On the Optimal Stopping of a One-dimensional Diffusion

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Abstract

We consider the one-dimensional diffusion X that satisfies the stochastic differential equation

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \tag{1}$$

in the interior $\operatorname{int}\mathcal{I} =]\alpha, \beta[$ of a given interval $\mathcal{I} \subseteq [-\infty, \infty]$, where $b, \sigma : \operatorname{int}\mathcal{I} \to \mathbb{R}$ are Borel-measurable functions and W is a standard one-dimensional Brownian motion. We allow for the endpoints α and β to be inaccessible or absorbing. Given a Borel-measurable function $r : \mathcal{I} \to \mathbb{R}_+$ that is uniformly bounded away from 0, we establish a new analytic representation of the $r(\cdot)$-potential of a continuous additive functional of X. Furthermore, we derive a complete characterisation of differences of two convex functions in terms of appropriate $r(\cdot)$-potentials, and we show that a function $F : \mathcal{I} \to \mathbb{R}_+$ is $r(\cdot)$-excessive if and only if it is the difference of two convex functions and $-\left(\frac{1}{2}\sigma^2 F'' + bF' - rF\right)$ is a positive measure. We use these results to study the optimal stopping problem that aims at maximising the performance index

$$\mathbb{E}_x\left[\exp\left(-\int_0^\tau r(X_t)\,dt\right) f(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] \tag{2}$$

over all stopping times τ, where $f : \mathcal{I} \to \mathbb{R}_+$ is a Borel-measurable function that may be unbounded. We derive a simple necessary and sufficient condition for the value function v of this problem to be real-valued. In the presence of this condition, we show that v is the difference of two convex functions, and we prove that it satisfies the variational inequality

$$\max\left\{\frac{1}{2}\sigma^2 v'' + bv' - rv,\ \overline{f} - v\right\} = 0 \tag{3}$$

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in the sense of distributions, where $\overline{f}$ identifies with the upper semicontinuous envelope of f in the interior $\operatorname{int}\mathcal{I}$ of $\mathcal{I}$. Conversely, we derive a simple necessary and sufficient condition for a solution to (3) to identify with the value function v. Furthermore, we establish several other characterisations of the solution to the optimal stopping problem, including a generalisation of the so-called "principle of smooth fit". In our analysis, we also make a construction that is concerned with pasting weak solutions to (1) at appropriate hitting times, which is an issue of fundamental importance to dynamic programming.

1 Introduction

We consider the one-dimensional diffusion X that satisfies the SDE (1) in the interior $\operatorname{int}\mathcal{I} =]\alpha, \beta[$ of a given interval $\mathcal{I} \subseteq [-\infty, \infty]$. We assume that $b, \sigma : \operatorname{int}\mathcal{I} \to \mathbb{R}$ are Borel-measurable functions satisfying appropriate local integrability and non-degeneracy conditions ensuring that (1) has a weak solution that is unique in the sense of probability law up to a possible explosion time at which X hits the boundary $\{\alpha, \beta\}$ of $\mathcal{I}$ (see Assumption 1 in Section 2). If the boundary point α (resp., β) is inaccessible, then the interval $\mathcal{I}$ is open from the left (resp., open from the right), while, if α (resp., β) is not inaccessible, then it is absorbing and the interval $\mathcal{I}$ is closed from the left (resp., closed from the right).

In the presence of Assumption 1, a weak solution to (1) can be obtained by first time-changing a standard one-dimensional Brownian motion and then making an appropriate state space transformation. This construction can be used to prove all of the results that we obtain by first establishing them assuming that the diffusion X identifies with a standard one-dimensional Brownian motion. However, such an approach would hardly simplify the formalism because the data b (resp., σ) appear in all of the analysis exclusively (resp., mostly) though the operators $\mathcal{L}$, $\mathcal{L}_{\mathrm{ac}}$ defined by (36)–(37) below. Furthermore, deriving the general results, which are important because many applications assume specific functional forms for the data b and σ, by means of this approach would require several time changes and state space transformations, which would lengthen the paper significantly.

Given a point $z \in \operatorname{int}\mathcal{I}$, we denote by L^z the right-sided local time process of X at level z (see Revuz and Yor [32, Section VI.1] for the precise definition of L^z and its properties). Also, we denote by $\mathcal{B}(\mathcal{J})$ the Borel σ-algebra on any given interval $\mathcal{J} \subseteq [-\infty, \infty]$. With each signed Radon measure μ on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ such that σ^{-2} is locally integrable with respect to $|\mu|$, we associate the continuous additive functional

$$\mathcal{A}^{\mu}_t = \int_{\alpha}^{\beta} \frac{L^z_t}{\sigma^2(z)}\, \mu(dz), \quad t \in [0, T_\alpha \wedge T_\beta[, \tag{4}$$

where T_α (resp., T_β) is the first hitting time of α (resp., β). It is worth noting that (4) provides a one-to-one correspondence between the continuous additive functionals of the Markov process X and the signed Radon measures on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ (see Theorem X.2.9, Corollary X.2.10 and the comments on Section 2 at the end of Chapter X in Revuz and

Yor [32, Section X.2]). We also consider a discounting rate function $r : \mathcal{I} \to \mathbb{R}_+$, we assume that this is a Borel-measurable function that is uniformly bounded away from 0 and satisfies a suitable local integrability condition (see Assumption 2 in Section 2), and we define

$$\Lambda_t \equiv \Lambda_t(X) = \int_0^t r(X_s)\, ds. \tag{5}$$

Given a signed Radon measure μ on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$, we consider the $r(\cdot)$-potential of the continuous additive functional A^μ, which is defined by

$$R_\mu(x) = \mathbb{E}_x\left[\int_0^{T_\alpha \wedge T_\beta} e^{-\Lambda_t}\, dA_t^\mu\right]. \tag{6}$$

We recall that a function $F : \operatorname{int}\mathcal{I} \to \mathbb{R}$ is the difference of two convex functions if and only if its left-hand side derivative F'_- exists and its second distributional derivative is a measure, and we define the measure $\mathcal{L}F$ by

$$\mathcal{L}F(dx) = \frac{1}{2}\sigma^2(x)F''(dx) + b(x)F'_-(x)\, dx - r(x)F(x)\, dx.$$

In the presence of a general integrability condition ensuring that the potential R_μ is well-defined, we show that it is the difference of two convex functions, the measures $\mathcal{L}R_\mu$ and $-\mu$ are equal, and

$$\begin{aligned} R_\mu(x) &= \frac{2}{C}\varphi(x)\int_{]\alpha,x[}\frac{\psi(s)}{\sigma^2(s)p'(s)}\,\mu(ds) + \frac{2}{C}\psi(x)\int_{[x,\beta[}\frac{\varphi(s)}{\sigma^2(s)p'(s)}\,\mu(ds) \\ &= \int_{]\alpha,\beta[}\frac{2\varphi(x)\psi(x)}{C\sigma^2(s)p'(s)}\min\left\{\frac{\psi(s)}{\psi(x)}, \frac{\varphi(s)}{\varphi(x)}\right\}\mu(ds), \end{aligned} \tag{7}$$

where $C > 0$ is an appropriate constant, $p : \operatorname{int}\mathcal{I} \to \mathbb{R}$ is the scale function of X, and $\varphi, \psi : \operatorname{int}\mathcal{I} \to]0, \infty[$ are C^1 functions with absolutely continuous with respect to the Lebesgue measure derivatives spanning the solution space of the ODE

$$\frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) = 0,$$

and such that φ (resp., ψ) is decreasing (resp., increasing) (see Theorem 6). If the signed measure μ^h is absolutely continuous with respect to the Lebesgue measure with Radon-Nikodym derivative given by a function h, then the potential R_{μ^h} admits the expressions

$$\begin{aligned} R_{\mu^h}(x) &= \mathbb{E}_x\left[\int_0^{T_\alpha \wedge T_\beta} e^{-\Lambda_t} h(X_t)\, dt\right] \\ &= \frac{2}{C}\varphi(x)\int_\alpha^x \frac{\psi(s)}{\sigma^2(s)p'(s)}h(s)\, ds + \frac{2}{C}\psi(x)\int_x^\beta \frac{\varphi(s)}{\sigma^2(s)p'(s)}h(s)\, ds \end{aligned} \tag{8}$$

(see Corollary 8 for this and other related results). Conversely, we show that, under a general growth condition, a difference of two convex functions $F : \operatorname{int}\mathcal{I} \to \mathbb{R}$ is such that (a) both limits $\lim_{y\downarrow\alpha} F(y)/\varphi(y)$ and $\lim_{y\uparrow\beta} F(y)/\psi(y)$ exist, (b) F admits the characterisation

$$F(x) = \lim_{y\downarrow\alpha} \frac{F(y)}{\varphi(y)}\varphi(x) + R_{-\mathcal{L}F}(x) + \lim_{y\uparrow\beta} \frac{F(y)}{\psi(y)}\psi(x), \tag{9}$$

and (c) an appropriate form of Dynkin's formula holds true (see Theorem 7). With a view to optimal stopping, we use these results to show that a function $F : \mathcal{I} \to \mathbb{R}_+$ is $r(\cdot)$-excessive if and only if it is the difference of two convex functions and $-\mathcal{L}F$ is a positive measure (see Theorem 9 for the precise result).

If r is constant, then general theory of Markov processes implies the existence of a transition kernel u^r such that $R_\mu(x) = \int_{]\alpha,\beta[} u^r(x,s)\,\mu(ds)$ (see Meyer [27] and Revuz [31]). If X is a standard Brownian motion, then

$$u^r(x,s) = \frac{1}{\sqrt{2r}} e^{-\sqrt{2r}|x-s|}$$

(see Revuz and Yor [32, Theorem X.2.8]). The general expression for this kernel provided by (7) is one of the contributions of this paper. On the other hand, the identity in (8) is well-known and can be found in several references (e.g., see Borodin and Salminen [8, II.4.24]). Also, Johnson and Zervos [20] prove that the potential given by (6) admits the analytic expression (7) and show that the measures $\mathcal{L}R_\mu$ and $-\mu$ are equal when both of the endpoints α and β are assumed to be inaccessible.

The representation of differences of two convex functions given by (9) is also new. Such a result is important for the solution to one-dimensional infinite time horizon stochastic control as well as optimal stopping problems using dynamic programming. Indeed, the analysis of several explicitly solvable problems involve such a representation among their assumptions. For constant r, Salminen [34] considered more general one-dimensional linear diffusions than the one given by (1) and used Martin boundary theory to show that every r-excessive function admits a representation that is similar to but much less straightforward than the one in (9). Since a function on an open interval is the difference of two convex functions if and only if it is the difference of two excessive functions (see Çinlar, Jacod, Protter and Sharpe [11]), the representation derived by Salminen [34] can be extended to differences of two convex functions. However, it is not straightforward to derive such an extension of the representation in Salminen [34] from (9) or vice-versa when the underlying diffusion satisfies (1) and r is constant.

The result that a function F is $r(\cdot)$-excessive if and only if it is the difference of two convex functions and $-\mathcal{L}F$ is a positive measure is perhaps the simplest possible characterisation of excessive functions because it involves only derivative operators. In fact, we show that this result is equivalent to the characterisations of excessive functions derived by Dynkin [15] and Dayanik [12] (see Corollary 10).

We use the results that we have discussed above to analyse the optimal stopping problem that aims at maximising the performance criterion given by (2) over all stopping times τ, assuming that the reward function f is a positive Borel-measurable function that may be unbounded (see Assumption 2 in Section 2). We first prove that the value function v is the difference of two convex functions and satisfies the variational inequality (3) in the sense of distributions, where $\overline{f}$ is defined by

$$\overline{f}(x) = \begin{cases} \limsup_{y \to x} f(y), & \text{if } x \in \operatorname{int} \mathcal{I}, \\ f(\alpha), & \text{if } \alpha \text{ is absorbing and } x = \alpha, \\ f(\beta), & \text{if } \beta \text{ is absorbing and } x = \beta \end{cases} \tag{10}$$

(see Definition 1 and Theorem 12.(I)–(II) in Section 6). This result provides simple criteria for deciding which parts of the interval $\mathcal{I}$ must be subsets of the so-called waiting region. Indeed, the derived regularity of v implies that all points at which the reward function f is discontinuous as well as all "minimal" intervals in which f cannot be expressed as the difference of two convex functions (e.g., intervals throughout which f has the regularity of a Brownian sample path) should be parts of the closure of the waiting region. Similarly, the support of the measure $(\mathcal{L}f)^+$ in all intervals in which $\mathcal{L}f$ is well-defined should also be a subset of the closure of the waiting region.

We then establish a verification theorem that is the strongest one possible because it involves only the optimal stopping problem's data. In particular, we derive a simple necessary and sufficient condition for a solution w to (3) in the sense of distributions to identify with the problem's value function (see Theorem 13.(I)–(II)).

These results establish a complete characterisation of the value function v in terms of the variational inequality (3). Indeed, they imply that the restriction of the optimal stopping problem's value function v in $\operatorname{int} \mathcal{I}$ identifies with the unique solution to the variational inequality (3) in the sense of Definition 1 that satisfies the boundary conditions

$$\lim_{y \in \operatorname{int} \mathcal{I},\, y \downarrow \alpha} \frac{v(y)}{\varphi(y)} = \limsup_{y \downarrow \alpha} \frac{f(y)}{\varphi(y)} \quad \text{and} \quad \lim_{y \in \operatorname{int} \mathcal{I},\, y \uparrow \beta} \frac{v(y)}{\psi(y)} = \limsup_{y \uparrow \beta} \frac{f(y)}{\psi(y)}.$$

It is worth noting that, if α (resp., β) is absorbing, then the corresponding boundary condition is equivalent to

$$\lim_{y \in \operatorname{int} \mathcal{I},\, y \downarrow \alpha} v(y) = \limsup_{y \downarrow \alpha} f(y) \qquad \left(\text{resp.,} \quad \lim_{y \in \operatorname{int} \mathcal{I},\, y \uparrow \beta} \psi(y) = \limsup_{y \uparrow \beta} f(y) \right)$$

(see (28)–(29)). Also, it is worth stressing the precise nature of these boundary conditions. The limits on the left-hand sides are taken from inside the interior $\operatorname{int} \mathcal{I}$ of $\mathcal{I}$ and they indeed exist. On the other hand, the limsups on the right-hand sides are taken from inside $\mathcal{I}$ itself. Therefore, if, e.g., α is absorbing, then we are faced either with

$$v(\alpha) = f(\alpha) = \lim_{y \in \operatorname{int} \mathcal{I},\, y \downarrow \alpha} v(y) = \limsup_{y \downarrow \alpha} f(y), \quad \text{if } f(\alpha) = \limsup_{y \downarrow \alpha} f(y) \geq \limsup_{y \in \operatorname{int} \mathcal{I},\, y \downarrow \alpha} f(y),$$

or with

$$v(\alpha) = f(\alpha) < \lim_{y \in \operatorname{int} \mathcal{I},\, y \downarrow \alpha} v(y) = \limsup_{y \downarrow \alpha} f(y), \quad \text{if } f(\alpha) < \limsup_{y \downarrow \alpha} f(y) = \limsup_{y \in \operatorname{int} \mathcal{I},\, y \downarrow \alpha} f(y).$$

Furthermore, we prove that

$$v(x) = \inf\big\{ A\varphi(x) + B\psi(x) \mid \ A, B \geq 0 \text{ and } A\varphi + B\psi \geq f \big\} \tag{11}$$

for all $x \in \operatorname{int} \mathcal{I}$ (see Theorem 13.(III)). In fact, this characterisation can be used as a verification theorem as well (see also the discussion further below).

In the generality that we consider, an optimal stopping time might not exist (see Examples 1–4 in Section 8). Moreover, the hitting time of the so-called "stopping region", which is given by

$$\tau^\star = \inf\big\{ t \geq 0 \mid \ v(X_t) = f(X_t) \big\}, \tag{12}$$

may not be optimal (see Examples 2 and 4). In particular, Example 2 shows that τ^* may not be optimal and that an optimal stopping time may not exist at all unless f satisfies appropriate boundary / growth conditions. Also, Example 4 reveals that $\tau^\star$ is not in general optimal if $f \neq \overline{f}$. In Theorem 12.(III), we obtain a simple sequence of ε-optimal stopping times if f is assumed to be upper semicontinuous, and we show that $\tau^\star$ is an optimal stopping time if f satisfies an appropriate growth condition.

Building on the general theory, we also consider a number of related results and characterisations. In particular, we obtain a generalisation of the so-called "principle of smooth fit" (see part (III) of Corollaries 15, 16 and 17 in Section 7).

In view of the version of Dynkin's formula (98) in Corollary 8, we can see that, if h is any function such that R_{μ^h} given by (8) is well-defined, then

$$\begin{aligned} \sup_\tau \mathbb{E}_x \left[\int_0^{\tau \wedge T_\alpha \wedge T_\beta} e^{-\Lambda_t} h(X_t)\, dt + e^{-\Lambda_{\tau \wedge T_\alpha \wedge T_\beta}} f(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right] & \\ = R_{\mu^h}(x) + \sup_\tau \mathbb{E}_x \left[e^{-\Lambda_{\tau \wedge T_\alpha \wedge T_\beta}} \left(f - R_{\mu^h} \right) (X_{\tau \wedge T_\alpha \wedge T_\beta}) \mathbf{1}_{\{\tau < \infty\}} \right] & \\ = R_{\mu^h}(x) + \sup_\tau \mathbb{E}_x \left[e^{-\Lambda_{\tau \wedge T_\alpha \wedge T_\beta}} \left(f - R_{\mu^h} \right)^+ (X_{\tau \wedge T_\alpha \wedge T_\beta}) \mathbf{1}_{\{\tau < \infty\}} \right]. & \end{aligned} \tag{13}$$

Therefore, all of the results on the optimal stopping problem that we consider generalise most trivially to account for the apparently more general optimal stopping problem associated with (13).

The various aspects of the optimal stopping theory have been developed in several monographs, including Shiryayev [35], Friedman [17, Chapter 16], Krylov [23], Bensoussan and Lions [7], El Karoui [16], Øksendal [28, Chapter 10] and Peskir and Shiryaev [30]. In particular, the solution of optimal stopping problems using classical solutions to variational inequalities has been extensively studied (e.g., see Friedman [17, Chapter 16], Krylov [23] and Bensoussan and Lions [7]). Results in this direction typically make strong regularity

assumptions on the problem data (e.g., the diffusion coefficients are assumed to be Lipschitz continuous). To relax such assumptions, Øksendal and Reikvam [29] and Bassan and Ceci [4] have considered viscosity solutions to the variational inequalities associated with the optimal stopping problems that they study. Closer to the spirit of this paper, Lamberton [24] proved that the value function of the finite version of the problem we consider here satisfies its associated variational inequality in the sense of distributions.

Relative to the optimal stopping problem that we consider here when r is constant, Dynkin [14] and Shiryaev [35, Theorem 3.3.1] prove that the value function v identifies with the smallest r-excessive function that majorises the reward function f if f is assumed to be lower semicontinuous. Also, Shiryaev [35, Theorem 3.3.3] proves that the stopping time $\tau^\star$ defined by (12) is optimal if f is assumed to be continuous and bounded, while Salminen [34] establishes the optimality of τ^* assuming that the smallest r-excessive majorant of f exists and f is upper semicontinuous. Later, Dayanik and Karatzas [13] and Dayanik [12], who also considers random discounting instead of discounting at a constant rate r, addressed the solution of the optimal stopping problem by means of a certain concave characterisation of excessive functions. In particular, they established a generalisation of the so-called "principle of smooth fit" that is similar to, though not the same as, the one we derive here.

There are numerous special cases of the general optimal stopping problem we consider that have been explicitly solved in the literature. Such special cases have been motivated by applications or have been developed as illustrations of various general techniques. In all cases, their analysis relies on some sort of a verification theorem. Existing verification theorems for solutions using dynamic programming and variational inequalities typically make strong assumptions that are either tailor-made or difficult to verify in practice. For instance, Theorem 10.4.1 in Øksendal [28] involves Lipschitz as well as uniform integrability assumptions, while, Theorem I.2.4 in Peskir and Shiryaev [30] assumes the existence of an optimal stopping time, for which, a sufficient condition is provided by Theorem I.2.7. Alternatively, they assume that the so-called stopping region is a set of a simple specific form (e.g., see Rüschendorf and Urusov [33] or Gapeev and Lerche [18]).

Using martingale and change of measure techniques, Beibel and Lerche [5, 6], Lerche and Urusov [26] and Christensen and Irle [10] developed an approach to determining an optimal stopping strategy at any given point in the interval $\mathcal{I}$. Similar techniques have also been extensively used by Alvarez [1, 2, 3], Lempa [25] and references therein. To fix ideas, we consider the following representative cases that can be associated with any given initial condition $x \in \mathcal{I}$. If there exists a point $d_1 > x$ such that

$$C_1 := \sup_{x \in \mathcal{I}} \frac{f(x)}{\psi(x)} = \frac{f(d_1)}{\psi(d_1)}, \tag{14}$$

then $v(x) = C_1\psi(x)$ and the first hitting time of $\{d_1\}$ is optimal. Alternatively, if there exist points $\kappa \in]0,1[$ and $c_2 < x < d_2$ such that

$$C_2 := \sup_{x \in \mathcal{I}} \frac{f(x)}{\kappa\psi(x) + (1-\kappa)\varphi(x)} = \frac{f(c_2)}{\kappa\psi(c_2) + (1-\kappa)\varphi(c_2)} = \frac{f(d_2)}{\kappa\psi(d_2) + (1-\kappa)\varphi(d_2)}, \tag{15}$$

then $v(x) = \kappa C_2\psi(x) + (1-\kappa)C_2\varphi(x)$ and the first hitting time of $\{c_2, d_2\}$ is optimal. On the other hand, if x is a global maximiser of the function $f/(A\psi + B\varphi)$, for some $A, B \geq 0$, then x is in the stopping region and $v(x) = f(x)$. It is straightforward to see that the conclusions associated with each of these cases follow immediately from the representation (11) of the value function v (see also Corollary 14 and part (II) of Corollaries 15, 16 and 17). Effectively, this approach, which is summarised by (11), is a verification theorem of a local character. Indeed, its application invariably involves "guessing" the structure of the waiting and the stopping regions. Also, e.g., (14) on its own does not allow for any conclusions for initial conditions $x > d_1$ (see Example 5). It is also worth noting that, if f is C^1, then this approach is effectively the same as application of the so-called "principle of smooth fit": first order conditions at d_1 (resp., c_2, d_2) and (14) (resp., (15)) yield the same equations for d_1, C_1 (resp. c_2, d_2, κ, C_2) as the one that the "principle of smooth fit" yields (see also the generalisations in part (III) of Corollaries 15, 16 and 17).

In stochastic analysis, a filtration can be viewed as a model for an information flow. Such an interpretation gives rise to the following modelling issue. Consider an observer whose information flow identifies with a filtration $(\mathcal{H}_t)$. At an $(\mathcal{H}_t)$-stopping time τ, the observer gets access to an additional information flow, modelled by a filtration $(\mathcal{G}_t)$, that "switches on" at time τ. In this context, we construct a filtration that aggregates the two information sources available to such an observer (see Theorem (19)). Building on this construction, we address the issue of pasting weak solutions to (1), or, more, generally, the issue of pasting stopping strategies for the optimal stopping problem that we consider, at an appropriate stopping time (see Theorem (20) and Corollary 21). Such a rather intuitive result is fundamental to dynamic programming and has been assumed by several authors in the literature (e.g., see the proof of Proposition 3.2 in Dayanik and Karatzas [13]).

The paper is organised as follows. In Section 2, we develop the context within which the optimal stopping problem that we study is defined and we list all of the assumptions we make. Section 3 is concerned with a number of preliminary results that are mostly of a technical nature. In Section 4, we derive the representation (7) for $r(\cdot)$-potentials and the characterisation (9) of differences of two convex functions as well as a number of related results. In Section 5, we consider analytic characterisations of $r(\cdot)$-excessive functions, while, in Section 6, we establish our main results on the optimal stopping problem that we consider. In Section 7, we present several ramifications of our general results on optimal stopping, including a generalisation of the "principle of smooth fit". In Section 8, we consider a number of illustrating examples. Finally, we develop the theory concerned with pasting weak solutions to (1) in the Appendix.

2 The underlying diffusion and the optimal stopping problem

We consider a one-dimensional diffusion with state space an interval of the form

$$\mathcal{I} =]\alpha, \beta[\quad \text{or} \quad \mathcal{I} = [\alpha, \beta[\quad \text{or} \quad \mathcal{I} =]\alpha, \beta] \quad \text{or} \quad \mathcal{I} = [\alpha, \beta], \tag{16}$$

for some endpoints $-\infty \leq \alpha < \beta \leq \infty$. Following Definition 5.20 in Karatzas and Shreve [21, Chapter 5], a weak solution to the SDE (1) in the interval $\mathcal{I}$ is a collection $\mathbb{S}_x = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x, W, X)$ such that $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_x)$ is a filtered probability space satisfying the usual conditions and supporting a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W and a continuous $(\mathcal{F}_t)$-adapted $\mathcal{I}$-valued process X. The process X satisfies

$$\int_0^{t\wedge T_{\bar{\alpha}}\wedge T_{\bar{\beta}}} \left[|b(X_u)| + \sigma^2(X_u)\right] du < \infty \tag{17}$$

and

$$X_{t\wedge T_{\bar{\alpha}}\wedge T_{\bar{\beta}}} = x + \int_0^{t\wedge T_{\bar{\alpha}}\wedge T_{\bar{\beta}}} b(X_u)\, du + \int_0^{t\wedge T_{\bar{\alpha}}\wedge T_{\bar{\beta}}} \sigma(X_u)\, dW_u \tag{18}$$

for all $t \geq 0$ and $\alpha < \bar{\alpha} < x < \bar{\beta} < \beta$, $\mathbb{P}_x$-a.s.. Here, as well as throughout the paper, we denote by T_y the first hitting time of the set $\{y\}$, which is defined by

$$T_y = \inf\{t \geq 0 \mid \ X_t = y\}, \quad \text{for } y \in [\alpha, \beta],$$

with the usual convention that $\inf \emptyset = \infty$. The actual choice of the interval $\mathcal{I}$ from among the four possibilities in (16) depends on the choice of the data b and σ through the resulting properties of the explosion time $T_\alpha \wedge T_\beta$ at which the process X hits the boundary $\{\alpha, \beta\}$ of the interval $\mathcal{I}$. If the boundary point α (resp., β) is inaccessible, i.e., if

$$\mathbb{P}_x\big(T_\alpha < \infty\big) = 0 \quad \big(\text{resp., } \mathbb{P}_x\big(T_\beta < \infty\big) = 0\big),$$

then the interval $\mathcal{I}$ is open from the left (resp., open from the right). If α (resp., β) is not inaccessible, then it is absorbing and the interval $\mathcal{I}$ is closed from the left (resp., closed from the right). In particular,

$$X_t = \begin{cases} \alpha, & \text{if } \lim_{u\to T_\alpha\wedge T_\beta} X_u = \alpha, \\ \beta, & \text{if } \lim_{u\to T_\alpha\wedge T_\beta} X_u = \beta, \end{cases} \quad \text{for all } t \geq T_\alpha \wedge T_\beta. \tag{19}$$

The following assumption ensures that the SDE (1) has a weak solution in $\mathcal{I}$, as described above, which is unique in the sense of probability law (see Theorem 5.15 in Karatzas and Shreve [21, Chapter 5]).

