



HAL
open science

Constant mean curvature spacelike hypersurfaces in Lorentzian manifolds with a timelike gradient conformal vector field

Magdalena Caballero, Alfonso Romero, Rafael M Rubio

► **To cite this version:**

Magdalena Caballero, Alfonso Romero, Rafael M Rubio. Constant mean curvature spacelike hypersurfaces in Lorentzian manifolds with a timelike gradient conformal vector field. *Classical and Quantum Gravity*, 2011, 28 (14), pp.145009. 10.1088/0264-9381/28/14/145009 . hal-00719838

HAL Id: hal-00719838

<https://hal.science/hal-00719838v1>

Submitted on 21 Jul 2012

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Constant Mean Curvature Spacelike Hypersurfaces in Lorentzian Manifolds with a Timelike Gradient Conformal Vector Field

Magdalena Caballero¹, Alfonso Romero² and Rafael M. Rubio¹

¹ Departamento de Matemáticas, Campus de Rabanales, Universidad de Córdoba, 14071 Córdoba, Spain

E-mail: magdalena.caballero@uco.es, rmrubio@uco.es

² Departamento de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain

E-mail: aromero@ugr.es

Abstract. A new technique to study spacelike hypersurfaces of constant mean curvature in a spacetime which admits a timelike gradient conformal vector field is introduced. As an application, the leaves of the natural spacelike foliation of such spacetimes are characterized in some relevant cases. The global structure of this class of spacetimes is analyzed and the relation with its well-known subfamily of Generalized Robertson-Walker spacetimes is exposed in detail. Moreover, some known uniqueness results for compact spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes are widely extended. Finally, and as a consequence, several Calabi-Bernstein problems are solved obtaining all the entire solutions on a compact Riemannian manifold to the constant mean curvature spacelike hypersurface equation, under natural geometric assumptions.

PACS numbers: 98.80.Jk, 04.20.-q, 02.40.Ky, 02.30.Jr

Keywords: Spacelike hypersurface, Constant mean curvature, Gradient conformally stationary spacetime, Generalized Robertson-Walker spacetime, Calabi-Bernstein problem.

1. Introduction

The notion of symmetry is clearly basic in Physics. In General Relativity, symmetry is usually based on the assumption of the existence of a one-parameter group of transformations generated by a Killing or, more generally, conformal vector field. In fact, an usual simplification for the search of exact solutions to the Einstein equation is to assume the existence a priori of such an infinitesimal symmetry (see [9, 11] for instance). A complete general approach to symmetries in General Relativity can be found in [22] (see also [10] and references therein). Although it is not always assumed

the same causal character for the infinitesimal symmetry, the timelike choice is natural, since the integral curves of such a timelike infinitesimal symmetry provide a privileged class of observers or test particles in the spacetime. Moreover, this choice is supported by very well-known examples of exact solutions.

A spacetime \overline{M} admitting a timelike Killing vector field is called stationary. It can be easily seen that if a spacetime \overline{M} has a timelike conformal vector field, then it is globally conformal to a stationary spacetime. This is a reason to call \overline{M} conformally stationary (CS). Clearly, a CS spacetime is time orientable. In general, the orthogonal distribution defined by a timelike conformal vector field K in a spacetime is not integrable. If the 1-form metrically equivalent to the vector field is closed, or equivalently, if K is locally the gradient of some function, then this distribution is integrable and provides the spacetime with a distinguished foliation by spacelike hypersurfaces. The presence of such a vector field is not enough to prevent the existence of closed nonspacelike curves, i.e., CS spacetimes fail to be causal, in general. However, if the timelike vector field K is globally the gradient of some smooth function (K is then called a gradient vector field) then the (clearly noncompact) spacetime admits a global time function. Therefore, it is stably causal [14], i.e., there is a fine C^0 neighborhood of the original metric of the spacetime such that any of its Lorentzian metrics is causal [6]. The existence of a gradient conformal vector field in a spacetime has been used to study certain cosmological models [19] and plays a relevant role for vacuum and perfect fluid spacetimes (see [9]). Along this paper, a spacetime \overline{M} with a timelike gradient conformal vector field will be called gradient conformally stationary (GCS) spacetime.

