

**THE DYNAMIC OF ABELIAN SUBGROUP OF $Diff^r(\mathbb{K}^n)$,
FIXING A POINT, ($\mathbb{K} = \mathbb{R}$ OR \mathbb{C})**

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ABSTRACT. In this paper, we study the action of any abelian subgroup G of $Diff^r(\mathbb{K}^n)$, $r \geq 1$ on \mathbb{K}^n , ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}). We prove that there exist a decreasing finite sequence F_0, F_1, \dots, F_q of invariant closed sets such that $\Omega_j = F_j \setminus F_{j-1}$ is an open subset (for the relative topology) of F_j in which every orbit is minimal in it.

1. Introduction

Denote by $Diff^r(\mathbb{K}^n)$, $r \geq 1$, ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}) the group of all C^r -diffeomorphisms of \mathbb{K}^n . Let G be an abelian subgroup of $Diff^r(\mathbb{K}^n)$, $r \geq 1$ such that $Fix(G) \neq \emptyset$, where $Fix(G) = \{x \in \mathbb{K}^n : f(x) = x, \forall f \in G\}$ be the global fixed point set of G . There is a natural action $G \times \mathbb{K}^n \rightarrow \mathbb{K}^n$. $(f, x) \mapsto f(x)$. For a point $x \in \mathbb{K}^n$, denote by $G(x) = \{f(x), f \in G\} \subset \mathbb{K}^n$ the orbit of G through x . A subset $E \subset \mathbb{K}^n$ is called G -invariant if $f(E) \subset E$ for any $f \in G$; that is E is a union of orbits. Denote by \overline{E} (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E . An orbit O is called minimal in any G -invariant subset Ω of \mathbb{K}^n if for every $x \in \overline{O} \cap \Omega$ we have $\overline{O} \cap \Omega = \overline{G(x)} \cap \Omega$.

In [2], S.Frimo proved that if G is an abelian subgroup of $Diff^1(\mathbb{R}^2)$ generated by any family of commuting diffeomorphisms of the plane which are C^1 -close to the identity and having a bounded orbit then $Fix(G) \neq \emptyset$. In [7], J. Franks, M. Handel and K. Parwani proved that if G is a finitely generated abelian subgroup of $Diff_+^1(\mathbb{R}^2)$ and if there is a compact G -invariant set $C \subset \mathbb{R}^2$, then $Fix(G)$ is non-empty.

This paper can be viewed as a generalization of the results given in [1] for abelian linear group. We study here, the action of an abelian subgroup G of $Diff(\mathbb{K}^n)$.

Our principal results can be stated as follows:

We generalize the result given in [1] for abelian subgroup of $GL(n, \mathbb{K})$:

Theorem 1.1 (Structure's Theorem). *Let G be an abelian subgroup of $Diff^r(\mathbb{K}^n)$, $r \geq 1$ such that $Fix(G) \neq \emptyset$. Then there exist a finite increasing sequence of G -invariant closed subsets of \mathbb{K}^n , $F_0 \subset \dots \subset F_q$ with the following properties:*

- (i) $F_0 = \emptyset$ and $F_q = \mathbb{K}^n \setminus \{0\}$.
- (ii) $\Omega_j = F_j \setminus F_{j-1}$ ($1 \leq j \leq q$) is a G -invariant open set of F_j .

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- (iii) There exists $p \leq q$ such that Ω_j is dense in F_j for every $1 \leq j \leq p$.
- (iv) Every orbit in Ω_j is minimal in Ω_j , $1 \leq j \leq q$.

Denote by $U = \Omega_q = \mathbb{K}^n \setminus F_{q-1}$.

Corollary 1.2. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $\text{Fix}(G) \neq \emptyset$. If G has a dense orbit then every orbit in U is dense in \mathbb{K}^n .*

For $n = 1$, we obtain the following result.