Assumption 1 The functions $b, \sigma : \operatorname{int}\mathcal{I} \to \mathbb{R}$ are Borel-measurable,

$$\sigma^2(x) > 0 \quad \text{for all } x \in \operatorname{int}\mathcal{I} \equiv]\alpha, \beta[, \tag{20}$$

and

$$\int_{\bar{\alpha}}^{\bar{\beta}} \frac{1 + |b(s)|}{\sigma^2(s)}\, ds < \infty \quad \text{for all } \alpha < \bar{\alpha} < \bar{\beta} < \beta. \tag{21}$$

□

This assumption also implies that, given $c \in \operatorname{int}\mathcal{I}$ fixed, the scale function p, given by

$$p(x) = \int_c^x \exp\left(-2\int_c^s \frac{b(u)}{\sigma^2(u)}\, du\right) ds, \quad \text{for } x \in \operatorname{int}\mathcal{I}, \tag{22}$$

is well-defined, and the speed measure m on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\mathcal{I})\big)$, given by

$$m(dx) = \frac{2}{\sigma^2(x)p'(x)}\, dx, \tag{23}$$

is a Radon measure. At this point, it is worth noting that Feller's test for explosions provides necessary and sufficient conditions that determine whether the solution of (1) hits one or the other or both of the boundary points α, β in finite time with positive probability (see Theorem 5.29 in Karatzas and Shreve [21, Chapter 5]).

We consider the optimal stopping problem, the value function of which is defined by

$$v(x) = \sup_{(\mathbb{S}_x,\tau)\in\mathcal{T}_x} \mathbb{E}_x\left[e^{-\Lambda_\tau} f(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] = \sup_{(\mathbb{S}_x,\tau)\in\mathcal{T}_x} J(\mathbb{S}_x,\tau), \quad \text{for } x \in \mathcal{I}, \tag{24}$$

where

$$J(\mathbb{S}_x,\tau) = \mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_\alpha\wedge T_\beta}} f(X_{\tau\wedge T_\alpha\wedge T_\beta})\mathbf{1}_{\{\tau<\infty\}}\right],$$

the discounting factor Λ is defined by (5) in the introduction, and the set of all stopping strategies $\mathcal{T}_x$ is the collection of all pairs $(\mathbb{S}_x, \tau)$ such that $\mathbb{S}_x$ is a weak solution to (1), as described above, and τ is an associated $(\mathcal{F}_t)$-stopping time.

We make the following assumption, which also implies the identity in (24).

Assumption 2 The reward function $f : \mathcal{I} \to \mathbb{R}_+$ is Borel-measurable. The discounting rate function $r : \mathcal{I} \to \mathbb{R}_+$ is Borel-measurable and uniformly bounded away from 0, i.e., $r(x) \geq r_0$ for all $x \in \mathcal{I}$, for some $r_0 > 0$. Also,

$$\int_{\bar{\alpha}}^{\bar{\beta}} \frac{r(s)}{\sigma^2(s)}\, ds < \infty \quad \text{for all } \alpha < \bar{\alpha} < \bar{\beta} < \beta. \tag{25}$$

□

In the presence of Assumptions 1 and 2, there exists a pair of C^1 with absolutely continuous first derivatives functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}_+$ such that φ (resp., ψ) is strictly decreasing (resp., increasing), and

$$\varphi(x) = \varphi(y)\mathbb{E}_x\left[e^{-\Lambda_{T_y}}\right] \equiv \varphi(y)\mathbb{E}_x\left[e^{-\Lambda_{T_y}}\mathbf{1}_{\{T_y<T_\beta\}}\right] \quad \text{for all } y < x, \tag{26}$$

$$\psi(x) = \psi(y)\mathbb{E}_x\left[e^{-\Lambda_{T_y}}\right] \equiv \psi(y)\mathbb{E}_x\left[e^{-\Lambda_{T_y}}\mathbf{1}_{\{T_y<T_\alpha\}}\right] \quad \text{for all } x < y, \tag{27}$$

for every solution $\mathbb{S}_x$ to (1). Also,

$$\text{if } \alpha \text{ is absorbing, then } \varphi(\alpha) := \lim_{x\downarrow\alpha}\varphi(x) < \infty \text{ and } \psi(\alpha) := \lim_{x\downarrow\alpha}\psi(x) = 0, \tag{28}$$

$$\text{if } \beta \text{ is absorbing, then } \varphi(\beta) := \lim_{x\uparrow\beta}\varphi(x) = 0 \text{ and } \psi(\beta) := \lim_{x\uparrow\beta}\psi(x) < \infty, \tag{29}$$

$$\text{and, if } \alpha \text{ (resp., } \beta\text{) is inaccessible, then } \lim_{x\downarrow\alpha}\varphi(x) = \infty \text{ (resp., } \lim_{x\uparrow\beta}\psi(x) = \infty). \tag{30}$$

An inspection of these facts reveals that, in all cases,

$$\lim_{y\downarrow\alpha}\frac{\psi(y)}{\varphi(y)} = \lim_{y\uparrow\beta}\frac{\varphi(y)}{\psi(y)} = 0. \tag{31}$$

The functions φ and ψ are classical solutions to the homogeneous ODE

$$\frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) = 0, \tag{32}$$

and satisfy

$$\varphi(x)\psi'(x) - \varphi'(x)\psi(x) = Cp'(x) \quad \text{for all } x \in \mathcal{I}, \tag{33}$$

where $C = \varphi(c)\psi'(c) - \varphi'(c)\psi(c)$ and p is the scale function defined by (22). Furthermore, given any solution $\mathbb{S}_x$ to (1),

$$\text{the processes } \left(e^{-\Lambda_t}\varphi(X_t)\right) \text{ and } \left(e^{-\Lambda_t}\psi(X_t)\right) \text{ are local martingales.} \tag{34}$$

The existence of these functions and their properties that we have listed can be found in several references, including Borodin and Salminen [8, Section II.1], Breiman [9, Chapter 16], and Itô and McKean [19, Chapter 4].

3 Preliminary considerations

Throughout this section, we assume that a weak solution $\mathbb{S}_x$ to (1) has been associated with each initial condition $x \in \operatorname{int}\mathcal{I}$. We first need to introduce some notation. To this end, we recall that, if $g : \operatorname{int}\mathcal{I} \to \mathbb{R}$ is a function that is the difference of two convex functions, then its left-hand side first derivative g'_- exists and is a function of finite variation, and its second distributional derivative g'' is a measure. We denote by

$$g''(dx) = g''_{\mathrm{ac}}(x)\,dx + g_{\mathrm{s}}(dx) \tag{35}$$

the Lebesgue decomposition of the second distributional derivative $g''(dx)$ into the measure $g''_{\rm ac}(x)\,dx$ that is absolutely continuous with respect to the Lebesgue measure and the measure $g''_{\rm s}(dx)$ that is mutually singular with the Lebesgue measure. Note that the function $g''_{\rm ac}$ identifies with the "classical" sense second derivative of g, which exists Lebesgue-a.e.. In view of these observations and notation, we define the measure $\mathcal{L}g$ on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ and the function $\mathcal{L}_{\rm ac}g : \operatorname{int}\mathcal{I} \to \mathbb{R}$ by

$$\mathcal{L}g(dx) = \frac{1}{2}\sigma^2(x)g''(dx) + b(x)g'_-(x)\,dx - r(x)g(x)\,dx \tag{36}$$

and

$$\mathcal{L}_{\rm ac}g(x) = \frac{1}{2}\sigma^2(x)g''_{\rm ac}(x) + b(x)g'_-(x) - r(x)g(x). \tag{37}$$

Given a Radon measure μ on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ such that σ^{-2} is locally integrable with respect to $|\mu|$, we consider the continuous additive functional A^μ defined by (4) in the introduction. Given any $t < T_\alpha \wedge T_\beta$, A^μ_t is well-defined and real-valued because $\alpha < \inf_{s\le t} X_s < \sup_{s\le t} X_s < \beta$ and the process L^z increases on the set $\{X_s = z\}$. Also, since L^z is an increasing process, A^μ (resp., $-A^\mu$) is an increasing process if μ (resp., $-\mu$) is a positive measure. The following result is concerned with various properties of the process A^μ that we will need.

Lemma 1 *Let μ be a Radon measure on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ such that σ^{-2} is locally integrable with respect to $|\mu|$, consider any increasing sequence of real-valued Borel-measurable functions (ζ_n) on $\mathcal{I}$ such that*

$$0 \le \zeta_n(z) \le 1 \quad \textit{and} \quad \lim_{n\to\infty} \zeta_n(z) = 1, \quad \mu\textit{-a.e.}, \tag{38}$$

and denote by μ_n the measure defined by

$$\mu_n(\Gamma) = \int_\Gamma \zeta_n(z)\,\mu(dz), \quad \textit{for } \Gamma \in \mathcal{B}(\operatorname{int}\mathcal{I}). \tag{39}$$

$A^{|\mu|}$ is a continuous increasing process,

$$A^\mu = -A^{-\mu} = A^{\mu^+} - A^{\mu^-}, \qquad A^{|\mu|} = A^{\mu^+} + A^{\mu^-}, \tag{40}$$

and

$$\lim_{n\to\infty} \mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta} e^{-\Lambda_t}\,dA^{|\mu_n|}_t\right] = \mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta} e^{-\Lambda_t}\,dA^{|\mu|}_t\right] \quad \textit{for all } x \in \operatorname{int}\mathcal{I}. \tag{41}$$

Proof. The process $A^{|\mu|}$ is continuous and increasing because this is true for the local time process L^z for all $z \in \mathcal{I}$. Also, (40) can be seen by a simple inspection of the definition (4) of A^μ. To prove (41), we have to show that, given any $x \in \operatorname{int}\mathcal{I}$,

$$\lim_{n\to\infty} \mathbb{E}_x\big[I^{(n)}_{T_\alpha\wedge T_\beta}\big] = \mathbb{E}_x\big[I_{T_\alpha\wedge T_\beta}\big], \tag{42}$$

where

$$I_t^{(n)} = \int_0^t e^{-\Lambda_u}\, dA_u^{|\mu_n|} \quad \text{and} \quad I_t = \int_0^t e^{-\Lambda_u}\, dA_u^{|\mu|}, \quad \text{for } t \in [0, T_\alpha \wedge T_\beta].$$

To this end, we note that (38) and the monotone convergence theorem imply that the sequence $(A_t^{|\mu_n|})$ increases to $A_t^{|\mu|}$ for all $t < T_\alpha \wedge T_\beta$ as $n \to \infty$, because

$$A_t^{|\mu_n|} = \int_\alpha^\beta \frac{L_t^z}{\sigma^2(z)}\, |\mu_n|(dz) = \int_\alpha^\beta \frac{L_t^z}{\sigma^2(z)} \zeta_n(z)\, |\mu|(dz), \quad \text{for } t \in [0, T_\alpha \wedge T_\beta[.$$

Also, we use the integration by parts formula to calculate

$$\int_0^t e^{-\Lambda_u}\, dA_u^{|\mu_n|} = e^{-\Lambda_t} A_t^{|\mu_n|} + \int_0^t e^{-\Lambda_u} r(X_u) A_u^{|\mu_n|}\, du, \quad \text{for } t \in [0, T_\alpha \wedge T_\beta[. \tag{43}$$

In view of these observations and the monotone convergence theorem, we can see that

$$0 \leq I_t^{(n)} \leq I_t^{(n+1)} \quad \text{for all } t \in [0, T_\alpha \wedge T_\beta] \text{ and } n \geq 1, \tag{44}$$

and

$$\lim_{n\to\infty} I_t^{(n)} = I_t \quad \text{for all } t \in [0, T_\alpha \wedge T_\beta[, \tag{45}$$

Combining these results with the fact that the positive processes $I^{(n)}$ are increasing, we can see that

$$I_{T_\alpha \wedge T_\beta} = \lim_{t \to T_\alpha \wedge T_\beta} I_t \geq \lim_{t \to T_\alpha \wedge T_\beta} I_t^{(n)} = I_{T_\alpha \wedge T_\beta}^{(n)} \quad \text{for all } n \geq 1$$

and

$$I_{T_\alpha \wedge T_\beta} = \lim_{t \to T_\alpha \wedge T_\beta} I_t = \lim_{t \to T_\alpha \wedge T_\beta} \lim_{n\to\infty} I_t^{(n)} \leq \lim_{n\to\infty} I_{T_\alpha \wedge T_\beta}^{(n)}.$$

It follows that $\lim_{n\to\infty} I_{T_\alpha \wedge T_\beta}^{(n)} = I_{T_\alpha \wedge T_\beta}$, which, combined with monotone convergence theorem, implies (42) and the proof is complete. □

We will need the results derived in the following lemma, the proof of which is based on the Itô-Tanaka-Meyer formula.

Lemma 2 *If $F : \operatorname{int} \mathcal{I} \to \mathbb{R}$ is a function that is the difference of two convex functions, then the following statements are true:*

(I) *The increasing process $A^{|\mathcal{L}F|}$ is real-valued, and*

$$e^{-\Lambda_t}F(X_t) = F(x) + \int_0^t e^{-\Lambda_u}\, dA_u^{\mathcal{L}F} + \int_0^t e^{-\Lambda_u}\sigma(X_u)F'_-(X_u)\, dW_u, \quad \text{for } t \in [0, T_\alpha \wedge T_\beta]. \tag{46}$$

(II) *If F is C^1 with absolutely continuous with respect to the Lebesgue measure first derivative, i.e., if $\mathcal{L}F(dx) = \mathcal{L}_{\mathrm{ac}}F(x)\, dx$ in the notation of (36)–(37), then*

$$\int_0^t e^{-\Lambda_u}\, dA_u^{\mathcal{L}F} = \int_0^t e^{-\Lambda_u}\mathcal{L}_{\mathrm{ac}}F(X_u)\, du, \quad \text{for } t \in [0, T_\alpha \wedge T_\beta]. \tag{47}$$

Proof. In view of the Lebesgue decomposition of the second distributional derivative $F''(dx)$ of F as in (35) and the occupation times formula

$$\int_\alpha^\beta L_t^z F''_{\mathrm{ac}}(z)\, dz = \int_0^t \sigma^2(X_u)F''_{\mathrm{ac}}(X_u)\, du,$$

we can see that the Itô-Tanaka-Meyer formula

$$F(X_t) = F(x) + \int_0^t b(X_u)F'_-(X_u)\, du + \frac{1}{2}\int_\alpha^\beta L_t^z\, F''(dz) + \int_0^t \sigma(X_u)F'_-(X_u)\, dW_u$$

implies that

$$\begin{aligned} F(X_t) = F(x) &+ \int_0^t \left[\frac{1}{2}\sigma^2(X_u)F''_{\mathrm{ac}}(X_u) + b(X_u)F'_-(X_u)\right] du + \frac{1}{2}\int_\alpha^\beta L_t^z\, F''_{\mathrm{s}}(dz) \\ &+ \int_0^t \sigma(X_u)F'_-(X_u)\, dW_u. \end{aligned} \tag{48}$$

Combining this expression with the definition (37) of $\mathcal{L}_{\mathrm{ac}}$, we can see that

$$\begin{aligned} F(X_t) = F(x) &+ \int_0^t r(X_u)F(X_u)\, du + \int_0^t \mathcal{L}_{\mathrm{ac}}F(X_u)\, du + \frac{1}{2}\int_\alpha^\beta L_t^z\, F''_{\mathrm{s}}(dz) \\ &+ \int_0^t \sigma(X_u)F'_-(X_u)\, dW_u. \end{aligned} \tag{49}$$

Using the occupation times formula once again and the definitions (36), (37) of $\mathcal{L}$, $\mathcal{L}_{\mathrm{ac}}$, we can see that

$$\begin{aligned} \int_0^t \mathcal{L}_{\mathrm{ac}}F(X_u)\, du + \frac{1}{2}\int_\alpha^\beta L_t^z\, F''_{\mathrm{s}}(dz) &= \int_\alpha^\beta \frac{L_t^z}{\sigma^2(z)}\mathcal{L}_{\mathrm{ac}}F(z)\, dz + \int_\alpha^\beta \frac{L_t^z}{\sigma^2(z)}\frac{1}{2}\sigma^2(z)\, F''_{\mathrm{s}}(dz) \\ &= \int_\alpha^\beta \frac{L_t^z}{\sigma^2(z)}\, \mathcal{L}F(dz) \\ &= A_t^{\mathcal{L}F}. \end{aligned} \tag{50}$$

The validity of Itô-Tanaka-Meyer's and the occupation times formulae and (49)–(50) imply that the process $A^{\mathcal{L}F}$ is well-defined and real-valued. Also, (46) follows from the definition (5) of the process Λ, (49)–(50) and an application of the integration by parts formula.

If $\mathcal{L}F(dx) = \mathcal{L}_{\mathrm{ac}}F(x)\,dx$, the definition of $A^{\mathcal{L}F}$ and the occupation times formula imply that

$$A_t^{\mathcal{L}F} = \int_0^t \mathcal{L}_{\mathrm{ac}}F(X_u)\,du,$$

and (47) follows. □

The next result is concerned with a form of Dynkin's formula that the functions φ, ψ satisfy as well as with a pair of expressions that become useful when explicit solutions to special cases of the general optimal stopping problem are explored (see Section 7).

Lemma 3 *The functions φ, ψ introduced by (26), (27) satisfy*

$$\varphi(x) = \mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}}\varphi(X_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}})\right] \quad \textit{and} \quad \psi(x) = \mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}}\psi(X_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}})\right] \tag{51}$$

for all stopping times τ and all points $\bar\alpha < x < \bar\beta$ in $\mathcal{I}$. Furthermore,

$$\mathbb{E}_x\left[e^{-\Lambda_{T_{\bar\alpha}}}\mathbf{1}_{\{T_{\bar\alpha}<T_{\bar\beta}\}}\right] = \frac{\varphi(\bar\beta)\psi(x) - \varphi(x)\psi(\bar\beta)}{\varphi(\bar\beta)\psi(\bar\alpha) - \varphi(\bar\alpha)\psi(\bar\beta)} \tag{52}$$

and

$$\mathbb{E}_x\left[e^{-\Lambda_{T_{\bar\beta}}}\mathbf{1}_{\{T_{\bar\beta}<T_{\bar\alpha}\}}\right] = \frac{\varphi(x)\psi(\bar\alpha) - \varphi(\bar\alpha)\psi(x)}{\varphi(\bar\beta)\psi(\bar\alpha) - \varphi(\bar\alpha)\psi(\bar\beta)}. \tag{53}$$

Proof. Combining (46) with the fact that $\mathcal{L}\varphi = 0$, we can see that

$$e^{-\Lambda_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}}\varphi(X_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}) = \varphi(x) + M_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}, \tag{54}$$

where

$$M_t = \int_0^t e^{-\Lambda_u}\sigma(X_u)\varphi'(X_u)\,dW_u.$$

In view of (28) and the fact that the positive function φ is decreasing, we can see that $\sup_{y\in[\bar\alpha,\bar\beta]}\varphi(y) < \infty$. Therefore, $M^{T_{\bar\alpha}\wedge T_{\bar\beta}}$ is a uniformly integrable martingale because it is a uniformly bounded local martingale. It follows that $\mathbb{E}_x\big[M_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}\big] = 0$ and (54) implies the first identity in (51). The second identity in (51) can be established using similar arguments.

Finally, (52) and (53) follow immediately once we observe that they are equivalent to the system of equations

$$\varphi(x) = \varphi(\bar\alpha)\mathbb{E}_x\left[e^{-\Lambda_{T_{\bar\alpha}}}\mathbf{1}_{\{T_{\bar\alpha}<T_{\bar\beta}\}}\right] + \varphi(\bar\beta)\mathbb{E}_x\left[e^{-\Lambda_{T_{\bar\beta}}}\mathbf{1}_{\{T_{\bar\beta}<T_{\bar\alpha}\}}\right]$$

and

$$\psi(x) = \psi(\bar{\alpha})\mathbb{E}_x\left[e^{-\Lambda_{T_{\bar{\alpha}}}}\mathbf{1}_{\{T_{\bar{\alpha}}<T_{\bar{\beta}}\}}\right] + \psi(\bar{\beta})\mathbb{E}_x\left[e^{-\Lambda_{T_{\bar{\beta}}}}\mathbf{1}_{\{T_{\bar{\beta}}<T_{\bar{\alpha}}\}}\right],$$

which holds true thanks to (51) for $\tau \equiv \infty$. □

We conclude this section with a necessary and sufficient condition for the value function of our optimal stopping problem to be finite.

Lemma 4 *Consider the optimal stopping problem formulated in Section 2, and let $\overline{f}$ be defined by (10) in the introduction. If*

$$\overline{f} : \mathcal{I} \to \mathbb{R}_+ \text{ is real-valued}, \quad \limsup_{y\downarrow\alpha} \frac{f(y)}{\varphi(y)} < \infty \quad \text{and} \quad \limsup_{y\uparrow\beta} \frac{f(y)}{\psi(y)} < \infty, \tag{55}$$

then $v(x) < \infty$ for all $x \in \mathcal{I}$,

$$\limsup_{y\downarrow\alpha} \frac{v(y)}{\varphi(y)} = \limsup_{y\downarrow\alpha} \frac{f(y)}{\varphi(y)} \quad \text{and} \quad \limsup_{y\uparrow\beta} \frac{v(y)}{\psi(y)} = \limsup_{y\uparrow\beta} \frac{f(y)}{\psi(y)}. \tag{56}$$

If any of the conditions in (55) is not true, then $v(x) = \infty$ for all $x \in \operatorname{int}\mathcal{I}$.

Proof. If (55) is true, then we can see that

$$\sup_{u\le y} \frac{f(u)}{\varphi(u)} < \infty \quad \text{and} \quad \sup_{u\ge y} \frac{f(u)}{\psi(u)} < \infty \quad \text{for all } y \in \mathcal{I}.$$

Also,

$$f(x) \le \sup_{u\le y} \frac{f(u)}{\varphi(u)}\varphi(x) + \sup_{u\ge y} \frac{f(u)}{\psi(u)}\psi(x) \quad \text{for all } x, y \in \mathcal{I}.$$

In view of (34), the processes $\left(e^{-\Lambda_t}\varphi(X_t)\right)$ and $\left(e^{-\Lambda_t}\psi(X_t)\right)$ are positive supermartingales. It follows that, given any stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{T}_x$,

$$\begin{aligned} J(\mathbb{S}_x, \tau) \le & \sup_{u\le y} \frac{f(u)}{\varphi(u)}\mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_\alpha\wedge T_\beta}}\varphi(X_{\tau\wedge T_\alpha\wedge T_\beta})\mathbf{1}_{\{\tau<\infty\}}\right] \\ & + \sup_{u\ge y} \frac{f(u)}{\psi(u)}\mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_\alpha\wedge T_\beta}}\psi(X_{\tau\wedge T_\alpha\wedge T_\beta})\mathbf{1}_{\{\tau<\infty\}}\right] \\ \le & \sup_{u\le y} \frac{f(u)}{\varphi(u)}\varphi(x) + \sup_{u\ge y} \frac{f(u)}{\psi(u)}\psi(x), \end{aligned} \tag{57}$$

which implies that $v(x) < \infty$.

To show the first identity in (56), we note that (57) implies that

$$\frac{v(x)}{\varphi(x)} \leq \sup_{u \leq y} \frac{f(u)}{\varphi(u)} + \sup_{u \geq y} \frac{f(u)}{\psi(u)} \frac{\psi(x)}{\varphi(x)}.$$

Combining this calculation with (31), we obtain

$$\limsup_{x \downarrow \alpha} \frac{v(x)}{\varphi(x)} \leq \sup_{u \leq y} \frac{f(u)}{\varphi(u)},$$

which implies that $\limsup_{y \downarrow \alpha} v(y)/\varphi(y) \leq \limsup_{y \downarrow \alpha} f(y)/\varphi(y)$. The reverse inequality follows immediately from the fact that $v \geq f$. The second identity in (56) can be established using similar arguments.

If the problem data is such that the first limit in (55) is infinite, then we consider any initial condition $x \in \operatorname{int} \mathcal{I}$ and any sequence (y_n) in $\mathcal{I}$ such that $y_n < x$ for all $n \geq 1$ and $\lim_{n \to \infty} f(y_n)/\varphi(y_n) = \infty$. We can then see that

$$v(x) \geq \lim_{n \to \infty} J(\mathbb{S}_x, T_{y_n}) \geq \lim_{n \to \infty} f(y_n) \mathbb{E}_x \left[e^{-\Lambda_{T_{y_n}}} \right] \overset{(26)}{=} \lim_{n \to \infty} \frac{f(y_n)\varphi(x)}{\varphi(y_n)} = \infty,$$

where $\mathbb{S}_x$ is any solution to (1). Similarly, we can see that $v(x) = \infty$ for all $x \in \operatorname{int} \mathcal{I}$ if the second limit in (55) is infinite or if there exists a point $y \in \operatorname{int} \mathcal{I}$ such that $\overline{f}(y) = \infty$. □

4 $r(\cdot)$-potentials and differences of two convex functions

Throughout this section, we assume that a weak solution $\mathbb{S}_x$ to (1) has been associated with each initial condition $x \in \operatorname{int} \mathcal{I}$. Accordingly, whenever we consider a stopping time τ, we refer to a stopping time of the filtration in the solution $\mathbb{S}_x$.

We first characterise the limiting behaviour at the boundary of $\mathcal{I}$ of a difference of two convex functions on $\operatorname{int} \mathcal{I}$, and we show that such a function satisfies Dynkin's formula under appropriate assumptions.

