Proof of Structure's Theorem.* By Proposition 3.4,  $U_t$  is a  $G$ -invariant open set of  $\mathbb{K}^n$  for every  $0 \leq t \leq r_G$  and it is dense in  $\mathbb{K}^n$  for every  $0 \leq t \leq \tilde{r}_G$ . Consider the distinct values  $r_0 = 0 < r_1 < \dots < r_q = r_G$  taken by the map  $r : \mathbb{K}^n \rightarrow \mathbb{N}$  given by  $x \mapsto r(x)$ ,  $x \in \mathbb{K}^n$  and let  $F_j = \{x \in \mathbb{K}^n / r(x) \leq r_j\}$ ,  $j = 0, 1, \dots, q$ . Evidently  $F_j$  ( $0 \leq j \leq q$ ) is the complementary of the  $G$ -invariant open set  $U_{r_j}$  and so the sequence  $F_0, \dots, F_q$  is an increasing sequence of closed subsets of  $\mathbb{K}^n$  and  $\Omega_j = F_j \setminus F_{j-1} = F_j \cap U_{r_{j-1}}$ . Yet, by Proposition 3.4,  $U_{r_{j-1}}$  is a  $G$ -invariant open set of  $\mathbb{K}^n$  and it is dense in  $\mathbb{K}^n$  if  $r_{j-1} \leq \tilde{r}_G$ , so  $\Omega_j$  is a dense open set of  $F_j$  if  $r_{j-1} \leq \tilde{r}_G$ , for the relative topology. Let  $p \geq 0$  such that  $r_j \leq \tilde{r}_G$  for every  $0 \leq j \leq p$ . This proves (i) and (ii).

*The proof of (iii):* Remark at first that  $\Omega_j = \{x \in \mathbb{K}^n, r(x) = r_j\}$ . Let  $x \in \Omega_j$  ( $1 \leq j \leq q$ ) and  $y \in \overline{G(x)} \cap \Omega_j$ . We have  $r(x) = r(y) = r_j$  since  $x, y \in \Omega_j$ . Then  $E(x) = E(y)$ , so by Lemma 3.2 there exists  $f \in DF(x)$  such that  $y = f(x)$ . By Proposition 2.3,  $\Phi_x : F(x) \rightarrow E(x)$  is an isomorphism, so  $(\Phi_x)^{-1}(y) = f \in (\Phi_x)^{-1}(\overline{G(x)}) = \overline{G/E(x)}$ . Thus  $f \in DF(x) \cap \overline{G/E(x)}$ , so

$f \in \tilde{G}$ . Hence  $y = f(x) \in \tilde{G}(x)$ . It follows that  $\tilde{G}(y) = \tilde{G}(x)$ . By Lemma 2.4,  $\overline{G(x)} = \overline{\tilde{G}(x)} = \overline{\tilde{G}(y)}$ , then  $\overline{G(y)} \cap \Omega_j = \overline{\tilde{G}(y)} \cap \Omega_j = \overline{G(x)} \cap \Omega_j$ .  $\square$

**Remark 3.7.** (i) See that  $q \leq r_G$  and  $p \leq \tilde{r}_G$ .

(ii) If  $\tilde{r}_G = r_G$  then  $p = q$  and so  $\Omega_j$  is dense in  $F_j$  for every  $j \leq q$ . In general,  $\tilde{r}_G < r_G$  (see example 4.2).

