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# Entropy for quantum fields in curved space time

Heide Narnhofer \*

Fakultät für Physik  
Universität Wien

## Abstract

We discuss that entropy can be assigned to local domains for quantum fields only if we relate it to two domains nested in one another such that the entropy includes a surface effect depending on the distance of the two domains. We give upper and lower limits for the corresponding expressions, based on assumptions on the nuclearity of the quantum field and on the existence of a scaling limit. We apply these estimates to local domains in flat space and in de Sitter space. We show that in both cases the total system is in a pure state with vanishing entropy, but also, that the entropy of domains with vanishing size tends to 0. For quantum fields on a black hole we consider the Schwarzschild space time and its extension to the Kruskal space time. The quantum field on the Schwarzschild space time has infinite entropy, even if we regularize over the horizon. Nevertheless for domains in the Schwarzschild space time the entropy tends to 0, if the size tends to 0. If however we consider domains that include the total Schwarzschild domain the entropy is always  $\infty$ .

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\* E-mail address: heide.narnhofer@univie.ac.at

# 1 Introduction

In [1] it is argued that a black hole has to have some entropy. In [2] it is shown that a quantum field on a Schwarzschild space time is in a temperature state where the temperature is determined by the size of the black hole. This inspired [3] to look for natural connections between gravitational forces and temperature as consequence of quantum theory. On the other hand it was argued [4] that the Hawking temperature is just an analogue to the Bisognano-Wichmann theorem [10] arising from mathematical equivalences and therefore is an effect related to an observer who in the Bisognano-Wichmann situation is an accelerated observer moving in the Rindler wedge and his acceleration determines the temperature. For the de Sitter space the observer moves in the wedge and observes a temperature depending on the curvature of the metric as well as on his acceleration [6] [7]. In the case of a black hole it is an observer following the time evolution of the Schwarzschild time. Altogether the temperature interpretation depends on the observer and so does the notion of an entropy, if we rely on its definition from thermodynamics.

In this note we take the conservative view point that entropy is just a property of a state and does not refer to any dynamics or any observer. We consider the state of a quantum field over a curved space, so that the quantum field itself cannot influence the geometry. We are interested in the entropy of this state restricted to some local domains and its dependence on the choice of the local domain. Already here problems arise and we have to choose the local domain carefully to ensure that the entropy is not necessarily infinite. To be more precise we are unable to fully characterize permitted local domains and as a consequence are also unable to give explicit numbers. However we are able to give upper limits on the entropy, where we need the assumption of nuclearity [5], and also lower limits, where we demand that the system admits a scaling law [8]. Both assumptions are satisfied for free fields and are taken to be necessary properties of physically meaningful theories. These upper and lower limits are strong enough to find qualitative differences between the entropy of local domains in flat space, in de Sitter space and in the Schwarzschild-Kruskal space time. In all these three situations our analysis is based on the existence of a local Killing vector field and the corresponding analyticity properties of the Wightman functions.. However it will also become clear that its existence is only a useful tool in the mathematical analysis that enables explicit calculations but suggests that the relevant behaviour will also hold in a more general context.

The main observations are the following: Given two local domains one included in the other with some distance between the boundaries, then it is possible to find a subalgebra containing all the observables corresponding to the smaller domain but still belonging to the observables of the larger domain in such a way, that the entropy of this algebra is finite. Its value increases with the size of the smaller domain but also with its inverse distance to the larger domain tending to infinity if the distance tends to 0. However it is possible to let the size of the smaller domain tend to zero together with the distance, where the distance has to be scaled appropriately. If we are in flat space we can arrange the scaling in such a way, that in the limit the entropy tends to 0. The same holds if we

are in de Sitter space. It also holds if we are in Schwarzschild space outside of the horizon. If however our local domain contains the black hole then we have to become careful in applying the scaling law. With an appropriate definition for the algebra corresponding to an arbitrary small neighborhood of the black hole its entropy becomes infinite.

Another question concerns how the entropy depends on the size of the smaller domain, when we keep the distance of the larger domain fixed. Here we can find a qualitative behaviour that distinguishes between flat space and curved space. The entropy of a finite domain depends on one hand on the size of the domain, on the other hand on the size of the boundary. In flat space the dominating term is the surface effect. In de Sitter space the lack of a global time evolution that corresponds to a Killing vector field only admits weaker estimates on the upper and lower limit of the entropy, and this is reflected on a dependence on the size of the domain. The entropy is far from being monotonically increasing with the size of the domain due to the interplay of its dependence on the size of the boundary but it becomes maximal when the domain approaches halfspace. A similar behaviour can be found in Schwarzschild space, when we consider domains outside of the horizon.

The paper is organized as follows: first we collect the relevant results obtained in flat space. Here we present also the necessary concepts as nuclearity and split property. These properties are known to be satisfied for quasifree quantum fields on flat space, and it is a reasonable assumption that they hold also for interacting quantum fields and also on curved space. We will argue that the essential features guaranteeing the finiteness of the entropy do not change or can find a replacement that is as powerful if we consider quantum fields in curved space away from singularities. We will consider first the similarities and changes in de Sitter space where we have a Killing horizon but no singularity. The main tool in our analysis will be the application of the modular theory offered by the existence of a Killing vector field. Finally we will apply the methods to Schwarzschild space time and its extension to Kruskal space time. Here the existence of a singularity is the cause that we can assign an entropy to the black hole.

