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# Semianalyticity of isoperimetric profiles

Renata Grimaldi, Stefano Nardulli and Pierre Pansu

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ABSTRACT: It is shown that, in dimensions  $< 8$ , isoperimetric profiles of compact real analytic Riemannian manifolds are semi-analytic.

RESUMÉ : On montre qu'en dimensions  $< 8$ , le profil isopérimétrique d'une variété riemannienne compacte est semi-analytique.

## 1 Introduction

### 1.1 The problem

Let  $M$  be a compact real analytic Riemannian manifold. We are concerned with the regularity of the *isoperimetric profile* of  $M$ .

Given  $0 < v < \text{vol}(M)$ , consider all integral currents in  $M$  with volume  $v$ . Define  $I_M(v)$  as the least upper bound of the boundary volumes of such currents. In this way, one gets a function  $I_M : (0, \text{vol}(M)) \rightarrow \mathbf{R}_+$  called the *isoperimetric profile* of  $M$ . In fact, for each  $0 < v < \text{vol}(M)$ , there exist currents in  $M$  with volume  $v$  and boundary volume  $I_M(v)$ . Such minimizing currents will be called *bubbles*, for short.

Here is a typical example. For  $a > 0$ , let  $S$  denote the circle of length  $2\pi$ . Let  $M = S \times S$ . Then the isoperimetric profile of  $M$  is easily computed to be

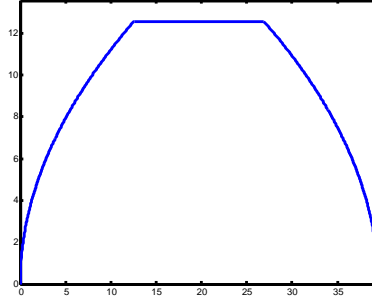
$$I_M(v) = \begin{cases} \sqrt{4\pi v} & \text{for } 0 < v \leq 4\pi, \\ 4\pi & \text{for } 4\pi \leq v \leq 4\pi(\pi - 1), \\ \sqrt{4\pi(4\pi^2 - v)} & \text{for } 4\pi(\pi - 1) \leq v < 4\pi^2. \end{cases}$$

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Profile of a flat torus

This is proven as follows. In 2 dimensions, the boundaries of these bubbles are smooth, they have constant geodesic curvature, therefore they lift to disjoint unions of circles of equal radii or lines in  $\mathbf{R}^2 = \tilde{M}$ . It follows that bubbles are either round disks or annuli bounded by parallel geodesics, or complements of such. There remains to minimize boundary length among these three families.

**Question 1** *For general real analytic manifolds, is it true that bubbles fall into finitely many analytic families, and that the profile is piecewise analytic ?*

This has been proven in [7] in dimension 2 only.

## 1.2 The results

First, in a neighborhood of zero.

**Theorem 1** *Let  $M$  be a compact real analytic Riemannian manifold. There exists  $\epsilon > 0$  such that  $I_M$  is real analytic on  $(0, \epsilon)$ .*

The isoperimetric profile of Euclidean space  $\mathbf{R}^n$  is  $I_{\mathbf{R}^n}(v) = n(\omega_n)^{1/n}v^{n-1/n}$ , where  $\omega_n$  is the volume of the unit ball in  $\mathbf{R}^n$ . In a curved manifold,  $I_M(v)^{n-1} \sim n(\omega_n)^{1/n}v^{n-1/n}$  as  $v$  tends to 0.

**Question 2** *For a compact analytic Riemannian  $n$ -manifold, is  $I_M(v)$  an analytic function of  $v^{1/n}$  on  $[0, \epsilon)$  ?*

We have only a partial answer.

**Theorem 2** *Let  $M$  be a compact real analytic Riemannian manifold. Assume that the absolute maxima of scalar curvature are nondegenerate critical points. Then there exists an analytic function  $f$  defined in a neighborhood of 0 such that  $I_M(v) = f(v^{1/n})$  for  $v$  small enough.*

Away from 0, our result also requires an extra assumption.