An interesting subclass of GCS spacetimes is the family of Generalized Robertson-Walker (GRW) spacetimes. A GRW spacetime is a warped product, with base a negatively defined line, fiber a general Riemannian manifold and arbitrary warping function. Note that, in this definition the fiber is not assumed to be of constant sectional curvature, in general. When this assumption holds and the dimension of the spacetime is 3, the GRW spacetime is a (classical) Robertson-Walker spacetime. Thus, GRW spacetimes widely extend Robertson-Walker spacetimes, and they include, for instance, the Einstein-de Sitter spacetime, Friedmann cosmological models, the static Einstein spacetime and the de Sitter spacetime. Observe that conformal changes of the metric of a GRW spacetime, with a conformal factor which only depends on universal time, produce new GRW spacetimes. Moreover, small deformations of the metric on the fiber of Robertson-Walker spacetimes also fit into the class of GRW spacetimes. Thus, a GRW spacetime is not necessarily spatially homogeneous, as in the classical cosmological models. Recall that spatial homogeneity seems appropriate just as a rough approach to consider the universe in the large. However, in order to consider it in a more accurate scale, this assumption is not realistic. Thus, GRW spacetimes could be suitable spacetimes to model universes with inhomogeneous spacelike geometry [18].

Spacelike hypersurfaces of constant mean curvature (CMC) in a spacetime are critical points of the area functional under a suitable volume constraint [5, 8]. Such hypersurfaces play an important role in General Relativity, since they can be used as

initial hypersurfaces where the constraint equations can be split into a linear system and a nonlinear elliptic equation (see [7] and references therein). A summary of other reasons justifying the study of CMC spacelike hypersurfaces can be found in [16]. In previous papers, [1, 2], CMC spacelike hypersurfaces in spatially closed GRW spacetimes were studied. The main tools were several Minkowski-type integral formulas, and some curvature assumptions on the ambient spacetime were needed. Later, with the same approach, compact CMC spacelike hypersurfaces were studied in CS spacetimes, [3]. With a very different starting point, the classical Bochner technique, several results in [1] were extended, changing the compactness of the spacelike hypersurface by the existence of a local maximum of a distinguished function [15]. Along the previous papers, the family of slices (i.e., the level hypersurfaces of the global time function) of a GRW spacetime was widely characterized. More recently, [4], it was proved that in a GRW spacetime with warping function f such that $-\log f$ is convex, the only compact CMC spacelike hypersurfaces are the slices. The authors used a different technique and they needed no curvature assumption.

The main aim of this paper is to introduce a new technique to study CMC spacelike hypersurfaces in GCS spacetimes and to give new uniqueness results for compact CMC spacelike hypersurfaces in these ambient spacetimes, both in the parametric and nonparametric cases. Its content is organized as follows. In Section 3, several results on the global structure of GCS spacetimes are given. Since the orthogonal distribution to the timelike conformal vector field is integrable, the leaves of the foliation are spacelike hypersurfaces. In this setting, our first result gives a characterization of the spacetimes which can be globally split as a GRW spacetime (Theorem 3.1).

A Lorentzian manifold (\bar{M}, \bar{g}) admits a global decomposition as a GRW spacetime if and only if it is a GCS spacetime with a timelike gradient conformal vector field, K , such that the flow of its normalized vector field, Z , is well defined and onto in a domain $I \times \mathcal{L}$, for some interval $I \subseteq \mathbb{R}$ and some leaf, \mathcal{L} , of the orthogonal foliation \mathcal{F}_K to K .

When the leaves of the foliation are compact, we can state Theorem 3.4.

Let (\bar{M}, \bar{g}) be a GCS spacetime and let K be a timelike gradient conformal vector field on \bar{M} . Suppose that the leaves of \mathcal{F}_K are compact. Then (\bar{M}, \bar{g}) admits a global decomposition as a GRW spacetime.

The role of any potential function of the timelike gradient conformal vector field is studied in Section 4. The results obtained are used in Section 5. On each compact spacelike hypersurface of the spacetime, we consider the restriction of a potential function. This function is constant if and only if the hypersurface is a leaf of the orthogonal foliation \mathcal{F}_K . Our technique allows us to obtain new results under very few assumptions. First, when the GCS spacetime is static, we can state, Corollary 5.3.