Lemma 5 *Consider any function $F : \operatorname{int} \mathcal{I} \to \mathbb{R}$ that is a difference of two convex functions and is such that*

$$\limsup_{y \downarrow \alpha} \frac{|F(y)|}{\varphi(y)} < \infty \quad \text{and} \quad \limsup_{y \uparrow \beta} \frac{|F(y)|}{\psi(y)} < \infty. \tag{58}$$

(I) *If $-\mathcal{L}F$ is a positive measure, then*

$$\mathbb{E}_x \left[\int_0^{T_\alpha \wedge T_\beta} e^{-\Lambda_t} \, dA_t^{|\mathcal{L}F|} \right] < \infty \quad \text{for all } x \in \operatorname{int} \mathcal{I}. \tag{59}$$

(II) *If F satisfies*

$$\mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta} e^{-\Lambda_t}\, dA_t^{|\mathcal{L}F|}\right] < \infty, \quad \textit{for some } x \in \operatorname{int}\mathcal{I}, \tag{60}$$

then both of the limits $\lim_{y\downarrow\alpha} F(y)/\varphi(y)$ and $\lim_{y\uparrow\beta} F(y)/\psi(y)$ exist.
(III) *Suppose that F satisfies (60),*

$$\lim_{y\downarrow\alpha}\frac{F(y)}{\varphi(y)} = 0 \quad \textit{and} \quad \lim_{y\uparrow\beta}\frac{F(y)}{\psi(y)} = 0. \tag{61}$$

If $x \in \operatorname{int}\mathcal{I}$ is an initial condition such that (60) is true, then

$$\begin{aligned}\mathbb{E}_x\left[e^{-\Lambda_\tau}F(X_\tau)\mathbf{1}_{\{\tau<T_\alpha\wedge T_\beta\}}\right] &= F(x) + \mathbb{E}_x\left[\int_0^{\tau\wedge T_\alpha\wedge T_\beta} e^{-\Lambda_t}\, dA_t^{\mathcal{L}F}\right] \\ &= \mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_\alpha\wedge T_\beta}}F(X_{\tau\wedge T_\alpha\wedge T_\beta})\mathbf{1}_{\{\tau\wedge T_\alpha\wedge T_\beta<\infty\}}\right]\end{aligned} \tag{62}$$

for every stopping time τ; in the last identity here, we assume that

$$F(\alpha) = \lim_{y\downarrow\alpha} F(y) = 0 \quad \left(\textit{resp., } F(\beta) = \lim_{y\uparrow\beta} F(y) = 0\right)$$

if α (resp., β) is absorbing, namely, if $\mathbb{P}_x(T_\alpha < \infty) > 0$ (resp., $\mathbb{P}_x(T_\beta < \infty) > 0$), consistently with (61).

Proof. Throughout the proof, τ denotes any stopping time. Recalling (46) in Lemma 2, we write

$$e^{-\Lambda_t}F(X_t) = F(x) + \int_0^t e^{-\Lambda_u}\, dA_u^{\mathcal{L}F} + M_t, \tag{63}$$

where M is the stochastic integral defined by

$$M_t = \int_0^t e^{-\Lambda_u}\sigma(X_u)F'_-(X_u)\, dW_u.$$

We consider any decreasing sequence (α_n) and any increasing sequence (β_n) such that

$$\alpha < \alpha_n < x < \beta_n < \beta \text{ for all } n \geq 1, \quad \lim_{n\to\infty}\alpha_n = \alpha \quad \text{and} \quad \lim_{n\to\infty}\beta_n = \beta. \tag{64}$$

Also, we define

$$\tau_\ell(\bar\alpha,\bar\beta) = \inf\left\{t \geq 0 \,\middle|\, \int_0^{t\wedge T_{\bar\alpha}\wedge T_{\bar\beta}} \sigma^2(X_u)\, du \geq \ell\right\} \wedge T_{\bar\alpha} \wedge T_{\bar\beta}, \tag{65}$$

where we adopt the usual convention that $\inf\emptyset = \infty$, and we note that the definition and the construction of a weak solution to (1) (see Definition 5.5.20 in Karatzas and Shreve [21]) imply that these stopping times satisfy

$$\tau_\ell(\bar\alpha,\bar\beta) > 0 \text{ for all } \ell \geq 1 \quad \text{and} \quad \lim_{\ell\to\infty} \tau_\ell(\bar\alpha,\bar\beta) = T_{\bar\alpha} \wedge T_{\bar\beta}. \tag{66}$$

The function F'_- is locally bounded because it is of finite variation. Therefore, we can use Itô's isometry to calculate

$$\begin{aligned}
\mathbb{E}_x\left[M^2_{\tau\wedge\tau_\ell(\alpha_m,\beta_n)}\right] &= \mathbb{E}_x\left[\int_0^{\tau\wedge\tau_\ell(\alpha_m,\beta_n)} \left[e^{-\Lambda_u}\sigma(X_u)F'_-(X_u)\right]^2 du\right] \\
&\leq \sup_{y\in[\alpha_n,\beta_n]}\left[F'_-(y)\right]^2 \mathbb{E}_x\left[\int_0^{\tau_\ell(\alpha_m,\beta_n)} \sigma^2(X_u)\,du\right] \\
&\leq \ell \sup_{y\in[\alpha_n,\beta_n]}\left[F'_-(y)\right]^2 \\
&< \infty,
\end{aligned} \tag{67}$$

which implies that the stopped process $M^{\tau\wedge\tau_\ell(\alpha_m,\beta_n)}$ is a uniformly integrable martingale. Combining this observation with (63), we can see that

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge\tau_\ell(\alpha_m,\beta_n)}}F(X_{\tau\wedge\tau_\ell(\alpha_m,\beta_n)})\right] = F(x) + \mathbb{E}_x\left[\int_0^{\tau\wedge\tau_\ell(\alpha_m,\beta_n)} e^{-\Lambda_u}\,dA^{\mathcal{L}F}_u\right].$$

In view of (66) and the local boundedness of F, we can pass to the limit using the dominated convergence theorem to obtain

$$\begin{aligned}
&F(x) + \lim_{\ell\to\infty}\mathbb{E}_x\left[\int_0^{\tau\wedge\tau_\ell(\alpha_m,\beta_n)} e^{-\Lambda_u}\,dA^{\mathcal{L}F}_u\right] \\
&\quad= \mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_{\alpha_m}\wedge T_{\beta_n}}}F(X_{\tau\wedge T_{\alpha_m}\wedge T_{\beta_n}})\right] \\
&\quad= \mathbb{E}_x\left[e^{-\Lambda_\tau}F(X_\tau)\mathbf{1}_{\{\tau\leq T_{\alpha_m}\wedge T_{\beta_n}\}}\right] + F(\alpha_m)\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha m}}}\mathbf{1}_{\{T_{\alpha_m}<\tau\wedge T_{\beta_n}\}}\right] \\
&\qquad + F(\beta_n)\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta_n}}}\mathbf{1}_{\{T_{\beta_n}<\tau\wedge T_{\alpha_m}\}}\right] \\
&\quad= \mathbb{E}_x\left[e^{-\Lambda_\tau}F(X_\tau)\mathbf{1}_{\{\tau\leq T_{\alpha_m}\wedge T_{\beta_n}\}}\right] + \varphi(x)\frac{F(\alpha_m)}{\varphi(\alpha_m)}\frac{\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha m}}}\mathbf{1}_{\{T_{\alpha_m}<\tau\wedge T_{\beta_n}\}}\right]}{\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha m}}}\right]} \\
&\qquad + \psi(x)\frac{F(\beta_n)}{\psi(\beta_n)}\frac{\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta_n}}}\mathbf{1}_{\{T_{\beta_n}<\tau\wedge T_{\alpha_m}\}}\right]}{\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta_n}}}\right]},
\end{aligned} \tag{68}$$

the last identity following thanks to (26)–(27).

Proof of (I). If $-\mathcal{L}F$ is a positive measure, then $-A^{\mathcal{L}F} = A^{-\mathcal{L}F} = A^{|\mathcal{L}F|}$ is an increasing process. Therefore, we can use (66), (64) and the monotone convergence theorem to calculate

$$\begin{aligned}\lim_{m,n\to\infty}\lim_{\ell\to\infty}\mathbb{E}_x\left[\int_0^{\tau_\ell(\alpha_m,\beta_n)} e^{-\Lambda_u}\,dA_u^{\mathcal{L}F}\right] &= \lim_{m,n\to\infty}\mathbb{E}_x\left[\int_0^{T_{\alpha m}\wedge T_{\beta n}} e^{-\Lambda_u}\,dA_u^{\mathcal{L}F}\right]\\ &= \mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta} e^{-\Lambda_u}\,dA_u^{\mathcal{L}F}\right].\end{aligned}$$

Combining this with assumption (58), the inequalities

$$0 < \frac{\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha m}}}\mathbf{1}_{\{T_{\alpha m}<T_{\beta n}\}}\right]}{\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha m}}}\right]} \le 1 \quad\text{and}\quad 0 < \frac{\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta n}}}\mathbf{1}_{\{T_{\beta n}<T_{\alpha m}\}}\right]}{\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta n}}}\right]} \le 1, \tag{69}$$

and (68) for $\tau=\infty$, we can see that

$$\begin{aligned}0 &\le \mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta} e^{-\Lambda_t}\,dA_t^{|\mathcal{L}F|}\right]\\ &= -\mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta} e^{-\Lambda_u}\,dA_u^{\mathcal{L}F}\right]\\ &= \lim_{m,n\to\infty}\left(F(x)-\varphi(x)\frac{F(\alpha_m)}{\varphi(\alpha_m)}\frac{\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha m}}}\mathbf{1}_{\{T_{\alpha m}<T_{\beta n}\}}\right]}{\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha m}}}\right]}-\psi(x)\frac{F(\beta_n)}{\psi(\beta_n)}\frac{\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta n}}}\mathbf{1}_{\{T_{\beta n}<T_{\alpha m}\}}\right]}{\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta n}}}\right]}\right)\\ &\le |F(x)|+\varphi(x)\limsup_{m\to\infty}\frac{|F(\alpha_m)|}{\varphi(\alpha_m)}+\psi(x)\limsup_{n\to\infty}\frac{|F(\beta_n)|}{\psi(\beta_n)}\\ &< \infty.\end{aligned} \tag{70}$$

Proof of (II). We now fix any initial condition $x\in\operatorname{int}\mathcal{I}$ such that (60) is true and we assume that the sequence (α_m) has been chosen so that

$$\lim_{m\to\infty}\frac{F(\alpha_m)}{\varphi(\alpha_m)} \text{ exists.} \tag{71}$$

In light of (40) in Lemma 1 and (66), we can see that the dominated convergence theorem implies that

$$\lim_{m,n\to\infty}\lim_{\ell\to\infty}\mathbb{E}_x\left[\int_0^{\tau\wedge\tau_\ell(\alpha_m,\beta_n)} e^{-\Lambda_u}\,dA_u^{\mathcal{L}F}\right] = \mathbb{E}_x\left[\int_0^{\tau\wedge T_\alpha\wedge T_\beta} e^{-\Lambda_u}\,dA_u^{\mathcal{L}F}\right]. \tag{72}$$

The continuity of F and (58) imply that there exists a constant $C_1>0$ such that

$$|F(y)| \le C_1\left[\varphi(y)+\psi(y)\right].$$

Also, (34) implies that the processes $\left(e^{-\Lambda_t}\varphi(X_t)\right)$ and $\left(e^{-\Lambda_t}\psi(X_t)\right)$ are positive supermartingales, therefore,

$$\mathbb{E}_x\left[e^{-\Lambda_\tau}\left[\varphi(X_\tau)+\psi(X_\tau)\right]\mathbf{1}_{\{\tau<\infty\}}\right] \leq C_1\left[\varphi(x)+\psi(x)\right] < \infty.$$

Since

$$e^{-\Lambda_\tau}\left|F(X_\tau)\right|\mathbf{1}_{\{\tau\leq T_{\alpha_m}\wedge T_{\beta_n}\}} \leq C_1 e^{-\Lambda_\tau}\left[\varphi(X_\tau)+\psi(X_\tau)\right]\mathbf{1}_{\{\tau<\infty\}} \quad \text{for all } m,n\geq 1,$$

we can see that the dominated convergence theorem implies that

$$\lim_{m\to\infty}\mathbb{E}_x\big[e^{-\Lambda_\tau}F(X_\tau)\mathbf{1}_{\{\tau\leq T_{\alpha_m}\wedge T_{\beta_n}\}}\big] = \mathbb{E}_x\big[e^{-\Lambda_\tau}F(X_\tau)\mathbf{1}_{\{\tau<T_\alpha\}\cap\{\tau\leq T_{\beta_n}\}}\big]$$

and

$$\lim_{m,n\to\infty}\mathbb{E}_x\big[e^{-\Lambda_\tau}F(X_\tau)\mathbf{1}_{\{\tau\leq T_{\alpha_m}\wedge T_{\beta_n}\}}\big] = \mathbb{E}_x\big[e^{-\Lambda_\tau}F(X_\tau)\mathbf{1}_{\{\tau<T_\alpha\wedge T_\beta\}}\big]. \tag{73}$$

In view of these results, we can pass to the limit $m\to\infty$ in (68) to obtain

$$\begin{aligned}
&F(x)+\mathbb{E}_x\left[\int_0^{\tau\wedge T_\alpha\wedge T_{\beta_n}} e^{-\Lambda_u}\,dA_u^{\mathcal{L}F}\right]\\
&\quad=\mathbb{E}_x\left[e^{-\Lambda_\tau}F(X_\tau)\mathbf{1}_{\{\tau<T_\alpha\}\cap\{\tau\leq T_{\beta_n}\}}\right]+\lim_{m\to\infty}\varphi(x)\frac{F(\alpha_m)}{\varphi(\alpha_m)}\frac{\mathbb{E}_x\big[e^{-\Lambda_{T_{\alpha_m}}}\mathbf{1}_{\{T_{\alpha_m}<\tau\wedge T_{\beta_n}\}}\big]}{\mathbb{E}_x\big[e^{-\Lambda_{T_{\alpha_m}}}\big]}\\
&\qquad+\lim_{m\to\infty}\psi(x)\frac{F(\beta_n)}{\psi(\beta_n)}\frac{\mathbb{E}_x\big[e^{-\Lambda_{T_{\beta_n}}}\mathbf{1}_{\{T_{\beta_n}<\tau\wedge T_{\alpha_m}\}}\big]}{\mathbb{E}_x\big[e^{-\Lambda_{T_{\beta_n}}}\big]}\\
&\quad=\mathbb{E}_x\left[e^{-\Lambda_\tau}F(X_\tau)\mathbf{1}_{\{\tau<T_\alpha\}\cap\{\tau\leq T_{\beta_n}\}}\right]+\varphi(x)\frac{\mathbb{E}_x\big[e^{-\Lambda_{T_\alpha}}\mathbf{1}_{\{T_\alpha\leq\tau\wedge T_{\beta_n}\}}\big]}{\mathbb{E}_x\big[e^{-\Lambda_{T_\alpha}}\big]}\lim_{m\to\infty}\frac{F(\alpha_m)}{\varphi(\alpha_m)}\\
&\qquad+\psi(x)\frac{F(\beta_n)}{\psi(\beta_n)}\frac{\mathbb{E}_x\big[e^{-\Lambda_{T_{\beta_n}}}\mathbf{1}_{\{T_{\beta_n}<\tau\wedge T_\alpha\}}\big]}{\mathbb{E}_x\big[e^{-\Lambda_{T_{\beta_n}}}\big]},
\end{aligned}\tag{74}$$

the second equality following by an application of the dominated convergence theorem. These identities prove that the limit $\lim_{y\downarrow\alpha}F(y)/\varphi(y)$ exists because (α_m) has been an arbitrary sequence satisfying (71) and the function F/φ is continuous.

Proving that the limit $\lim_{y\uparrow\beta}F(y)/\psi(y)$ exists follows similar symmetric arguments.

Proof of (III). The event $\{T_\alpha<\infty\}$ has strictly positive probability if and only if α is an absorbing boundary point, in which case, (28) and (61) imply that $\lim_{y\downarrow\alpha}F(y)=0$. In view of this observation and a similar one concerning the boundary point β, we can see that the first identity in (62) holds true. Finally, the second identity in (62) follows immediately once we combine (61) with (69) and (72)–(74). □

The assumptions of the previous lemma involve the measure $\mathcal{L}F$ that we can associate with a function on $\operatorname{int}\mathcal{I}$ that is the difference of two convex functions. We now address the following inverse problem: given a signed measure μ on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$, determine a function F on $\operatorname{int}\mathcal{I}$ such that F is the difference of two convex functions and $\mathcal{L}F = -\mu$. Plainly, the solution to this problem is not unique because $\mathcal{L}\varphi = \mathcal{L}\psi = 0$. In view of this observation, the solution R_μ that we now derive and identifies with the $r(\cdot)$-potential of the continuous additive functional A^μ is "minimal" in the sense that it has the limiting behaviour captured by (80).

Theorem 6 *A signed Radon measure μ on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ satisfies*

$$\int_{]\alpha,x[} \frac{\psi(s)}{\sigma^2(s)p'(s)}\,|\mu|(ds) + \int_{[x,\beta[} \frac{\varphi(s)}{\sigma^2(s)p'(s)}\,|\mu|(ds) < \infty \tag{75}$$

for all $x \in \mathcal{I}$, if and only if

$$\int_{\bar\alpha}^{\bar\beta} \frac{1}{\sigma^2(s)}\,|\mu|(ds) < \infty \quad \textit{and} \quad \mathbb{E}_x\left[\int_0^{T_\alpha \wedge T_\beta} e^{-\Lambda_t}\,dA_t^{|\mu|}\right] < \infty \tag{76}$$

for all $\alpha < \bar\alpha < \bar\beta < \beta$ and all $x \in \mathcal{I}$. In the presence of these integrability conditions, the function $R_\mu : \operatorname{int}\mathcal{I} \to \mathbb{R}$ defined by

$$R_\mu(x) = \frac{2}{C}\varphi(x)\int_{]\alpha,x[} \frac{\psi(s)}{\sigma^2(s)p'(s)}\,\mu(ds) + \frac{2}{C}\psi(x)\int_{[x,\beta[} \frac{\varphi(s)}{\sigma^2(s)p'(s)}\,\mu(ds), \tag{77}$$

where $C > 0$ is the constant appearing in (33), identifies with the $r(\cdot)$-potential of A^μ, namely,

$$R_\mu(x) = \mathbb{E}_x\left[\int_0^{T_\alpha \wedge T_\beta} e^{-\Lambda_t}\,dA_t^{\mu}\right], \tag{78}$$

it is the difference of two convex functions, and

$$\mathcal{L}R_\mu(dx) = -\mu(dx) \quad \textit{and} \quad \mathcal{L}R_{|\mu|}(dx) = -|\mu|(dx). \tag{79}$$

Furthermore,

$$\lim_{y\downarrow\alpha} \frac{|R_\mu(y)|}{\varphi(y)} = \lim_{y\uparrow\beta} \frac{|R_\mu(y)|}{\psi(y)} = \lim_{y\downarrow\alpha} \frac{R_{|\mu|}(y)}{\varphi(y)} = \lim_{y\uparrow\beta} \frac{R_{|\mu|}(y)}{\psi(y)} = 0. \tag{80}$$

Proof. First, we note that, if the integrability condition (75) is true for some $x \in \mathcal{I}$, then it is true for all $x \in \mathcal{I}$. If μ is a measure on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ satisfying (75), then the function R_μ given by (77) is well-defined, it is the difference of two convex functions, and it satisfies the

corresponding identity in (79). To see these claims, we consider the left-continuous function $H : \operatorname{int}\mathcal{I} \to \mathbb{R}$ given by $H(\gamma) = 0$ and

$$H(x) = \begin{cases} -\int_{]x,\gamma[} \frac{2}{C\sigma^2(s)p'(s)}\,\mu(ds), & \text{if } x \in]\alpha,\gamma[, \\ \int_{[\gamma,x[} \frac{2}{C\sigma^2(s)p'(s)}\,\mu(ds), & \text{if } x \in]\gamma,\beta[, \end{cases}$$

where γ is any constant in $\operatorname{int}\mathcal{I}$. Given any points $\bar\alpha, \bar\beta \in \operatorname{int}\mathcal{I}$ such that $\bar\alpha < \gamma < \bar\beta$, we can use the integration by parts formula to see that

$$-H(\bar\alpha)\psi(\bar\alpha) - \int_{\bar\alpha}^{x} \psi'(s)H(s)\,ds = -H(x)\psi(x) + \int_{[\bar\alpha,x[} \frac{2\psi(s)}{C\sigma^2(s)p'(s)}\,\mu(ds),$$

$$H(\bar\beta)\varphi(\bar\beta) - \int_{x}^{\bar\beta} \varphi'(s)H(s)\,ds = H(x)\varphi(x) + \int_{[x,\bar\beta[} \frac{2\varphi(s)}{C\sigma^2(s)p'(s)}\,\mu(ds)$$

for all $x \in [\bar\alpha, \bar\beta]$. It follows that the function R_μ defined by (77) admits the expression

$$\begin{aligned} R_\mu(x) = {} & \left[\frac{2}{C}\int_{]\alpha,\bar\alpha[} \frac{\psi(s)}{\sigma^2(s)p'(s)}\,\mu(ds) - H(\bar\alpha)\psi(\bar\alpha)\right]\varphi(x) \\ & + \left[\frac{2}{C}\int_{[\bar\beta,\beta[} \frac{\varphi(s)}{\sigma^2(s)p'(s)}\,\mu(ds) + H(\bar\beta)\varphi(\bar\beta)\right]\psi(x) \\ & - \varphi(x)\int_{\bar\alpha}^{x} \psi'(s)H(s)\,ds - \psi(x)\int_{x}^{\bar\beta} \varphi'(s)H(s)\,ds \end{aligned} \tag{81}$$

for all $\alpha < \bar\alpha \le x \le \bar\beta < \beta$. This result, the left-continuity of H and (33) imply that

$$\begin{aligned} (R_\mu)'_-(x) = {} & \left[\frac{2}{C}\int_{]\alpha,\bar\alpha[} \frac{\psi(s)}{\sigma^2(s)p'(s)}\,\mu(ds) - H(\bar\alpha)\psi(\bar\alpha)\right]\varphi'(x) \\ & + \left[\frac{2}{C}\int_{[\bar\beta,\beta[} \frac{\varphi(s)}{\sigma^2(s)p'(s)}\,\mu(ds) + H(\bar\beta)\varphi(\bar\beta)\right]\psi'(x) - Cp'(x)H(x) \\ & - \varphi'(x)\int_{\bar\alpha}^{x} \psi'(s)H(s)\,ds - \psi'(x)\int_{x}^{\bar\beta} \varphi'(s)H(s)\,ds \end{aligned} \tag{82}$$

for all $\alpha < \bar\alpha \le x \le \bar\beta < \beta$. Furthermore, we can see that the restriction of the measure $(R_\mu)''$ in $\big(]\bar\alpha,\bar\beta[, \mathcal{B}(]\bar\alpha,\bar\beta[)\big)$ has Lebesgue decomposition that is given by

$$\begin{aligned} (R_\mu)''_{\mathrm{ac}}(x) = {} & \left[\frac{2}{C}\int_{]\alpha,\bar\alpha[} \frac{\psi(s)}{\sigma^2(s)p'(s)}\,\mu(ds) - H(\bar\alpha)\psi(\bar\alpha)\right]\varphi''(x) \\ & + \left[\frac{2}{C}\int_{[\bar\beta,\beta[} \frac{\varphi(s)}{\sigma^2(s)p'(s)}\,\mu(ds) + H(\bar\beta)\varphi(\bar\beta)\right]\psi''(x) - Cp''(x)H(x) - \frac{2\mu_{\mathrm{ac}}(x)}{\sigma^2(x)} \\ & - \varphi''(x)\int_{\bar\alpha}^{x} \psi'(s)H(s)\,ds - \psi''(x)\int_{x}^{\bar\beta} \varphi'(s)H(s)\,ds, \\ (R_\mu)''_{\mathrm{s}}(dx) = {} & -\frac{2}{\sigma^2(x)}\,\mu_{\mathrm{s}}(dx), \end{aligned}$$

in the notation of (35). Combining these expressions with (81)–(82) and the definition (22) of the scale function p, we can see that the restrictions of the measures $\mathcal{L}R_\mu$ and $-\mu$ in $\big(]\bar\alpha,\bar\beta[,\mathcal{B}(]\bar\alpha,\bar\beta[)\big)$ are equal. It follows that the measures $\mathcal{L}R_\mu$ and $-\mu$ on $\big(\operatorname{int}\mathcal{I},\mathcal{B}(\operatorname{int}\mathcal{I})\big)$ are equal because $\bar\alpha<\bar\beta$ have been arbitrary points in $\operatorname{int}\mathcal{I}$. Similarly, we can check that the function $R_{|\mu|}$ that is defined by (77) with $|\mu|$ in place of μ is the difference of two convex functions and satisfies the corresponding identity in (79).

To proceed further, we consider any Radon measure μ on $\big(\operatorname{int}\mathcal{I},\mathcal{B}(\operatorname{int}\mathcal{I})\big)$. Given monotone sequences (α_n) and (β_n) as in (64), we define

$$\zeta_n(z)=\begin{cases}0, & \text{if } z<\alpha_n \text{ or } z>\beta_n,\\ 1, & \text{if } \sigma^2(z)\geq\frac{1}{n} \text{ and } \alpha_n\leq z\leq\beta_n,\\ \sigma^2(z), & \text{if } \sigma^2(z)<\frac{1}{n} \text{ and } \alpha_n\leq z\leq\beta_n,\end{cases}$$

and we consider the sequence of measures (μ_n) that are defined by (39). The functions $R_{|\mu_n|}$, defined by (77) with $|\mu_n|$ in place of μ, are real-valued and satisfy

$$R_{|\mu_n|}(x)=\begin{cases}\frac{2}{C}\psi(x)\int_{[\alpha_n,\beta_n]}\frac{\varphi(s)}{\sigma^2(s)p'(s)}\zeta_n(s)\,|\mu|(ds), & \text{if } x<\alpha_n,\\ \frac{2}{C}\varphi(x)\int_{[\alpha_n,\beta_n]}\frac{\psi(s)}{\sigma^2(s)p'(s)}\zeta_n(s)\,|\mu|(ds), & \text{if } x>\beta_n.\end{cases}$$

Combining this calculation with (31), we can see that $R_{|\mu_n|}$ satisfies the corresponding limits in (80). Since $-\mathcal{L}R_{|\mu_n|}=|\mu_n|=|\mathcal{L}R_{\mu_n}|$ is a positive measure, part (I) of Lemma 5 implies that

$$\mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta}e^{-\Lambda_t}\,dA_t^{|\mu_n|}\right]<\infty\quad\text{for all } x\in\mathcal{I},$$

while, (62) in Lemma 5 with $\tau=T_\alpha\wedge T_\beta$ implies that

$$R_{|\mu_n|}(x)=-\mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta}e^{-\Lambda_t}\,dA_t^{\mathcal{L}R_{|\mu_n|}}\right].$$

This identity, the fact that $\mathcal{L}R_{|\mu_n|}=-|\mu_n|$ and (40) imply that the function $R_{|\mu_n|}$ that is defined as in (77) satisfies

$$R_{|\mu_n|}(x)=\mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta}e^{-\Lambda_t}\,dA_t^{|\mu_n|}\right].\tag{83}$$

Since the sequence of functions (ζ_n) is monotonically increasing to the identity function, the monotone convergence theorem implies that

$$\begin{aligned}R_{|\mu|}(x)=\lim_{n\to\infty}\bigg(&\frac{2}{C}\varphi(x)\int_{]\alpha,x[}\frac{\psi(s)}{\sigma^2(s)p'(s)}\zeta_n(s)\,|\mu|(ds)\\&+\frac{2}{C}\psi(x)\int_{[x,\beta[}\frac{\varphi(s)}{\sigma^2(s)p'(s)}\zeta_n(s)\,|\mu|(ds)\bigg).\end{aligned}\tag{84}$$

If (75) is satisfied, then σ^{-2} is locally integrable with respect to $|\mu|$, namely, the first condition in (76) holds true, thanks to the continuity of the functions φ, ψ and p'. In this case, (41) in Lemma 1 and (83) imply that

$$\lim_{n\to\infty} R_{|\mu_n|}(x) = \lim_{n\to\infty} \mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta} e^{-\Lambda_t}\, dA_t^{|\mu_n|}\right] = \mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta} e^{-\Lambda_t}\, dA_t^{|\mu|}\right] \tag{85}$$

because (ζ_n) satisfies (38). Combining this result with (84) and the fact that (75) implies that $R_{|\mu|}(x) < \infty$, we can see that

$$R_{|\mu|}(x) = \mathbb{E}_x\left[\int_0^{T_\alpha\wedge T_\beta} e^{-\Lambda_t}\, dA_t^{|\mu|}\right] < \infty, \tag{86}$$

and the second condition in (76) follows. Thus, we have proved that (75) implies (76). Conversely, if (76) is satisfied, then (41) in Lemma 1 and (83) imply that (85) is true. Combining (76) with (84) and (85), we can see that $R_{|\mu|}(x) < \infty$, and (75) follows.