**Theorem 3** *Let  $M$  be a compact real analytic Riemannian manifold. Let  $0 < v_0 < \text{vol}(M)$ . Assume that all bubbles of volume  $v_0$  are smooth. Then  $I_M$  is semi-analytic on a neighborhood of  $v_0$ .*

Since bubbles are known to be smooth in dimensions  $< 8$ , [1], it follows that

**Corollary 4** *If the dimension of  $M$  is less than 8,  $I_M$  is semianalytic on  $[0, \text{vol}(M)]$ .*

**Question 3** *Our method of proof relies on the regularity of bubbles. Can this be circumvented ?*

## 2 Proof of Theorem 1

It relies on results from [5]. There, it is shown that small bubbles is a subset of a smooth finite dimensional family of domains called *pseudo-balls*. We merely need show that if the metric is real analytic, pseudo-balls form a compact, finite dimensional real analytic set, on which the volume and boundary volume functions are real analytic.

Pseudo-balls are solutions of a differential equations which is weaker than constancy of mean curvature, but to which the implicit function theorem can be applied. Specifically, for  $k \geq 0$ , consider the bundle  $\mathcal{F}^k \rightarrow M$  whose fiber at  $p \in M$  consists in  $C^{k,\alpha}$  functions on the unit sphere in the tangent space  $T_p M$ . There is a smooth map  $\Phi : \mathbf{R} \times \mathcal{F}^2 \rightarrow \mathcal{F}^0$  with the following properties.

1. Let  $r > 0$ ,  $p \in M$  and  $x \in \mathcal{F}_p^2$ . If the graph, in polar coordinates, of  $r(1+x)$  has constant mean curvature, then  $\Phi(r, p, x) = 0$ .
2. For all  $p \in M$ ,  $\Phi(0, p, 0) = 0$ .
3. The differential of  $\Phi$  restricted to the fibers is an isomorphism.

**Lemma 1**  $\Phi : \mathbf{R} \times \mathcal{F}^2 \rightarrow \mathcal{F}^0$  is a real analytic map.

**Proof:**  $\Phi(r, p, x)$  has two components,  $\Phi(r, p, x) = (\mathcal{A}(r, p, x), Q \circ \Psi(r, p, x))$ . The fact that  $\Psi$  is smooth is shown in [5], Proposition 1.1. The map  $\mathcal{A}$  is defined implicitly, by an equation whose smoothness is established in [5], Lemma 2.1. When the metric is real analytic, the same proofs show that  $\mathcal{A}$  and  $\Psi$  are real analytic. ■

From the implicit function theorem, it follows that for all  $p \in M$  and for  $r$  small enough, the equation  $\Phi(r, p, x) = 0$  has a unique small solution  $x = x(r, p)$  which depends analytically on  $(r, p) \in \mathbf{R} \times M$ . Theorem 6 of [5] asserts that there exist  $r_0 > 0$  and  $v_0 > 0$  such that every bubble of volume less than  $v_0$  coincides with the domain  $\mathcal{N}^+(p, r)$  bounded by the graph, in polar coordinates, of the function  $r(1 + x(r, p))$ , for some  $r \leq r_0$  and some  $p \in M$ . Therefore, for  $v < v_0$ ,

$$I_M(v) = \min\{\text{vol}(\partial\mathcal{N}^+(p, r)) \mid p \in M, 0 \leq r \leq r_0, \text{vol}(\mathcal{N}^+(p, r)) = v\}.$$

Define the *lower contour*  $c(A)$  of a subset  $A \subset \mathbf{R}^2$  as the function  $v \mapsto \inf\{w \in \mathbf{R} \mid (v, w) \in A\}$ . Then the restriction of  $I_M$  to  $[0, v_0]$  coincides with the lower contour of the image of the real analytic map

$$\Omega : [0, r_0] \times M \rightarrow \mathbf{R}^2, \quad (r, p) \mapsto (\text{vol}(\partial\mathcal{N}(p, r)), \text{vol}(\partial\mathcal{N}^+(p, r))).$$

Since  $[0, r_0] \times M$  is semi-analytic and compact,  $\Omega([0, r_0] \times M)$  is a compact subanalytic set. Its lower contour is a subanalytic function. According to S. Lojasiewicz (Theorem 6.1 in [3]), subanalytic functions on the real line are semi-analytic. Semi-analytic functions are piecewise analytic, thus there exists  $\epsilon > 0$  such that  $I_M$  is analytic on  $(0, \epsilon)$ . This completes the proof of Theorem 1.

