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Semianalyticity of isoperimetric profiles

Renata Grimaldi, Stefano Nardulli and Pierre Pansu

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ABSTRACT: It is shown that, in dimensions < 8, isoperimetric profiles of compact real analytic Riemannian manifolds are semi-analytic.

RESUMÉ : On montre qu'en dimensions < 8, le profil isopérimétrique d'une variété riemannienne compacte est semi-analytique.

1 Introduction

1.1 The problem

Let M be a compact real analytic Riemannian manifold. We are concerned with the regularity of the *isoperimetric profile* of M.

Given $0 < v < \mathrm{vol}(M)$, consider all integral currents in M with volume v. Define $I_M(v)$ as the least upper bound of the boundary volumes of such currents. In this way, one gets a function $I_M : (0, \mathrm{vol}(M)) \to \mathbf{R}_+$ called the *isoperimetric profile* of M. In fact, for each $0 < v < \mathrm{vol}(M)$, there exist currents in M with volume v and boundary volume $I_M(v)$. Such minimizing currents will be called *bubbles*, for short.

Here is a typical example. For $a > 0$, let S denote the circle of length 2π. Let $M = S \times S$. Then the isoperimetric profile of M is easily computed to be

$$I_M(v) = \begin{cases} \sqrt{4\pi v} & \text{for } 0 < v \leq 4\pi, \\ 4\pi & \text{for } 4\pi \leq v \leq 4\pi(\pi - 1), \\ \sqrt{4\pi(4\pi^2 - v)} & \text{for } 4\pi(\pi - 1) \leq v < 4\pi^2. \end{cases}$$

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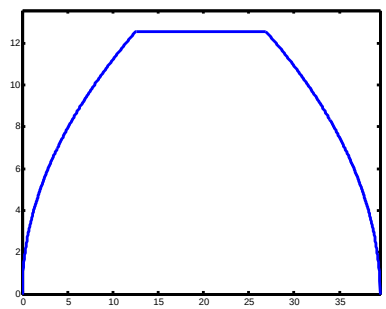

Profile of a flat torus

This is proven as follows. In 2 dimensions, the boundaries of these bubbles are smooth, they have constant geodesic curvature, therefore they lift to disjoint unions of circles of equal radii or lines in $\mathbf{R}^2 = \tilde{M}$. It follows that bubbles are either round disks or annuli bounded by parallel geodesics, or complements of such. There remains to minimize boundary length among these three families.

Question 1 *For general real analytic manifolds, is it true that bubbles fall into finitely many analytic families, and that the profile is piecewise analytic ?*

This has been proven in [7] in dimension 2 only.

1.2 The results

First, in a neighborhood of zero.

Theorem 1 *Let M be a compact real analytic Riemannian manifold. There exists $\epsilon > 0$ such that I_M is real analytic on $(0, \epsilon)$.*

The isoperimetric profile of Euclidean space $\mathbf{R}^n$ is $I_{\mathbf{R}^n}(v) = n(\omega_n)^{1/n} v^{n-1/n}$, where ω_n is the volume of the unit ball in $\mathbf{R}^n$. In a curved manifold, $I_M(v)^{n-1} \sim n(\omega_n)^{1/n} v^{n-1/n}$ as v tends to 0.

Question 2 *For a compact analytic Riemannian n-manifold, is $I_M(v)$ an analytic function of $v^{1/n}$ on $[0, \epsilon)$?*

We have only a partial answer.

Theorem 2 *Let M be a compact real analytic Riemannian manifold. Assume that the absolute maxima of scalar curvature are nondegenerate critical points. Then there exists an analytic function f defined in a neighborhood of 0 such that $I_M(v) = f(v^{1/n})$ for v small enough.*

Away from 0, our result also requires an extra assumption.