If μ satisfies the integrability conditions (75)–(76), then the function R_μ given by (77) is well-defined and real-valued. Furthermore, it satisfies (78) thanks to (40), (86) with μ^+ and μ^- in place of $|\mu|$, and the linearity of integrals.

To establish (80), we consider any sequences (α_n), (β_n) as in (64), and we calculate

$$\begin{aligned}
0 &\overset{(78)}{=} R_{|\mu|}(x) - \lim_{m,n\to\infty} \mathbb{E}_x\left[\int_0^{T_{\alpha_m}\wedge T_{\beta_n}} e^{-\Lambda_u}\, dA_u^{|\mu|}\right] \\
&\overset{(79)}{=} R_{|\mu|}(x) + \lim_{m,n\to\infty} \mathbb{E}_x\left[\int_0^{T_{\alpha_m}\wedge T_{\beta_n}} e^{-\Lambda_u}\, dA_u^{\mathcal{L}R_{|\mu|}}\right] \\
&= \lim_{m,n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_m}\wedge T_{\beta_n}}} R_{|\mu|}(X_{T_{\alpha_m}\wedge T_{\beta_n}})\right] \\
&= \lim_{m,n\to\infty} R_{|\mu|}(\alpha_m)\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_m}}}\mathbf{1}_{\{T_{\alpha_m}<T_{\beta_n}\}}\right] + \lim_{m,n\to\infty} R_{|\mu|}(\beta_n)\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta_n}}}\mathbf{1}_{\{T_{\beta_n}<T_{\alpha_m}\}}\right],
\end{aligned}$$

the third identity following from (62) for $\tau = T_{\alpha_m}\wedge T_{\beta_n}$. Since $R_{|\mu|}$ is a positive function, each of the two limits on the right-hand side of this expression is equal to 0. We can therefore see that the first of these limits implies that

$$\begin{aligned}
0 &= \lim_{m\to\infty}\lim_{n\to\infty} R_{|\mu|}(\alpha_m)\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_m}}}\mathbf{1}_{\{T_{\alpha_m}<T_{\beta_n}\}}\right] \\
&= \lim_{m\to\infty} R_{|\mu|}(\alpha_m)\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_m}}}\mathbf{1}_{\{T_{\alpha_m}<T_{\beta}\}}\right] \\
&\overset{(26)}{=} \lim_{m\to\infty} \frac{R_{|\mu|}(\alpha_m)\varphi(x)}{\varphi(\alpha_m)},
\end{aligned}$$

which proves that $\lim_{y\downarrow\alpha} R_{|\mu|}(y)/\varphi(y) = 0$ because (α_m) has been arbitrary. We can show that $\lim_{x\uparrow\beta} R_{|\mu|}(x)/\psi(x) = 0$ using similar arguments. Finally, the function $|R_\mu|$ satisfies the corresponding limits in (80) because $|R_\mu| \le R_{|\mu|}$. □

The result we have just established and Lemma 5 imply the following representation of differences of two convex functions that involves the operator $\mathcal{L}$ and the functions φ, ψ.

Theorem 7 *Consider any function $F : \operatorname{int}\mathcal{I} \to \mathbb{R}$ that is the difference of two convex functions, and suppose that*

$$\limsup_{y\downarrow\alpha} \frac{|F(y)|}{\varphi(y)} < \infty \quad \textit{and} \quad \limsup_{y\uparrow\beta} \frac{|F(y)|}{\psi(y)} < \infty,$$

and that the measure $\mathcal{L}F$ satisfies the equivalent integrability conditions (75)–(76) (see also Remark 1 below). In this context, the limits $\lim_{y\downarrow\alpha} F(y)/\varphi(y)$ and $\lim_{y\uparrow\beta} F(y)/\psi(y)$ both exist, and the function F admits the representation

$$F(x) = \lim_{y\downarrow\alpha} \frac{F(y)}{\varphi(y)}\varphi(x) + R_{-\mathcal{L}F}(x) + \lim_{y\uparrow\beta} \frac{F(y)}{\psi(y)}\psi(x), \tag{87}$$

where $R_{-\mathcal{L}F}$ is given by (77)–(78). Furthermore, given any points $\bar{\alpha} < x < \bar{\beta}$ in $\mathcal{I}$ and any stopping time τ,

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_{\bar{\alpha}}\wedge T_{\bar{\beta}}}} F(X_{\tau\wedge T_{\bar{\alpha}}\wedge T_{\bar{\beta}}})\right] = F(x) + \mathbb{E}_x\left[\int_0^{\tau\wedge T_{\bar{\alpha}}\wedge T_{\bar{\beta}}} e^{-\Lambda_u}\, dA_u^{\mathcal{L}F}\right], \tag{88}$$

in which expression, we denote

$$F(\alpha) = \lim_{y\downarrow\alpha} F(y) \quad \left(\textit{resp., } F(\beta) = \lim_{y\uparrow\beta} F(y)\right)$$

if α (resp., β) is absorbing, namely, if $\mathbb{P}_x(T_\alpha < \infty) > 0$ (resp., $\mathbb{P}_x(T_\beta < \infty) > 0$).

Proof. In the presence of the assumption that $\mathcal{L}F$ satisfies (75)–(76), Lemma 5.(II) implies that the limits $\lim_{y\downarrow\alpha} F(y)/\varphi(y)$ and $\lim_{y\uparrow\beta} F(y)/\psi(y)$ exist, while Theorem 6 implies that the function $R_{-\mathcal{L}F}$ is well-defined. In particular, (79) implies that $\mathcal{L}\,(F - R_{-\mathcal{L}F}) = 0$. It follows that

$$F - R_{-\mathcal{L}F} = A\varphi + B\psi,$$

for some constants $A, B \in \mathbb{R}$. Combining (31) with (80), we can see that the constants A and B are as in (87). Finally, (88) follows from the representation (87) of F, (51) in Lemma 3, (62) in Lemma 5 and (80) in Theorem 6. □

Remark 1 In view of Lemma 5.(I), the positivity of the measure $-\mathcal{L}F$ is a sufficient condition for $\mathcal{L}F$ to satisfy the integrability conditions (75)–(76). Also, if F is C^1 with first derivative that is absolutely continuous with respect to the Lebesgue measure, then $\mathcal{L}F(dx) = \mathcal{L}_{\mathrm{ac}}F(x)\,dx$, where $\mathcal{L}_{\mathrm{ac}}$ is defined by (37). This observation and part (II) of Lemma 2 imply that, in this case (75)–(76) are equivalent to (89)–(90) below for $h = \mathcal{L}_{\mathrm{ac}}F$. Furthermore, $R_{-\mathcal{L}F}$ admits the expressions (91)–(92) below for $h = -\mathcal{L}_{\mathrm{ac}}F$. □

The measure $\mathcal{L}F$ and the potential $R_{-\mathcal{L}F}$ have central roles in the characterisation of differences of two convex functions we have established above. The following result is concerned with the potential $R_{-\mathcal{L}F}$ when $\mathcal{L}F$ is absolutely continuous with respect to the Lebesgue measure.

Corollary 8 *Consider any function $h : \mathcal{I} \to \mathbb{R}$ that is locally integrable with respect to the Lebesgue measure, and let μ^h be the measure on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ defined by*

$$\mu^h(\Gamma) = \int_\Gamma h(s)\,ds, \quad \text{for } \Gamma \in \mathcal{B}(\operatorname{int}\mathcal{I}).$$

If μ^h satisfies the equivalent integrability conditions (75)–(76), which are equivalent to

$$\int_\alpha^x \frac{\psi(s)}{\sigma^2(s)p'(s)}|h(s)|\,ds + \int_x^\beta \frac{\varphi(s)}{\sigma^2(s)p'(s)}|h(s)|\,ds < \infty, \tag{89}$$

$$\mathbb{E}_x\left[\int_0^{T_\alpha \wedge T_\beta} e^{-\Lambda_t}\big|h(X_t)\big|\,dt\right] < \infty, \tag{90}$$

then the function $R_{\mu^h} : \operatorname{int}\mathcal{I} \to \mathbb{R}$ defined by (77) or, equivalently, by

$$R_{\mu^h}(x) = \frac{2}{C}\varphi(x)\int_\alpha^x \frac{\psi(s)}{\sigma^2(s)p'(s)}h(s)\,ds + \frac{2}{C}\psi(x)\int_x^\beta \frac{\varphi(s)}{\sigma^2(s)p'(s)}h(s)\,ds, \tag{91}$$

admits the probabilistic expression

$$R_{\mu^h}(x) = \mathbb{E}_x\left[\int_0^{T_\alpha \wedge T_\beta} e^{-\Lambda_t}h(X_t)\,dt\right]. \tag{92}$$

This function, as well as the function defined by

$$\tilde{R}_{\mu^h}(x) = \mathbb{E}_x\left[\int_0^{\infty} e^{-\Lambda_t}h(X_t)\,dt\right], \quad \text{for } x \in \operatorname{int}\mathcal{I}, \tag{93}$$

is C^1 with absolutely continuous first derivative and satisfies the ODE

$$\mathcal{L}_{\mathrm{ac}}g(x) + h(x) \equiv \frac{1}{2}\sigma^2(x)g''(x) + b(x)g'(x) - r(x)g(x) + h(x) = 0. \tag{94}$$

The functions R_{μ^h} and $\tilde{R}_{\mu^h}$ satisfy

$$\tilde{R}_{\mu^h}(x) = \frac{h(\alpha)}{r(\alpha)}\frac{\varphi(x)}{\varphi(\alpha)}\mathbb{I}_\alpha + R_{\mu^h}(x) + \frac{h(\beta)}{r(\beta)}\frac{\psi(x)}{\psi(\beta)}\mathbb{I}_\beta, \tag{95}$$

$$\lim_{y\downarrow\alpha}\frac{\tilde{R}_{\mu^h}(y)}{\varphi(y)} = \lim_{y\downarrow\alpha}\frac{R_{\mu^h}(y)}{\varphi(y)} + \frac{h(\alpha)}{r(\alpha)\varphi(\alpha)}\mathbb{I}_\alpha = \frac{h(\alpha)}{r(\alpha)\varphi(\alpha)}\mathbb{I}_\alpha, \tag{96}$$

$$\lim_{y\uparrow\beta}\frac{\tilde{R}_{\mu^h}(y)}{\psi(y)} = \lim_{y\uparrow\beta}\frac{R_{\mu^h}(y)}{\psi(y)} + \frac{h(\beta)}{r(\beta)\psi(\beta)}\mathbb{I}_\beta = \frac{h(\beta)}{r(\beta)\psi(\beta)}\mathbb{I}_\beta, \tag{97}$$

where

$$\mathbb{I}_\alpha = \begin{cases} 1, & \text{if } \alpha \text{ is absorbing,} \\ 0, & \text{if } \alpha \text{ is inaccessible,} \end{cases} \quad \text{and} \quad \mathbb{I}_\beta = \begin{cases} 1, & \text{if } \beta \text{ is absorbing,} \\ 0, & \text{if } \beta \text{ is inaccessible.} \end{cases}$$

Furthermore,

$$R_{\mu^h}(x) = \mathbb{E}_x \left[\int_0^{\tau \wedge T_\alpha \wedge T_\beta} e^{-\Lambda_t} h(X_t)\, dt + e^{-\Lambda_{\tau \wedge T_\alpha \wedge T_\beta}} R_{\mu^h}(X_{\tau \wedge T_\alpha \wedge T_\beta}) \mathbf{1}_{\{\tau \wedge T_\alpha \wedge T_\beta < \infty\}} \right] \tag{98}$$

for every stopping time τ and all initial conditions $x \in \operatorname{int} \mathcal{I}$, in which expression, $R_{\mu^h}(\alpha) = 0$ (resp., $R_{\mu^h}(\beta) = 0$) if α (resp., β) is absorbing, consistently with (96)–(97).

Proof. It is straightforward to check that the function R_{μ^h} defined by (91) is C^1 with absolutely continuous first derivative and satisfies the ODE (94). This observation and (95) imply the corresponding statements for $\tilde{R}_{\mu^h}$. The equivalence of (76) (resp., (78)) with (90) (resp., (92)) is a consequence of part (II) of Lemma 2 and the identities $\mu^h(dx) = -\mathcal{L} R_{\mu^h}(dx) = -\mathcal{L}_{\mathrm{ac}} R_{\mu^h}(x)\, dx = h(x)\, dx$. Also, these identities, part (II) of Lemma 2 and (62) imply (98), while the limits in (96)–(97) follow from (80) and (95).

To prove (95), we first note that

$$\tilde{R}_{\mu^h}(x) = \mathbb{E}_x \left[\mathbf{1}_{\{T_\alpha < T_\beta\}} \int_{T_\alpha}^\infty e^{-\Lambda_t}\, dt \right] h(\alpha) + R_{\mu^h}(x) + \mathbb{E}_x \left[\mathbf{1}_{\{T_\beta < T_\alpha\}} \int_{T_\beta}^\infty e^{-\Lambda_t}\, dt \right] h(\beta).$$

In view of the definition (5) of Λ, we can see that, if α is absorbing, then

$$\begin{aligned} \mathbb{E}_x \left[\mathbf{1}_{\{T_\alpha < T_\beta\}} \int_{T_\alpha}^\infty e^{-\Lambda_t}\, dt \right] &= \mathbb{E}_x \left[\mathbf{1}_{\{T_\alpha < T_\beta\}} e^{-\Lambda_{T_\alpha}} \int_{T_\alpha}^\infty e^{-r(\alpha)(t - T_\alpha)}\, dt \right] \\ &= \frac{1}{r(\alpha)} \mathbb{E}_x \left[e^{-\Lambda_{T_\alpha}} \mathbf{1}_{\{T_\alpha < T_\beta\}} \right] \\ &\overset{(26)}{=} \frac{1}{r(\alpha)} \frac{\varphi(x)}{\varphi(\alpha)}, \end{aligned}$$

otherwise, this expectation is plainly 0. Similarly, we can see that

$$\mathbb{E}_x \left[\mathbf{1}_{\{T_\beta < T_\alpha\}} \int_{T_\beta}^\infty e^{-\Lambda_t}\, dt \right] = \frac{1}{r(\beta)} \frac{\psi(x)}{\psi(\beta)} \mathbb{I}_\beta,$$

and (95) follows. □

5 Analytic characterisations of $r(\cdot)$-excessive functions

The following is the main result of this section.

Theorem 9 *A function $F : \mathcal{I} \to \mathbb{R}_+$ is $r(\cdot)$-excessive, namely, it satisfies*

$$\mathbb{E}_x \left[e^{-\Lambda_\tau} F(X_\tau) \mathbf{1}_{\{\tau<\infty\}} \right] \leq F(x) \tag{99}$$

for all stopping times τ and all initial conditions $x \in \mathcal{I}$, if and only if the following statements are both true:
(I) *the restriction of F in the interior $\operatorname{int}\mathcal{I}$ of $\mathcal{I}$ is the difference of two convex functions and the associated measure $-\mathcal{L}F$ on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ is positive;*
(II) *if α (resp., β) is an absorbing boundary point, then $F(\alpha) \leq \liminf_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} F(y)$ (resp., $F(\beta) \leq \liminf_{y\in\operatorname{int}\mathcal{I},\, y\uparrow\beta} F(y)$).*

Proof. First, we consider any function $F : \mathcal{I} \to \mathbb{R}_+$ with the properties listed in (I)–(II). The assumption that $-\mathcal{L}F$ is a positive measure implies that $-A^{\mathcal{L}F} = A^{-\mathcal{L}F}$ is an increasing process. Therefore, (88) in Theorem 7 implies that, given any points $\bar\alpha < x < \bar\beta$ in $\mathcal{I}$ and any stopping time τ such that $\bar\alpha = \alpha$ and $\tau = \tau \wedge T_\alpha$ (resp., $\bar\beta = \beta$ and $\tau = \tau \wedge T_\beta$) if α (resp., β) is absorbing,

$$\begin{aligned}
F(x) &\geq \mathbb{E}_x \left[e^{-\Lambda_\tau} F(X_\tau) \mathbf{1}_{\{\tau<T_{\bar\alpha}\wedge T_{\bar\beta}\}} \right] + F(\bar\alpha)\mathbb{E}_x \left[e^{-\Lambda_{T_{\bar\alpha}}} \mathbf{1}_{\{T_{\bar\alpha}\leq\tau\wedge T_{\bar\beta}\}} \right] (1 - \mathbb{I}_\alpha) \\
&\quad + \lim_{y\downarrow\alpha} F(y)\mathbb{E}_x \left[e^{-\Lambda_{T_{\bar\alpha}}} \mathbf{1}_{\{T_{\bar\alpha}\leq\tau\wedge T_{\bar\beta}\}} \right] \mathbb{I}_\alpha + F(\bar\beta)\mathbb{E}_x \left[e^{-\Lambda_{T_{\bar\beta}}} \mathbf{1}_{\{T_{\bar\beta}\leq\tau\wedge T_{\bar\alpha}\}} \right] (1 - \mathbb{I}_\beta) \\
&\quad + \lim_{y\uparrow\beta} F(y)\mathbb{E}_x \left[e^{-\Lambda_{T_{\bar\beta}}} \mathbf{1}_{\{T_{\bar\beta}\leq\tau\wedge T_{\bar\alpha}\}} \right] \mathbb{I}_\beta \\
&\geq \mathbb{E}_x \left[e^{-\Lambda_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}} F(X_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}) \right],
\end{aligned} \tag{100}$$

the second inequality following from the assumption that F satisfies the inequalities in (II). If α (resp., β) is inaccessible, then we can pass to the limit $\bar\alpha \downarrow \alpha$ (resp., $\bar\beta \uparrow \beta$) using Fatou's lemma to obtain (99) thanks to the choices of $\bar\alpha$ and $\bar\beta$ that we have made. It follows that F is $r(\cdot)$-excessive.

To establish the reverse implication, we first show that an $r(\cdot)$-excessive function is lower semicontinuous and its restriction in $\operatorname{int}\mathcal{I}$ is continuous. Given an initial condition $x \in \operatorname{int}\mathcal{I}$ and a point $y \in \mathcal{I}$, we can use (99) to calculate

$$F(x) \geq \mathbb{E}_x \left[e^{-\Lambda_{T_y}} \right] F(y) \overset{(26)-(27)}{=} \min\left\{ \frac{\psi(x)}{\psi(y)}, \frac{\varphi(x)}{\varphi(y)} \right\} F(y).$$

This calculation and the continuity of the functions φ, ψ imply that $F(x) \geq \limsup_{y\to x} F(y)$, which proves that F is upper semicontinuous in $\operatorname{int}\mathcal{I}$. The same arguments but with points $x \in \mathcal{I}$ and $y \in \operatorname{int}\mathcal{I}$ and their roles reversed imply that

$$F(y) \geq \min\left\{ \frac{\psi(y)}{\psi(x)}, \frac{\varphi(y)}{\varphi(x)} \right\} F(x).$$

It follows that $F(x) \leq \liminf_{y \in \operatorname{int} \mathcal{I},\, y \to x} F(y)$, and the lower semicontinuity of F in $\mathcal{I}$ has been established. In particular, part (II) of the proposition is true.

To prove that an $r(\cdot)$-excessive function satisfies (I), we define the function F_q by

$$F_q(x) = q\mathbb{E}_x \left[\int_0^\infty e^{-qt-\Lambda_t} F(X_t)\, dt \right], \quad \text{for } x \in \mathcal{I}, \tag{101}$$

where $q > 0$ is a constant, and we note that

$$0 \leq F_q(x) \overset{(99)}{\leq} q \int_0^\infty e^{-qt} F(x)\, dt = F(x) \quad \text{for all } x \in \mathcal{I}. \tag{102}$$

If we consider the change of variables $u = qt$, then we can see that

$$F_q(x) = \mathbb{E}_x \left[\int_0^\infty e^{-u-\Lambda_{u/q}} F(X_{u/q})\, du \right].$$

In view of (102), the continuity properties of the function F and the continuity of the process X, this expression implies that

$$\lim_{q \to \infty} F_q(x) = F(x) \quad \text{for all } x \in \mathcal{I}. \tag{103}$$

Given its definition in (101), Corollary 8 implies that the function F_q is C^1 with absolutely continuous first derivative and that it satisfies the ODE

$$\frac{1}{2}\sigma^2(x) F_q''(x) + b(x) F_q'(x) - (q + r(x))\, F_q(x) + qF(x) = 0$$

in the interior of $\mathcal{I}$. In view of (102), we can see that

$$\frac{1}{2}\sigma^2(x) F_q''(x) + b(x) F_q'(x) - r(x) F_q(x) = -q\,[F(x) - F_q(x)] \leq 0.$$

This inequality implies that

$$\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{F_q(x)}{p'(x)} \right) - F_q(x) \frac{d}{dx} \frac{1}{p'(x)} \right) - \frac{2r(x) F_q(x)}{\sigma^2(x) p'(x)} \leq 0, \tag{104}$$

where p is the scale function of the diffusion X, which is defined by (22).

To proceed further, we introduce the antiderivatives $\mathcal{A}^1$ and $\mathcal{A}^2$ of a function g that is locally integrable in $\mathcal{I}$, which are defined by

$$\mathcal{A}^1 g(x) = \int_c^x g(y)\, dy \quad \text{and} \quad \mathcal{A}^2 g(x) = \int_c^x \int_c^y g(z)\, dz\, dy,$$

respectively, where $c \in \mathcal{I}$ is a fixed point that we can take to be the same as the point appearing in the definition (22) of the scale function p. Inequality (104) then implies that

the function $F_q/p' - \mathcal{A}^1\left((1/p')'F_q\right) - \mathcal{A}^2\left((2rF_q)/(\sigma^2 p')\right)$ is concave, which, combined with (103), implies that the function $G := F/p' - \mathcal{A}^1\left((1/p')'F\right) - \mathcal{A}^2\left((2rF)/(\sigma^2 p')\right)$ is concave. The concavity of G and the equality

$$\frac{F}{p'} = G + \mathcal{A}^1\left(\left(\frac{1}{p'}\right)' F\right) + \mathcal{A}^2\left(\frac{2rF}{\sigma^2 p'}\right)$$

imply that F/p' is absolutely continuous and

$$F'_-(x) = p'(x)\left(G'_-(x) + \mathcal{A}^1\left(\frac{2rF}{\sigma^2 p'}\right)(x)\right).$$

This expression shows that F' has finite variation. Furthermore, taking distributional derivatives, we can see that

$$\frac{2}{\sigma^2(x)}\mathcal{L}F(dx) \equiv F''(dx) + \frac{2b(x)}{\sigma^2(x)}F'_-(x)\,dx - \frac{2r(x)}{\sigma^2(x)}F(x)\,dx = p'(x)G''(dx),$$

which proves that F has the properties listed in part (I) thanks to the concavity of G. $\square$

In the spirit of Dynkin [15, Theorems 15.10 and 16.4], Dayanik [12] proves that a function F is $r(\cdot)$-excessive if and only if the function F/φ is (ψ/φ)-concave (equivalently, the function F/ψ is $(-\varphi/\psi)$-concave), and he shows that such concavity assumptions imply that the function $-D^-_{\psi/\varphi}(F/\varphi)$ defined by (105) is increasing (equivalently, the right-continuous modification $D^+_{\varphi/\psi}(F/\psi)$ of the function defined by (106) is increasing) (see Proposition 3.1 and Remarks 3.1–3.3 of Dayanik [12] for the precise statements). Such a result, which focuses on the functions $-D^-_{\psi/\varphi}(F/\varphi)$, $D^+_{\varphi/\psi}(F/\psi)$, follows immediately from our analysis above.

Corollary 10 *A function $F : \mathcal{I} \to \mathbb{R}_+$ is $r(\cdot)$-excessive if and only if the following statements are both true:*
(I) *the function $-D^-_{\psi/\varphi}(F/\varphi)$ given by*

$$-D^-_{\psi/\varphi}(F/\varphi)(x) = -\lim_{y\uparrow x}\frac{(F/\varphi)(x) - (F/\varphi)(y)}{(\psi/\varphi)(x) - (\psi/\varphi)(y)}, \quad \text{for } x \in \operatorname{int}\mathcal{I}, \tag{105}$$

is well-defined, real-valued and increasing; equivalently, the function $D^-_{\varphi/\psi}(F/\psi)$ given by

$$D^-_{\varphi/\psi}(F/\psi)(x) = \lim_{y\uparrow x}\frac{(F/\psi)(y) - (F/\psi)(x)}{(\varphi/\psi)(y) - (\varphi/\psi)(x)}, \quad \text{for } x \in \operatorname{int}\mathcal{I}, \tag{106}$$

is well-defined, real-valued and increasing, and
(II) *if α (resp., β) is an absorbing boundary point, then $F(\alpha) \le \liminf_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} F(y)$ (resp., $F(\beta) \le \liminf_{y\in\operatorname{int}\mathcal{I},\, y\uparrow\beta} F(y)$).*

Proof. Given a measure μ on $\big(\text{int}\,\mathcal{I}, \mathcal{B}(\text{int}\,\mathcal{I})\big)$, we mean that $-\mu$ is a positive measure whenever we write $\mu(dx) \leq 0$ in the proof below. In view of Theorem 9, the result will follow if we show that either of the functions given by (105), (106) is well-defined, real-valued and increasing if and only if the restriction of F in $\text{int}\,\mathcal{I}$ is the difference of two convex functions and $\mathcal{L}F \leq 0$. To this end, we note that the functions given by (105), (106) are well-defined and real-valued if and only if F'_- exists and is real-valued, in which case,

$$-D^-_{\psi/\varphi}(F/\varphi)(x) = -\frac{\varphi(x)F'_-(x) - \varphi'(x)F(x)}{\varphi(x)\psi'(x) - \varphi'(x)\psi(x)} \overset{(33)}{=} -\frac{\varphi(x)F'_-(x) - \varphi'(x)F(x)}{Cp'(x)},$$
$$D^-_{\varphi/\psi}(F/\psi)(x) = \frac{\psi(x)F'_-(x) - \psi'(x)F(x)}{\varphi'(x)\psi(x) - \varphi(x)\psi'(x)} \overset{(33)}{=} -\frac{\psi(x)F'_-(x) - \psi'(x)F(x)}{Cp'(x)}.$$

The function $-D^-_{\psi/\varphi}(F/\varphi)$ is increasing if and only if its first distributional derivative is a positive measure, namely, if and only if the second distributional derivative of F is a measure and

$$\frac{\varphi(x)}{Cp'(x)}F''(dx) - \frac{\varphi''(x)}{Cp'(x)}F(x)\,dx - \big[\varphi(x)F'_-(x) - \varphi'(x)F(x)\big]\frac{p''(x)}{C\big[p'(x)\big]^2}\,dx \leq 0.$$

In view of the definition (22) of the scale function p and the fact that p and C are both strictly positive, we can see that this is true if and only if

$$\varphi(x)\frac{1}{2}\sigma^2(x)F''(dx) + \varphi(x)b(x)F'_-(x)\,dx - \left[\frac{1}{2}\sigma^2(x)\varphi''(x) + b(x)\varphi'(x)\right]F(x)\,dx \leq 0,$$

which is true if and only if $-\mathcal{L}F \geq 0$, thanks to the fact that $\varphi > 0$ satisfies the ODE (32). Similarly, we can see that the function $D^-_{\varphi/\psi}(F/\psi)$ is increasing if and only if $-\mathcal{L}F \geq 0$. $\square$

6 The solution of the optimal stopping problem

Before addressing the main results of the section, we prove that the value function v is excessive.