## 2 Facts in flat space

The algebra of quantum fields [9] can be considered to be built by Wightman fields  $W(f)$  where  $f(x)$  are functions over  $\mathbf{R}^4$ . Local subalgebras  $\mathcal{A}_\Lambda$  with  $\Lambda$  some compact domain in  $\mathbf{R}^4$  are built by  $W(f)$  with  $\text{supp}(f) \subset \Lambda$ . A special example is the algebra on the Rindler wedge  $\Lambda_R$  where

$$\Lambda_R = \{x \in \mathbf{R}^4; x_1 \geq 0, x_1 < |x_0|\} \quad (1)$$

The vacuum state over the quantum field reduced to the Rindler wedge is a temperature state with respect to the Lorenz boost

$$\tau(s) = \begin{bmatrix} \cosh s & \sinh s & 0 & 0 \\ \sinh s & \cosh s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2)$$

at temperature  $2\pi$  [10]. Under the assumption of a scaling law where the limit of the Wightman functions

$$\lim_{\lambda \rightarrow 0} \lambda^2 \omega(\lambda x_1, \dots, \lambda x_n) = \omega_0(x_1, \dots, x_n) \quad (3)$$

exists and corresponds again to a quantum field theory that we may even expect to be massless then [8] has shown that the vacuum state over the causal completion of a compact domain in  $\mathbf{R}^3$ , e.g. a double cone, is again a state constructed over a cyclic and separating vector and therefore a temperature state with respect to its modular evolution, and that further this modular evolution has the same spectral properties as the generator of the boost. But if the modular evolution has continuous spectrum then necessarily the algebra is of type  $III_1$  and the entropy of any state over such an algebra is  $\infty$ .

In order to find a qualitative expression to the fact that in physically reasonable systems the phase space over a finite domain should again be essentially finite with its size limited by its energy [5] formulated the nuclearity condition: We assume that we can express

$$e^{-\beta H} A |\Omega\rangle = \sum_n \phi_n(A) |\Psi_n\rangle \quad \forall A \in \mathcal{A}_\Lambda \quad (4)$$

with  $\|\Psi_n\|$  normalized vectors and  $\phi_n(A)$  linear functionals over  $\mathcal{A}_\Lambda$ , depending on  $\beta$ . Nuclearity holds with respect to some nuclearity norm that e.g. is defined as

$$\nu_p(\Lambda, \beta) = \inf \left[ \sum_n |\phi_n|^p \right]^{1/p} \quad (5)$$

if this norm is finite. The infimum in (5) is taken over all possible  $\{\phi_n, \Psi_n\}$ . Variations in the norm are possible and can be adjusted to the special demands, especially we can also consider  $\nu_{ln}$  defined with respect to  $xlnx$ .

For free massless theories this nuclearity index  $\nu$  (for various choices of the index  $p$ ) was estimated in [12] by

$$\nu_p(\Lambda_r, \beta) \leq c_p e^{r/\beta}, \quad r < \beta \quad (6)$$

where  $\Lambda_r$  is the causal completion of a ball of radius  $r$  in  $\mathbf{R}^3$ . Of course we have to keep in mind that the fact that a quantum field allows finite nuclearity indices is an additional assumption and should not be taken for granted.

Based on the assumption of nuclearity [11] showed that for  $A \in \mathcal{A}_{\Lambda_1}$  and  $B \in \mathcal{A}_{\Lambda_2^C}$  with a distance  $d(\Lambda_1, \Lambda_2^C) = \inf_{x \in \Lambda_1, y \in \Lambda_2^C} \|x - y\|$  the vacuum state satisfies

$$\omega(AB) = \sum \phi_{1,k}(A) \phi_{2,k}(B). \quad (7)$$

Here  $\phi_{1,k}$  are linear functionals on  $\mathcal{A}_{\Lambda_1}$  and  $\phi_{2,k}$  linear functionals on  $\mathcal{A}_{\Lambda_2^C}$  implemented by  $\Psi_n$  as they entered in the estimation of the nuclearity index. If we fix  $|\phi_{2k}| = 1$  then they satisfy

$$\sum_k |\phi_{1,k}|_p < c_p \nu_p(\Lambda_1, d(\Lambda_1, \Lambda_2)) \quad (8)$$

for all choices of the norm, including the one given by  $xlnx$ . Notice that the parameter  $\beta$  that in the definition was related to the Hamiltonian has been shown to be essentially linearly related to the distance [11].

Based on this fact it can be shown, that it is possible to find algebras that are local but also have finite entropy. We start with an inclusion  $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$  where  $\Lambda_2^C$  is the causal complement to  $\Lambda_2$  and  $d(\Lambda_1, \Lambda_2^C) > 0$ . Then it is possible to construct some algebra  $\mathcal{N}$  such that

$$\mathcal{A}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{A}_{\Lambda_2} \quad (9)$$

and  $\mathcal{N}$  is a type I algebra in such a way that the vacuum state restricted to  $\mathcal{N}$  can be expressed by a density matrix belonging to  $\mathcal{N}$ . The algebra  $\mathcal{N}$  is not unique, however relating the construction of the algebra to the decomposition (7) the nuclearity index gives a bound on the entropy [13],[15]. To assign a well defined entropy to the inclusion  $\Lambda_1 \subset \Lambda_2$  there are several possibilities. One strategy was followed in [13]: Though  $S(\mathcal{A}_{\Lambda_1}) = \infty$  ( we will surpress that the entropy depends on the state since all our considerations concentrate on the vacuum state or on a well defined state for curved space time )and also the entanglement of formation  $E(\mathcal{A}_{\Lambda_1}) = \infty$  (considered with respect to  $\mathcal{A}_{\Lambda_2}$  and expressing the quantum correlations between the domain  $\Lambda_1$  and its causal complement in  $\Lambda_2$  ) the difference [14]

$$H_{\mathcal{A}_{\Lambda_2}}(\mathcal{A}_{\Lambda_1}) = \sup \sum_{\sum \lambda_i \omega_i = \omega} \lambda_i S(\omega | \omega_i)_{\mathcal{A}_{\Lambda_1}} \quad (10)$$

(the supremum is taken over all possible decompositions of  $\omega$  into states  $\omega_i$  in  $\mathcal{A}_{\Lambda_2}$ ) stays finite under the appropriate definition of the nuclearity index  $\nu$  (5) with respect to the norm given by  $-\sum_n |\Phi_n| \ln |\Phi_n|$  as it enters in all entropy expressions.