Corollary 1.3. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K})$, $r \geq 1$ such that $\text{Fix}(G) \neq \emptyset$. If G has a dense orbit then the orbit of every $x \in \mathbb{K} \setminus \text{Fix}(G)$, is dense in \mathbb{K}^n .*

This paper is organized as follows: In Section 2, we introduce some properties of G -invariant vector space. In Section 3, we prove the Structure's Theorem (Theorem 1.1) and the Corollary 1.2. The section 4, is devoted to study particular cases, we prove the Corollary 1.3 and we give an example for $n = 2$.

2. G -invariant vector space

For a subset $E \subset \mathbb{K}^n$, denote by $\text{vect}(E)$ the vector subspace of \mathbb{K}^n generated by all elements of E . Set $\mathcal{A}(G)$ be the algebra generated by G .

For every $x \in \mathbb{K}^n$, We consider the vector space $E(x) = \text{vect}(G(x))$ generated by the orbit $G(x)$.

Lemma 2.1. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ and $x \in \mathbb{K}^n$. Then $E(x)$ is G -invariant.*

Proof. Suppose that $E(x)$ is generated by $f_1(x), \dots, f_p(x)$, with $f_k \in G$, $k = 1, \dots, p$. Let $y = \sum_{k=1}^p \alpha_k f_k(x) \in E(x)$ and $f \in G$, then $y = g(x)$, with $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$. Therefore $f(y) = f \circ g(x) = \Phi_x(f \circ g) \in E(x)$, since $f \circ g \in \mathcal{A}(G)$. \square

For a fixed vector $x \in \mathbb{K}^n$, denote by:
- $F(x) = \mathcal{A}(G|_{E(x)})$ the algebra generated by the restriction $G|_{E(x)}$ of G to $E(x)$.
We will show that $F(x)$ has a finite dimension (Proposition 2.3).

Lemma 2.2. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$. Then $g(0) = 0$ for every $g \in \mathcal{A}(G)$.*

Proof. Let $g = \sum_{k=1}^p \alpha_k f_k \in \mathcal{A}(G)$ with $f_k \in G$, $\alpha_k \in \mathbb{K}$, so $g(0) = \sum_{k=1}^p \alpha_k f_k(0) = 0$.
Now, let $f_1, \dots, f_m, g_1, \dots, g_q \in \mathcal{A}(G)$ such that $f_k(0) = g_j(0) = 0$, $1 \leq k \leq m$,

$1 \leq j \leq q$, so for every $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_m \in \mathbb{K}$ we have

$$\begin{aligned} \left(\sum_{j=1}^q \beta_j g_j \right) \circ \left(\sum_{k=1}^m \alpha_k f_k \right) (0) &= \sum_{j=1}^q \beta_j g_j \left(\sum_{k=1}^m \alpha_k f_k(0) \right) \\ &= \sum_{j=1}^q \beta_j g_j(0) \\ &= 0 \end{aligned}$$

Since $\mathcal{A}(G)$ is the algebra generated by G , so it is stable by composition and by linear combinations, hence we obtain the results. \square

Denote by $\Phi_x : F(x) \rightarrow \Phi_x(F(x)) \subset \mathbb{K}^n$ the linear map given by $\Phi_x(f) = f(x)$. So $E(x) = \Phi_x(F(x))$.

Proposition 2.3. *Let G be an abelian subgroup of $Diff^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in Fix(G)$ and $x \in \mathbb{K}^n$. Then Φ_x is a linear isomorphism. In particular, $F(x)$ has a finite dimension.*

Proof. Let $f \in Ker(\Phi_x)$, so $f(x) = 0$. Let $y \in E(x)$, since $\Phi_x : F(x) \rightarrow E(x)$ is surjective there exists $g \in \mathcal{A}(G)$ such that $y = g(x)$. As G is abelian, $\mathcal{A}(G)$ is also. Then $g \circ f = f \circ g$. Therefore $f(y) = f(g(x)) = g(f(x)) = g(0)$. By Lemma 2.2, $g(0) = 0$, so $f(y) = 0$. It follows that $f = 0$. Hence $Ker(\Phi_x) = \{0\}$. This completes the proof. \square

Denote by:

- $DF(x)$ the group of all diffeomorphisms of $F(x)$.
- $\overline{G}_{/E(x)}$ the closure of the restriction $G_{/E(x)}$ in $F(x)$ for the relative topology of $\mathcal{C}^r(\mathbb{K}^n, \mathbb{K}^n)$ (the vector space of all C^r -differentiable maps from \mathbb{K}^n to \mathbb{K}^n).
- $\tilde{G} = \overline{G}_{/E(x)} \cap DF(x)$. It is a closed subgroup of $DF(x)$.
- $\tilde{G}(x) = \Phi_x(\tilde{G})$.