### 3 Proof of Theorem 2

For  $(p, \rho) \in M \times \mathbf{R}_+$ , let  $\beta(p, \rho)$  denote the pseudo-ball defined by

$$\beta(p, \text{vol}(\mathcal{N}^+(p, r))^{1/n}) = \mathcal{N}^+(p, r).$$

Since  $\text{vol}(\mathcal{N}^+(p, r))^{1/n} \sim \omega_n^{1/n} r$  is a 1 to 1 analytic function of  $r$ , the notation is unambiguous. Let

$$f(p, \rho) = f(\text{vol}(\partial\beta(p, \rho))) = f_\rho(p).$$

Then  $f$  is real analytic. Furthermore, among pseudo-balls of volume  $v = \rho^n$ , bubbles are characterized as minima of  $f_\rho$ . The following expansion

$$f_\rho(p) = c_n \rho^{n-1} \left(1 - \frac{1}{2n(n+2)} \omega_n^{-2/n} Sc(p) \rho^2 + \mathcal{O}(\rho^4)\right)$$

is computed in [5], Lemma 3.6, compare [9]. If the absolute maxima  $p_1, \dots, p_k$  of the scalar curvature function  $Sc$  are non degenerate critical points, then each of them deforms into a critical point  $p_i(\rho)$  of  $f_\rho$  that depends analytically on  $\rho$ . Therefore (Theorem 8 in [5]),

$$I_M(\rho^n) = \min_{i=1, \dots, k} f_\rho(p_i(\rho)).$$

There exists  $\epsilon > 0$  and  $i$  such that the minimum is equal to  $f_\rho(p_i(\rho))$  for all  $\rho \in [0, \epsilon)$ . Indeed, otherwise, some function  $f_\rho(p_i(\rho)) - f_\rho(p_j(\rho))$  would change sign infinitely many times near 0, contradiction. Thus the right hand side is analytic on  $[0, \epsilon)$ . This completes the proof of Theorem 2.

## 4 Proof of Theorem 3

We follow M. Tamm's strategy, [8]. We aim at including bubbles in a parametrized analytic variety. We shall first do this in a neighborhood of a smooth bubble  $B$  with volume  $v_0$ . Our first candidate is the set of domains whose boundary is a graph in normal exponential coordinates to  $\partial B$  and has constant mean curvature. To decide whether this set is a submanifold in some function space, let us examine the mean curvature operator and its linearization.

### 4.1 Pseudo-bubbles

Let  $B$  be a smooth bubble with volume  $v_0$ . Let  $H_B : C^{2,\alpha}(\partial B) \rightarrow C^{0,\alpha}(\partial B)$  denote the operator which to a function  $u$  on  $\partial B$  associates the mean curvature of the graph of  $u$  in normal exponential coordinates to  $\partial B$ . In particular,  $H_B(0) = H(\partial B) = h_B$  is the constant mean curvature of  $\partial B$ . Let  $L_B : C^{2,\alpha}(\partial B) \rightarrow C^{0,\alpha}(\partial B)$  denote its linearization at 0 (sometimes called the *Jacobi operator*).

**Lemma 2** *For all  $v \in C^{2,\alpha}(\partial B)$*

$$L_B(v) = -\Delta_{\partial B}v - (||II_{\partial B}||^2 + Ric(\nu))v, \quad (1)$$

where  $\Delta_{\partial B}v = \text{div}(\nabla v)$  is the Laplace operator on  $\partial B$  (with negative spectrum when taken on the round sphere),  $||II_{\partial B}||^2$  is the Hilbert-Schmidt squared norm ( $\text{tr}(A^t A)$  for a square matrix  $A$ ) of the second fundamental form of  $\partial B$  and  $Ric(\nu)$  is the Ricci curvature of the ambient manifold  $M$  in the direction  $\nu$  of the unit outward normal vector to  $\partial B$  evaluated at a point of  $\partial B$ .

**Proof:** We recall here formula (6) of 3.3 of [6]

$$\begin{aligned} H(u) &= -\text{div}_{(\mathbb{S}^{n-1}, g_u)}\left(\frac{\vec{\nabla}_{g_u} u}{W_u}\right) - \frac{1}{W_u^2} \langle \nabla_{\vec{\nabla}_{g_u} u}}\left(\frac{u \vec{\nabla}_{g_u} u}{W_u}\right), \vec{\nabla}_{g_u} u \rangle_{g_u} \\ &+ \frac{u^2}{W_u^3} II_{\theta}^u(\vec{\nabla}_{g_u} u, \vec{\nabla}_{g_u} u) \\ &- \frac{1}{W_u} H_{\theta}^u(u) + \frac{1}{W_u} \langle \vec{\nabla}_{g_u}\left(\frac{1}{W_u}\right), u \frac{\vec{\nabla}_{g_u} u}{W_u} \rangle_{g_u}. \end{aligned} \quad (2)$$