An alternative is to consider the usual expression for the entropy  $S(\rho) = -Tr \rho \ln \rho$  and define

$$S(\Lambda_1, \Lambda_2) = \inf_{\mathcal{A}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{A}_{\Lambda_2}} S(\mathcal{N}) \geq H_{\mathcal{A}_{\Lambda_2}}(\mathcal{A}_{\Lambda_1}) \quad (11)$$

However it is not clear that this optimization already defines a unique algebra  $\mathcal{N}$ . The results in [15] give an upper bound for some explicitly constructed algebra  $\mathcal{N}$  that is sufficient for our purpose.

The available estimates permit to consider

$$\lim_{\Lambda_1 \rightarrow \mathbf{R}^3} \lim_{\Lambda_2 \rightarrow \mathbf{R}^3} H_{\mathcal{A}_{\Lambda_2}}(\mathcal{A}_{\Lambda_1}) = \lim_{\Lambda_1 \rightarrow \mathbf{R}^3} \lim_{\Lambda_2 \rightarrow \mathbf{R}^3} S(\Lambda_1, \Lambda_2) = 0 \quad (12)$$

in agreement with the fact that the vacuum is the ground state for the quantum field and therefore pure [13]. But the same estimates also allow to consider

$$\lim_{\Lambda_1 \subset \Lambda_2 \rightarrow 0} H_{\mathcal{A}_{\Lambda_2}}(\mathcal{A}_{\Lambda_1}) = \lim_{\Lambda_1 \subset \Lambda_2 \rightarrow 0} S(\Lambda_1, \Lambda_2) = 0 \quad (13)$$

where we have to specify how the larger domain tends to 0 in comparison to the smaller domain. This specification can be done on the basis of the nuclearity condition that provides an upper limit. However estimates to obtain bounds on the nuclearity index are rather rough and we expect they can be improved. This belief is supported by the behaviour of the lower limit.

Here we use the relative entropy

$$S(\omega \otimes \omega | \omega)_{\mathcal{A}_{\Lambda_1} \otimes \mathcal{A}'_{\Lambda_2}}. \quad (14)$$

That this expression makes sense is based on the fact that  $\mathcal{A}_{\Lambda_1 \cup \Lambda_2} = \mathcal{A}_{\Lambda_1} \otimes \mathcal{A}'_{\Lambda_2}$  due to the results in [11]. The relative entropy is monotonically increasing in  $\mathcal{A}_{\Lambda_1}$  and in  $\mathcal{A}'_{\Lambda_2}$ , therefore monotonically decreasing in  $\mathcal{A}_{\Lambda_2}$ . Especially the relative entropy is monotonically increasing with the algebra so that

$$S(\omega \otimes \omega | \omega)_{\mathcal{A}_{\Lambda_1} \otimes \mathcal{A}'_{\Lambda_2}} \leq S(\omega \otimes \omega | \omega)_{\mathcal{N} \otimes \mathcal{N}'} = S(\mathcal{N}) + S(\mathcal{N}') - S(\mathcal{N} \otimes \mathcal{N}') = 2S(\mathcal{N}) \quad (15)$$

provided  $\mathcal{N}$  is a factor. Otherwise there is the inequality for type I algebras

$$S(\mathcal{N}) \leq S(\omega \otimes \omega | \omega)_{\mathcal{N} \vee \mathcal{N}'} \leq 2S(\mathcal{N}) \quad (16)$$

where the left side becomes an equality if  $\mathcal{N}$  is abelian. From this view point, combined with the results in [15] (15) can be read that the nuclearity index gives an upper bound on the relative entropy or that the relative entropy gives a lower bound for the nuclearity index.

The algebras  $\mathcal{A}_{\Lambda_1}$  and  $\mathcal{A}'_{\Lambda_2}$  are type III algebras. Therefore the relative entropy cannot be expressed in terms of density matrices but instead we can use the Kosaki-formula [16]

$$S(\omega | \phi)_{\mathcal{A}} = \sup \int \left[ \frac{\omega(1)}{1+t} - \omega(y^*(t)y(t)) - \frac{1}{t} \phi(x(t)x^*(t)) \right] \frac{dt}{t} \quad (17)$$

where the supremum is taken over  $x(t) \in \mathcal{A}$  and  $y(t) = 1 - x(t)$  as operator valued step functions. Especially for states  $\omega, \phi$  we have the general inequality

$$S(\omega | \phi) \geq \frac{1}{2} \|\omega - \phi\|^2.$$

Therefore the entropy can only tend to 0 if we choose  $\Lambda_2$  sufficiently large with respect to  $\Lambda_1$  so that  $\omega$  nearly factorizes. We concentrate on shrinking balls. The nuclearity index (6) tends to 1 as long as  $\beta \gg r$ , but still  $\beta \rightarrow 0$  is possible so that

$$S(\Lambda_r, \Lambda_\beta) \rightarrow 0. \quad (18)$$

Next we are interested how  $S(\Lambda_r, \Lambda_{r+\beta})$  depends on  $r, \beta$ , i.e. how far the entropy is a volume effect and how far it is a surface effect. The upper limit was discussed in [13], but the estimates are rather rough. We rely more on the lower limit that is based on an analysis of (17). The motivation comes from the additivity of the relative entropy for tensor products. This additivity is however a bit hidden in (17).