*Proof of Corollary 1.2.* We can assume that  $0 \in \text{Fix}(G)$ , leaving to replace  $G$  by  $T_{-a} \circ G \circ T_a$  for some  $a \in \text{Fix}(G)$ . Let  $L$  be an orbit of  $G$  and  $\gamma_0, \gamma_1, \dots, \gamma_k$  be  $k$  orbits of  $G$  with  $\overline{\gamma_0} \subset \overline{\gamma_1} \subset \dots \subset \overline{\gamma_k} = \overline{L}$  and  $\overline{\gamma_i} \neq \overline{\gamma_j}$ , for every  $0 \leq i, j \leq k$ ,  $i \neq j$ . By Theorem 1.1, (i), we have  $\mathbb{K}^n = \bigcup_{i=1}^q \Omega_i$ , then  $\overline{\gamma_{k-1}} = \bigcup_{i=1}^q (\overline{\gamma_{k-1}} \cap \Omega_i)$ . By Theorem 1.1, if  $\overline{\gamma_{k-1}} \cap \Omega_i \neq \emptyset$  then  $\overline{\gamma_{k-1}} \cap \Omega_i = \overline{L} \cap \Omega_i$ . Since  $\overline{\gamma_{k-1}} \neq \overline{L}$  there exists  $i_1 \in \{1, \dots, q\}$  such that  $\overline{\gamma_{k-1}} \cap \Omega_{i_1} = \emptyset$ . In the same way, since  $\overline{\gamma_{k-2}} \neq \overline{\gamma_{k-1}}$  there exists  $i_2 \in \{1, \dots, q\} \setminus \{i_1\}$  such that  $\overline{\gamma_{k-2}} \cap \Omega_{i_2} = \overline{\gamma_{k-2}} \cap \Omega_{i_1} = \emptyset$ . By induction on  $1 \leq j \leq q$ , there exists  $i_j \in \{1, \dots, q\} \setminus \{i_1, \dots, i_{j-1}\}$  such that  $\overline{\gamma_{k-j}} \cap \Omega_{i_j} = \dots = \overline{\gamma_{k-2}} \cap \Omega_{i_2} = \overline{\gamma_{k-2}} \cap \Omega_{i_1} = \emptyset$ . It follows that  $k \leq q \leq n$ , otherwise, for  $j = q$ , we obtain  $\overline{\gamma_{k-q}} \cap \Omega_{i_q} = \dots = \overline{\gamma_{k-2}} \cap \Omega_{i_2} = \overline{\gamma_{k-2}} \cap \Omega_{i_1} = \emptyset$ , so  $\overline{\gamma_{k-q}} = \emptyset$  with  $k - q \neq 0$ , a contradiction. Hence,  $L$  is at level  $\leq q \leq n$ . We conclude that  $ht(G) \leq q \leq n$ .  $\square$

*Proof of Corollary 1.3.* Since  $\text{Fix}(G)$  is a  $G$ -invariant closed subset of  $\mathbb{K}^n$ ,  $W = \mathbb{K}^n \setminus \text{Fix}(G)$  is open and  $G$ -invariant. Let  $\gamma_0, \dots, \gamma_l$  of  $G|_U$  ( $l \geq 0$ ) such that  $\overline{\gamma_0^U} \subset \dots \subset \overline{\gamma_l^U}$  where  $\overline{\gamma_i^U} = \overline{\gamma_i} \cap U$  and  $\overline{\gamma_i^U} \neq \overline{\gamma_j^U}$ ,  $0 \leq i \neq j \leq l$ . So  $\overline{\gamma_0} \subset \dots \subset \overline{\gamma_l}$  with  $\overline{\gamma_i} \neq \overline{\gamma_j}$ ,  $0 \leq i \neq j \leq l$ . By Corollary 1.2,  $ht(G) \leq q \leq n$ , hence  $l \leq q$ . This proves that  $G|_U$  has a minimal set in  $U$ .  $\square$

*Proof of Corollary 1.4.* Suppose that  $\overline{G(x)} = \mathbb{K}^n$  and let  $y \in U$ . We have  $y \in \overline{G(x)} \cap U$  and  $E(x) = E(y) = \mathbb{K}^n$ , so by Lemma 3.2, there exists  $f \in DF(x)$  such that  $G(y) = f(G(x))$ . Hence  $\overline{G(y)} = f(\overline{G(x)}) = f(\mathbb{K}^n) = \mathbb{K}^n$ . The proof is complete.  $\square$

*Proof of Corollary 1.4.* Suppose that  $0 \in \text{Fix}(G)$  (leaving to replace  $G$  by  $T_{-a} \circ G \circ T_a$ , for some  $a \in \text{Fix}(G)$ ). Set  $x \in \mathbb{K}$  with a dense orbit. Then  $x \notin \text{Fix}(G)$ , so  $V = \mathbb{K} \setminus \text{Fix}(G) \neq \emptyset$  and  $x \neq 0$ . See that  $V$  is an open subset of  $\mathbb{K}$  contained in  $U = \mathbb{K} \setminus \{0\}$ . For every  $y \in V$ , we have  $y \in U = \overline{G(x)} \cap U$ , so by Corollary 4.1,  $\overline{G(y)} \cap U = \overline{G(x)} \cap U = U$ , hence  $\overline{G(y)} = \mathbb{K}$ . This completes the proof.  $\square$