Lemma 2.4. *We have $\overline{\tilde{G}(x)} = \overline{G(x)}$.*

Proof. Recall that $G(x) = \Phi_x(G_{/E(x)})$ and $\tilde{G}(x) = \Phi_x(\tilde{G})$. Then $\overline{G(x)} \subset \overline{\tilde{G}(x)}$. Let $y \in \overline{\tilde{G}(x)}$, so $y = \lim_{m \rightarrow +\infty} f_m(x)$ for some sequence $(f_m)_{m \in \mathbb{N}}$ in \tilde{G} . Therefore, for every $m \in \mathbb{N}$, there exists a sequence $(f_{m,k})_{k \in \mathbb{N}}$ in $G_{/E(x)}$ tending to f_m . As Φ_x is continuous, $\lim_{k \rightarrow +\infty} \Phi_x(f_{m,k}) = \Phi_x(f_m)$, so $\lim_{k \rightarrow +\infty} f_{m,k}(x) = f_m(x)$. Thus for every $\varepsilon > 0$, there exists $M > 0$ and for every $m \geq M$, there exists $k_m > 0$, such that for every $k \geq k_m$, we have $\|f_m(x) - y\| < \frac{\varepsilon}{2}$ and $\|f_{m,k}(x) - f_m(x)\| < \frac{\varepsilon}{2}$. Then, for every $m > M$,

$$\|f_{m,k_m}(x) - y\| \leq \|f_{m,k_m}(x) - f_m(x)\| + \|f_m(x) - y\| < \varepsilon.$$

Hence $\lim_{m \rightarrow +\infty} f_{m,k_m}(x) = y$, so $y \in \overline{G(x)}$. It follows that $\overline{\tilde{G}(x)} \subset \overline{G(x)}$. The proof is completed. \square

3. Proof of structure's Theorem

For a fixed point $x \in \mathbb{K}^n$, denote by:

- $p = \dim(F(x))$.
- $\varphi_x : F(x) \longrightarrow \varphi_x(F(x)) \subset M_p(\mathbb{K})$ the linear map given by $\varphi_x(f) = Df(0)$, where $Df(0)$ is the differential of f on 0.

Lemma 3.1. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$. Then φ_x is a homomorphism of groups. In particular, $\varphi_x(DF(x))$ is an abelian subgroup of $GL(p, \mathbb{K})$.*

Proof. Let $f, g \in DF(x)$, so $\varphi_x(f \circ g) = D(f \circ g)(0) = Df(g(0)).Dg(0)$. By Lemma 2.2, $g(0) = 0$, so $\varphi_x(f \circ g) = D(f)(0).Dg(0) = \varphi_x(f).\varphi_x(g)$. The proof is completed. \square

Lemma 3.2. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$. Let $x, y \in \mathbb{K}^n$ such that $E(x) = E(y)$. Then $y = f(x)$ for some $f \in DF(x)$. In particular, $G(y) = f(G(x))$.*

Proof. Since $E(y) = E(x)$, we have $G_{/E(x)} = G_{/E(y)}$, so $F(x) = F(y)$. Then there exist $f, h \in F(x)$ such that $f(x) = y$ and $h(y) = x$. So $f \circ h(y) - y = 0$. As $f \circ h - id_{E(x)} \in F(y)$, where $id_{E(x)}$ is the identity of $E(x)$, we obtain $\Phi_y(f \circ h - id_{E(x)}) = 0$. By Proposition 2.3, $\Phi_y : F(y) \longrightarrow E(y)$ is an isomorphism, so $f \circ h - id_{E(x)} = 0$. Hence $f \circ h = h \circ f = id_{E(x)}$. It follows that $f \in DF(x)$. This completes the proof since $f \circ g = g \circ f$ for every $g \in G$. \square