Having chosen some  $x(t)$  for (17) we can optimize its size in the Kosaki formula with

$$\inf_{f(t)} [\omega((1 - f(t)x(1 - f(t)x^*) + \phi(x^* x f^2(t)))/t]$$

which is attained for

$$f(t) = \frac{t(\omega(x) + \omega(x^*))}{2(\phi(x^*x) + t\omega(xx^*))} \quad (19)$$

This gives a contribution

$$\begin{aligned} \int \left[ \frac{\omega(1)}{t+1} - \omega(1) + \frac{t\omega(x+x^*)^2}{2(\phi(x^*x) + t\omega(xx^*))^2} - \frac{t\omega(x+x^*)^2}{4(\phi(x^*x) + t\omega(xx^*))} \right] \frac{dt}{t} = \\ = \int \left[ \frac{\omega(x+x^*)^2}{4(\phi(x^*x) + t\omega(xx^*))} - \frac{1}{1+t} \right] dt \end{aligned} \quad (20)$$

For large  $t$  the main contribution comes from

$$\frac{\omega(x+x^*)^2}{4t\omega(xx^*)} - \frac{1}{t}$$

which can only be positive if  $x = 1$ . Large contributions to the relative entropy can only be obtained for small  $t$  where we can obtain

$$\frac{\omega(x+x^*)^2}{\phi(x^*x)} \gg c \quad (21)$$

Scaling cannot increase this value. But we can recover the additivity behaviour of the relative entropy for tensor products  $\omega = \otimes \omega_i$  and  $\phi = \otimes \phi_i$  if we choose  $x = \sum_i^n x_i$  with  $\omega_i(x_i + x_i^*)^2 - \phi_i(x_i^*x_i) = c$  and also  $\phi_i(x_i) = 0$  for the individual contributions so that

$$\frac{\omega(\sum_i^n (x_i + x_i^*))^2}{4\phi(\sum_i^n x_i^* \sum_i^n x_i)} = c^2 n \quad (22)$$

If we do not work with tensor products the same scaling behaviour remains as long as  $\phi(\sum_{i,j} x_i x_j)$  does not contribute as  $n^2$ , as it happens if the state nearly factorizes with increasing  $|i - j|$ .

If we therefore consider a ball or a similar domain and divide it into cells then the cells inside the ball will nearly factorize with the cells outside the ball and the main contribution will come from the cells at the boundary coupled to the corresponding cell outside but close. Therefore the entropy apart from a negligible contribution from the volume will essentially scale like its surface with a parameter depending on  $\beta^{-2}$ .

### 3 Estimates in de Sitter space

We consider the de Sitter space as subspace of a five dimensional space satisfying  $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2 = R^2$ . On this space we have Killing vector fields  $L_{0,i} = x_0 \partial_i - x_i \partial_0$  very similar to the boost in the Rindler wedge of flat space. Like the boosts they do not represent global time evolution, and as the boost they give rise to a horizon:  $\{x_1 = |x_0|, x_2^2 + x_3^2 + x_4^2 = R^2\}$ .

We consider a quantum field on de Sitter space with the condition that its state is invariant under the full de Sitter group and that in the scaling limit it reduces to a free



massless theory [6]. Then the state is a temperature state with respect to the Killing vector field  $L_{0,i}$  with temperature  $1/2\pi R$ . The Killing flow can be most easily expressed if we change to the coordinates

$$x_0 = (R^2 - x_2^2 - x_3^2 - x_4^2)^{1/2} \sinh t, x_1 = (R^2 - x_2^2 - x_3^2 - x_4^2)^{1/2} \cosh t, x_2, x_3, x_4 \quad (23)$$

where it corresponds to the shift in  $t$ . We observe that the domain

$$\mathcal{W} = \{t, x_2, x_3, x_4; x_1 \geq 0\} \quad (24)$$

is invariant under the Killing flow and corresponds to the Rindler wedge. Reduced to  $\mathcal{A}_{\mathcal{W}}$  the state is a temperature state with respect to  $L_{0,1}$ . Extending the state to the whole algebra we can use the modular conjugation:

$$JA|\Omega \rangle = e^{-H/2} A^* |\Omega \rangle = JAJ|\Omega \rangle \quad (25)$$

where  $A$  belongs to  $\mathcal{A}_{\mathcal{W}}$  and  $JAJ$  belongs to  $\mathcal{A}'_{\mathcal{W}}$ ,  $H$  implements the boost (i.e. the modular evolution) on  $\mathcal{A}_{\mathcal{W}}$  and  $J$  is the modular conjugation that maps  $\mathcal{A}_{\mathcal{W}}$  into  $\mathcal{A}'_{\mathcal{W}}$ . If we consider operators  $A = \Pi_k W(f_k(x))$  we can express the effect of the boost as  $e^{iHt} A e^{-iHt} = \Pi_k W(f_k(L_t x))$ . We know that the Wightman functions defining the state are analytic with respect to the Killing parameter. We apply this analyticity to  $L_t x$  and consider

$$\sinh t \rightarrow \sinh(t + i\pi), \quad y \cosh t \rightarrow y \cosh(t + i\pi)$$

For  $t = 0$  we obtain  $\mathcal{W} = \{x > 0, t\} \rightarrow \{x < 0, t\}$  so that the wedge is mapped into its causal complement. The algebras on the two wedges generate the whole algebra. Therefore our state is pure and the algebra satisfies Haag duality.