Here  $\theta$  denotes the gradient of the signed distance function to  $\partial B$ . For the meaning of the other terms involved in (2), see [6]. The operator  $L_B$  satisfies then  $H(tv) = H_{\partial B} + tL_B + \mathcal{O}(t^2)$ . At first observe that only the first and fourth term of (2) contribute to  $L_B$ . Denoting by  $U(r) = \nabla\theta$  the shape operator of the equidistant hypersurfaces to  $\partial B$  at distance  $r$ , we have  $U(r) = U_0 + U_1 r + \dots$ , hence by using the Riccati equation satisfied by  $U$  (see [4]) we can compute  $U_1 = -U_0^2 - R$  where  $R$  is the curvature tensor. Now taking traces we get  $H_{\theta}^{tv} = \text{tr}(-U) = H_{\partial B, ext} + (\text{tr}(U_0^2) + Ric)t + \dots$  where  $H_{\partial B, ext}$  is the outward mean curvature of the boundary of  $B$ . Finally (1) follows easily. ■

$L_B$  is a selfadjoint elliptic operator, which has a discrete spectrum. Let  $K_B = \text{kernel}(L_B)$ ,  $m(B) = \text{dim}(K_B)$ .

If  $L_B$  were invertible (i.e.  $m(B) = 0$ ), the implicit function theorem would imply that nearby domains with constant mean curvature boundary come in one analytic family parametrized by the value of mean curvature.

Unfortunately,  $L_B$  is not always invertible. Therefore, instead of solving  $H(u) = h$ ,  $h \in \mathbf{R}$ , we shall solve

$$\Phi_B(u, h) = P_B(H_B(u) - h) = 0,$$

where  $P_B$  is the orthogonal projection onto the  $L^2$ -orthogonal complement  $K_B^\perp$  of  $K_B$  in  $C^{0,\alpha}(\partial B)$ . Then

$$\Phi_B : C^{2,\alpha}(\partial B) \times \mathbf{R} \rightarrow K_B^\perp$$

is a real analytic map, whose linearization at 0 is  $P_B \circ L_B$ . By construction, it is onto. In fact, the restriction of  $P_B \circ L_B$  to  $K_B^\perp$  is an isomorphism (see for example [2], page 464). Note that  $\Phi_B(0, h_B) = 0$ . The following variant of the implicit function theorem provides us with an open neighborhood  $U_B$  of  $(0, h_B)$  in  $C^{2,\alpha}(\partial B) \times \mathbf{R}$  in which the solutions of  $\Phi_B(u, h) = 0$  form a real analytic submanifold. We shall call such solutions *B-pseudo-bubbles*.

**Lemma 3** *Let  $E$ ,  $P$  and  $F$  be real analytic Banach manifolds, let  $e_0 \in E$ ,  $f_0 \in B$ ,  $p_0 \in P$  be such that  $\Phi(e_0, p_0) = f_0$ . Let  $\Phi : E \times P \rightarrow B$  be a real analytic map. Assume that the differential  $d\Phi$  of  $\Phi$  at  $(e_0, p_0)$  in the direction of  $E$  has a finite dimensional kernel  $K \subset T_{e_0}E$ , which admits a closed complement  $K^\perp$ . Assume that the restriction of  $d\Phi$  to  $K^\perp$  is invertible. Then, in a neighborhood of  $(e_0, p_0)$  the solutions of equation  $\Phi(e, p) = f_0$  form a real analytic submanifold parametrized by a neighborhood of  $(0, p_0)$  in  $K \times P$ .*

**Proof:** Apply the implicit function theorem to  $\Psi : E \times P \rightarrow F \times K$  defined by  $\Psi(e, p) = (\Phi(e, p), \pi_K(e))$  where  $\pi_K$  is a local submersion onto  $K$ . ■

## 4.2 Compactness in $C^{2,\alpha}$ -topology

Let us define the  $C^{2,\alpha}$ -topology on the space of domains with smooth boundary as follows: as neighborhoods of a smooth domain  $\beta$ , take all domains  $S$  whose boundary is the graph, in normal exponential coordinates, of a  $C^{2,\alpha}$ -small function on  $\partial\beta$ . Using a result from [6], we show that on smooth bubbles with volume close to  $v_0$ , the topologies induced by the  $C^{2,\alpha}$ -topology on smooth domains and the flat topology on currents coincide.