For every $x \in \mathbb{K}^n$, denote by:

- $r(x) = \dim(E(x))$ is called the rank of $xy \in \mathbb{K}^n$ over G .
- $r_G = \max \{r(y), \quad y \in \mathbb{K}^n\}$.
- $\tilde{\Phi}_x : \varphi_x(F(x)) \longrightarrow E(x)$ the linear map given by $\tilde{\Phi}_x(A) = Ax, A \in \varphi_x(F(x))$.
- $\tilde{E}_x = \tilde{\Phi}_x \circ \varphi_x(F(x))$.

Lemma 3.3. *Let G be an abelian subgroup of $\text{Diff}^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in \text{Fix}(G)$. Then the linear map $\tilde{\Phi}_y : \varphi_y(F(y)) \longrightarrow \tilde{E}(y)$ is an isomorphism.*

Proof. The proof results directly by applying the Proposition 2.3 to $\varphi_y(F(y))$, because all matrices in $\varphi_y(F(y))$ fixed 0 and commute two by two and φ_y is surjective. \square

Denote by:

- $U = \{y \in \mathbb{K}^n, \quad r(y) = r_G\}$.
- $U_t = \{y \in \mathbb{K}^n / r(y) \geq t\}$ for every $0 < t \leq r_G$.

Proposition 3.4. *Let G be an abelian subgroup of $Diff^r(\mathbb{K}^n)$, $r \geq 1$ such that $0 \in Fix(G)$. Then for every $0 \leq t \leq r_G$, U_t is a G -invariant open subset of \mathbb{K}^n . In particular, is U . Moreover, for every $0 \leq t \leq \tilde{r}_G$, U_t is dense in \mathbb{K}^n .*

To prove Proposition 3.4, we need the following notations. For a fixed point $x \in \mathbb{K}^n$, denote by:

- $\tilde{L} = \varphi_x(\tilde{G})$.
- $\tilde{L}(x) = \tilde{\Phi}_x \circ \varphi_x(\tilde{G})$, so \tilde{E}_x is the vector space generated by $\tilde{L}(x)$.
- $\tilde{r}_y = \dim(\tilde{E}(y))$.
- $\tilde{r}_G = \max\{\tilde{r}_y, y \in \mathbb{K}^n\}$.
- $\tilde{U} = \{y \in \mathbb{K}^n : \dim(\tilde{E}(y)) = \tilde{r}_G\}$.
- $\tilde{U}_t = \{y \in \mathbb{K}^n / \tilde{r}_y \geq t\}$ for every $0 \leq t \leq \tilde{r}_G$.

By applying Lemma 3.11 given in [5] to the abelian linear group \tilde{L} , we found the following result:

Lemma 3.5. ([5], Lemma 3.11) \tilde{U}_t is a \tilde{G} -invariant dense open subset of \mathbb{K}^n .

Lemma 3.6. $\dim(E(y)) \geq \dim(\tilde{E}_y)$ for every $y \in \mathbb{K}^n$. In particular, for every $0 \leq t \leq \tilde{r}_G$, we have $\tilde{U}_t \subset U_t$.

Proof. Let $y \in \mathbb{K}^n$. Denote by $r = \tilde{r}_y$ and let $f_1, \dots, f_r \in F(y)$ such that $Df_1(0)(y), \dots, Df_r(0)(y)$ generate \tilde{E}_y . By Lemma 3.3, $\tilde{\Phi}_y : \varphi_y(F(y)) \rightarrow \tilde{E}_y$ is an isomorphism, then $Df_1(0), \dots, Df_r(0)$ generate $\varphi_y(F(y))$. Let's show that f_1, \dots, f_r are linear independent in $F(y)$: If $\alpha_1 f_1 + \dots + \alpha_r f_r = 0$ for some $\alpha_1, \dots, \alpha_r \in \mathbb{K}$ then $\alpha_1 Df_1(0) + \dots + \alpha_r Df_r(0) = 0$, hence $\alpha_1 = \dots = \alpha_r = 0$. It follows that $\dim(F(y)) \geq r$. By Proposition 2.3, $\Phi_y : F(y) \rightarrow E(y)$ is an isomorphism, so $\dim(E(y)) \geq r$. The proof is completed. \square