Theorem 3 *Let M be a compact real analytic Riemannian manifold. Let $0 < v_0 < \mathrm{vol}(M)$. Assume that all bubbles of volume v_0 are smooth. Then I_M is semi-analytic on a neighborhood of v_0.*

Since bubbles are known to be smooth in dimensions < 8, [1], it follows that

Corollary 4 *If the dimension of M is less than 8, I_M is semianalytic on $[0, \mathrm{vol}(M)]$.*

Question 3 *Our method of proof relies on the regularity of bubbles. Can this be circumvented ?*

2 Proof of Theorem 1

It relies on results from [5]. There, it is shown that small bubbles is a subset of a smooth finite dimensional family of domains called *pseudo-balls*. We merely need show that if the metric is real analytic, pseudo-balls form a compact, finite dimensional real analytic set, on which the volume and boundary volume functions are real analytic.

Pseudo-balls are solutions of a differential equations which is weaker than constancy of mean curvature, but to which the implicit function theorem can be applied. Specificly, for $k \geq 0$, consider the bundle $\mathcal{F}^k \to M$ whose fiber at $p \in M$ consists in $C^{k,\alpha}$ functions on the unit sphere in the tangent space T_pM. There is a smooth map $\Phi : \mathbf{R} \times \mathcal{F}^2 \to \mathcal{F}^0$ with the following properties.

1. Let $r > 0$, $p \in M$ and $x \in \mathcal{F}^2_p$. If the graph, in polar coordinates, of $r(1 + x)$ has constant mean curvature, then $\Phi(r, p, x) = 0$.
2. For all $p \in M$, $\Phi(0, p, 0) = 0$.
3. The differential of Φ restricted to the fibers is an isomorphism.

Lemma 1 $\Phi : \mathbf{R} \times \mathcal{F}^2 \to \mathcal{F}^0$ *is a real analytic map.*

Proof: $\Phi(r,p,x)$ has two components, $\Phi(r,p,x) = (\mathcal{A}(r,p,x), Q \circ \Psi(r,p,x))$. The fact that Ψ is smooth is shown in [5], Proposition 1.1. The map $\mathcal{A}$ is defined implicitely, by an equation whose smoothness is established in [5], Lemma 2.1. When the metric is real analytic, the same proofs show that $\mathcal{A}$ and Ψ are real analytic. ■

From the implicit function theorem, it follows that for all $p \in M$ and for r small enough, the equation $\Phi(r,p,x) = 0$ has a unique small solution $x = x(r,p)$ which depends analytically on $(r,p) \in \mathbf{R} \times M$. Theorem 6 of [5] asserts that there exist $r_0 > 0$ and $v_0 > 0$ such that every bubble of volume less than v_0 coincides with the domain $\mathcal{N}^+(p,r)$ bounded by the graph, in polar coordinates, of the function $r(1 + x(r,p))$, for some $r \leq r_0$ and some $p \in M$. Therefore, for $v < v_0$,

$$I_M(v) = \min\{\mathrm{vol}(\partial\mathcal{N}^+(p,r))\,|p \in M,\, 0 \leq r \leq r_0,\, \mathrm{vol}(\mathcal{N}^+(p,r)) = v\}.$$

Define the *lower contour* $c(A)$ of a subset $A \subset \mathbf{R}^2$ as the function $v \mapsto \inf\{w \in \mathbf{R}\,|\,(v,w) \in A\}$. Then the restriction of I_M to $[0, v_0]$ coincides with the lower contour of the image of the real analytic map

$$\Omega : [0, r_0] \times M \to \mathbf{R}^2, \quad (r,p) \mapsto (\mathrm{vol}(\partial\mathcal{N}(p,r)), \mathrm{vol}(\partial\mathcal{N}^+(p,r))).$$

Since $[0, r_0] \times M$ is semi-analytic and compact, $\Omega([0, r_0] \times M)$ is a compact subanalytic set. Its lower contour is a subanalytic function. According to S. Lojasiewicz (Theorem 6.1 in [3]), subanalytic functions on the real line are semi-analytic. Semi-analytic functions are piecewise analytic, thus there exists $\epsilon > 0$ such that I_M is analytic on $(0, \epsilon)$. This complete the proof of Theorem 1.