The only compact CMC spacelike hypersurfaces in a GCS spacetime whose timelike gradient conformal vector field, K , is Killing, are the totally geodesic leaves of the foliation \mathcal{F}_K .

On the other hand, if the timelike gradient conformal vector field is homothetic, we can characterize the compact CMC spacelike hypersurfaces of the spacetime, Corollary 5.7.

Let (\bar{M}, \bar{g}) be a GCS spacetime with a timelike gradient homothetic vector field K . The only compact CMC spacelike hypersurfaces in \bar{M} on which the length of K is constant are the leaves of the foliation \mathcal{F}_K .

Recall that if Z is the reference frame obtained from a timelike conformal vector field K on an $(n + 1)$ -dimensional spacetime, its divergence satisfies $\text{div}(Z) = n\rho/\sqrt{-\bar{g}(K, K)}$, where ρ is the function such that $L_K \bar{g} = 2\rho\bar{g}$. In consequence, if $\rho > 0$, then $\text{div}(Z) > 0$, and so, the observers in Z are, on average, spreading apart. If $\rho < 0$, then $\text{div}(Z) < 0$ and, consequently, the observers in Z come together. Let γ be an observer in Z and assume $(\rho/\sqrt{-\bar{g}(K, K)}) \circ \gamma$ is decreasing. In this case, the existence of t_0 such that $(\rho/\sqrt{-\bar{g}(K, K)})(\gamma(t_0)) < 0$, implies that $(\rho/\sqrt{-\bar{g}(K, K)})(\gamma(t)) < (\rho/\sqrt{-\bar{g}(K, K)})(\gamma(t_0)) < 0$, for all $t > t_0$. Thus, the assumption that the divergence is negative for at least one proper time t_0 seem to be relevant when analyzing a contracting universe towards a black hole (Theorem 5.9).

Let (\bar{M}, \bar{g}) be a GCS spacetime and let K be a timelike gradient conformal vector field on \bar{M} . If $(\log \sqrt{-\bar{g}(K, K)}(\gamma(t)))' \leq 0$ holds for each integral curve γ of Z , then the only compact CMC spacelike hypersurfaces in \bar{M} are the leaves of the foliation \mathcal{F}_K , which are totally umbilical with mean curvature $H = \frac{-\rho}{\sqrt{-\bar{g}(K, K)}}$.

In particular, our uniqueness results apply to compact CMC spacelike hypersurfaces in GRW spacetimes. We characterize the slices as the only compact CMC spacelike hypersurfaces in those spacetimes, under certain assumption on the warping function and without imposing any condition on the fiber (Corollary 5.10). As previously said, this result was obtained in [4], with a different technique which only works for GRW spacetimes. Let us remark that the same problem was before studied in [1], using Minkowski-type integral techniques, but assuming more restrictive curvature conditions.

In Section 6, we consider as setting an Einstein GRW spacetime. We extend and improve the uniqueness results on compact CMC spacelike hypersurfaces obtained in [2]. Theorems 6.1 and 6.2 close completely the open problems in the subject. Finally, in Section 7, we deal with several Calabi-Bernstein type problems on compact Riemannian manifolds (Theorems 7.2, 7.3).

Let (F, g) be a compact Riemannian manifold whose Ricci curvature is a non positive constant c . Let H be a real constant and let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be one of the functions in the cases from two to six of the table in Section 6. The only entire solutions $u \in C^\infty(F)$, $u(F) \subset I$, to the CMC spacelike hypersurface equation (E) are the constant functions $u = u_0$ such that $-f'(u_0)/f(u_0) = H$.