Proof of Proposition 3.4. Remark that the rank is constant on the orbits, so U_t is G -invariant for every $0 \leq t \leq r_G$. Let's show that U_t is an open set: Let $y \in U_t$ and $r = r_y$, so $r \geq t$. Then there exist $f_1, \dots, f_r \in F(y)$ such that the r vectors $f_1(y), \dots, f_r(y)$ are linearly independent in $E(y)$. For all $z \in \mathbb{K}^n$, we consider the Gram's determinant

$$\Delta(z) = \det(\langle f_i(z) | f_j(z) \rangle)_{1 \leq i, j \leq r}$$

of the vectors $f_1(z), \dots, f_r(z)$ where $\langle \cdot | \cdot \rangle$ denotes the scalar product in \mathbb{K}^n . It is well known that these vectors are independent if and only if $\Delta(z) \neq 0$, in particular $\Delta(y) \neq 0$. Let

$$V_y = \{z \in \mathbb{K}^n, \Delta(z) \neq 0\}$$

The set V_y is open in \mathbb{K}^n , because the map $z \mapsto \Delta(z)$ is continuous. Now $\Delta(y) \neq 0$, and so $y \in V_y \subset U_t$. On the other hand, for every $0 \leq t \leq \tilde{r}_G$, we have $\tilde{U}_t \subset U_t$ (Lemma 3.6) and by Lemma 3.5, \tilde{U}_t is dense in \mathbb{K}^n , so is U_t . The proof is completed. \square

Proof of Structure's Theorem. By Proposition 3.4, U_t is a G -invariant open set of \mathbb{K}^n for every $0 \leq t \leq r_G$ and it is dense in \mathbb{K}^n for every $0 \leq t \leq \tilde{r}_G$. Consider the distinct values $r_0 = 0 < r_1 < \dots < r_q = r_G$ taken by the map $r : \mathbb{K}^n \rightarrow \mathbb{N}$ given by $x \mapsto r(x)$, $x \in \mathbb{K}^n$ and let $F_j = \{x \in \mathbb{K}^n / r(x) \leq r_j\}$, $j = 0, 1, \dots, q$. Evidently F_j ($0 \leq j \leq q$) is the complementary of the G -invariant open set U_{r_j} and so the sequence F_0, \dots, F_q is an increasing sequence of closed subsets of \mathbb{K}^n and $\Omega_j = F_j \setminus F_{j-1} = F_j \cap U_{r_{j-1}}$. Yet, by Proposition 3.4, $U_{r_{j-1}}$ is a G -invariant open set of \mathbb{K}^n and it is dense in \mathbb{K}^n if $r_{j-1} \leq \tilde{r}_G$, so Ω_j is a dense open set of F_j if $r_{j-1} \leq \tilde{r}_G$, for the relative topology. Let $p \geq 0$ such that $r_j \leq \tilde{r}_G$ for every $0 \leq j \leq p$. This proves (i), (ii) and (iii).

The proof of (iv): Remark at first that $\Omega_j = \{x \in \mathbb{K}^n, r(x) = r_j\}$. Let $x \in \Omega_j$ ($1 \leq j \leq q$) and $y \in \overline{G(x)} \cap \Omega_j$. We have $r(x) = r(y) = r_j$ since $x, y \in \Omega_j$. Then $E(x) = E(y)$, so by Lemma 3.2 there exists $f \in DF(x)$ such that $y = f(x)$. By Proposition 2.3, $\Phi_x : F(x) \rightarrow E(x)$ is an isomorphism, so $(\Phi_x)^{-1}(y) = f \in (\Phi_x)^{-1}(\overline{G(x)}) = \overline{G/E(x)}$. Thus $f \in DF(x) \cap \overline{G/E(x)}$, so $f \in \tilde{G}$. Hence $y = f(x) \in \tilde{G}(x)$. It follows that $\tilde{G}(y) = \tilde{G}(x)$. By Lemma 2.4, $\overline{G(x)} = \overline{\tilde{G}(x)} = \overline{\tilde{G}(y)}$, then $\overline{G(y)} \cap \Omega_j = \overline{\tilde{G}(y)} \cap \Omega_j = \overline{G(x)} \cap \Omega_j$. \square