We are interested in the entropy of algebras over local domains. For these local domains we choose local domains in  $\mathbf{R}^4$  with  $t = 0$  and their causal completion. Again the scaling law implies that the state on these algebras is a KMS state with respect to a modular automorphism whose generator has the same spectrum as the Killing generator and therefore the algebra is type III and has infinite entropy. Again we consider the replacement  $\mathcal{A}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{A}_{\Lambda_2}$ . In order to obtain an upper bound the nuclearity condition of [5] cannot be used because a global time like vector field is never a Killing vector field and therefore does not define a unitary group resp. a hamiltonian. However for Minkowski space [8] has shown that the nuclearity condition guarantees that for an inclusion  $\Lambda_1 \subset \Lambda_2$  we can consider the modular evolution implemented by  $e^{iH_2 t}$  with respect to  $\mathcal{A}_{\Lambda_2}$  and can obtain a nuclear bound for the set

$$e^{-\lambda H_2} A |\Omega \rangle, \quad A \in \mathcal{A}_{\Lambda_1}, \|A\| = 1, 0 < \lambda < 1/2. \quad (26)$$

The restriction on  $\lambda$  stems from the fact that differently to the Hamiltonian in full flat space  $H_2$  is not bounded from below and we can only be sure that  $A|\Omega \rangle$  belongs to the domain of  $e^{-\lambda H_2}$  for  $0 \leq \lambda < 1/2$ . Otherwise the definition of the nuclearity index follows the definition (5). The optimal value for the nuclearity bound is obtained for

$\lambda = 1/4$ . Further from the order relation between the modular operators (the forms of the corresponding modular operators are defined on increasing domains) it follows that  $\nu(\Lambda_2)$  is decreasing with increasing  $\Lambda_2$ . If  $\Lambda_2$  approaches  $\Lambda_1$  then the nuclearity bound diverges.

The nuclearity bound in Minkowski space was introduced as a measure on something that replaces the size of the phase space in classical mechanics. Giving such an interpretation to a nuclearity bound for inclusions might be done on the basis of a scaling at a point far away from the horizon, where modular automorphism, boost and time evolution nearly coincide in first order in the appropriate scaling. However this is not very convincing. Its relation to entanglement might give some insight: since the nuclearity index decreases with increasing  $\Lambda_2$  this implies that also the entanglement with  $\Lambda_1$  and the causal complement of  $\Lambda_2$  though existing gets smaller. Thus the entanglement between  $\mathcal{A}_\Lambda$  and  $\mathcal{A}'_\Lambda$  is concentrated on the boundary as a consequence of the nuclearity condition.

For the de Sitter space we can consider the inclusion of a small local domain into a wedge that we choose such that the local domain is sufficiently separated from the boundary. Then we can use the Killing vector field to examine whether nuclearity holds. We assume that this is satisfied. For free fields modifications of the estimates in [12] are possible. Now take two domains that are causally independent. They will stay causally independent when one domain is moved by the Killing flow for a sufficiently small interval. Therefore as in [13] the methods of [11] can be generalized to guarantee that also the nuclearity condition with respect to a general inclusion  $\Lambda_1 \subset \Lambda_2$  holds. As in Minkowski space also in de Sitter space we can find a type I algebra  $\mathcal{A}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{A}_{\Lambda_2}$  with  $S(\mathcal{N}) < \infty$ .

The construction of the quantum field is based on the scaling law. Therefore again

$$\lim_{\Lambda_1 \subset \Lambda_2 \rightarrow 0} \inf_{\mathcal{A}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{A}_{\Lambda_2}} S(\mathcal{N}) = 0 \quad (27)$$

with the same conditions on the distance as in Minkowski space. Similarly

$$\lim_{\Lambda_2 \rightarrow deSitterspace} H_{\mathcal{A}_{\Lambda_2}}(\mathcal{A}_{\Lambda_1}) = 0 \quad \forall \Lambda_1$$

Since  $H_{\mathcal{A}_{\Lambda_2}}$  is monotonically decreasing and obtained as a supremum, we get this result as a consequence that the state is pure.

It remains to look for a difference between the behaviour in Minkowski space and in de Sitter space. In Minkowski space we have argued that the entropy of a local domain depends essentially on its surface. The argument was based on the lower estimate where we assumed that only operators at the boundary contribute to (17) whereas for operators concentrated inside of the domain the nuclearity (6) index can be regarded with respect to the boundary, i.e. with large  $\beta$  in (4). Now we can only estimate the nuclearity with respect to the boost and for limited values of  $\beta$ . We examine again which contributions we have to expect in (17). We divide our local domain into cells. The cells at the boundary will contribute similarly as in Minkowski space being sensitive to the distance of the outer domain. For the cell inside of the local domain we can only use the nuclearity bound up

to  $\lambda < 1/2$ . For an operator  $A$  in such a cell we will find some  $JAJ$  in the other wedge such that

$$\langle \Omega | JAJA | \Omega \rangle = \langle \Omega | A^* e^{-Hw/2} A | \Omega \rangle \neq | \langle \Omega | A | \Omega \rangle |^2. \quad (28)$$

How much it differs determines how much it can contribute to the relative entropy. Of course the definition of the local entropy is done without referring to a wedge. Obviously we have to choose the wedge such that  $\| (e^{-Hw/2} A - A) | \Omega \rangle \|$  is minimal. This means, that the horizon of  $\mathcal{W}$  should be far away. This can be achieved in flat space, but not in de Sitter space.