3 Proof of Theorem 2

For $(p,\rho) \in M \times \mathbf{R}_+$, let $\beta(p,\rho)$ denote the pseudo-ball defined by

$$\beta(p, \mathrm{vol}(\mathcal{N}^+(p,r))^{1/n}) = \mathcal{N}^+(p,r).$$

Since $\mathrm{vol}(\mathcal{N}^+(p,r))^{1/n} \sim \omega_n^{1/n} r$ is a 1 to 1 analytic function of r, the notation is unambiguous. Let

$$f(p,\rho) = f(\mathrm{vol}(\partial\beta(p,\rho))) = f_\rho(p).$$

Then f is real analytic. Furthermore, among pseudo-balls of volume $v = \rho^n$, bubbles are characterized as minima of f_ρ. The following expansion

$$f_\rho(p) = c_n \rho^{n-1}(1 - \frac{1}{2n(n+2)}\omega_n^{-2/n} Sc(p)\rho^2 + \mathcal{O}(\rho^4))$$

is computed in [5], Lemma 3.6, compare [9]. If the absolute maxima p_1,...,p_k of the scalar curvature function Sc are non degenerate critical points, then each of them deforms into a critical point $p_i(\rho)$ of f_ρ that depends analytically on ρ. Therefore (Theorem 8 in [5]),

$$I_M(\rho^n) = \min_{i=1,\ldots,k} f_\rho(p_i(\rho)).$$

There exists $\epsilon > 0$ and i such that the minimum is equal to $f_\rho(p_i(\rho))$ for all $\rho \in [0, \epsilon)$. Indeed, otherwise, some function $f_\rho(p_i(\rho)) - f_\rho(p_j(\rho))$ would change sign infinitely many times near 0, contradiction. Thus the right hand side is analytic on $[0, \epsilon)$. This completes the proof of Theorem 2.

4 Proof of Theorem 3

We follow M. Tamm's strategy, [8]. We aim at including bubbles in a parametrized analytic variety. We shall first do this in a neighborhood of a smooth bubble B with volume v_0. Our first candidate is the set of domains whose boundary is a graph in normal exponential coordinates to ∂B and has constant mean curvature. To decide wether this set is a submanifold in some function space, let us examine the mean curvature operator and its linearization.

4.1 Pseudo-bubbles

Let B be a smooth bubble with volume v_0. Let $H_B : C^{2,\alpha}(\partial B) \to C^{0,\alpha}(\partial B)$ denote the operator which to a function u on ∂B associates the mean curvature of the graph of u in normal exponential coordinates to ∂B. In particular, $H_B(0) = H(\partial B) = h_B$ is the constant mean curvature of ∂B. Let $L_B : C^{2,\alpha}(\partial B) \to C^{0,\alpha}(\partial B)$ denote its linearization at 0 (sometimes called the *Jacobi operator*).

Lemma 2 *For all $v \in C^{2,\alpha}(\partial B)$*

$$L_B(v) = -\Delta_{\partial B} v - (||II_{\partial B}||^2 + Ric(\nu))v, \tag{1}$$

where $\Delta_{\partial B} v = div(\nabla v)$ is the Laplace operator on ∂B (with negative spectrum when taken on the round sphere), $||II_{\partial B}||^2$ is the Hilbert-Schmidt squared norm ($tr(A^t A)$ for a square matrix A) of the second fundamental form of ∂B and $Ric(\nu)$ is the Ricci curvature of the ambient manifold M in the direction ν of the unit outward normal vector to ∂B evaluated at a point of ∂B.