*Proof of Corollary 1.5.* Let  $x \in \mathbb{K}^n$  such that  $\overline{G(x)} \neq \emptyset$ , so  $r(x) = n$  and so  $x \in U$ . Denote by  $C$  the connected component of  $U$  meeting  $G(x)$ , say that  $x \in C$ , (otherwise we let  $y \in G(x) \cap C$ ). Now, let's show that  $B := (\overline{G(x)} \cap C) \setminus \overline{G(x)}$  is an open and closed subset of  $C$ : By construction,  $B$  is  $G$ -invariant and closed in  $C$ . Let  $y \in B$ , so  $G(y) \subset \overline{G(x)} \cap U$ . By Theorem 1.1,  $\overline{G(y)} \cap U = \overline{G(x)} \cap U$ , so

$y \in \overline{G(y)} \subset B$ , hence  $B$  is an open and closed subset of  $C$ . It follows that  $B = \emptyset$ , since  $x \in C \setminus B$ . Then  $\overline{oG(x)} \cap C$  is a non empty open and closed subset of  $C$ , so  $G(x)$  is dense in  $C$  and by Theorem 1.1, for every  $z \in C$ , we have  $\overline{G(z)} \cap U = \overline{G(x)} \cap U$ , so  $G(z)$  is dense in  $C$ . This completes the proof.  $\square$

#### 4. Particular cases

**4.1. Results for the dimension  $n=1$ .** For every abelian subgroup  $G$  of  $\mathcal{H}(1, \mathbb{R})$ , we have  $\tilde{L}(x) \neq \{0\}$ ,  $\forall x \in \mathbb{K}^n \setminus \{0\}$ , so  $\tilde{r}_G = r_G$ . Therefore, the Theorem 1.1 can be stated as follow:

**Corollary 4.1.** *Let  $G$  be an abelian subgroup of  $\text{Diff}^r(\mathbb{K})$ ,  $r \geq 1$  such that  $\text{Fix}(G) \neq \emptyset$ . Then there exists  $a \in \text{Fix}(G)$  such that every orbit in  $U = \mathbb{K} \setminus \{a\}$  is minimal in  $U$ .*

*Proof.* Suppose that  $a = 0$  (leaving to replace  $G$  by  $T_{-a} \circ G \circ T_a$ ). So for every  $x \in \mathbb{K} \setminus \{0\}$ ,  $\tilde{L}(x) \neq \{0\}$ , so  $\tilde{r}_x = 1$ . Hence  $r_G = 1$ . Therefore,  $U = \mathbb{K} \setminus \{0\}$  and the proof results directly from Theorem 4.1 and Remark 3.7.  $\square$

*Proof of Corollary 1.6.* Suppose that  $0 \in \text{Fix}(G)$  (leaving to replace  $G$  by  $T_{-a} \circ G \circ T_a$ , for some  $a \in \text{Fix}(G)$ ). Let  $x \in \mathbb{K}$  such that  $G(x)$  is dense in  $\mathbb{K}$ . So  $r(x) = 1$ ,  $x \in U$  and  $U$  is dense in  $\mathbb{K}$ . Here, we recall the following sets given in the proof of Theorem 1.1:  $F_0 = \emptyset$ ,  $F_1 = \{0\}$ ,  $F_2 = \{y \in \mathbb{K}, r(y) \leq 1\} = \mathbb{K}$  since  $E(0) = \{0\}$  and for every  $y \in \mathbb{K} \setminus \{0\}$ , we have  $E(y) = \mathbb{K}$ . If  $\text{Fix}(G)$  contains a point  $a \neq 0$ , then  $a \in F_2 \setminus F_1$ , so  $a \in U = F_2 \setminus F_1$ . Hence,  $U = \mathbb{K} \setminus \{0\}$ . By Corollary 1.6, every orbit of  $\mathbb{K} \setminus \{0\}$  is dense in  $\mathbb{K}$ , in particular,  $\overline{G(a)} = \mathbb{K}$ , a contradiction because  $G(a) = \{a\}$ . We conclude that  $\text{Fix}(G) = \{0\}$ . This completes the proof.  $\square$

#### 4.2. Example in the dimension $n=2$ .

**Example 4.2.** Let  $G$  be the group generated by  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the diffeomorphism given by  $f(x, y) = \left( x - \frac{y-1}{2((y-1)^2+1)} - \frac{1}{4}, y - \frac{x+1}{2((x+1)^2+1)} + \frac{1}{4} \right)$  with  $f(0, 0) = (0, 0)$ . Then  $r_G = 2$  and  $\tilde{r}_G = 1$ .