**Lemma 4** *Let  $B$  be a bubble of volume  $v_0$ . For all  $\delta > 0$ , there exist  $\epsilon > 0$  such that if  $\beta$  is a bubble of volume  $\in [v_0 - \epsilon, v_0 + \epsilon]$  with  $\text{vol}(\beta\Delta B) < \epsilon$ , then there exists a smooth function  $u$  on  $\partial B$  with  $\|u\|_{C^{2,\alpha}} < \delta$  such that  $\partial\beta$  is the graph in normal exponential coordinates of  $u$ . Conversely, the graph of a  $C^{2,\alpha}$ -small function on  $\partial B$  bounds a current which is close to  $B$  volumewise.*

**Proof:** By contradiction. Otherwise, there exists a sequence  $\beta_j$  of bubbles with  $\text{vol}(\beta_j) \rightarrow v_0$  and  $\text{vol}(\beta_j\Delta B) \rightarrow 0$  such that  $\partial\beta_j$  is not the normal exponential graph of a  $C^{2,\alpha}$ -small function on  $\partial B$ .

Theorem 1 of [6] asserts that for  $j$  large enough,  $\partial\beta_j$  is the graph in normal exponential coordinates of a function  $u_j$  on  $\partial B$  whose  $C^{2,\alpha}$ -norm tends to zero, contradiction. The last statement is obvious. ■

**Lemma 5** *Assume that all volume  $v_0$  bubbles in  $M$  are smooth. Then there exists  $\epsilon > 0$  such that if  $\beta$  is a bubble of volume  $\in [v_0 - \epsilon, v_0 + \epsilon]$ , then  $\beta$  is smooth, and  $(\beta, h_\beta)$  belongs to the open set  $U(B)$  for some volume  $v_0$  bubble  $B$ .*

**Proof:** By contradiction. Otherwise, there exists a sequence  $\beta_j$  of bubbles with  $\text{vol}(\beta_j) \rightarrow v_0$  that avoids all  $U(B)$ . By compactness of integral currents of bounded boundary volume, we can assume that  $\beta_j$  converges in flat norm to some integral current  $B$ . By continuity of volume and semi-continuity of boundary volume,  $B$  is a bubble of volume  $v_0$ . By assumption,  $B$  is smooth. Theorem 1 of [6] asserts that for  $j$  large enough,  $\partial\beta_j$  is the graph in normal exponential coordinates of a  $C^{2,\alpha}$ -small smooth function  $u_j$  on  $\partial B$ . Therefore  $\beta_j$  is smooth and  $(\beta_j, h_{\beta_j})$  belongs to  $U(B)$ , contradiction. ■

**Lemma 6** *Assume that all volume  $v_0$  bubbles in  $M$  are smooth. Then there exists  $\epsilon > 0$  such that the set  $\mathcal{B}$  of pseudo-bubbles with volumes  $\in [v_0 - \epsilon, v_0 + \epsilon]$  is contained in a finite union of compact semi-analytic pieces of finite dimensional real analytic manifolds, on which the volume and boundary volume functions are real analytic.*

**Proof:** It was just proven that the set  $\mathcal{B}$  of bubbles with volumes  $\in [v_0 - \epsilon, v_0 + \epsilon]$  is compact in flat topology and the set  $\mathcal{BH} = \{(B, h_B) \mid B \in \mathcal{B}\}$  is covered by the sets  $U(B)$ . According to Lemma 4, it is compact in  $C^{2,\alpha}$ -topology as well. Therefore,  $\mathcal{BH}$  can be covered with finitely many open sets  $U(B_1), \dots, U(B_N)$ . The set  $\Psi B_i$  of  $B_i$ -pseudo-bubbles in  $U(B_i)$  is an analytic submanifold. There exist compact semi-analytic subsets  $W_i \subset \Psi B_i$  which suffice to cover  $\mathcal{BH}$ . ■

The proof of Theorem 3 is completed in the same manner as the proof of Theorem 1. In dimensions less than 8, F. Almgren has shown that all bubbles are smooth. Therefore the profile is semi-analytic in a neighborhood of every point of the closed interval  $[0, \text{vol}(M)]$ . It follows that it is semi-analytic on this interval. This proves Corollary 4.

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