Proof: We recall here formula (6) of 3.3 of [6]

$$\begin{aligned} H(u) &= -div_{(\mathbb{S}^{n-1},g_u)}(\frac{\overrightarrow{\nabla}_{g_u} u}{W_u}) - \frac{1}{W_u^2} < \nabla_{\overrightarrow{\nabla}_{g_u} u}(\frac{u\overrightarrow{\nabla}_{g_u} u}{W_u}), \overrightarrow{\nabla}_{g_u} u >_{g_u} \\ &+ \frac{u^2}{W_u^3} II_\theta^u(\overrightarrow{\nabla}_{g_u} u, \overrightarrow{\nabla}_{g_u} u) \\ &- \frac{1}{W_u} H_\theta^u(u) + \frac{1}{W_u} < \overrightarrow{\nabla}_{g_u}(\frac{1}{W_u}), u\frac{\overrightarrow{\nabla}_{g_u} u}{W_u}) >_{g_u} . \end{aligned} \tag{2}$$

Here θ denotes the gradient of the signed distance function to ∂B. For the meaning of the other terms involved in (2), see [6]. The operator L_B satisfies then $H(tv) = H_{\partial B} + tL_B + \mathcal{O}(t^2)$. At first observe that only the first and fourth term of (2) contribute to L_B. Denoting by $U(r) = \nabla\theta$ the shape operator of the equidistant hypersurfaces to ∂B at distance r, we have $U(r) = U_0 + U_1 r + \cdots$, hence by using the Riccati equation satisfied by U (see [4]) we can compute $U_1 = -U_0^2 - R$ where R is the curvature tensor. Now taking traces we get $H_\theta^{tv} = tr(-U) = H_{\partial B,ext} + (tr(U_0^2) + Ric)t + \cdots$ where $H_{\partial B,ext}$ is the outward mean curvature of the boundary of B. Finally (1) follows easily. ■

L_B is a selfadjoint elliptic operator, which has a discrete spectrum. Let $K_B = \text{kernel}(L_B)$, $m(B) = \dim(K_B)$.

If L_B were invertible (i.e. $m(B) = 0$), the implicit function theorem would imply that nearby domains with constant mean curvature boundary come in one analytic family parametrized by the value of mean curvature.

Unfortunately, L_B is not always invertible. Therefore, instead of solving $H(u) = h$, $h \in \mathbf{R}$, we shall solve

$$\Phi_B(u, h) = P_B(H_B(u) - h) = 0,$$

where P_B is the orthogonal projection onto the L^2-orthogonal complement $K_B^\perp$ of K_B in $C^{0,\alpha}(\partial B)$. Then

$$\Phi_B : C^{2,\alpha}(\partial B) \times \mathbf{R} \to K_B^\perp$$

is a real analytic map, whose linearization at 0 is $P_B \circ L_B$. By construction, it is onto. In fact, the restriction of $P_B \circ L_B$ to $K_B^\perp$ is an isomorphism (see for example [2], page 464). Note that $\Phi_B(0, h_B) = 0$. The following variant of the implicit function theorem provides us with an open neighborhood U_B of $(0, h_B)$ in $C^{2,\alpha}(\partial B) \times \mathbf{R}$ in which the solutions of $\Phi_B(u, h) = 0$ form a real analytic submanifold. We shall call such solutions *B-pseudo-bubbles.*

Lemma 3 *Let E, P and F be real analytic Banach manifolds, let $e_0 \in E$, $f_0 \in B$, $p_0 \in P$ be such that $\Phi(e_0, p_0) = f_0$. Let $\Phi : E \times P \to B$ be a real analytic map. Assume that the differential $d\Phi$ of Φ at (e_0, p_0) in the direction of E has a finite dimensional kernel $K \subset T_{e_0}E$, which admits a closed complement $K^\perp$. Assume that the restriction of $d\Phi$ to $K^\perp$ is invertible. Then, in a neighborhood of (e_0, p_0) the solutions of equation $\Phi(e, p) = f_0$ form a real analytic submanifold parametrized by a neighborhood of $(0, p_0)$ in $K \times P$.*

Proof: Apply the implicit function theorem to $\Psi : E \times P \to F \times K$ defined by $\Psi(e, p) = (\Phi(e, p), \pi_K(e))$ where π_K is a local submersion onto K. ■

4.2 Compactness in $C^{2,\alpha}$-topology

Let us define the $C^{2,\alpha}$-topology on the space of domains with smooth boundary as follows: as neighborhoods of a smooth domain β, take all domains S whose boundary is the graph, in normal exponential coordinates, of a $C^{2,\alpha}$-small function on $\partial\beta$. Using a result from [6], we show that on smooth bubbles with volume close to v_0, the topologies induced by the $C^{2,\alpha}$-topology on smooth domains and the flat topology on currents coincide.