*Proof.* Let  $u = (1, -1)$ ,  $g = T_u \circ f \circ T_{-u}$  and  $G' = T_u \circ f \circ T_{-u}$ , where  $T_u$  denotes the translation by  $u$ . Then  $g(x, y) = \left( x - \frac{y}{2(y^2+1)} - \frac{1}{4}, y - \frac{x}{2(x^2+1)} + \frac{1}{4} \right)$ . We will show that  $g$  is a diffeomorphism,  $r_{G'} = 2$  and  $\tilde{r}_{G'} = 1$ . Let  $\varphi : x \mapsto \frac{-x}{2(x^2+1)} - \frac{1}{4}$  and  $\psi : x \mapsto \varphi(x) + \frac{1}{2}$ , be two differentiable map of  $\mathbb{R}$  satisfying  $|\varphi'(x)| = |\psi'(x)| < \frac{1}{2}$  for every  $x \in \mathbb{R}$ , since  $\varphi'(x) = \frac{x^2-1}{2(x^2+1)^2}$ . We have  $g(x, y) = (x + \varphi(y), y + \psi(x))$ . Firstly, we will show that  $g$  is bijective: Let  $(a, b), (x, y) \in \mathbb{R}^2$  then

$$g(x, y) = (a, b) \iff \begin{cases} x + \varphi(y) = a \\ y + \psi(x) = b \end{cases} \iff \begin{cases} x + \varphi(b - \psi(x)) = a \\ y = b - \psi(x) \end{cases}$$



Consider the map  $\chi_b : x \mapsto x + \varphi(b - \psi(x))$ , then  $\chi_b$  is continuous, differentiable and  $\chi_b'(x) = 1 - \psi'(x)\varphi'(b - \psi(x))$ , so  $\chi_b'(x) > 0$  because  $|\psi'(x)\varphi'(b - \psi(x))| < 1$ . Therefore,  $\chi_b$  is strictly increasing. Moreover  $\varphi$  and  $\psi$  are  $\frac{1}{2}$  Lipschitz:  $|\varphi(t) - \varphi(0)| \leq \frac{|t|}{2}$  and  $|\psi(t) - \psi(0)| \leq \frac{|t|}{2}$ , then  $|\varphi(t)| \leq \frac{|t|}{2} + |\varphi(0)|$  and  $|\psi(t)| \leq \frac{|t|}{2} + |\psi(0)|$ , so

$$\begin{aligned} |\varphi(b - \psi(x))| &\leq \frac{1}{2}|b - \psi(x)| - \frac{1}{4} \\ &\leq \frac{1}{2}|\psi(x)| + \frac{|b|}{2} - \frac{1}{4} \\ &\leq \frac{1}{4}|x| + \frac{|b|}{2} - \frac{1}{8} \end{aligned}$$

therefore,

$$\chi_b(x) \geq (1 - \frac{1}{4})x - \left(\frac{|b|}{2} - \frac{1}{8}\right) \rightarrow +\infty, \text{ if } x \rightarrow +\infty$$

and

$$\chi_b(x) \leq (1 - \frac{1}{4})x + \left(\frac{|b|}{2} - \frac{1}{8}\right) \rightarrow -\infty, \text{ if } x \rightarrow -\infty$$

Thus  $\chi_b$  realizes a bijection from  $\mathbb{R}$  unto  $\mathbb{R}$ . As a consequence:

$$g(x, y) = (a, b) \iff \begin{cases} x = \chi_b^{-1}(a) \\ y = b - \psi(\chi_b^{-1}(a)) \end{cases}$$

This implies that  $g$  is bijective.

Secondly, we will show that  $g$  is a local diffeomorphism: The Jacobien matrix of  $g$  on any point  $(x, y) \in \mathbb{R}^2$  is given by

$$Dg_{(x,y)} = \begin{bmatrix} 1 & \psi'(x) \\ \varphi'(y) & 1 \end{bmatrix}.$$

Then  $\det(Dg_{(x,y)}) = 1 - \varphi'(x)\psi'(y) \neq 0$  for every  $(x, y) \in \mathbb{R}^2$ . Since  $g$  is bijective, it follows that  $g$  is a diffeomorphism, so is  $f$ . Now  $g(u) = u$  and  $Dg_u = I_2$ . As a consequence, we have  $f(0,0) = (0,0)$  and  $Df_{(0,0)} = Dg_u = I_2$ . For  $v = (1,0)$ , we have  $f(v) = (1, \frac{1}{20})$ . Since  $(v, f(v))$  is a basis of  $\mathbb{R}^2$ , so  $E(v) = \mathbb{R}^2$ , hence  $r_G = 2$ . On the other hand,  $\tilde{G} = G$  and  $L = \varphi_v(G) = \{I_2\}$ , so  $L(x, y) = \mathbb{R}(x, y)$ , so  $\dim(L(x, y)) = 1$ , hence  $\tilde{r}_G = 1$ .  $\square$

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