As a consequence the state does not cluster approximately and therefore we can use such  $A$  for a contribution in (17) as indicated in (22). Therefore the entropy of a local domain will also depend on its size and not only on its surface. Notice however, that tacitly we assumed that  $\mathcal{N}$  belongs to the wedge which gives a limit on the including algebra. If the including algebra approaches the total algebra again the entropy of the smaller algebra will vanish. Notice however keeping the algebras small the amount of the entropy will depend on  $R$  and will decrease with  $R$ , in correspondence to the observation in flat space, that the boost far away from the horizon has to be scaled to estimate the local domain.

## 4 Schwarzschild-Kruskal space time

We consider quantum fields on the Schwarzschild space time

$$g = \frac{1}{1 - r/r_0} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - (1 - r/r_0) dt^2 \quad (29)$$

together with its extension to the Kruskal space time with the coordinates

$$u = \left(\frac{r}{r_0} - 1\right)^{1/2} e^{r/2r_0} Ch \frac{t}{2r_0} \quad v = \left(\frac{r}{r_0} - 1\right)^{1/2} e^{r/2r_0} Sh \frac{t}{2r_0} \quad (30)$$

with

$$g = \frac{4r_0^3}{r} e^{-r/r_0} (du^2 - dv^2) + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (31)$$

where now  $u, v \in \mathbf{R}$ ,  $u^2 - v^2 > -1$  and the domain  $u^2 - v^2 > 0$  corresponds to the Schwarzschild domain.

We concentrate on a free quantum field over the Schwarzschild -Kruskal space time. According to our description with respect to the local structure we consider the algebra to be built by Weyl operators  $W(f)$  (now not Wightman fields but their exponent so that  $W(f)$  are bounded operators) with  $\text{supp}(f) \subset \Lambda$  satisfying the commutation relations

$$W(f_1)W(f_2) = e^{-\sigma(f_1, f_2)} W(f_1 + f_2) \quad (32)$$

where  $\sigma(f_1, f_2) = -\sigma(f_2, f_1)$  is a symplectic form

$$\sigma(f, g) = \int f_1(x) G(x, x') f_2(x') d\Omega_x d\Omega_{x'} \quad (33)$$

where

$$\begin{aligned} &(-\partial_\mu g^{\mu\nu} \sqrt{-g} \partial_\nu + m^2 \sqrt{-g})G(x, x') = \\ &\sqrt{-g}(-\square_x + m^2)G(x, x') = \sqrt{-g}(-\square_{x'} + m^2)G(x, x') = 0 \end{aligned} \quad (34)$$

with  $G(x, x') = -G(x', x)$  and  $G(x, x') = 0$  if  $x$  and  $x'$  are relatively spacelike and normalized such that they reflect the commutation relations. With

$$h_1(x) = \int f(x')G(x, x')d\Omega_{x'}, \quad h_2 = \int f(x')\partial_\mu G(x, x')d\Omega_{x'} \quad (35)$$

$h_1, h_2$  can serve as the initial data determining uniquely  $W(f)$  in accordance to the equivalence relation  $W(f_1) = W(f_2)$  if  $\sigma(f_1, f_3) = \sigma(f_2, f_3) \quad \forall f_3$ . According to [2] we choose the state so that for the free quantum field it is a quasifree state satisfying for the two point function the conditions formulated in [17]

$$w^{(2)}(x_1, x_2) = \frac{\alpha}{s} + \beta \ln |s| + \gamma \quad (36)$$

where  $\alpha, \beta, \gamma$  are smooth functions of  $x, x'$  and  $s$  is the square of the geodesic distance. Notice that this state has been constructed as a global state on the Kruskal space time and therefore we have left the Schwarzschild domain.

Based on (30) in the Schwarzschild domain we have again the same analyticity properties as in the Rindler wedge in Minkowski space or in the wedge in de Sitter space. We have the time evolution, that is now a Killing flow and therefore implemented by a unitary one parameter group. With its extension into the complex plane we can map the Schwarzschild domain  $u > 0, u^2 - v^2 > 0$  into the causal complement  $u < 0, u^2 > v^2$ . Therefore again the state is a temperature state with temperature  $T = 1/4\pi r_0$  in the Schwarzschild domain with respect to the time evolution. Considered as a state over the whole Kruskal space time the state is pure and satisfies Haag duality  $\mathcal{A}'_\Lambda = \mathcal{A}_{\Lambda^C}$  with  $\Lambda^C$  the causal complement to  $\Lambda$ . Notice that the causal complement to  $\{u > 0, u^2 - v^2 > 0\} \cup \{v > 0, 0 < v^2 - u^2 < 1\}$  is empty. Therefore the quantum field considered as acting on the physical space with  $r \geq 0$  is again in a pure state.

In the Schwarzschild domain we can apply the same considerations as in de Sitter space. Due to the scaling law that now is a consequence of (36) local algebras with compact  $\Lambda$  have modular operators with continuous spectrum, are type III and have infinite entropy. Due to the decrease properties of the two point function we can find algebras containing  $\mathcal{A}_\Lambda$ , remaining localized but with finite entropy. Again as in de Sitter space the entropy will not only scale with the surface but due to the fact that the state is a temperature state we have again to apply (28) telling us that also an operator far away from the horizon is quantum mechanically correlated to an operator in the commutant. Therefore the entropy of a local domain will contain a term depending on the size of the domain, not only on its boundary. From this view point we can observe the effect of the Hawking temperature even if we are far away from the horizon and therefore the Hawking temperature is not only an analogue to the temperature with respect to the boost in Rindler space, since now the temperature is given by the time evolution and not by a boost and therefore no scaling is necessary.