Lemma 4 *Let B be a bubble of volume v_0. For all $\delta > 0$, there exist $\epsilon > 0$ such that if β is a bubble of volume $\in [v_0 - \epsilon, v_0 + \epsilon]$ with $\mathrm{vol}(\beta \Delta B) < \epsilon$, then there exists a smooth function u on ∂B with $\| u \|_{C^{2,\alpha}} < \delta$ such that $\partial\beta$ is the graph in normal exponential coordinates of u. Conversely, the graph of a $C^{2,\alpha}$-small function on ∂B bounds a current which is close to B volumewise.*

Proof: By contradiction. Otherwise, there exists a sequence β_j of bubbles with $\mathrm{vol}(\beta_j) \to v_0$ and $\mathrm{vol}(\beta_j \Delta B) \to 0$ such that $\partial\beta_j$ is not the normal exponential graph of a $C^{2,\alpha}$-small function on ∂B.

Theorem 1 of [6] asserts that for j large enough, $\partial\beta_j$ is the graph in normal exponential coordinates of a function u_j on ∂B whose $C^{2,\alpha}$-norm tends to zero, contradiction. The last statement is obvious. ■

Lemma 5 *Assume that all volume v_0 bubbles in M are smooth. Then there exists $\epsilon > 0$ such that if β is a bubble of volume $\in [v_0 - \epsilon, v_0 + \epsilon]$, then β is smooth, and (β, h_β) belongs to the open set $U(B)$ for some volume v_0 bubble B.*

Proof: By contradiction. Otherwise, there exists a sequence β_j of bubbles with $\mathrm{vol}(\beta_j) \to v_0$ that avoids all $U(B)$. By compactness of integral currents of bounded boundary volume, we can assume that β_j converges in flat norm to some integral current B. By continuity of volume and semi-continuity of boundary volume, B is a bubble of volume v_0. By assumption, B is smooth. Theorem 1 of [6] asserts that for j large enough, $\partial\beta_j$ is the graph in normal exponential coordinates of a $C^{2,\alpha}$-small smooth function u_j on ∂B. Therefore β_j is smooth and (β_j, h_{β_j}) belongs to $U(B)$, contradiction. ■

Lemma 6 *Assume that all volume v_0 bubbles in M are smooth. Then there exists $\epsilon > 0$ such that the set $\mathcal{B}$ of pseudo-bubbles with volumes $\in [v_0 - \epsilon, v_0 + \epsilon]$ is contained in a finite union of compact semi-analytic pieces of finite dimensional real analytic manifolds, on which the volume and boundary volume functions are real analytic.*

Proof: It was just proven that the set $\mathcal{B}$ of bubbles with volumes $\in [v_0 - \epsilon, v_0 + \epsilon]$ is compact in flat topology and the set $\mathcal{BH} = \{(B, h_B) \,|\, B \in \mathcal{B}\}$ is covered by the sets $U(B)$. According to Lemma 4, it is compact in $C^{2,\alpha}$-topology as well. Therefore, $\mathcal{BH}$ can be covered with finitely many open sets $U(B_1)$,...,$U(B_N)$. The set ΨB_i of B_i-pseudo-bubbles in $U(B_i)$ is an analytic submanifold. There exist compact semi-analytic subsets $W_i \subset \Psi B_i$ which suffice to cover $\mathcal{BH}$. ■

The proof of Theorem 3 is completed in the same manner as the proof of Theorem 1. In dimensions less than 8, F. Almgren has shown that all bubbles are smooth. Therefore the profile is semi-analytic in a neighborhood of every point of the closed interval $[0, \mathrm{vol}(M)]$. It follows that it is semi-analytic on this interval. This proves Corollary 4.

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