Remark 3.7. (i) See that $q \leq r_G$ and $p \leq \tilde{r}_G$.
(ii) If $\tilde{r}_G = r_G$ then $p = q$ and so Ω_j is dense in F_j for every $j \leq q$. In general, $\tilde{r}_G < r_G$ (see example 4.2).

Proof of Corollary 1.2. Suppose that $\overline{G(x)} = \mathbb{K}^n$ and let $y \in U$. We have $y \in \overline{G(x)} \cap U$ and $E(x) = E(y) = \mathbb{K}^n$, so by Lemma 3.2, there exists $f \in DF(x)$ such that $G(y) = f(G(x))$. Hence $\overline{G(y)} = \overline{f(G(x))} = f(\mathbb{K}^n) = \mathbb{K}^n$. The proof is complete. \square

4. Particular cases

4.1. Results for the dimension $n=1$. For every abelian subgroup G of $\mathcal{H}(1, \mathbb{R})$, we have $\tilde{L}(x) \neq \{0\}$, $\forall x \in \mathbb{K}^n \setminus \{0\}$, so $\tilde{r}_G = r_G$. Therefore, the Theorem 1.1 can be stated as follow:

Corollary 4.1. *Let G be an abelian subgroup of $Diff^r(\mathbb{K})$, $r \geq 1$ such that $Fix(G) \neq \emptyset$. Then there exists $a \in Fix(G)$ such that every orbit in $U = \mathbb{K} \setminus \{a\}$ is minimal in U .*

Proof. Suppose that $a = 0$ (leaving to replace G by $T_{-a} \circ G \circ T_a$). So for every $x \in \mathbb{K} \setminus \{0\}$, $\tilde{L}(x) \neq \{0\}$, so $\tilde{r}_x = 1$. Hence $r_G = 1$. Therefore, $U = \mathbb{K} \setminus \{0\}$ and the proof results directly from Theorem 4.1 and Remark 3.7. \square

Proof of Corollary 1.3. Suppose that $a = 0$ (leaving to replace G by $T_{-a} \circ G \circ T_a$). Set $x \in \mathbb{K}$ with a dense orbit. Then $x \notin \text{Fix}(G)$, so $V = \mathbb{K} \setminus \text{Fix}(G) \neq \emptyset$ and $x \neq 0$. See that V is an open subset of \mathbb{K} contained in $U = \mathbb{K} \setminus \{0\}$. For every $y \in V$, we have $y \in U = \overline{G(x)} \cap U$, so by Corollary 4.1, $\overline{G(y)} \cap U = \overline{G(x)} \cap U = U$, hence $\overline{G(y)} = \mathbb{K}$. This completes the proof. \square

4.2. Example in the dimension $n=2$.

Example 4.2. Let G be the group generated by $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the diffeomorphism given by $f(x, y) = \left(x - \frac{y-1}{2((y-1)^2+1)} - \frac{1}{4}, y - \frac{x+1}{2((x+1)^2+1)} + \frac{1}{4} \right)$. Then $r_G = 2$ and $\tilde{r}_G = 1$.