Lemma 11 *Consider the optimal stopping problem formulated in Section 2 and suppose that its value function is real-valued. The value function v is $r(\cdot)$-excessive, i.e.,*

$$\mathbb{E}_x\left[e^{-\Lambda_\tau}v(X_\tau)\mathbf{1}_{\{\tau<\infty\}}\right] \leq v(x), \tag{107}$$

for all initial conditions $x \in \mathcal{I}$ and every stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{T}_x$. Also,

$$v(x) = \sup_{(\mathbb{S}_x,\tau)\in\mathcal{T}_x} \mathbb{E}_x\big[e^{-\Lambda_{\tau\wedge T_\alpha\wedge T_\beta}}\overline{f}(X_{\tau\wedge T_\alpha\wedge T_\beta})\mathbf{1}_{\{\tau<\infty\}}\big] \quad \textit{for all } x \in \mathcal{I}, \tag{108}$$

where $\overline{f}$ is given by (10).

Proof. To prove the $r(\cdot)$-excessivity of v, we first show that v is continuous in $\operatorname{int}\mathcal{I}$ and lower semicontinuous in $\mathcal{I}$. To this end, we consider any points $x, y \in \operatorname{int}\mathcal{I}$. Given the stopping strategy $(\mathbb{S}_x, T_y) \in \mathcal{T}_x$ and any stopping strategy $(\mathbb{S}_y, \tau) \in \mathcal{T}_y$, we denote by $(\hat{\mathbb{S}}_x, \hat{\tau})$ a stopping strategy that is as in Corollary 21, so that

$$\begin{aligned} v(x) \geq J(\hat{\mathbb{S}}_x, \hat{\tau}) &= \mathbb{E}_x\left[e^{-\Lambda_{T_\alpha\wedge T_\beta}} f(X_{T_\alpha\wedge T_\beta})\mathbf{1}_{\{T_\alpha\wedge T_\beta<T_y\}}\right] + \mathbb{E}_x\left[e^{-\Lambda_{T_y}}\mathbf{1}_{\{T_y<T_\alpha\wedge T_\beta\}}\right] J(\mathbb{S}_y,\tau) \\ &\geq \mathbb{E}_x\left[e^{-\Lambda_{T_y}}\mathbf{1}_{\{T_y<T_\alpha\wedge T_\beta\}}\right] J(\mathbb{S}_y,\tau). \end{aligned}$$

Since $(\mathbb{S}_y, \tau)$ is arbitrary, we can use the dominated convergence theorem to see that this inequality implies that

$$v(x) \geq \lim_{y\to x} \mathbb{E}_x\left[e^{-\Lambda_{T_y}}\mathbf{1}_{\{T_y<T_\alpha\wedge T_\beta\}}\right] \limsup_{y\to x} v(y) = \limsup_{y\to x} v(y),$$

which proves that v is upper semicontinuous in $\operatorname{int}\mathcal{I}$.

Repeating the same arguments with the roles of the points $x, y \in \operatorname{int}\mathcal{I}$ reversed, we can see that

$$\liminf_{y\to x} v(y) \geq \lim_{y\to x} \mathbb{E}_y\left[e^{-\Lambda_{T_x}}\mathbf{1}_{\{T_x<T_\alpha\wedge T_\beta\}}\right] v(x) = v(x).$$

If both α and β are absorbing, then we can use (26)–(29) to calculate

$$\begin{aligned} \liminf_{x\in\operatorname{int}\mathcal{I},\, x\downarrow\alpha} v(x) &\geq \liminf_{x\in\operatorname{int}\mathcal{I},\, x\downarrow\alpha} \left(f(\alpha)\mathbb{E}_x\left[e^{-\Lambda_{T_\alpha}}\mathbf{1}_{\{T_\alpha<\mathcal{T}_\beta\}}\right] + f(\beta)\mathbb{E}_x\left[e^{-\Lambda_{T_\beta}}\mathbf{1}_{\{T_\beta<T_\alpha\}}\right]\right) \\ &= \liminf_{x\downarrow\alpha}\left(\frac{f(\alpha)\varphi(x)}{\varphi(\alpha)} + \frac{f(\beta)\psi(x)}{\psi(\beta)}\right) = f(\alpha) = v(\alpha), \end{aligned}$$

while, if α is absorbing and β is inaccessible, then

$$\liminf_{x\in\operatorname{int}\mathcal{I},\, x\downarrow\alpha} v(x) \geq \liminf_{x\in\operatorname{int}\mathcal{I},\, x\downarrow\alpha} f(\alpha)\mathbb{E}_x\left[e^{-\Lambda_{T_\alpha}}\right] = \liminf_{x\downarrow\alpha} \frac{f(\alpha)\varphi(x)}{\varphi(\alpha)} = f(\alpha) = v(\alpha).$$

If β is absorbing, then we can see that $\liminf_{x\in\operatorname{int}\mathcal{I},\, x\uparrow\beta} v(x) \geq v(\beta)$ similarly. It follows that v is lower semicontinuous in $\mathcal{I}$.

To show that v satisfies (107), we consider any stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{T}_x$. We assume that $X_\tau\mathbf{1}_{\{\tau<T_\alpha\wedge T_\beta\}}$ takes values in a finite set $\{a_1, \ldots, a_n\} \subset \operatorname{int}\mathcal{I}$. For each $i = 1, \ldots, n$, we consider an ε-optimal strategy $(\mathbb{S}^\varepsilon_{a_i}, \tau^\varepsilon_i) \in \mathcal{T}_{a_i}$. If we denote by $(\mathbb{S}^\varepsilon_x, \tau^\varepsilon) \in \mathcal{T}_x$ a stopping strategy that is as in Corollary 21, then

$$\begin{aligned} v(x) \geq J(\mathbb{S}^\varepsilon_x, \tau^\varepsilon) &= \mathbb{E}_x\left[e^{-\Lambda_{T_\alpha\wedge T_\beta}} f(X_{T_\alpha\wedge T_\beta})\mathbf{1}_{\{T_\alpha\wedge T_\beta<\tau\}}\right] + \sum_{i=1}^n \mathbb{E}_x\left[e^{-\Lambda_\tau}\mathbf{1}_{\{X_\tau=a_i\}}\right] J(\mathbb{S}^\varepsilon_{a_i}, \tau^\varepsilon_i) \\ &\geq \mathbb{E}_x\left[e^{-\Lambda_{T_\alpha\wedge T_\beta}} v(X_{T_\alpha\wedge T_\beta})\mathbf{1}_{\{T_\alpha\wedge T_\beta<\tau\}}\right] + \sum_{i=1}^n \mathbb{E}_x\left[e^{-\Lambda_\tau}\mathbf{1}_{\{X_\tau=a_i\}}\right] [v(a_i) - \varepsilon], \end{aligned}$$

where the last inequality follows from the fact that $f(X_{T_\alpha \wedge T_\beta}) = v(X_{T_\alpha \wedge T_\beta})$ and the ε-optimality of the strategies $(\mathbb{S}^\varepsilon_{a_i}, \tau^\varepsilon_i)$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$v(x) \geq \mathbb{E}_x \left[e^{-\Lambda_{\tau \wedge T_\alpha \wedge T_\beta}} v(X_{\tau \wedge T_\alpha \wedge T_\beta}) \mathbf{1}_{\{\tau < \infty\}} \right],$$

and (107) follows in this case.

Now, we consider any stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{T}_x$, and we define

$$\tau_n = \inf \{ t \geq \tau \mid \ X_t \in \{a_1, \ldots, a_n\} \},$$

where (a_n) is any sequence that is dense in $\operatorname{int} \mathcal{I}$. Such a sequence of stopping times is such that

$$\tau_n \mathbf{1}_{\{T_\alpha \wedge T_\beta \leq \tau\}} = \infty \mathbf{1}_{\{T_\alpha \wedge T_\beta \leq \tau\}} \text{ for all } n \geq 1 \quad \text{and} \quad \lim_{n \to \infty} \tau_n \mathbf{1}_{\{\tau < T_\alpha \wedge T_\beta\}} = \tau \mathbf{1}_{\{\tau < T_\alpha \wedge T_\beta\}}.$$

Therefore, $\lim_{n \to \infty} \tau_n \wedge T_\alpha \wedge T_\beta = \tau \wedge T_\alpha \wedge T_\beta$. Our analysis above has established that (107) holds true for each of the stopping strategies $(\mathbb{S}_x, \tau_n) \in \mathcal{T}_x$. Combining this observation with Fatou's lemma and the fact that v is lower semicontinuous, we can see that

$$v(x) \geq \liminf_{n \to \infty} \mathbb{E}_x \left[e^{-\Lambda_{\tau_n \wedge T_\alpha \wedge T_\beta}} v(X_{\tau_n \wedge T_\alpha \wedge T_\beta}) \mathbf{1}_{\{\tau_n < \infty\}} \right] \geq \mathbb{E}_x \left[e^{-\Lambda_{\tau \wedge T_\alpha \wedge T_\beta}} v(X_{\tau \wedge T_\alpha \wedge T_\beta}) \mathbf{1}_{\{\tau < \infty\}} \right],$$

which establishes (107).

Finally, we note that the continuity properties of v and the inequality $v \geq f$ imply that $v \geq \overline{f}$. This observation and the $r(\cdot)$-excessivity of v imply that

$$\begin{aligned} v(x) &= \sup_{(\mathbb{S}_x, \tau) \in \mathcal{T}_x} \mathbb{E}_x \left[e^{-\Lambda_\tau} v(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right] \\ &\geq \sup_{(\mathbb{S}_x, \tau) \in \mathcal{T}_x} \mathbb{E}_x \left[e^{-\Lambda_\tau} \overline{f}(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right] \geq \sup_{(\mathbb{S}_x, \tau) \in \mathcal{T}_x} \mathbb{E}_x \left[e^{-\Lambda_\tau} f(X_\tau) \mathbf{1}_{\{\tau < \infty\}} \right] = v(x), \end{aligned}$$

and (108) follows. □

Our main results in this section involve solutions to the variational inequality

$$\max \left\{ \mathcal{L} v, \ \overline{f} - v \right\} = 0 \tag{109}$$

in the following sense.

Definition 1 A function $v : \mathcal{I} \to \mathbb{R}_+$ satisfies the variational inequality (109) if its restriction in $\operatorname{int} \mathcal{I}$ is the difference of two convex functions,

$$-\mathcal{L} v \text{ is a positive measure on } \left(\operatorname{int} \mathcal{I}, \mathcal{B}(\operatorname{int} \mathcal{I}) \right), \tag{110}$$

$$f(x) \leq v(x) \text{ for all } x \in \operatorname{int} \mathcal{I}, \tag{111}$$

$$\text{and the measure } \mathcal{L} v \text{ does not charge the open set } \{x \in \operatorname{int} \mathcal{I} \mid \ v(x) > \overline{f}(x)\}, \tag{112}$$

where $\mathcal{L}$ is defined by (36) and $\overline{f}$ is defined by (10). □

We now prove that the value function v satisfies the variational inequality (109) in the sense of this definition. Also, we establish sufficient conditions for the existence of ε-optimal as well as optimal stopping strategies. It is worth noting that the requirements (118)–(119) are not really needed: the only reason we have adopted them is to simplify the exposition of the proof.

Theorem 12 *Consider the optimal stopping problem formulated in Section 2. The following statements are true.*
(I) *If the problem data is such that*

$$\overline{f}(y) = \infty, \text{ for some } y \in \mathcal{I}, \quad \text{or} \quad \limsup_{y\downarrow\alpha} \frac{f(y)}{\varphi(y)} = \infty \quad \text{or} \quad \limsup_{y\uparrow\beta} \frac{f(y)}{\psi(y)} = \infty,$$

then $v(x) = \infty$ for all $x \in \mathcal{I}$, otherwise, $v(x) < \infty$ for all $x \in \mathcal{I}$.
(II) *If the problem data is such that*

$$\overline{f}(y) < \infty \text{ for all } y \in \mathcal{I}, \quad \limsup_{y\downarrow\alpha} \frac{f(y)}{\varphi(y)} < \infty \quad \text{and} \quad \limsup_{y\uparrow\beta} \frac{f(y)}{\psi(y)} < \infty, \tag{113}$$

then the value function v satisfies the variational inequality (109) in the sense of Definition 1,

$$\lim_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} \frac{v(y)}{\varphi(y)} = \limsup_{y\downarrow\alpha} \frac{f(y)}{\varphi(y)}, \qquad \lim_{y\in\operatorname{int}\mathcal{I},\, y\uparrow\beta} \frac{v(y)}{\psi(y)} = \limsup_{y\uparrow\beta} \frac{f(y)}{\psi(y)} \tag{114}$$

and

$$v(\alpha) = f(\alpha) \ \big(\text{resp., } v(\beta) = f(\beta)\big) \text{ if } \alpha \ \big(\text{resp., } \beta\big) \text{ is absorbing.} \tag{115}$$

(III) *Suppose that (113) is true and that $f = \overline{f}$. Given an initial condition $x \in \operatorname{int}\mathcal{I}$ consider any monotone sequences (α_n), (β_n) in $\mathcal{I}$ such that*

$$\alpha_1 < x < \beta_1, \quad \lim_{n\to\infty} \alpha_n = \alpha, \quad \lim_{n\to\infty} \beta_n = \beta, \tag{116}$$

$$\lim_{n\to\infty} \frac{f(\alpha_n)}{\varphi(\alpha_n)} = \limsup_{y\downarrow\alpha} \frac{f(y)}{\varphi(y)}, \quad \lim_{n\to\infty} \frac{f(\beta_n)}{\psi(\beta_n)} = \limsup_{y\uparrow\beta} \frac{f(y)}{\psi(y)}, \tag{117}$$

$$\text{if } \alpha \text{ is absorbing and } f(\alpha) = \limsup_{y\downarrow\alpha} f(y), \text{ then } \alpha_n = \alpha \text{ for all } n \geq 1, \tag{118}$$

$$\text{and if } \beta \text{ is absorbing and } f(\beta) = \limsup_{y\uparrow\beta} f(y), \text{ then } \beta_n = \beta \text{ for all } n \geq 1. \tag{119}$$

Also, let $\mathbb{S}_x$ be any weak solution to (1), and define the associated stopping times

$$\tau^\star = \inf\{t \geq 0 \mid \ v(X_t) = f(X_t)\} \quad \text{and} \quad \tau_n^\star = \tau^\star \wedge T_{\alpha_n} \wedge T_{\beta_n}. \tag{120}$$

Then

$$v(x) = \lim_{n\to\infty} \mathbb{E}_x \left[e^{-\Lambda_{\tau_n^\star}} f(X_{\tau_n^\star}) \right]. \tag{121}$$

Furthermore, the stopping strategy $(\mathbb{S}_x, \tau^\star) \in \mathcal{T}_x$ *is optimal if*

$$\limsup_{y\downarrow\alpha} \frac{f(y)}{\varphi(y)} = 0 \text{ if } \alpha \text{ is inaccessible}, \quad \limsup_{y\uparrow\beta} \frac{f(y)}{\psi(y)} = 0 \text{ if } \beta \text{ is inaccessible}, \tag{122}$$

$$f(\alpha) = \limsup_{y\downarrow\alpha} f(y) \text{ if } \alpha \text{ is absorbing} \quad \text{and} \quad f(\beta) = \limsup_{y\uparrow\beta} f(y) \text{ if } \beta \text{ is absorbing}. \tag{123}$$

Proof. We have established part (I) of the theorem in Lemma 4, so we assume that (113) holds in what follows. In view of (56) and the fact that $-\mathcal{L}v$ is a positive measure on $\big(\operatorname{int}\mathcal{I}, \mathcal{B}(\operatorname{int}\mathcal{I})\big)$ (see Theorem 9.(I) and Lemma 11), we can see that the restriction of v in $\operatorname{int}\mathcal{I}$ satisfies all of the assumptions of Theorem 7. Therefore, the limits of v/φ and v/ψ in (114) exist,

$$v(x) = \lim_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} \frac{v(y)}{\varphi(y)}\varphi(x) + R_{-\mathcal{L}v}(x) + \lim_{y\in\operatorname{int}\mathcal{I},\, y\uparrow\beta} \frac{v(y)}{\psi(y)}\psi(x) \quad \text{for all } x \in \operatorname{int}\mathcal{I}, \tag{124}$$

and, given any stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{T}_x$,

$$\mathbb{E}_x \left[e^{-\Lambda_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}} \tilde{v}(X_{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}}) \right] = v(x) + \mathbb{E}_x \left[\int_0^{\tau\wedge T_{\bar\alpha}\wedge T_{\bar\beta}} e^{-\Lambda_u}\, dA_u^{\mathcal{L}v} \right] \tag{125}$$

for all $\bar\alpha < x < \bar\beta$ in $\mathcal{I}$, where

$$\tilde{v}(x) = \begin{cases} v(x), & \text{if } x \in \operatorname{int}\mathcal{I}, \\ \lim_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} v(y), & \text{if } \alpha \text{ is absorbing and } x = \alpha, \\ \lim_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} v(y), & \text{if } \beta \text{ is absorbing and } x = \beta. \end{cases} \tag{126}$$

If α (resp., β) is absorbing, then (115) plainly holds true and

$$f(\alpha) = v(\alpha) \le \liminf_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} v(y) \quad \left(\text{resp., } f(\beta) = v(\beta) \le \liminf_{y\in\operatorname{int}\mathcal{I},\, y\uparrow\beta} v(y) \right),$$

thanks to the $r(\cdot)$-excessivity of v (see Theorem 9.(II) and Lemma 11). Combining this observation with the fact that the limit of v/φ in (114) exists, we can see that

$$\lim_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} \frac{v(y)}{\varphi(y)} = \limsup_{y\downarrow\alpha} \frac{v(y)}{\varphi(y)} \overset{(56)}{=} \limsup_{y\downarrow\alpha} \frac{f(y)}{\varphi(y)}.$$

We can establish the second identity in (114) similarly.

With each initial condition $x \in \operatorname{int}\mathcal{I}$, we associate any monotone sequences (α_n), (β_n) in $\mathcal{I}$ such that (116)–(119) hold true. If α (resp., β) is absorbing and $\alpha_n = \alpha$ (resp., $\beta_n = \beta$), then (114)–(115) and (118)–(119) imply that

$$v(\alpha) = \lim_{y \in \operatorname{int}\mathcal{I},\, y \downarrow \alpha} v(y) \quad \left(\text{resp., } v(\beta) = \lim_{y \in \operatorname{int}\mathcal{I},\, y \uparrow \beta} v(y) \right).$$

This observation, the definition of $\tilde{v}$ in (126) and (125) imply that

$$\mathbb{E}_x \left[e^{-\Lambda_{\tau \wedge T_{\alpha_n} \wedge T_{\beta_n}}} v(X_{\tau \wedge T_{\alpha_n} \wedge T_{\beta_n}}) \right] = v(x) + \mathbb{E}_x \left[\int_0^{\tau \wedge T_{\alpha_n} \wedge T_{\beta_n}} e^{-\Lambda_u} \, dA_u^{\mathcal{L}v} \right] \tag{127}$$

for every stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{T}_x$. Furthermore, (114) and (117) imply that

$$\lim_{n\to\infty} \frac{v(\alpha_n)}{\varphi(\alpha_n)} = \lim_{n\to\infty} \frac{\overline{f}(\alpha_n)}{\varphi(\alpha_n)} = \lim_{n\to\infty} \frac{f(\alpha_n)}{\varphi(\alpha_n)} \tag{128}$$

and

$$\lim_{n\to\infty} \frac{v(\beta_n)}{\psi(\beta_n)} = \lim_{n\to\infty} \frac{f(\beta_n)}{\psi(\beta_n)} = \lim_{n\to\infty} \frac{\overline{f}(\beta_n)}{\psi(\beta_n)}. \tag{129}$$

Given a stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{T}_x$ such that $\tau = \tau \wedge T_\alpha \wedge T_\beta$, we can use (26)–(27) to calculate

$$\begin{aligned}
&\mathbb{E}_x \left[e^{-\Lambda_{\tau \wedge T_{\alpha_n} \wedge T_{\beta_n}}} \left[v(X_{\tau \wedge T_{\alpha_n} \wedge T_{\beta_n}}) - \overline{f}(X_{\tau \wedge T_{\alpha_n} \wedge T_{\beta_n}}) \right] \right] \\
&\quad = \mathbb{E}_x \left[e^{-\Lambda_\tau} \left[v(X_\tau) - \overline{f}(X_\tau) \right] \mathbf{1}_{\{\tau \le T_{\alpha_n} \wedge T_{\beta_n}\}} \right] + \left[v(\alpha_n) - \overline{f}(\alpha_n) \right] \mathbb{E}_x \left[e^{-\Lambda_{T_{\alpha_n}}} \mathbf{1}_{\{T_{\alpha_n} < \tau \wedge T_{\beta_n}\}} \right] \\
&\qquad + \left[v(\beta_n) - \overline{f}(\beta_n) \right] \mathbb{E}_x \left[e^{-\Lambda_{T_{\beta_n}}} \mathbf{1}_{\{T_{\beta_n} < \tau \wedge T_{\alpha_n}\}} \right] \\
&\quad = \mathbb{E}_x \left[e^{-\Lambda_\tau} \left[v(X_\tau) - \overline{f}(X_\tau) \right] \mathbf{1}_{\{\tau \le T_{\alpha_n} \wedge T_{\beta_n}\}} \right] + \varphi(x) \frac{v(\alpha_n) - \overline{f}(\alpha_n)}{\varphi(\alpha_n)} \frac{\mathbb{E}_x \left[e^{-\Lambda_{T_{\alpha_n}}} \mathbf{1}_{\{T_{\alpha_n} < \tau \wedge T_{\beta_n}\}} \right]}{\mathbb{E}_x \left[e^{-\Lambda_{T_{\alpha_n}}} \right]} \\
&\qquad + \psi(x) \frac{v(\beta_n) - \overline{f}(\beta_n)}{\psi(\beta_n)} \frac{\mathbb{E}_x \left[e^{-\Lambda_{T_{\beta_n}}} \mathbf{1}_{\{T_{\beta_n} < \tau \wedge T_{\alpha_n}\}} \right]}{\mathbb{E}_x \left[e^{-\Lambda_{T_{\beta_n}}} \right]}.
\end{aligned}$$

Combining this calculation with (128)–(129) and the monotone convergence theorem, we can see that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}_x \left[e^{-\Lambda_{\tau \wedge T_{\alpha_n} \wedge T_{\beta_n}}} \left[v(X_{\tau \wedge T_{\alpha_n} \wedge T_{\beta_n}}) - \overline{f}(X_{\tau \wedge T_{\alpha_n} \wedge T_{\beta_n}}) \right] \right] \\
&\qquad\qquad = \mathbb{E}_x \left[e^{-\Lambda_\tau} \left[v(X_\tau) - \overline{f}(X_\tau) \right] \mathbf{1}_{\Gamma(\tau)} \right],
\end{aligned} \tag{130}$$

where

$$\Gamma(\tau) = \begin{cases} \{\tau < T_\alpha \wedge T_\beta\}, & \text{if } \alpha < \alpha_n < \beta_n < \beta, \\ \{\tau < T_\beta\}, & \text{if } \alpha_n = \alpha \text{ and } \beta_n < \beta, \\ \{\tau < T_\alpha\}, & \text{if } \alpha < \alpha_n \text{ and } \beta_n = \beta, \\ \Omega, & \text{if } \alpha_n = \alpha \text{ and } \beta_n = \beta, \end{cases}$$

(see also (118)–(119)).