Let (F, g) be a compact Riemannian manifold with positive constant Ricci curvature c , and let f be the function defined by

$$f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)}e^{-bt},$$

where \bar{c} and a are positive constants and $b = \sqrt{\frac{\bar{c}}{n}}$. Let H be a real number. Then

- i) If $H \in \mathbb{R} \setminus]-b, b[$, there exists no solution to the CMC spacelike hypersurface equation (E).
- ii) If $H \in]-b, b[$ and $u \in C^\infty(F)$, $u(F) \subset I$, is a solution to the CMC spacelike hypersurface equation (E), then

$$\frac{1}{2b} \log \left(\frac{cn(b+H)}{4a^2\bar{c}(n-1)(b-H)} \right) \in u(F).$$

2. Preliminaries

A vector field K on a Lorentzian manifold (\bar{M}, \bar{g}) is called conformal if the Lie derivative of \bar{g} with respect to K satisfies

$$L_K \bar{g} = 2\rho \bar{g}, \tag{1}$$

where ρ is a (smooth) function. When $\rho = \text{constant}$, the vector field is called homothetic and if, in particular, $\rho = 0$ then K is called Killing. The existence of a (nontrivial) conformal vector field is a symmetry assumption for the metric tensor \bar{g} . In Relativity this assumption has been widely used to obtain exact solutions of the Einstein equation. Of course, a conformal vector field does not have a fixed causal character, but for spacetimes it is natural to assume that K is timelike. In such a case the integral curves of

$$Z = \frac{1}{\sqrt{-\bar{g}(K, K)}} K \tag{2}$$

provide a family of privileged observers in spacetime. From now on we will denote the length of K , $\sqrt{-\bar{g}(K, K)}$, with the letter \bar{h} .

For a given vector field K on \bar{M} , let ω be the 1-form on \bar{M} which is \bar{g} -equivalent to K , i.e., $\omega(X) := \bar{g}(K, X)$ for any $X \in \mathfrak{X}(\bar{M})$. When ω is closed, then it is locally exact or equivalently, K is locally a gradient. Such a K is called closed and each smooth function ϕ , defined on an open subset U of \bar{M} such that $K = \bar{\nabla}\phi$ on U is called a potential function of K on U , $\bar{\nabla}$ being the gradient on (\bar{M}, \bar{g}) . If we have $K = \bar{\nabla}\phi$ globally, then K is called a gradient vector field and ϕ is a potential function of K . In this case, if \bar{M} is connected, then any other potential function of K is $\phi + d$, where $d \in \mathbb{R}$.

Let (\bar{M}, \bar{g}) be a GCS spacetime and K a timelike gradient conformal vector field. It is not difficult to show

$$\bar{\nabla}_X K = \rho X \tag{3}$$

for any $X \in \mathfrak{X}(\overline{M})$, $\overline{\nabla}$ being the Levi-Civita connection of \overline{M} .

A direct computation from the previous formula gives us the following expression of ρ

$$\rho = \frac{K(\overline{h})}{\overline{h}} = Z(\overline{h}), \quad (4)$$

Z being the unit timelike vector field given in (2). From which we get that

$$\overline{\nabla}_Z Z = 0. \quad (5)$$

On the other hand, the global Frobenius theorem guarantees the existence of a n -foliation \mathcal{F}_K on \overline{M} , such that K is normal to its leaves. It is not difficult to see that the length of K , \overline{h} , as well as the function ρ are constant on each leaf of \mathcal{F}_K . Using (3) and the fact that \overline{h} is constant, we deduce

$$\overline{\nabla}_Y Z = \frac{\rho}{\overline{h}} Y \quad (6)$$

for any vector field Y such that $\overline{g}(Y, K) = 0$. And so,

$$L_Z \overline{g}(X, Y) = 2 \frac{\rho}{\overline{h}} \overline{g}(X, Y) \quad (7)$$

for any vector fields X, Y such that $\overline{g}(X, Z) = \overline{g}(Y, Z) = 0$. Combining (5) and (6), we get $L_Z \omega' = 0$, where ω' is the 1-form on \overline{M} which is \overline{g} -equivalent to Z . Consequently, any (local) flow of Z maps homothetically leaves of \mathcal{F}_K on leaves of \mathcal{F}_K , being each leaf totally umbilical with constant mean curvature

$$H = \frac{-\rho}{\overline{h}}, \quad (8)$$

where $H = -(1/n)\text{trace}A$, A being the shape operator associated to $-Z$.

3. Global structure of GCS spacetimes and characterization of GRW spacetimes

In this section we study the structure of GCS spacetimes. We will see that they are a natural extension of the GRW spacetimes. This assertion is reaffirmed by the first theorem of the section, which is a characterization of the spacetimes that admit a global decomposition as a GRW spacetime.

First, let us notice some general facts about GCS spacetimes. The setting is the same one as in the preliminaries.