Proof. Let $u = (1, -1)$, $g = T_u \circ f \circ T_{-u}$ and $G' = T_u \circ f \circ T_{-u}$, where T_u denotes the translation by u . Then $g(x, y) = \left(x - \frac{y}{2(y^2+1)} - \frac{1}{4}, y - \frac{x}{2(x^2+1)} + \frac{1}{4} \right)$. We will show that g is a diffeomorphism, $r_{G'} = 2$ and $\tilde{r}_{G'} = 1$. Let $\varphi : x \mapsto \frac{-x}{2(x^2+1)} - \frac{1}{4}$ and $\psi : x \mapsto \varphi(x) + \frac{1}{2}$, be two differentiable map of \mathbb{R} satisfying $|\varphi'(x)| = |\psi'(x)| < \frac{1}{2}$ for every $x \in \mathbb{R}$, since $\varphi'(x) = \frac{x^2-1}{2(x^2+1)^2}$. We have $g(x, y) = (x + \varphi(y), y + \psi(x))$. Firstly, we will show that g is bijective: Let $(a, b), (x, y) \in \mathbb{R}^2$ then

$$g(x, y) = (a, b) \iff \begin{cases} x + \varphi(y) = a \\ y + \psi(x) = b \end{cases} \iff \begin{cases} x + \varphi(b - \psi(x)) = a \\ y = b - \psi(x) \end{cases}$$

Consider the map $\chi_b : x \mapsto x + \varphi(b - \psi(x))$, then χ_b is continuous, differentiable and $\chi'_b(x) = 1 - \psi'(x)\varphi'(b - \psi(x))$, so $\chi'_b(x) > 0$ because $|\psi'(x)\varphi'(b - \psi(x))| < 1$. Therefore, χ_b is strictly increasing. Moreover φ and ψ are $\frac{1}{2}$ Lipschitz: $|\varphi(t) - \varphi(0)| \leq \frac{|t|}{2}$ and $|\psi(t) - \psi(0)| \leq \frac{|t|}{2}$, then $|\varphi(t)| \leq \frac{|t|}{2} + |\varphi(0)|$ and $|\psi(t)| \leq \frac{|t|}{2} + |\psi(0)|$, so

$$\begin{aligned} |\varphi(b - \psi(x))| &\leq \frac{1}{2}|b - \psi(x)| - \frac{1}{4} \\ &\leq \frac{1}{2}|\psi(x)| + \frac{|b|}{2} - \frac{1}{4} \\ &\leq \frac{1}{4}|x| + \frac{|b|}{2} - \frac{1}{8} \end{aligned}$$

therefore,

$$\chi_b(x) \geq \left(1 - \frac{1}{4}\right)x - \left(\frac{|b|}{2} - \frac{1}{8}\right) \rightarrow +\infty, \text{ if } x \rightarrow +\infty$$

and

$$\chi_b(x) \leq \left(1 - \frac{1}{4}\right)x + \left(\frac{|b|}{2} - \frac{1}{8}\right) \rightarrow -\infty, \text{ if } x \rightarrow -\infty$$

Thus χ_b realizes a bijection from \mathbb{R} unto \mathbb{R} . As a consequence:

$$g(x, y) = (a, b) \iff \begin{cases} x = \chi_b^{-1}(a) \\ y = b - \psi(\chi_b^{-1}(a)) \end{cases}$$

This implies that g is bijective.

Secondly, we will show that g is a local diffeomorphism: The Jacobien matrix of g on any point $(x, y) \in \mathbb{R}^2$ is given by

$$Dg_{(x,y)} = \begin{bmatrix} 1 & \psi'(x) \\ \varphi'(y) & 1 \end{bmatrix}.$$

Then $\det(Dg_{(x,y)}) = 1 - \varphi'(x)\psi'(y) \neq 0$ for every $(x, y) \in \mathbb{R}^2$. Since g is bijective, it follows that g is a diffeomorphism, so is f . Now $g(u) = u$ and $Dg_u = I_2$. As a consequence, we have $f(0, 0) = (0, 0)$ and $Df_{(0,0)} = Dg_u = I_2$. For $v = (1, 0)$, we have $f(v) = (1, \frac{1}{20})$. Since $(v, f(v))$ is a basis of \mathbb{R}^2 , so $E(v) = \mathbb{R}^2$, hence $r_G = 2$. On the other hand, $\tilde{G} = G$ and $L = \varphi_v(G) = \{I_2\}$, so $L(x, y) = \mathbb{R}(x, y)$, so $\dim(L(x, y)) = 1$, hence $\tilde{r}_G = 1$. \square

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