With each initial condition $x \in \operatorname{int} \mathcal{I}$, we associate any sequence of stopping strategies $(\mathbb{S}_x^\ell, \tau_\ell) \in \mathcal{T}_x$ such that $\tau_\ell = \tau_\ell \wedge T_\alpha \wedge T_\beta$ and

$$v(x) - \frac{1}{2\ell} \leq \mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell}} \overline{f}(X_{\tau_\ell}) \mathbf{1}_{\{\tau_\ell < \infty\}} \right] \quad \text{for all } \ell \geq 1$$

(see (108) in Lemma 11). If α is absorbing and $\alpha < \alpha_n$ (see (118)), then we may assume without loss of generality that $\tau_\ell < T_\alpha$, $\mathbb{P}_x^\ell$-a.s.. To see this claim, suppose that α is absorbing and $\alpha < \alpha_n$, which is the case when $f(\alpha) < \limsup_{y \downarrow \alpha} f(y)$. Since $\tau_\ell = \tau_\ell \wedge T_\alpha \wedge T_\beta$,

$$\begin{aligned} \bigcap_{n=1}^\infty \{T_{\alpha_n} < \tau_\ell\} &= \bigcap_{n=1}^\infty \{T_{\alpha_n} < \tau_\ell \wedge T_\alpha\} \cap \{T_{\alpha_n} < T_\beta\} \\ &= \{T_\alpha \leq \tau_\ell \wedge T_\alpha\} \cap \{T_\alpha < T_\beta\} \\ &= \{\tau_\ell = T_\alpha\} \cap \{T_\alpha < T_\beta\}. \end{aligned}$$

In view of this observation and the dominated convergence theorem, we can see that

$$\begin{aligned} &\lim_{n \to \infty} \mathbb{E}\big[e^{-\Lambda_{\tau_\ell}} \big[f(\alpha_n) - f(X_{\tau_\ell}) \big] \mathbf{1}_{\{T_{\alpha_n} < \tau_\ell\}} \mathbf{1}_{\{\tau_\ell < \infty\}} \big] \\ &\qquad = \left[\limsup_{y \downarrow \alpha} f(y) - f(\alpha) \right] \mathbb{E}\left[e^{-\Lambda_{\tau_\ell}} \mathbf{1}_{\{\tau_\ell = T_\alpha\} \cap \{T_\alpha < T_\beta\}} \right]. \end{aligned}$$

If $\mathbb{P}_x^\ell(\tau_\ell = T_\alpha) > 0$, then the right-hand side of this identity is strictly positive, and there exists $k \geq 1$ such that

$$f(\alpha_k) \mathbb{E}\big[e^{-\Lambda_{\tau_\ell}} \mathbf{1}_{\{T_{\alpha_k} < \tau_\ell\}} \big] \geq \mathbb{E}\big[e^{-\Lambda_{\tau_\ell}} f(X_{\tau_\ell}) \mathbf{1}_{\{T_{\alpha_k} < \tau_\ell\}} \mathbf{1}_{\{\tau_\ell < \infty\}} \big].$$

Given such a k, we can see that

$$\begin{aligned} &\mathbb{E}\big[e^{-\Lambda_{\tau_\ell \wedge T_{\alpha_k}}} f(X_{\tau_\ell \wedge T_{\alpha_k}}) \mathbf{1}_{\{\tau_\ell \wedge T_{\alpha_k} < \infty\}} \big] \\ &\qquad = \mathbb{E}\left[e^{-\Lambda_{\tau_\ell}} f(X_{\tau_\ell}) \mathbf{1}_{\{\tau_\ell \leq T_{\alpha_k}\} \cap \{\tau_\ell < \infty\}} \right] + f(\alpha_k) \mathbb{E}\left[e^{-\Lambda_{T_{\alpha_k}}} \mathbf{1}_{\{T_{\alpha_k} < \tau_\ell\}} \right] \\ &\qquad \geq \mathbb{E}\left[e^{-\Lambda_{\tau_\ell}} f(X_{\tau_\ell}) \mathbf{1}_{\{\tau_\ell \leq T_{\alpha_k}\} \cap \{\tau_\ell < \infty\}} \right] + \mathbb{E}\big[e^{-\Lambda_{\tau_\ell}} f(X_{\tau_\ell}) \mathbf{1}_{\{T_{\alpha_k} < \tau_\ell\}} \mathbf{1}_{\{\tau_\ell < \infty\}} \big] \\ &\qquad = \mathbb{E}\left[e^{-\Lambda_{\tau_\ell}} f(X_{\tau_\ell}) \mathbf{1}_{\{\tau_\ell < \infty\}} \right], \end{aligned}$$

and the claim follows. Similarly, we may assume that $\tau_\ell < T_\beta$, $\mathbb{P}_x^\ell$-a.s., if β is absorbing and $\beta_n < \beta$.

In light of the above observations and (118)–(119), we can use the monotone convergence theorem to calculate

$$\begin{aligned}\liminf_{n\to\infty} \mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell\wedge T_{\alpha_n}\wedge T_{\beta_n}}} \overline{f}(X_{\tau_\ell\wedge T_{\alpha_n}\wedge T_{\beta_n}})\right] &\geq \lim_{n\to\infty} \mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell}} \overline{f}(X_{\tau_\ell}) \mathbf{1}_{\{\tau_\ell\leq T_{\alpha_n}\wedge T_{\beta_n}\}}\right] \\ &= \mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell}} \overline{f}(X_{\tau_\ell}) \mathbf{1}_{\{\tau_\ell<\infty\}}\right],\end{aligned}$$

which implies that, for all $\ell \geq 1$, there exists n_ℓ such that

$$\mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell}} \overline{f}(X_{\tau_\ell}) \mathbf{1}_{\{\tau_\ell<\infty\}}\right] \leq \mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell\wedge T_{\alpha_{n_\ell}}\wedge T_{\beta_{n_\ell}}}} \overline{f}(X_{\tau_\ell\wedge T_{\alpha_{n_\ell}}\wedge T_{\beta_{n_\ell}}})\right] + \frac{1}{2\ell}.$$

It follows that, if we define

$$\tau_\ell^\circ = \tau_\ell \wedge T_{\alpha_{n_\ell}} \wedge T_{\beta_{n_\ell}}, \tag{131}$$

then the stopping strategy $(\mathbb{S}_x^\ell, \tau_\ell^\circ) \in \mathcal{T}_x$ satisfies

$$v(x) - \mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell^\circ}} \overline{f}(X_{\tau_\ell^\circ})\right] \leq \frac{1}{\ell}. \tag{132}$$

In view of (127) and (131), we can see that

$$v(x) - \mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell^\circ}} \overline{f}(X_{\tau_\ell^\circ})\right] = \mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell^\circ}} \left[v(X_{\tau_\ell^\circ}) - \overline{f}(X_{\tau_\ell^\circ})\right]\right] + \mathbb{E}_x^\ell \left[-\int_0^{\tau_\ell^\circ} e^{-\Lambda_u}\, dA_u^{\mathcal{L}v}\right]. \tag{133}$$

The first term on the right-hand side of this identity is clearly positive, while, the second one is positive because $-\mathcal{L}v$ is a positive measure and $-A^{\mathcal{L}v}$ is an increasing process (see also (40) in Lemma 1). This observation and (132)–(133) imply that

$$\lim_{\ell\to\infty} \mathbb{E}_x^\ell \left[e^{-\Lambda_{\tau_\ell^\circ}} \left[v(X_{\tau_\ell^\circ}) - \overline{f}(X_{\tau_\ell^\circ})\right]\right] = \lim_{\ell\to\infty} \mathbb{E}_x^\ell \left[-\int_0^{\tau_\ell^\circ} e^{-\Lambda_u}\, dA_u^{\mathcal{L}v}\right] = 0. \tag{134}$$

Proof of (II). To prove that v satisfies the variational inequality (109) in the sense of Definition 1, and thus complete the proof of part (II) of the theorem, we have to show that (112) holds true because $v \geq \overline{f}$ and $-\mathcal{L}v$ is a positive measure. To this end, we consider any interval $[\tilde{\alpha}, \tilde{\beta}] \subseteq \{x \in \operatorname{int}\mathcal{I} \mid\ v(x) > \overline{f}(x)\}$ and we note that there exists $\xi > 0$ such that

$$\xi \leq \min_{x\in[\tilde{\alpha},\tilde{\beta}]} \left[v(x) - \overline{f}(x)\right] \leq \max_{x\in[\tilde{\alpha},\tilde{\beta}]} v(x) \leq \xi^{-1}$$

because the restrictions of $v - \overline{f}$ and v in $\operatorname{int}\mathcal{I}$ are lower semicontinuous and continuous, respectively. In view of this observation, we can see that

$$e^{-\Lambda_{\tau_\ell^\circ}} \left[v(X_{\tau_\ell^\circ}) - \overline{f}(X_{\tau_\ell^\circ})\right] \geq \xi e^{-\Lambda_{\tau_\ell^\circ}} \mathbf{1}_{\{\tau_\ell^\circ<T_{\tilde{\alpha}}\wedge T_{\tilde{\beta}}\}} \geq \xi e^{-\Lambda_{T_{\tilde{\alpha}}}} \mathbf{1}_{\{\tau_\ell^\circ<T_{\tilde{\alpha}}<T_{\tilde{\beta}}\}} + \xi e^{-\Lambda_{T_{\tilde{\beta}}}} \mathbf{1}_{\{\tau_\ell^\circ<T_{\tilde{\beta}}<T_{\tilde{\alpha}}\}}$$

and

$$e^{-\Lambda_{\tau_\ell^\circ}}\left[v(X_{\tau_\ell^\circ})-\overline{f}(X_{\tau_\ell^\circ})\right]\geq \xi e^{-\Lambda_{\tau_\ell^\circ}}\mathbf{1}_{\{\tau_\ell^\circ<T_{\tilde\alpha}\wedge T_{\tilde\beta}\}}\geq \xi^2 e^{-\Lambda_{\tau_\ell^\circ}}v(X_{\tau_\ell^\circ})\mathbf{1}_{\{\tau_\ell^\circ<T_{\tilde\alpha}\wedge T_{\tilde\beta}\}}.$$

These inequalities and (134) imply that

$$\lim_{\ell\to\infty}\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\alpha}}}\mathbf{1}_{\{\tau_\ell^\circ<T_{\tilde\alpha}<T_{\tilde\beta}\}}\right]=0,\qquad \lim_{\ell\to\infty}\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\beta}}}\mathbf{1}_{\{\tau_\ell^\circ<T_{\tilde\beta}<T_{\tilde\alpha}\}}\right]=0,$$

$$\lim_{\ell\to\infty}\mathbb{E}_x^\ell\left[e^{-\Lambda_{\tau_\ell^\circ}}v(X_{\tau_\ell^\circ})\mathbf{1}_{\{\tau_\ell^\circ<T_{\tilde\alpha}\wedge T_{\tilde\beta}\}}\right]=0\quad\text{and}\quad \lim_{\ell\to\infty}\mathbb{E}_x^\ell\left[-\int_0^{\tau_\ell^\circ\wedge T_{\tilde\alpha}\wedge T_{\tilde\beta}}e^{-\Lambda_u}\,dA_u^{\mathcal{L}v}\right]=0. \quad (135)$$

The first of these limits implies that

$$\lim_{\ell\to\infty}\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\alpha}}}\mathbf{1}_{\{T_{\tilde\alpha}\leq\tau_\ell^\circ\wedge T_{\tilde\beta}\}}\right]=\lim_{\ell\to\infty}\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\alpha}}}\mathbf{1}_{\{T_{\tilde\alpha}<T_{\tilde\beta}\}}\right] \quad (136)$$

because $\{\tau_\ell^\circ<T_{\tilde\alpha}<T_{\tilde\beta}\}=\{T_{\tilde\alpha}<T_{\tilde\beta}\}\setminus\{T_{\tilde\alpha}\leq\tau_\ell^\circ\wedge T_{\tilde\beta}\}$. Similarly, the second limit implies that

$$\lim_{\ell\to\infty}\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\beta}}}\mathbf{1}_{\{T_{\tilde\beta}\leq\tau_\ell^\circ\wedge T_{\tilde\alpha}\}}\right]=\lim_{\ell\to\infty}\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\beta}}}\mathbf{1}_{\{T_{\tilde\beta}<T_{\tilde\alpha}\}}\right]. \quad (137)$$

Now, (127) and (131) imply that

$$\begin{aligned}v(x)&=\mathbb{E}_x^\ell\left[e^{-\Lambda_{\tau_\ell^\circ\wedge T_{\tilde\alpha}\wedge T_{\tilde\beta}}}v(X_{\tau_\ell^\circ\wedge T_{\tilde\alpha}\wedge T_{\tilde\beta}})\right]+\mathbb{E}_x^\ell\left[-\int_0^{\tau_\ell^\circ\wedge T_{\tilde\alpha}\wedge T_{\tilde\beta}}e^{-\Lambda_u}\,dA_u^{\mathcal{L}v}\right]\\&=\mathbb{E}_x^\ell\left[e^{-\Lambda_{\tau_\ell^\circ}}v(X_{\tau_\ell^\circ})\mathbf{1}_{\{\tau_\ell^\circ<T_{\tilde\alpha}\wedge T_{\tilde\beta}\}}\right]+v(\tilde\alpha)\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\alpha}}}\mathbf{1}_{\{T_{\tilde\alpha}\leq\tau_\ell^\circ\wedge T_{\tilde\beta}\}}\right]\\&\quad+v(\tilde\beta)\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\beta}}}\mathbf{1}_{\{T_{\tilde\beta}\leq\tau_\ell^\circ\wedge T_{\tilde\alpha}\}}\right]+\mathbb{E}_x^\ell\left[-\int_0^{\tau_\ell^\circ\wedge T_{\tilde\alpha}\wedge T_{\tilde\beta}}e^{-\Lambda_u}\,dA_u^{\mathcal{L}v}\right].\end{aligned}$$

In view of (135)–(137), we can pass to the limit as $\ell\to\infty$ to obtain

$$\begin{aligned}v(x)&=\lim_{\ell\to\infty}\left\{v(\tilde\alpha)\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\alpha}}}\mathbf{1}_{\{T_{\tilde\alpha}<T_{\tilde\beta}\}}\right]+v(\tilde\beta)\mathbb{E}_x^\ell\left[e^{-\Lambda_{T_{\tilde\beta}}}\mathbf{1}_{\{T_{\tilde\beta}<T_{\tilde\alpha}\}}\right]\right\}\\&=v(\tilde\alpha)\frac{\varphi(\tilde\beta)\psi(x)-\varphi(x)\psi(\tilde\beta)}{\varphi(\tilde\beta)\psi(\tilde\alpha)-\varphi(\tilde\alpha)\psi(\tilde\beta)}+v(\tilde\beta)\frac{\varphi(x)\psi(\tilde\alpha)-\varphi(\tilde\alpha)\psi(x)}{\varphi(\tilde\beta)\psi(\tilde\alpha)-\varphi(\tilde\alpha)\psi(\tilde\beta)},\end{aligned}$$

the second identity following from (52)–(53) in Lemma 3. Since this identity is true for all $x\in\,]\tilde\alpha,\tilde\beta[$ and $\mathcal{L}\varphi=\mathcal{L}\psi=0$, it follows that the restriction of the measure $\mathcal{L}v$ in $x\in\,]\tilde\alpha,\tilde\beta[$ vanishes, which establishes (112).

Proof of (III). We now assume that $f=\overline{f}$ and we consider the stopping times $\tau^\star$ and $\tau_n^\star$ that are defined by (120) on any given weak solution $\mathbb{S}_x$ to (1). In view of (127) and the fact that v satisfies (112), we can see that

$$v(x)-\mathbb{E}_x\left[e^{-\Lambda_{\tau_n^\star}}f(X_{\tau_n^\star})\right]=\mathbb{E}_x\left[e^{-\Lambda_{\tau_n^\star}}\left[v(X_{\tau_n^\star})-f(X_{\tau_n^\star})\right]\right].$$

Combining this result with the identities

$$\lim_{n\to\infty} \mathbb{E}_x\big[e^{-\Lambda_{\tau_n^\star}}\left[v(X_{\tau_n^\star}) - f(X_{\tau_n^\star})\right]\big] \overset{(130)}{=} \mathbb{E}_x\left[e^{-\Lambda_{\tau^\star}}\left[v(X_{\tau^\star}) - f(X_{\tau^\star})\right]\mathbf{1}_{\Gamma(\tau^\star)}\right] = 0,$$

we obtain (121).

To establish the optimality of $(\mathbb{S}_x, \tau^\star)$ if $f = \overline{f}$ and (122)–(123) are satisfied, we first note that if α is inaccessible, then

$$0 \leq \lim_{n\to\infty} f(\alpha_n)\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_n}}}\mathbf{1}_{\{T_{\alpha_n}<\tau^\star\wedge T_{\beta_n}\}}\right] \leq \lim_{n\to\infty} f(\alpha_n)\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_n}}}\right] \overset{(26)}{=} \lim_{n\to\infty}\frac{f(\alpha_n)\varphi(x)}{\varphi(\alpha_n)} = 0.$$

Similarly, if β is inaccessible, then

$$\lim_{n\to\infty} f(\beta_n)\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta_n}}}\mathbf{1}_{\{T_{\beta_n}<\tau^\star\wedge T_{\alpha_n}\}}\right] = 0.$$

In view of (118)–(119) and (123), we can see that, if α (resp., β) is absorbing, then $\alpha_n = \alpha$ (resp., $\beta_n = \beta$) and

$$\{T_{\alpha_n} < \tau^* \wedge T_{\beta_n}\} = \{T_\alpha < \tau^* \wedge T_{\beta_n}\} = \emptyset \quad \big(\text{resp., } \{T_{\beta_n} < \tau^\star \wedge T_{\alpha_n}\} = \emptyset\big).$$

In light of these observations and the monotone convergence theorem, we can see that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau_n^\star}}f(X_{\tau_n^\star})\right] &= \lim_{n\to\infty}\Big(\mathbb{E}_x\left[e^{-\Lambda_{\tau^\star}}f(X_{\tau^\star})\mathbf{1}_{\{\tau^\star\leq T_{\alpha_n}\wedge T_{\beta_n}\}}\right] \\ &\qquad + f(\alpha_n)\mathbb{E}_x\left[e^{-\Lambda_{T_{\alpha_n}}}\mathbf{1}_{\{T_{\alpha_n}<\tau^\star\wedge T_{\beta_n}\}}\right] + f(\beta_n)\mathbb{E}_x\left[e^{-\Lambda_{T_{\beta_n}}}\mathbf{1}_{\{T_{\beta_n}<\tau^\star\wedge T_{\alpha_n}\}}\right]\Big) \\ &= \mathbb{E}_x\big[e^{-\Lambda_{\tau^\star}}f(X_{\tau^\star})\mathbf{1}_{\{\tau^\star<\infty\}}\big],\end{aligned}$$

and the optimality of $(\mathbb{S}_x, \tau^\star)$ follows thanks to (121). □

It is straightforward to see that the variational inequality (109) does not have a unique solution. In the previous result, we proved that the value function v satisfies (109) as well as the boundary / growth conditions (114). We now establish a converse result, namely a verification theorem, which shows that v is the minimal solution to (109).

Theorem 13 *Consider the optimal stopping problem formulated in Section 2 and suppose that (113) holds true. The following statements are true.*
(I) *If a function $w : \operatorname{int}\mathcal{I} \to \mathbb{R}_+$ is the difference of two convex functions such that $-\mathcal{L}w$ is a positive measure, $w(x) \geq f(x)$ for all $x \in \operatorname{int}\mathcal{I}$,*

$$\limsup_{y\downarrow\alpha}\frac{w(y)}{\varphi(y)} < \infty \quad \textit{and} \quad \limsup_{y\uparrow\beta}\frac{w(y)}{\psi(y)} < \infty,$$

then $v(x) \leq w(x)$ *for all* $x \in \operatorname{int}\mathcal{I}$.
(II) *If a function* $w : \operatorname{int}\mathcal{I} \to \mathbb{R}_+$ *is a solution to the variational inequality (109) in the sense of Definition 1 that satisfies*

$$\limsup_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} \frac{w(y)}{\varphi(y)} = \limsup_{y\in\mathcal{I},\, y\downarrow\alpha} \frac{f(y)}{\varphi(y)} \quad \textit{and} \quad \limsup_{y\in\operatorname{int}\mathcal{I},\, y\uparrow\beta} \frac{w(y)}{\psi(y)} = \limsup_{y\in\mathcal{I},\, y\uparrow\beta} \frac{f(y)}{\psi(y)}, \tag{138}$$

then $v(x) = w(x)$ *for all* $x \in \operatorname{int}\mathcal{I}$.
(III) *The value function* v *admits the characterisation*

$$v(x) = \inf\bigl\{A\varphi(x) + B\psi(x) \mid \ A, B \geq 0 \text{ and } A\varphi(y) + B\psi(y) \geq f(y) \text{ for all } y \in \operatorname{int}\mathcal{I}\bigr\} \tag{139}$$

for all $x \in \operatorname{int}\mathcal{I}$. *Furthermore, if* $c < d$ *are any points in* $\mathcal{I}$ *such that* $v(x) > \overline{f}(x)$ *for all* $x \in]c, d[$, *then there exist constants* $\tilde{A}$, $\tilde{B}$ *such that*

$$v(x) = \tilde{A}\varphi(x) + \tilde{B}\psi(x) \text{ and } \tilde{A}\varphi(y) + \tilde{B}\psi(y) \geq f(y) \text{ for all } x \in]c, d[\text{ and } y \in \operatorname{int}\mathcal{I}. \tag{140}$$

Proof. A function $w : \operatorname{int}\mathcal{I} \to \mathbb{R}_+$ that is as in the statement of part (I) of the theorem satisfies all of the requirements of Theorem 7. Therefore, if $\mathcal{I}$ is not open and we identify w with its extension on $\mathcal{I}$ that is given by $w(\alpha) = \lim_{y\downarrow\alpha} w(y)$ (resp., $w(\beta) = \lim_{y\uparrow\beta} w(y)$) if α (resp., β) is absorbing, then

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_{\alpha_n}\wedge T_{\beta_n}}} w(X_{\tau\wedge T_{\alpha_n}\wedge T_{\beta_n}})\right] = w(x) + \mathbb{E}_x\left[\int_0^{\tau\wedge T_{\alpha_n}\wedge T_{\beta_n}} e^{-\Lambda_u}\, dA_u^{\mathcal{L}w}\right] \tag{141}$$

for every stopping strategy $(\mathbb{S}_x, \tau) \in \mathcal{T}_x$, where (α_n), (β_n) are any monotone sequences in $\mathcal{I}$ satisfying (116). Combining this identity with the fact that $-\mathcal{L}w$ is a positive measure, which implies that $-A^{\mathcal{L}w}$ is an increasing process, we can see that

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_{\alpha_n}\wedge T_{\beta_n}}} w(X_{\tau\wedge T_{\alpha_n}\wedge T_{\beta_n}})\right] \leq w(x). \tag{142}$$

This inequality and Fatou's lemma imply that

$$\mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_\alpha\wedge T_\beta}} w(X_{\tau\wedge T_\alpha\wedge T_\beta})\right] \leq \liminf_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau\wedge T_{\alpha_n}\wedge T_{\beta_n}}} w(X_{\tau\wedge T_{\alpha_n}\wedge T_{\beta_n}})\right] \leq w(x),$$

which, combined with the inequality $w \geq f$, proves that $v(x) \leq w(x)$.

If the function w satisfies (138) as well, then we choose any monotone sequences (α_n), (β_n) as in (116)–(119) and we note that (128)–(129) hold true with the extension of w on $\mathcal{I}$ considered at the beginning of the proof in place of v. If we consider the stopping strategies $(\mathbb{S}_x, \tau_n^\star) \in \mathcal{T}_x$, where

$$\tau_n^\star = \bigl(\inf\bigl\{t \geq 0 \mid \ w(X_t) = \overline{f}(X_t)\bigr\}\bigr) \wedge T_{\alpha_n} \wedge T_{\beta_n},$$

then we can see that (130) with w in place of v and (141) imply that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau_n^\star}}\overline{f}(X_{\tau_n^\star})\right] &= w(x) + \lim_{n\to\infty} \mathbb{E}_x\left[e^{-\Lambda_{\tau_n^\star}}\left[\overline{f}(X_{\tau_n^\star}) - w(X_{\tau_n^\star})\right]\right] \\ &= w(x) + \mathbb{E}_x\left[e^{-\Lambda_{\tau^\star}}\left[\overline{f}(X_{\tau^\star}) - w(X_{\tau^\star})\right]\mathbf{1}_{\Gamma(\tau^\star)}\right] \\ &= w(x).\end{aligned}$$

It follows that $v(x) \geq w(x)$ thanks to (108) in Lemma 11, which, combined with the inequality $v(x) \leq w(x)$ that we have established above, implies that $v(x) = w(x)$.

To show part (III) of the theorem, we first note that, given any constants $A, B \in \mathbb{R}$, the function $A\varphi + B\psi$ satisfies the variational inequality (109) if and only if $A\varphi + B\psi \geq f$. Combining this observation with part (I) of the theorem, we can see that $v(x)$ is less than or equal to the right-hand side of (139). To establish the reverse inequality, we first use (51) in Lemma 3 and (127) with $\tau \equiv \infty$ to obtain

$$\begin{aligned}&\mathbb{E}_{\bar{x}}\left[e^{-\Lambda_{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}}}\left[v(X_{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}}) - A\varphi(X_{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}}) - B\psi(X_{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}})\right]\right] \\ &\qquad = v(\bar{x}) - A\varphi(\bar{x}) - B\psi(\bar{x}) + \mathbb{E}_{\bar{x}}\left[\int_0^{T_{\bar{\alpha}}\wedge T_{\bar{\beta}}} e^{-\Lambda_u}\, dA_u^{\mathcal{L}v}\right] \qquad (143)\end{aligned}$$

for all points $\bar{\alpha} < \bar{x} < \bar{\beta}$ in $\operatorname{int}\mathcal{I}$ and all constants $A, B \in \mathbb{R}$. Also, we fix any point $x \in \operatorname{int}\mathcal{I}$ and we consider any monotone sequences (α_n), (β_n) in $\operatorname{int}\mathcal{I}$ such that

$$\alpha_n < x < \beta_n \text{ for all } n \geq 1 \quad \text{and} \quad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = x. \qquad (144)$$

If we define

$$A_n = \frac{v(\beta_n)\psi(\alpha_n) - v(\alpha_n)\psi(\beta_n)}{\varphi(\beta_n)\psi(\alpha_n) - \varphi(\alpha_n)\psi(\beta_n)} \quad \text{and} \quad B_n = \frac{\varphi(\beta_n)v(\alpha_n) - \varphi(\alpha_n)v(\beta_n)}{\varphi(\beta_n)\psi(\alpha_n) - \varphi(\alpha_n)\psi(\beta_n)},$$

then we can check that

$$A_n\varphi(\alpha_n) + B_n\psi(\alpha_n) = v(\alpha_n) \quad \text{and} \quad A_n\varphi(\beta_n) + B_n\psi(\beta_n) = v(\beta_n),$$

and observe that the identity

$$0 = v(x) - A_n\varphi(x) - B_n\psi(x) + \mathbb{E}_x\left[\int_0^{T_{\alpha_n}\wedge T_{\beta_n}} e^{-\Lambda_u}\, dA_u^{\mathcal{L}v}\right]$$

follows immediately from (143) for $\bar{\alpha} = \alpha_n$, $\bar{x} = x$ and $\bar{\beta} = \beta_n$. Since $-A^{\mathcal{L}v} = A^{-\mathcal{L}v}$ is a continuous increasing process, this identity, (144) and the dominated convergence theorem imply that

$$v(x) \geq A_n\varphi(x) + B_n\psi(x) \quad \text{and} \quad v(x) = \lim_{n\to\infty}\left[A_n\varphi(x) + B_n\psi(x)\right]. \qquad (145)$$

Also, given any $y \in]\beta_n, \beta[$, we can see that (143) with $\bar{\alpha} = \alpha_n$, $\bar{x} = \beta_n$ and $\bar{\beta} = y$ yields

$$\bigl[v(y) - A_n\varphi(y) - B_n\psi(y)\bigr]\mathbb{E}_{\beta_n}\left[e^{-\Lambda_{T_y}}\mathbf{1}_{\{T_y<T_{\alpha_n}\}}\right] = \mathbb{E}_{\beta_n}\left[\int_0^{T_{\alpha_n}\wedge T_y} e^{-\Lambda_u}\,dA_u^{\mathcal{L}v}\right],$$

which implies that

$$A_n\varphi(y) + B_n\psi(y) \geq v(y) \quad \text{for all } y \in]\beta_n, \beta[. \tag{146}$$

Similarly,

$$A_n\varphi(y) + B_n\psi(y) \geq v(y) \quad \text{for all } y \in]\alpha, \alpha_n[. \tag{147}$$

Combining these results with (31), we can see that

$$A_n \geq \lim_{y\in\operatorname{int}\mathcal{I},\, y\downarrow\alpha} \frac{v(y)}{\varphi(y)} \geq 0 \quad \text{and} \quad B_n \geq \lim_{y\in\operatorname{int}\mathcal{I},\, y\uparrow\beta} \frac{v(y)}{\psi(y)} \geq 0 \quad \text{for all } n \geq 1.$$

If we consider any sequence (n_ℓ) such that $\lim_{\ell\to\infty} A_{n_\ell}$ exists, then the positivity of the constants A_n, B_n and (145) imply that $\lim_{\ell\to\infty} B_{n_\ell}$ also exists and that both limits are positive and finite. In particular, (145) and (146)–(147) imply that

$$v(x) = \lim_{\ell\to\infty} A_{n_\ell}\varphi(x) + \lim_{\ell\to\infty} B_{n_\ell}\psi(x)$$

and

$$v(y) \leq \lim_{\ell\to\infty} A_{n_\ell}\varphi(y) + \lim_{\ell\to\infty} B_{n_\ell}\psi(y) \quad \text{for all } y \in \operatorname{int}\mathcal{I} \setminus \{x\}.$$

It follows that $v(x)$ is greater than or equal to the right-hand side of (139).