Our main interest however concerns the behaviour on the horizon and inside of the horizon. First we ask whether again the entropy of a point vanishes. In flat space we argued with the scaling law. This scaling law is evident for our free field described by (32) and (33). We consider functions  $f$  with finite support. With scaling  $f(x_0 + x) \rightarrow f(x_0 + \lambda x)$  their support shrinks with  $\lambda \rightarrow \infty$ . This scaling can be absorbed in the integration  $d\Omega_x$  and in the Green function  $G(x, x')$ . In this sense we obtain some  $f_\lambda(x_0 + x)$  that is close to  $\lambda^3 c f(x_0 + \lambda x)$  where  $c$  depends on the point  $x_0$  around which we scale. The weight of  $f$  is irrelevant for our description, only its support characterizes whether the Weyl-operator belongs to the local domain. Therefore as in flat space we can conclude that it is possible to arrange the scaling in such a way that the local domain shrinks and so does its surrounding, though with a slightly different scaling that demands a relatively increasing distance if we approach the singularity. The entropy with respect to this inclusion tends to zero, also inside the horizon. Of course we have to stay away from the singularity  $u^2 - v^2 = -1$ .

Concentrating on the horizon we can argue that every local region that is sufficiently smeared out and stays away from the singularity  $r = 0$  has an entropy determined by its size respectively the size of its surface. This holds also for a covering of the horizon, so that for such a covering the entropy is proportional to  $r_0^2$ .

Our main interest however is the behaviour of the local entropy inside of the Schwarzschild radius or inside of an arbitrary small neighborhood of the black hole. We consider as physically relevant domains those that lie in the future of the Schwarzschild space time, i.e. the domain  $\{u^2 - v^2 > -1, v > 0\}$ . If we consider the local algebra  $\mathcal{A}_\Lambda$  with  $\Lambda$  the causal completion of the domain  $\{\delta < r < \epsilon, \theta, \phi\}$  that we interpret as included into the total space as  $\{u = 0, (1 - \delta)^2 > v > (1 - \epsilon)^2, \theta, \phi\}$  this domain shrinks and if we rely on the scaling law the entropy tends to zero if we scale  $\delta$  together with  $\epsilon$  in such a way that the boundary remains sufficiently large. However the choice of this domain is rather artificial and of no relevance for any observation. We can go back to flat space to recall, why the choice of double cones was taken to be the natural basic description. The starting point is the assumption that the observables are essentially located in 3-dimensional space and that these observables form a von Neumann algebra. Observables located in domains that are space like separated commute with one another. Therefore the von Neumann algebra is obtained as  $\mathcal{A}''_\Lambda$ , the double commutant of the original algebra, and this justifies the choice of causally closed domains. If however we choose domains in the Kruskal domain including  $0 < r < r_1 < r_0$  and compare with a domain  $r_1 < r < r_0$  the corresponding observables will not commute, since the points are not space like any more.

The other observation in flat space uses Haag duality: If we evaluate the entropy for some inclusion  $\Lambda_1 \subset \Lambda_2$  i.e. via

$$S(\Lambda_1, \Lambda_2) = \inf_{\mathcal{A}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{A}_{\Lambda_2}} S(\mathcal{N}) \quad (37)$$

then the fact that we are in a pure state implies  $S(\mathcal{N}) = S(\mathcal{N}')$  so that

$$S(\Lambda_1, \Lambda_2) = S(\Lambda_2^C, \Lambda_1^C). \quad (38)$$

As a consequence the entropy of the causal complement of a finite domain is finite. However the entropy of the wedge is infinite, even when we consider not the wedge itself but any algebra containing the wedge and belonging to a shifted wedge, again based on the estimates (28). According to our construction Haag duality is also satisfied for the Kruskal-Schwarzschild metric, i.e. the quantum field is in a pure state. Instead of evaluating  $S(\mathcal{A}_{0 \leq r < r_1})$  we can evaluate  $S(\mathcal{A}_{r_1 < r < \infty})$ , i.e. we consider as relevant what can be observed in the total domain that just stays away from the singularity. Next we recall that in flat space a diamond can be considered to be the causal completion of small diamonds covering a spatial cut of the large diamond. In this spirit we construct the complement of a neighborhood of the black hole. We specify which algebra we take for  $\mathcal{A}_{r_1 < r < \infty} = \mathcal{A}_{> r_1}$ : A possible choice is  $\mathcal{A}_{r_1 < r < \infty} = (\mathcal{A}_{r_1 < r < r_0, u=0}'' \vee \mathcal{A}_{r_0 < r < \infty})''$  where  $\mathcal{A}_{r_1 < r < r_0, u=0}''$  is a diamond in the region  $\{v > 0, v^2 - u^2 > 0\}$ . Its commutant is not  $\mathcal{A}_{0 < r < r_1, u=0}''$  but the von Neumann algebra is located over an infinitely extended domain in  $\{u^2 - v^2 > 0, u < 0\}$ . Therefore, based on the Kosaki formula, most of the local subalgebras of the Schwarzschild algebra  $(\mathcal{A}_{r_0 < r < \infty})'' = \mathcal{A}_S$  will contribute to the relative entropy and will make it infinite. Therefore

$$S(\mathcal{A}_{> r_2}, \mathcal{A}_{> r_1}) = \infty, \quad 0 < r_1 < r_2 \quad (39)$$

With our identification  $\mathcal{A}_{r < r_1} = \mathcal{A}'_{> r_1}$  this gives an infinite entropy to the black hole. Notice however, that in these considerations the horizon  $r = r_0$  is not of special significance.