The existence of constants $\tilde{A}$, $\tilde{B}$ such that the identity in (140) holds true follows from the fact that the measure $\mathcal{L}v$ does not charge the interval $]c, d[$. If $[d, \beta[$ is not empty, then, given any $\bar{\alpha} < \bar{x}$ in $]c, d[$ and $y \in [d, \beta[$, we can see that (143) with $\bar{\beta} = y$ yields

$$\bigl[v(y) - \tilde{A}\varphi(y) - \tilde{B}\psi(y)\bigr]\mathbb{E}_{\bar{x}}\left[e^{-\Lambda_{T_y}}\mathbf{1}_{\{T_y<T_{\bar{\alpha}}\}}\right] = \mathbb{E}_{\bar{x}}\left[\int_0^{T_{\bar{\alpha}}\wedge T_y} e^{-\Lambda_u}\,dA_u^{\mathcal{L}v}\right].$$

It follows that $\tilde{A}\varphi(y)+\tilde{B}\psi(y) \geq v(y) \geq f(y)$ because $-A^{\mathcal{L}v} = A^{-\mathcal{L}v}$ is a continuous increasing process. We can show that $\tilde{A}\varphi(y) + \tilde{B}\psi(y) \geq f(y)$ for all $y \in]\alpha, c]$, if $]\alpha, c] \neq \emptyset$, similarly, and the inequality in (140) has been established. □

7 Ramifications including a generalisation of the "principle of smooth fit"

Throughout the section, we assume that (113) is true, so that the value function is real-valued, and that $f = \overline{f}$. We can express the so-called waiting region $\mathcal{W}$ as a countable union

of pairwise disjoint open intervals because it is an open subset of $\operatorname{int}\mathcal{I}$. In particular, we write

$$\mathcal{W} = \{x \in \mathcal{I} \mid \ v(x) > f(x)\} = \bigcup_{\ell=1}^{\infty} \mathcal{W}_\ell, \tag{148}$$

where

$$\mathcal{W}_\ell =]c_\ell, d_\ell[, \quad \text{for some } c_\ell, d_\ell \in \mathcal{I} \cup \{\alpha, \beta\} \text{ such that } c_\ell \leq d_\ell,$$

and we adopt the usual convention that $]c, c[= \emptyset$ for $c \in \mathcal{I} \cup \{\alpha, \beta\}$. Since the measure $\mathcal{L}v$ does not charge the waiting region $\mathcal{W}$,

$$v(x) = A_\ell \varphi(x) + B_\ell \psi(x) \quad \text{for all } x \in \mathcal{W}_\ell, \tag{149}$$

for some constants A_ℓ and B_ℓ.

Our first result in this section is concerned with a characterisation of the value function if the problem data is such that $\mathcal{W} = \operatorname{int}\mathcal{I}$. Example 1 in Section 8 provides an illustration of this case.

Corollary 14 *Consider the optimal stopping problem formulated in Section 2, and suppose that (113) is true and $f = \overline{f}$. If $\mathcal{W}_1 =]\alpha, \beta[$ and $\mathcal{W}_\ell = \emptyset$ for $\ell > 1$, then*

$$A_1 = \limsup_{y \downarrow \alpha} \frac{f(y)}{\varphi(y)} \quad \textit{and} \quad B_1 = \limsup_{y \uparrow \beta} \frac{f(y)}{\psi(y)}. \tag{150}$$

Proof. The result follows immediately from the fact that $v(x) = A_1\varphi(x) + B_1\psi(x)$ for all $x \in \operatorname{int}\mathcal{I}$, (31) and (114). □

We next study the special case that arises when a portion of the general problem's value function has the features of the value function of a perpetual American call option, which has been extensively studied in the literature.

Corollary 15 *Consider the optimal stopping problem formulated in Section 2, and suppose that (113) is true and $f = \overline{f}$. If $\mathcal{W}_\ell =]\alpha, d_\ell[$, for some $\ell \geq 1$ and $d_\ell \in \operatorname{int}\mathcal{I}$, then*

$$A_\ell = \limsup_{y \downarrow \alpha} \frac{f(y)}{\varphi(y)}, \qquad B_\ell = \frac{1}{\psi(d_\ell)}\big[f(d_\ell) - A_\ell\varphi(d_\ell)\big] \tag{151}$$

and

$$\frac{f(x)}{A_\ell\varphi(x) + B_\ell\psi(x)} \begin{cases} < 1 & \textit{for all } x \in]\alpha, d_\ell[, \\ = 1 & \textit{for } x = d_\ell, \\ \leq 1 & \textit{for all } x > d_\ell. \end{cases} \tag{152}$$

Proof. The identities in (151) follow immediately from the fact that $v(x)$ is given by (149) for all $x \in \mathcal{W}_\ell =]\alpha, d_\ell[$, (31) and (114). The first two inequalities in (152) are trivial. Given any $x \in]\alpha, d_\ell[$, the fact that $v(x)$ is given by (149) and part (III) of Theorem 13 imply that

$$A_\ell \varphi(x) + B_\ell \psi(x) \geq f(y) \quad \text{for all } y \in \operatorname{int} \mathcal{I},$$

and the last inequality in (152) follows. □

Using similar symmetric arguments, we can establish the following result that arises in the context of a perpetual American put option.

Corollary 16 *Consider the optimal stopping problem formulated in Section 2, and suppose that (113) is true and $f = \overline{f}$. If $\mathcal{W}_\ell =]c_\ell, \beta[$, for some $\ell \geq 1$ and $c_\ell \in \operatorname{int} \mathcal{I}$, then*

$$A_\ell = \frac{1}{\varphi(c_\ell)} \big[f(c_\ell) - B_\ell \psi(c_\ell) \big], \qquad B_\ell = \limsup_{y \uparrow \beta} \frac{f(y)}{\psi(y)} \tag{153}$$

and

$$\frac{f(x)}{A_\ell \varphi(x) + B_\ell \psi(x)} \begin{cases} \leq 1 & \text{for all } x < c_\ell, \\ = 1 & \text{for } x = c_\ell, \\ < 1 & \text{for all } x \in]c_\ell, \beta[. \end{cases} \tag{154}$$

The final result in this section focuses on a special case in which a component of the waiting region $\mathcal{W}$ has compact closure in $\operatorname{int} \mathcal{I}$, which is a case that can arise in the context of the valuation of a perpetual American straddle option.

Corollary 17 *Consider the optimal stopping problem formulated in Section 2, and suppose that (113) is true and $f = \overline{f}$. If $\mathcal{W}_\ell =]c_\ell, d_\ell[$, for some $\ell \geq 1$ and $c_\ell, d_\ell \in \operatorname{int} \mathcal{I}$, then*

$$A_\ell = \frac{f(d_\ell)\psi(c_\ell) - f(c_\ell)\psi(d_\ell)}{\varphi(d_\ell)\psi(c_\ell) - \varphi(c_\ell)\psi(d_\ell)}, \qquad B_\ell = \frac{\varphi(d_\ell)f(c_\ell) - \varphi(c_\ell)f(d_\ell)}{\varphi(d_\ell)\psi(c_\ell) - \varphi(c_\ell)\psi(d_\ell)} \tag{155}$$

and

$$\frac{f(x)}{A_\ell \varphi(x) + B_\ell \psi(x)} \begin{cases} < 1 & \text{for all } x \in]c_\ell, d_\ell[, \\ = 1 & \text{for } x = c_\ell \text{ and } x = d_\ell, \\ \leq 1 & \text{for all } x \leq c_\ell \text{ and } x \geq d_\ell. \end{cases} \tag{156}$$

Proof. The expressions in (155) follow immediately from the continuity of the value function. The first two inequalities in (156) are a consequence of the definition of the waiting region $\mathcal{W}$, while the last one is an immediate consequence of part (II) of Theorem 13. □

Our final result is concerned with a generalisation of the "principle of smooth fit".

Corollary 18 *Consider the optimal stopping problem formulated in Section 2, and suppose that (113) is true and $f = \overline{f}$. Also, consider any point $y \in \operatorname{int}\mathcal{I}$ such that $y \notin \mathcal{W}$. If f admits right and left-hand derivatives at y, then*

$$f'_+(y) \leq v'_+(y) \leq v'_-(y) \leq f'_-(y). \tag{157}$$

Proof. The inequality $v'_+(y) \leq v'_-(y)$ is an immediate consequence of the fact that $\mathcal{L}v \leq 0$. The inequalities $f'_+(y) \leq v'_+(y)$ and $v'_-(y) \leq f'_-(y)$ follow from the fact that $v - f$ has a local minimum at y. □

8 Examples

We assume that an appropriate weak solution $\mathbb{S}_x$ to (1) has been associated with each initial condition $x \in \operatorname{int}\mathcal{I}$ in all of the examples that we discuss in this section. The following example shows that an optimal stopping time may not exist if (122) is not satisfied. In this example, the stopping region $\mathcal{I} \setminus \mathcal{W}$ is empty.

Example 1 Suppose that $\mathcal{I} =]0, \infty[$ and X is a geometric Brownian motion, so that

$$dX_t = bX_t\,dt + \sigma X_t\,dW_t,$$

for some constants b and σ. Also, suppose that r is a constant. In this case, it is well-known that

$$\varphi(x) = x^m \quad \text{and} \quad \psi(x) = x^n,$$

where $m < 0 < n$ are the solutions to the quadratic equation

$$\frac{1}{2}\sigma^2 k^2 + \left(b - \frac{1}{2}\sigma^2\right) k - r = 0. \tag{158}$$

In this context, if the reward function f is given by

$$f(x) = \begin{cases} \kappa(x^m - x), & \text{if } x \in]0, 1], \\ \lambda(x^n - x^{-1}), & \text{if } x > 1, \end{cases}$$

for some constants $\kappa, \lambda > 0$, then

$$\begin{aligned} v(x) &= \kappa x^m + \lambda x^n \\ &= \lim_{j\to\infty} \mathbb{E}_x\left[e^{-r(T_{\alpha_j} \wedge T_{\beta_j})} f(X_{T_{\alpha_j} \wedge T_{\beta_j}})\right] \quad \text{for all } x > 0, \end{aligned} \tag{159}$$

where (α_j) and (β_j) are any sequences in $]0, \infty[$ such that

$$\alpha_j < x < \beta_j \text{ for all } j, \quad \lim_{j\to\infty} \alpha_j = 0 \quad \text{and} \quad \lim_{j\to\infty} \beta_j = \infty. \tag{160}$$

In particular, there exists no optimal stopping time. □

The next example shows that an optimal stopping time may not exist if (122) is not satisfied, while the stopping region $\mathcal{I} \setminus \mathcal{W}$ is not empty.

Example 2 In the context of the previous example, suppose that the reward function f is given by

$$f(x) = \begin{cases} 0, & \text{if } x \in]0,1[, \\ 1, & \text{if } x = 1, \\ x^n - x^{-1}, & \text{if } x > 1. \end{cases}$$

In view of straightforward considerations, we can see that

$$v(x) = x^n \quad \text{for all } x > 0.$$

In this case,

$$\tau^\star \equiv \inf\{t \geq 0 \mid \ v(X_t) = f(X_t)\} = T_1,$$

i.e., $\tau^\star$ is the first hitting time of $\{1\}$, and

$$v(x) = \lim_{j\to\infty} \mathbb{E}_x \left[e^{-r\left(T_1 \wedge T_{\beta_j}\right)} f(X_{T_1 \wedge T_{\beta_j}}) \right] > x^m = \mathbb{E}_x \left[e^{-r\tau_\star} f(X_{\tau_\star}) \right] \quad \text{for all } x > 1,$$

where (β_j) is any sequence in $]x, \infty[$ such that $\lim_{j\to\infty} \beta_j = \infty$. □

The following example shows that an optimal stopping time may not exist if (123) is not satisfied. In this example, the stopping region $\mathcal{I} \setminus \mathcal{W}$ is empty.

Example 3 Suppose that $\mathcal{I} = \mathbb{R}_+$, X is a standard one-dimensional Brownian motion starting from $x > 0$ and absorbed at 0 and r is a constant. In this case, we can see that

$$\varphi(x) = e^{-\sqrt{2r}x} \quad \text{and} \quad \psi(x) = e^{\sqrt{2r}x} - e^{-\sqrt{2r}x}.$$

If the reward function f is given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ e^{-2\sqrt{2r}x}, & \text{if } x > 0, \end{cases}$$

then we can see that

$$v(x) = \begin{cases} 0, & \text{if } x = 0, \\ e^{-\sqrt{2r}x}, & \text{if } x > 0. \end{cases}$$

In particular,

$$v(x) = \lim_{j\to\infty} \mathbb{E}_x \left[e^{-r(T_{\alpha_j} \wedge T_{\beta_j})} f(X_{T_{\alpha_j} \wedge T_{\beta_j}}) \right] \quad \text{for all } x > 0,$$

where (α_j), (β_j) are any sequences in $]0, \infty[$ satisfying (160), and there exists no optimal stopping time. □

The following example shows that an optimal stopping time may not exist if $f \neq \overline{f}$. In particular, the first hitting time $\tau^\star$ of the stopping region $\mathcal{I} \setminus \mathcal{W}$ may not be optimal.

Example 4 Suppose that X is a standard Brownian motion, namely, $\mathcal{I} = \mathbb{R}$ and $dX_t = dW_t$, and that $r = \frac{1}{2}$. In this context, it is straightforward to verify that

$$\varphi(x) = e^{-x} \quad \text{and} \quad \psi(x) = e^{x}.$$

Also, consider the reward function

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x \in]0,1], \\ 2, & \text{if } x > 1, \end{cases}$$

which is not upper semicontinuous. In this case, we can see that

$$v(x) = \begin{cases} e^{x}, & \text{if } x \leq 0, \\ \frac{e-2}{e-e^{-1}} e^{-x} + \frac{2-e^{-1}}{e-e^{-1}} e^{x}, & \text{if } x \in]0,1], \\ 2, & \text{if } x > 1. \end{cases}$$

Given an initial condition $x < 1$ and an associated solution $\mathbb{S}_x$ to (1), we note that

$$\tau^\star \equiv \inf\big\{t \geq 0 \mid \ v(X_t) = f(X_t)\big\} = \inf\big\{t \geq 0 \mid \ X_t > 1\big\}$$

defines an $(\mathcal{F}_t)$-stopping time because we have assumed that the filtration in $\mathbb{S}_x$ satisfies the usual conditions. However, $X_{\tau^\star} = 1$, $\mathbb{P}_x$-a.s., and

$$\mathbb{E}_x\left[e^{-r\tau^\star} f(X_{\tau^\star})\right] = e^{x-1} < v(x) \quad \text{for all } x < 1.$$

In view of these considerations, we can see that there is no optimal stopping time for initial conditions $x < 1$. □

The final example that we consider illustrates that a characterisation such as the one provided by (152) in Corollary 15 has a local rather than global character.

Example 5 In the context of the previous example, we consider the reward function

$$f(x) = \begin{cases} e^{2x}, & \text{if } x < 0, \\ 1, & \text{if } x \in [0,1], \\ 1 + (x-1)^2, & \text{if } x > 1, \end{cases}$$

and we note that the calculation

$$\frac{d}{dx}\frac{f(x)}{\psi(x)} = \begin{cases} e^{x}, & \text{if } x < 0, \\ -e^{-x}, & \text{if } x \in [0,1], \\ -(x-2)^2 e^{-x}, & \text{if } x > 1 \end{cases}$$

implies that the function f/ψ is strictly increasing in $]-\infty, 0[$ and strictly decreasing in $]0, \infty[$. A first consideration of the associated optimal stopping problem suggests that the value function v could identify with the function u given by

$$u(x) = \begin{cases} e^x, & \text{if } x < 0, \\ 1, & \text{if } x \in [0, 1], \\ 1 + (x-1)^2, & \text{if } x > 1. \end{cases}$$

In particular, we can check that

$$\frac{u(x)}{u(y)} \geq \min\left\{\frac{\varphi(x)}{\varphi(y)}, \frac{\psi(x)}{\psi(y)}\right\} \quad \text{for all } x, y \in \mathbb{R}.$$

However, the function u is not excessive because

$$\mathcal{L}u(dx) \equiv \frac{1}{2}u''(dx) - \frac{1}{2}u(x)\,dx = -\delta_0(dx) - \frac{1}{2}\left(\mathbf{1}_{[0,1]}(x) + x(x-2)\mathbf{1}_{[1,\infty[}(x)\right)dx,$$

where δ_0 is the Dirac probability measure that assigns mass 1 on the set $\{0\}$, which implies that

$$\mathcal{L}u\big([c, d]\big) > 0 \quad \text{for all } 1 \leq c < d \leq 2,$$

and suggests that $[1, 2]$ should be a subset of the waiting region $\mathcal{W}$. In this example, the value function v is given by

$$v(x) = \begin{cases} e^x, & \text{if } x < 0, \\ 1, & \text{if } x \in [0, a_l], \\ \frac{1}{2}e^{a_l - x} + \frac{1}{2}e^{-a_l + x}, & \text{if } x \in]a_l, a_r[, \\ 1 + (x-1)^2, & \text{if } x > a_r, \end{cases} \tag{161}$$

where

$$a_l = 1 + \sqrt{2} + 2\ln\left(\sqrt{2} - 1\right) \simeq 0.651 \quad \text{and} \quad a_r = 1 + \sqrt{2} \simeq 2.414.$$

These values for the boundary points a_l, a_r arise by the requirements that $a_l \in]0, 1[$, $a_r > 2$ and v should be C^1 along a_l, a_r (see Corollary 18), which are associated with the system of equations

$$\begin{aligned} a_l &= a_r + 2\ln(a_r - 2), \\ a_r^4 - 4a_r^3 + 4a_r^2 - 1 &\equiv (a_r - 1)^2\left(a_r - 1 - \sqrt{2}\right)\left(a_r - 1 + \sqrt{2}\right) = 0. \end{aligned}$$

In particular, we can check that the function given by the right-hand side of (161) satisfies all of the requirements of the verification Theorem 13.(II) and therefore identifies with the value function v. □

Appendix: pasting weak solutions of SDEs

The next result is concerned with aggregating two filtrations, one of which "switches on" at a stopping time of the other one.

Theorem 19 *Consider a measurable space $(\Omega, \mathcal{F})$ and two filtrations $(\mathcal{H}_t)$, $(\mathcal{G}_t)$ such that $\mathcal{H}_\infty \cup \mathcal{G}_\infty \subseteq \mathcal{F}$. Also, suppose that $(\mathcal{G}_t)$ is right-continuous and let τ be an $(\mathcal{H}_t)$-stopping time. If we define*

$$\mathcal{F}_t = \big\{ A \in \mathcal{H}_\infty \vee \mathcal{G}_\infty \mid A \cap \{t < \tau\} \in \mathcal{H}_t \vee \mathcal{G}_0 \text{ and } A \cap \{s \leq \tau\} \in \mathcal{H}_t \vee \mathcal{G}_{t-s} \text{ for all } s \in [0,t] \big\}, \tag{162}$$

then $(\mathcal{F}_t)$ is a filtration such that

$$\mathcal{F}_{\tau+t} = \mathcal{H}_{\tau+t} \vee \mathcal{G}_t \quad \text{for all } t \geq 0 \tag{163}$$

and

$$\mathcal{F}_{t\wedge\tau} = \mathcal{H}_{t\wedge\tau} \vee \mathcal{G}_0 \quad \text{for all } t \geq 0. \tag{164}$$

Proof. First, it is straightforward to verify that

$$\mathcal{H}_t \subseteq \mathcal{F}_t \quad \text{and} \quad \mathcal{G}_0 \subseteq \mathcal{F}_t \quad \text{for all } t \geq 0. \tag{165}$$

To prove that $(\mathcal{F}_t)$ is indeed a filtration, we consider any times $u < t$ and any event $A \in \mathcal{F}_u$. Using the definition of $\mathcal{F}_u$, we can see that

$$\begin{aligned} A \cap \{t < \tau\} = A \cap \{u < \tau\} \cap \{t < \tau\} &\in \mathcal{H}_u \vee \mathcal{G}_0 \vee \mathcal{H}_t \subseteq \mathcal{H}_t \vee \mathcal{G}_0, \\ A \cap \{s \leq \tau\} &\in \mathcal{H}_u \vee \mathcal{G}_{u-s} \subseteq \mathcal{H}_t \vee \mathcal{G}_{t-s} \quad \text{for all } s \in [0,u], \end{aligned}$$

and

$$A \cap \{s \leq \tau\} = A \cap \{u \leq \tau\} \cap \{s \leq \tau\} \in \mathcal{H}_u \vee \mathcal{G}_0 \vee \mathcal{H}_s \subseteq \mathcal{H}_t \vee \mathcal{G}_{t-s} \quad \text{for all } s \in [u,t].$$

It follows that $A \in \mathcal{F}_t$.

To establish (163), we first show that $\mathcal{G}_t \subseteq \mathcal{F}_{\tau+t}$, which amounts to proving that, given any $t \geq 0$ and $A \in \mathcal{G}_t$,

$$A \cap \{\tau + t \leq u\} = A \cap \{\tau \leq u - t\} \in \mathcal{F}_u \quad \text{for all } u \geq 0. \tag{166}$$

To this end, we note that

$$A \cap \{\tau \leq u - t\} \cap \{u < \tau\} = \emptyset \in \mathcal{H}_u \vee \mathcal{G}_0 \quad \text{for all } u \geq 0.$$

Also, given any $s, u \geq 0$ such that $s \in]u-t, u]$,

$$A \cap \{\tau \leq u-t\} \cap \{s \leq \tau\} = \emptyset \in \mathcal{H}_u \vee \mathcal{G}_{u-s},$$

while, given any $s, u \geq 0$ such that $s \in [0, u-t]$,

$$A \cap \{\tau \leq u-t\} \cap \{s \leq \tau\} = A \cap \{s \leq \tau \leq u-t\} \in \mathcal{H}_{u-t} \vee \mathcal{G}_t \subseteq \mathcal{H}_u \vee \mathcal{G}_{u-s}.$$

These observations and the definition (162) of $(\mathcal{F}_t)$ imply that (166) holds true and $\mathcal{G}_t \subseteq \mathcal{F}_{\tau+t}$. Combining this result with the fact that $\mathcal{H}_{\tau+t} \subseteq \mathcal{F}_{\tau+t}$, which follows from (165), we can see that $\mathcal{H}_{\tau+t} \vee \mathcal{G}_t \subseteq \mathcal{F}_{\tau+t}$.

To prove that $\mathcal{F}_{\tau+t} \subseteq \mathcal{H}_{\tau+t} \vee \mathcal{G}_t$, we consider any $A \in \mathcal{F}_{\tau+t}$ and we show that

$$A \cap \{\tau + t \leq u\} \in \mathcal{H}_u \vee \mathcal{G}_t \quad \text{for all } u \geq 0. \tag{167}$$

Since $A \cap \{\tau + t \leq \bar{u}\} \in \mathcal{F}_{\bar{u}}$ for all $\bar{u} \geq 0$, $A \cap \{\tau \leq \bar{u}\} \in \mathcal{F}_{\bar{u}+t}$ for all $\bar{u} \geq 0$. Combining this observation with the definition (162) of $(\mathcal{F}_t)$ we can see that

$$A \cap \{\tau \leq \bar{u}\} \cap \{s \leq \tau\} \in \mathcal{H}_{\bar{u}+t} \vee \mathcal{G}_{\bar{u}+t-s} \quad \text{for all } \bar{u} \geq 0 \text{ and } s \leq \bar{u}+t. \tag{168}$$

In particular,

$$A \cap \{\bar{u} - \varepsilon \leq \tau \leq \bar{u}\} \in \mathcal{H}_{\bar{u}+t} \vee \mathcal{G}_{t+\varepsilon} \quad \text{for all } \bar{u} > 0 \text{ and } \varepsilon \in [0, \bar{u}].$$

In view of this result, we can see that, given any $u > t$,

$$A \cap \{\tau + t \leq u\} = \bigcup_{j=0}^{n-1} A \cap \left\{ \frac{j(u-t)}{n} \leq \tau \leq \frac{(j+1)(u-t)}{n} \right\} \in \mathcal{H}_u \vee \mathcal{G}_{t+\frac{(u-t)}{n}}.$$

It follows that

$$A \cap \{\tau + t \leq u\} \in \bigcap_{n=1}^{\infty} \mathcal{H}_u \vee \mathcal{G}_{t+\frac{(u-t)}{n}} = \mathcal{H}_u \vee \mathcal{G}_t \quad \text{for all } u > t,$$

the equality being true thanks to the right continuity of $(\mathcal{G}_t)$. Combining this result with the fact that

$$A \cap \{\tau + t \leq t\} \in \mathcal{H}_t \vee \mathcal{G}_t,$$

which follows from (168) for $\bar{u} = s = 0$, we obtain (167).