We have to admit that these arguments do not offer a way to assign an entropy to the black hole that scales like  $r_0^2$  with a scaling factor that is justified by the smearing of the surface that should be chosen of the size of the Planck length. However such a behaviour cannot be extracted from the model, because on one hand an infinite region is in a temperature state and has therefore infinite entropy, whereas we assume that the total system is in a pure state, so that also the complement of this region has infinite entropy. This discrepancy with the usual arguments might be the result that our quantum field was constructed on a fixed background without the possibility that the gravitational field can react on the quantum field whereas in the usual approach the entropy is obtained by the possibility of an energy exchange and entropy exchange of the black hole with the quantum field. Imagine several quantum fields corresponding to different non interacting particle species, built over the gravitational field. For such a model the local entropies are additive, and therefore necessarily our entropies are model dependent. Therefore any scaling parameter cannot be model independent. Notice that the gravitational constant disappeared in the calculations and only  $r_0$  appears. If we express  $r_0$  given by the mass of the black hole and the gravitational constant we can scale these parameters keeping  $r_0$  fixed. That the quantum field cannot effect the gravitational background corresponds that we take the gravitational constant to be 0 together with the mass of the black hole  $M$  being  $\infty$  whereas  $r_0$  is kept constant. This would be in agreement with the fact that the entropy does not scale with  $r_0$  but with  $M$ . Our considerations were determined by the analyticity properties of the quantum field together with Haag duality. Of course the assumption to be in a pure state was essential. Another question is whether the precise form of the state in the Schwarzschild region was essential or if it is possible to reduce

the entropy in the Schwarzschild region by appropriate correlations between the horizon and infinity without violating the regularity conditions at the horizon.

## 5 Conclusion

The fact that we can assign a temperature to the state of quantum fields over a curved background has to refer to an automorphism group on the algebra. This can be done only in special cases in such a way, that we can give a geometrical interpretation to the automorphism group. These cases include the boost in the Rindler wedge, the boosts in the wedges of de Sitter space and time evolution in Schwarzschild space time.

In contrast a local entropy, now without interpretation with respect to thermodynamics but considered as property of a state, can be defined under some mild restrictions for inclusions of local algebras. Also this local entropy shows different behaviour on a qualitative level, reflecting the properties of the underlying quantum field. In all cases the entropy vanishes for shrinking regular domains. In flat space the entropy is mainly determined by the surface of the domain, reflecting the entanglement of domains that is concentrated on the surface. This contribution remains unchanged also in gravitational fields. In the Sitter space the temperature that we can assign to the wedge induces that the entropy of local domains scales roughly also with their volume. The same is true for Schwarzschild space time. From this view point the temperature is more than an analogue to the Rindler wedge, where the temperature is observed by an accelerated observer and the temperature is only determined by the acceleration. In addition we have to extend the quantum field over the Schwarzschild space time if we want to keep Haag duality, i.e. to consider a pure state. The obvious extension is the extension to Kruskal space. Here the origin corresponds to a singularity. We give an interpretation to the quantum field around this singularity as the complement, i.e. commutant to the algebra outside the singularity. With this interpretation the algebra around the origin has infinite entropy without the possibility to regularize it by smearing.

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## References

- [1] Bekenstein J.D.: Phys.Rev. D **7** 2333-2346 (1973); "Do We Understand Black hole entropy" in Jantzen, R.T. and Mac Keiser, G.,eds., Proceedings of the VII Marcel Grossman Meeting. 39-58, (World Scietific Press, singapore, 1996)
- [2] Hartle, J., Hawking, S.W.: Phys.Rev.D.**60**,2960 (1983)
- [3] Verlinde, E.: On the origin of Gravity and the Laws of Newton, arXiv: 1001.0785v1/hep-th

- [4] Sewell, G.L. :Phys. Rev. Lett.**79A** 23, (1980)
- [5] Buchholz, D., Wichmann, E.H. : Commun. Math. Phys. **106** 321 (1986)
- [6] Bros, J., Moschella, U.: Rev. Math. Phys. **8** 327 (1996)
- [7] Narnhofer, H, Peter, I, Thirring, W.: Int. Journ. of Modern Physics B **10** 1507, (1996)
- [8] Fredenhagen, K., Haag, R. : Commun. Math. Phys. **108** 91 (1989)
- [9] Haag, R. : Local Quantum Physics, Springer Berlin (1992)
- [10] Bisognano, J.J., Wichmann, E.H. : J. Math. Phys.**16**, 985 (1975); J. Math. Phys. **17**, 303 (1976)
- [11] Buchholz, D., D'Antoni, C., Fredenhagen, K.: Commun. Math. Phys. **111** 123 (1987)
- [12] Buchholz, D., Jacobi, P. : Lett. Math. Phys. **13** 313 (1987)
- [13] Narnhofer, H.: Rev. Math. Phys. bf6/5a 1127 (1994)
- [14] Connes, A. Narnhofer, H. Thirring, W.: Commun. Math. Phys. **112** 691 (1987)
- [15] Narnhofer, H.: Local type I algebras and their entropy in quantum field theory, to be published
- [16] Kosaki, H.: J. Operator Theory **16** 335 (1986)
- [17] Wald R.M. : General Relativity. Chicago: Univ. of Chicago Press (1984)