To prove (164), we first note that (165) implies that $\mathcal{H}_{t\wedge\tau} \vee \mathcal{G}_0 \subseteq \mathcal{F}_{t\wedge\tau}$. To establish the reverse inclusion, we consider any $A \in \mathcal{F}_{t\wedge\tau}$ and we show that

$$A \cap \{t \wedge \tau \leq u\} \in \mathcal{H}_u \vee \mathcal{G}_0 \quad \text{for all } u \geq 0.$$

Since $A \cap \{t \wedge \tau \leq \bar{u}\} \in \mathcal{F}_{\bar{u}}$ for all $\bar{u} \geq 0$, the definition (162) of $(\mathcal{F}_t)$ implies that

$$A \cap \{t \wedge \tau \leq \bar{u}\} \cap \{s \leq \tau\} \in \mathcal{H}_{\bar{u}} \vee \mathcal{G}_{\bar{u}-s} \quad \text{for all } \bar{u} \geq 0 \text{ and } s \in [0, \bar{u}]. \tag{169}$$

For $\bar{u} = s = 0$, this implies immediately that

$$A \cap \{t \wedge \tau \leq 0\} \in \mathcal{H}_0 \vee \mathcal{G}_0. \tag{170}$$

Also, it implies that

$$A \cap \{\bar{u} - \varepsilon \leq \tau \leq \bar{u}\} \in \mathcal{H}_{\bar{u}} \vee \mathcal{G}_{\varepsilon} \quad \text{for all } \bar{u} \in [0, t[\text{ and } \varepsilon \in [0, \bar{u}].$$

In view of this observation, we can see that

$$A \cap \{t \wedge \tau \leq u\} = \bigcup_{j=0}^{n-1} A \cap \left\{ \frac{ju}{n} \leq \tau \leq \frac{(j+1)u}{n} \right\} \in \mathcal{H}_u \vee \mathcal{G}_{\frac{u}{n}} \quad \text{for all } u \in]0, t[.$$

It follows that

$$A \cap \{t \wedge \tau \leq u\} \equiv A \cap \{\tau \leq u\} \in \bigcap_{n=1}^{\infty} \mathcal{H}_u \vee \mathcal{G}_{\frac{u}{n}} = \mathcal{H}_u \vee \mathcal{G}_0 \quad \text{for all } u \in]0, t[$$

because $(\mathcal{G}_t)$ is right-continuous. In particular, this implies that

$$A \cap \{\tau < t\} = \bigcup_{n=1}^{\infty} A \cap \left\{ \tau \leq \frac{n}{n+1} t \right\} \in \mathcal{H}_t \vee \mathcal{G}_0.$$

Combining this observation with the fact that

$$A \cap \{t \leq \tau\} \in \mathcal{H}_t \vee \mathcal{G}_0,$$

which follows from (169) for $\bar{u} = s = t$, we can see that

$$A \cap \{t \wedge \tau \leq u\} = A = A \cap \{\tau < t\} \cup A \cap \{t \leq \tau\} \in \mathcal{H}_t \vee \mathcal{G}_0 \subseteq \mathcal{H}_u \vee \mathcal{G}_0 \quad \text{for all } u \geq t. \tag{171}$$

From (170)–(171), it follows that $A \in \mathcal{H}_{t \wedge \tau} \vee \mathcal{G}_0$. □

The following result is concerned with the pasting of two stopping strategies, in particular, two weak solutions to (1), at an appropriate stopping time.

Theorem 20 *Consider initial conditions $x_0, x_1 \in \operatorname{int} \mathcal{I}$ and stopping strategies*

$$(\mathbb{S}^0_{x_0}, \tau^0) = \big(\big(\Omega^0, \mathcal{F}^0, \mathcal{F}^0_t, \mathbb{P}^0_{x_0}, W^0, X^0\big), \tau^0\big) \text{ and } (\mathbb{S}^1_{x_1}, \tau^1) = \big(\big(\Omega^1, \mathcal{F}^1, \mathcal{F}^1_t, \mathbb{P}^1_{x_1}, W^1, X^1\big), \tau^1\big).$$

Given any event $A \in \mathcal{F}^0_{\tau^0}$ such that $A \subseteq \{X^0_{\tau^0} \mathbf{1}_{\{\tau^0 < \infty\}} = x_1\}$, there exists a stopping strategy $(\mathbb{S}_{x_0}, \tau^{0,1}) = \big(\big(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_{x_0}, W, X\big), \tau^{0,1}\big) \in \mathcal{T}_{x_0}$ such that

$$J(\mathbb{S}_{x_0}, \tau^{0,1}) = J(\mathbb{S}^0_{x_0}, \tau^0_{A^c}) + \mathbb{E}^0_{x_0}\left[e^{-\Lambda_{\tau^0}(X^0)} \mathbf{1}_A \right] J(\mathbb{S}_{x_1}, \tau^1), \tag{172}$$

where $\tau^0_{A^c}$ is the $(\mathcal{F}^0_t)$-stopping time defined by $\tau^0_{A^c} = \tau^0 \mathbf{1}_{A^c} + \infty \mathbf{1}_A$.

Proof. Without loss of generality, we assume that $\{X^0_{\tau^0} = x_1\} \neq \emptyset$. For $j = 0, 1$, we define on the product space $\big(\Omega, \mathcal{F}, \mathbb{P}_{x_0}\big) = \big(\Omega^0 \times \Omega^1, \mathcal{F}^0 \otimes \mathcal{F}^1, \mathbb{P}^0_{x_0} \otimes \mathbb{P}^1_{x_1}\big)$ the independent filtrations $(\tilde{\mathcal{F}}^j_t)$ given by $\tilde{\mathcal{F}}^0_t = \mathcal{F}^0_t \otimes \{\Omega^1, \emptyset\}$ and $\tilde{\mathcal{F}}^1_t = \{\Omega^0, \emptyset\} \otimes \mathcal{F}^1_t$, the $(\tilde{\mathcal{F}}^j_t)$-stopping times $\tilde{\tau}^j$ given by $\tilde{\tau}^j(\omega^0, \omega^1) = \tau^j(\omega^j)$, the $(\tilde{\mathcal{F}}^j_t)$-Brownian motions $\tilde{W}^j$ given by $\tilde{W}^j_t(\omega^0, \omega^1) = W^j_t(\omega^j)$, and the continuous $(\tilde{\mathcal{F}}^j_t)$-adapted processes $\tilde{X}^j$ given by $\tilde{X}^j_t(\omega^0, \omega^1) = X^j_t(\omega^j)$. Also, we denote by $\tilde{T}^j_y$ the first hitting time of $\{y\}$ by $\tilde{X}^j$, for $y \in \mathcal{I}$ and $j = 0, 1$. In particular, we note that each of the collections $\big(\Omega, \mathcal{F}, \tilde{\mathcal{F}}^j, \mathbb{P}_{x_0}, \tilde{W}^j, \tilde{X}^j\big)$ is a weak solution to the SDE (1) with initial condition x_j.

We next consider the filtration $(\mathcal{F}_t)$ that is defined by (162) in Proposition 19 above with $(\mathcal{H}_t) = (\tilde{\mathcal{F}}^0_t)$, $(\mathcal{G}_t) = (\tilde{\mathcal{F}}^1_t)$ and $\tau = \tilde{\tau}^0$, so that

$$\mathcal{F}_{\tilde{\tau}^0+t} = \tilde{\mathcal{F}}^0_{\tilde{\tau}^0+t} \vee \tilde{\mathcal{F}}^1_t \quad \text{and} \quad \mathcal{F}_{t\wedge\tilde{\tau}^0} = \tilde{\mathcal{F}}^0_{t\wedge\tilde{\tau}^0} \vee \tilde{\mathcal{F}}^1_0, \tag{173}$$

and we define

$$\tilde{A} = A \times \Omega^1 \quad \text{and} \quad \tilde{A}^c = A^c \times \Omega^1. \tag{174}$$

The independence of $(\tilde{\mathcal{F}}^0_t)$, $(\tilde{\mathcal{F}}^1_t)$ and (164) imply that the processes $\big(\tilde{W}^0_{t\wedge\tilde{\tau}^0},\ t \geq 0\big)$ and $\big((\tilde{W}^0_{t\wedge\tilde{\tau}^0})^2 - t \wedge \tilde{\tau}^0,\ t \geq 0\big)$ are $(\mathcal{F}_t)$-martingales. On the other hand, (173) and the fact that $(\tilde{\mathcal{F}}^0_t)$, $(\tilde{\mathcal{F}}^1_t)$ are independent imply that $\tilde{W}^1$ is an $(\mathcal{F}_{\tilde{\tau}^0+t})$-Brownian motion. Since $(t-\tilde{\tau}^0)^+$ is an $(\mathcal{F}_{\tilde{\tau}^0+t})$-stopping time for all $t \geq 0$ and $\tilde{\tau}^0 + (t-\tilde{\tau}^0)^+ = \tilde{\tau}^0 \vee t$, the time-changed processes $\big(\tilde{W}^1_{(t-\tilde{\tau}^0)^+},\ t \geq 0\big)$ and $\big((\tilde{W}^1_{(t-\tilde{\tau}^0)^+})^2 - (t-\tilde{\tau}^0)^+,\ t \geq 0\big)$ are $(\mathcal{F}_{\tilde{\tau}^0\vee t})$-martingales, while the $(\mathcal{F}_{\tilde{\tau}^0\vee t})$-adapted process $\big(\tilde{X}^1_{(t-\tilde{\tau}^0)^+},\ t \geq 0\big)$ satisfies

$$\begin{aligned}
\tilde{X}^1_{(t-\tilde{\tau}^0)^+} &= x_1 + \int_0^{(t-\tilde{\tau}^0)^+} b\big(\tilde{X}^1_s\big)\, ds + \int_0^{(t-\tilde{\tau}^0)^+} \sigma\big(\tilde{X}^1_s\big)\, d\tilde{W}^1_s \\
&= x_1 + \int_0^t b\big(\tilde{X}^1_{(s-\tilde{\tau}^0)^+}\big)\, d(s-\tilde{\tau}^0)^+ + \int_0^t \sigma\big(\tilde{X}^1_{(s-\tilde{\tau}^0)^+}\big)\, d\tilde{W}^1_{(s-\tilde{\tau}^0)^+} \\
&= x_1 + \int_0^t \mathbf{1}_{\{\tilde{\tau}^0 \leq s\}} b\big(\tilde{X}^1_{(s-\tilde{\tau}^0)^+}\big)\, ds + \int_0^t \mathbf{1}_{\{\tilde{\tau}^0 \leq s\}} \sigma\big(\tilde{X}^1_{(s-\tilde{\tau}^0)^+}\big)\, d\tilde{W}^1_s.
\end{aligned} \tag{175}$$

In fact, all of these processes are $(\mathcal{F}_t)$-adapted. To see this, we consider, e.g., the process $\big(\tilde{X}^1_{(t-\tilde{\tau}^0)^+},\ t \geq 0\big)$, and we note that $\big\{\tilde{X}^1_{(t-\tilde{\tau}^0)^+} \in C\big\} \in \mathcal{F}_{\tilde{\tau}^0\vee t}$ implies that

$$\big\{\tilde{X}^1_{(t-\tilde{\tau}^0)^+} \in C\big\} \cap \{\tilde{\tau}^0 \vee t \leq u\} \in \mathcal{F}_u \quad \text{for all } u \geq 0.$$

Therefore,

$$\big\{\tilde{X}^1_{(t-\tilde{\tau}^0)^+} \in C\big\} \cap \{\tilde{\tau}^0 < t\} = \bigcup_{n=1}^{\infty} \big\{\tilde{X}^1_{(t-\tilde{\tau}^0)^+} \in C\big\} \cap \left\{\tilde{\tau}^0 \vee t \leq \frac{nt}{n+1}\right\} \in \mathcal{F}_t.$$

It follows that

$$\big\{\tilde{X}^1_{(t-\tilde{\tau}^0)^+} \in C\big\} = \big\{\tilde{X}^1_0 \in C\big\} \cap \{t \leq \tilde{\tau}^0\} \cup \big\{\tilde{X}^1_{(t-\tilde{\tau}^0)^+} \in C\big\} \cap \{\tilde{\tau}^0 < t\} \in \mathcal{F}_t,$$

because $\tilde{X}^1_0 = x_1$ is a constant, which establishes the claim. Furthermore, $\big(\tilde{W}^1_{(t-\tilde{\tau}^0)^+},\ t \geq 0\big)$ and $\big((\tilde{W}^1_{(t-\tilde{\tau}^0)^+})^2 - (t-\tilde{\tau}^0)^+,\ t \geq 0\big)$ are in fact $(\mathcal{F}_t)$-martingales. Indeed, given $s < t$, we can check, e.g., that

$$\begin{aligned}
&\mathbb{E}_{x_0}\left[\tilde{W}^1_{(t-\tilde{\tau}^0)^+} \mid \mathcal{F}_s\right] \\
&\quad = \mathbb{E}_{x_0}\left[\mathbb{E}_{x_0}\left[\tilde{W}^1_{(t-\tilde{\tau}^0)^+} \mid \mathcal{F}_{\tilde{\tau}^0 \vee s}\right] \mid \mathcal{F}_s\right] = \mathbb{E}_{x_0}\left[\tilde{W}^1_{(s-\tilde{\tau}^0)^+} \mid \mathcal{F}_s\right] = \tilde{W}^1_{(s-\tilde{\tau}^0)^+},
\end{aligned} \tag{176}$$

the last equality following because $\big(\tilde{W}^1_{(t-\tilde{\tau}^0)^+},\ t \geq 0\big)$ is $(\mathcal{F}_t)$-adapted. For future reference, we also note that

$$\begin{aligned}
&\mathbb{E}_{x_0}\left[\tilde{W}^0_{t\wedge\tilde{\tau}^0}\tilde{W}^1_{(t-\tilde{\tau}^0)^+} \mid \mathcal{F}_s\right] \\
&= \mathbb{E}_{x_0}\left[\mathbb{E}_{x_0}\left[\tilde{W}^0_{\tilde{\tau}^0}\tilde{W}^1_{(t-\tilde{\tau}^0)^+} \mid \mathcal{F}_{\tilde{\tau}^0 \vee s}\right] \mid \mathcal{F}_s\right] = \mathbb{E}_{x_0}\left[\tilde{W}^0_{\tilde{\tau}^0}\tilde{W}^1_{(s-\tilde{\tau}^0)^+} \mid \mathcal{F}_s\right] = \tilde{W}^0_{s\wedge\tilde{\tau}^0}\tilde{W}^1_{(s-\tilde{\tau}^0)^+}.
\end{aligned} \tag{177}$$

In view of (173) and (174), the process Y defined by $Y_t = \mathbf{1}_{\tilde{A}}\mathbf{1}_{\{\tilde{\tau}^1 < t\}}$ is $(\mathcal{F}_{\tilde{\tau}^0+t})$-adapted. Using arguments similar to the ones we developed above, we can see that the time-changed process given by

$$Y_{(t-\tilde{\tau}^0)^+} = \mathbf{1}_{\tilde{A}}\mathbf{1}_{\{\tilde{\tau}^0+\tilde{\tau}^1 < t\}}, \quad t \geq 0,$$

is $(\mathcal{F}_t)$-adapted, which proves that the random variable $(\tilde{\tau}^0+\tilde{\tau}^1)\mathbf{1}_{\tilde{A}}+\infty\mathbf{1}_{\tilde{A}^c}$ is an $(\mathcal{F}_t)$-stopping time. It follows that the random variable

$$\tau^{0,1} = \min\left\{\tilde{\tau}^0\mathbf{1}_{\tilde{A}^c} + \infty\mathbf{1}_{\tilde{A}},\ (\tilde{\tau}^0+\tilde{\tau}^1)\mathbf{1}_{\tilde{A}} + \infty\mathbf{1}_{\tilde{A}^c}\right\} = \tilde{\tau}^0\mathbf{1}_{\tilde{A}^c} + (\tilde{\tau}^0+\tilde{\tau}^1)\mathbf{1}_{\tilde{A}} \tag{178}$$

is an $(\mathcal{F}_t)$-stopping time.

To proceed further, we define

$$W_t = \tilde{W}^0_{t\wedge\tilde{\tau}^0} + \tilde{W}^1_{(t-\tilde{\tau}^0)^+}$$

and

$$\begin{aligned}
X_t &= \tilde{X}^0_{t\wedge\tilde{\tau}^0} + \left(\tilde{X}^0_t - \tilde{X}^0_{\tilde{\tau}^0}\right)\mathbf{1}_{\tilde{A}^c}\mathbf{1}_{\{\tilde{\tau}^0 \leq t\}} + \left(\tilde{X}^1_{(t-\tilde{\tau}^0)^+} - x_1\right)\mathbf{1}_{\tilde{A}}\mathbf{1}_{\{\tilde{\tau}^0 \leq t\}} \\
&\equiv \tilde{X}^0_t\mathbf{1}_{\{t<\tilde{\tau}^0\}} + \tilde{X}^0_t\mathbf{1}_{\tilde{A}^c}\mathbf{1}_{\{\tilde{\tau}^0 \leq t\}} + \tilde{X}^1_{(t-\tilde{\tau}^0)^+}\mathbf{1}_{\tilde{A}}\mathbf{1}_{\{\tilde{\tau}^0 \leq t\}},
\end{aligned} \tag{179}$$

and we note that

$$X_{\tau^{0,1}} = \tilde{X}^0_{\tilde{\tau}^0}\mathbf{1}_{\tilde{A}^c} + \tilde{X}^1_{\tilde{\tau}^1}\mathbf{1}_{\tilde{A}}. \tag{180}$$

Given $y \in \mathcal{I}$, if we denote by T_y the first hitting time of $\{y\}$ by X, then

$$T_\alpha\mathbf{1}_{\tilde{A}^c} = \tilde{T}^0_\alpha\mathbf{1}_{\tilde{A}^c}, \quad T_\beta\mathbf{1}_{\tilde{A}^c} = \tilde{T}^0_\beta\mathbf{1}_{\tilde{A}^c}, \tag{181}$$

$$T_\alpha\mathbf{1}_{\tilde{A}} = \left(\tilde{\tau}^0 + \tilde{T}^1_\alpha\right)\mathbf{1}_{\tilde{A}} \quad \text{and} \quad T_\beta\mathbf{1}_{\tilde{A}} = \left(\tilde{\tau}^0 + \tilde{T}^1_\beta\right)\mathbf{1}_{\tilde{A}} \tag{182}$$

because $\tilde\tau^0 \mathbf{1}_{\tilde A} < \big(\tilde T^0_\alpha + \tilde T^1_\beta\big)\mathbf{1}_{\tilde A}$. Since W is the sum of two $(\mathcal{F}_t)$-martingales, it is an $(\mathcal{F}_t)$-martingale. Furthermore, since

$$W_t^2 - t = \left\{\Big(\tilde W^0_{t\wedge\tilde\tau^0}\Big)^2 - t\wedge\tilde\tau^0\right\} + \left\{\Big(\tilde W^1_{(t-\tilde\tau^0)^+}\Big)^2 - (t-\tilde\tau^0)^+\right\} + 2\tilde W^0_{t\wedge\tilde\tau^0}\tilde W^1_{(t-\tilde\tau^0)^+},$$

and the three processes identified on the right-hand side of this identity are $(\mathcal{F}_t)$-martingales (see (176)–(177)), the process $(W_t^2 - t)$ is an $(\mathcal{F}_t)$-martingale. From Lévy's characterisation theorem, it follows that W is an $(\mathcal{F}_t)$-Brownian motion. Also, combining (179) with (175) and the fact that $\tilde X^0$ satisfies the SDE (1), we can see that

$$\begin{aligned}
X_t &= x_0 + \int_0^t \mathbf{1}_{\{s<\tilde\tau^0\}} b\big(\tilde X^0_s\big)\,ds + \int_0^t \mathbf{1}_{\{s<\tilde\tau^0\}}\sigma\big(\tilde X^0_s\big)\,d\tilde W^0_s \\
&\quad + \mathbf{1}_{\tilde A^c}\int_0^t \mathbf{1}_{\{\tilde\tau^0\le s\}} b\big(\tilde X^0_s\big)\,ds + \mathbf{1}_{\tilde A^c}\int_0^t \mathbf{1}_{\{\tilde\tau^0\le s\}}\sigma\big(\tilde X^0_s\big)\,d\tilde W^0_s \\
&\quad + \mathbf{1}_{\tilde A}\int_0^t \mathbf{1}_{\{\tilde\tau^0\le s\}} b\big(\tilde X^1_{(s-\tilde\tau^0)^+}\big)\,ds + \mathbf{1}_{\tilde A}\int_0^t \mathbf{1}_{\{\tilde\tau^0\le s\}}\sigma\big(\tilde X^1_{(s-\tilde\tau^0)^+}\big)\,d\tilde W^1_s \\
&= x_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s.
\end{aligned}$$

This calculation and the preceding considerations show that

$$(\mathbb{S}_{x_0}, \tau^{0,1}) = \big(\big(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}_{x_0}, W, X\big), \tau^{0,1}\big) \in \mathcal{T}_{x_0}.$$

To complete the proof, we use the definition (5) of Λ, (178)–(179) and (181)–(182) to calculate

$$\mathbf{1}_{\tilde A}\Lambda_{\tau^{0,1}\wedge T_\alpha\wedge T_\beta}(X) = \mathbf{1}_{\tilde A}\int_0^{(\tilde\tau^0+\tilde\tau^1)\wedge(\tilde\tau^0+\tilde T^1_\alpha)\wedge(\tilde\tau^0+\tilde T^1_\beta)} r(X_s)\,ds = \mathbf{1}_{\tilde A}\left[\Lambda_{\tilde\tau^0}(\tilde X^0) + \Lambda_{\tilde\tau^1\wedge\tilde T^1_\alpha\wedge\tilde T^1_\beta}(\tilde X^1)\right]$$

In view of this observation, (173)–(174), (178)–(182) and the independence of $(\tilde{\mathcal{F}}_t^0)$, $(\tilde{\mathcal{F}}_t^1)$, we can see that

$$\begin{aligned}
J(\mathbb{S}_{x_0},\tau^{0,1}) &= \mathbb{E}_{x_0}\left[e^{-\Lambda_{\tau^{0,1}\wedge T_\alpha\wedge T_\beta}(X)} f(X_{\tau^{0,1}\wedge T_\alpha\wedge T_\beta})\mathbf{1}_{\{\tau^{0,1}<\infty\}}\right] \\
&= \mathbb{E}_{x_0}\left[e^{-\Lambda_{\tilde\tau^0\wedge\tilde T^0_\alpha\wedge\tilde T^0_\beta}(\tilde X^0)} f(\tilde X^0_{\tilde\tau^0\wedge\tilde T^0_\alpha\wedge\tilde T^0_\beta})\mathbf{1}_{\tilde A^c}\mathbf{1}_{\{\tilde\tau^0<\infty\}}\right] \\
&\quad + \mathbb{E}_{x_0}\left[e^{-\Lambda_{\tilde\tau^0}(\tilde X^0)}\mathbb{E}_{x_0}\left[e^{-\Lambda_{\tilde\tau^1\wedge\tilde T^1_\alpha\wedge\tilde T^1_\beta}(\tilde X^1)} f(\tilde X^1_{\tilde\tau^1\wedge\tilde T^1_\alpha\wedge\tilde T^1_\beta})\mathbf{1}_{\{\tilde\tau^1<\infty\}} \mid \tilde{\mathcal{F}}^0_{\tilde\tau^0}\vee\tilde{\mathcal{F}}^1_0\right]\mathbf{1}_{\tilde A}\right] \\
&= \mathbb{E}^0_{x_0}\left[e^{-\Lambda_{\tau^0\wedge T^0_\alpha\wedge T^0_\beta}(X^0)} f(X^0_{\tau^0\wedge T^0_\alpha\wedge T^0_\beta})\mathbf{1}_{A^c}\mathbf{1}_{\{\tau^0<\infty\}}\right] \\
&\quad + \mathbb{E}_{x_0}\left[e^{-\Lambda_{\tilde\tau^0}(\tilde X^0)}\mathbb{E}_{x_0}\left[e^{-\Lambda_{\tilde\tau^1\wedge\tilde T^1_\alpha\wedge\tilde T^1_\beta}(\tilde X^1)} f(\tilde X^1_{\tilde\tau^1\wedge\tilde T^1_\alpha\wedge\tilde T^1_\beta})\mathbf{1}_{\{\tilde\tau^1<\infty\}} \mid \tilde{\mathcal{F}}^1_0\right]\mathbf{1}_{\tilde A}\right] \\
&= J(\mathbb{S}^0_{x_0},\tau^0_{A^c}) + \mathbb{E}_{x_0}\left[e^{-\Lambda_{\tilde\tau^0}(\tilde X^0)}\mathbf{1}_{\tilde A}\right]\mathbb{E}_{x_0}\left[e^{-\Lambda_{\tilde\tau^1\wedge\tilde T^1_\alpha\wedge\tilde T^1_\beta}(\tilde X^1)} f(\tilde X^1_{\tilde\tau^1\wedge\tilde T^1_\alpha\wedge\tilde T^1_\beta})\mathbf{1}_{\{\tilde\tau^1<\infty\}}\right] \\
&= J(\mathbb{S}^0_{x_0},\tau^0_{A^c}) + \mathbb{E}^0_{x_0}\left[e^{-\Lambda_{\tau^0}(X^0)}\mathbf{1}_A\right]\mathbb{E}^1_{x_1}\left[e^{-\Lambda_{\tau^1\wedge T^1_\alpha\wedge T^1_\beta}(X^1)} f(X^1_{\tau^1\wedge T^1_\alpha\wedge T^1_\beta})\mathbf{1}_{\{\tau^1<\infty\}}\right],
\end{aligned}$$

and (172) follows. □

Iterating the construction above, we obtain the following result.

Corollary 21 *Fix an initial condition $x\in\operatorname{int}\mathcal{I}$ and any distinct points $a_1,\dots,a_n\in\operatorname{int}\mathcal{I}$. Given stopping strategies*

$$(\mathbb{S}^0_x,\tau^0)=\big(\big(\Omega^0,\mathcal{F}^0,\mathcal{F}^0_t,\mathbb{P}^0_x,W^0,X^0\big),\tau^0\big)\quad\textit{and}\quad(\mathbb{S}^i_{a_i},\tau^i)=\big(\big(\Omega^i,\mathcal{F}^i,\mathcal{F}^i_t,\mathbb{P}^i_{a_i},W^i,X^i\big),\tau^i\big),$$

$i=1,\dots,n$, define $A=\big\{X^0_{\tau^0}\mathbf{1}_{\{\tau^0<\infty\}}\in\{a_1,\dots,a_n\}\big\}\in\mathcal{F}^0_{\tau^0}$. Then, there exists a stopping strategy $(\mathbb{S}_x,\tau)\in\mathcal{T}_x$ such that

$$J(\mathbb{S}_x,\tau)=J(\mathbb{S}^0_x,\tau^0_{A^c})+\sum_{i=1}^n\mathbb{E}^0_x\left[e^{-\Lambda_{\tau^0}(X^0)}\mathbf{1}_{\{X^0_{\tau^0}=a_i\}}\right]J(\mathbb{S}^i_{a_i},\tau^i),$$

where $\tau^0_{A^c}$ is the $(\mathcal{F}^0_t)$-stopping time defined by $\tau^0_{A^c}=\tau^0\mathbf{1}_{A^c}+\infty\mathbf{1}_A$.

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