Any GCS spacetime is stably causal. This follows from the fact that each global potential function of K is a smooth global time function, see [6, p. 64] (we are considering the time-orientation given by $-K$). Even more, global potential functions of K are constant along the leaves of \mathcal{F}_K .

From (6) it can be easily deduced that Z is irrotational, i.e., $d\omega'(X, Y) = 0$ for any X, Y orthogonal to Z . Therefore, if K is Killing, the GCS spacetime is static.

Let f be a positive smooth function defined on an open interval I of \mathbb{R} and let (F, g) be a n -dimensional ($n \geq 2$) Riemannian manifold. Consider $I \times F$ endowed with the Lorentzian metric

$$\langle \cdot, \cdot \rangle = -\pi_I^*(dt^2) + f(\pi_I)^2\pi_F^*(g), \quad (9)$$

where π_I and π_F denote the projections onto I and F , respectively. The Lorentzian manifold $(I \times F, \langle \cdot, \cdot \rangle)$ is a warped product in the sense of [17, p. 204], with base $(I, -dt^2)$, fiber (F, g) and warping function f . Following [1] we will refer to $(I \times F, \langle \cdot, \cdot \rangle)$ as a Generalized Robertson-Walker (GRW) spacetime.

The vector field $K = f(\pi_I)\partial_t$ is timelike and satisfies (3) with $\rho = f'(\pi_I)$, thus K is a timelike gradient conformal vector field with $\phi = -P_f \circ \pi_I$, where P_f is a primitive function of f . Therefore, GRW spacetimes lie in the family of GCS spacetimes. Locally, the converse is also true: i.e., for each point of a GCS spacetime there exists an open neighborhood which is isometric to a GRW spacetime, see [13, 21]. Indeed, the family of semi-Riemannian manifolds shown in [20, Sec. 3], whose members are called local warped product manifolds, contains a wide subfamily of GCS spacetimes constructed on certain fiber bundles over 1-dimensional basis.

Even more, if $(\overline{M}, \overline{g})$ is a GCS spacetime with timelike gradient conformal vector field K , we know that the normalized vector field Z is a timelike unit vector field that is spatially conformal, (7). From (5) and (6) it is easy to check that it is also closed and $\overline{\nabla} \operatorname{div} Z$ is proportional to Z . Using these facts, we arrive to the following characterization.

Theorem 3.1 *A Lorentzian manifold $(\overline{M}, \overline{g})$ admits a global decomposition as a GRW spacetime if and only if it is a GCS spacetime with a timelike gradient conformal vector field, K , such that the flow of its normalized vector field, Z , is well defined and onto in a domain $I \times \mathcal{L}$, for some interval $I \subseteq \mathbb{R}$ and some leaf of the orthogonal foliation to K , \mathcal{L} .*

Proof. We know that the necessary condition is true. Let us focus on the converse. It is clear that if we assume that Z is complete, from [12, Cor. 2.3, Rem. 2.4] and [17, Prop. 7.7], the flow of Z

$$\varphi : (\mathbb{R} \times \mathcal{L}_p, -dt^2 + f^2(t)\overline{g}|_{\mathcal{L}_p}) \longrightarrow \overline{M}$$

is a normal Lorentzian covering for all $p \in \overline{M}$, where $f(t) = \exp\left(\int_0^t \frac{\operatorname{div} Z(\varphi_s(p))}{n} ds\right)$ and \mathcal{L}_p is the leaf of \mathcal{F}_Z through p . Even more, \overline{M} is isometric to the quotient of $\mathbb{R} \times \mathcal{L}_p$ by the group of deck transformations of φ .

Observe that the proofs of [12, Cor. 2.3, Rem. 2.4] remain true when Z is not complete, but its flow is well defined and onto in a domain $I \times \mathcal{L}$, for some interval $I \subseteq \mathbb{R}$ and some leaf \mathcal{L} of the orthogonal foliation to K , \mathcal{F}_K .

The only remaining detail is proving that the group of deck transformations of φ is trivial. This can be directly deduced from the following two facts. Any global potential

function of K is a smooth global time function, and therefore is strictly decreasing on any integral curve of Z . Any global potential function of K is constant along the leaves of \mathcal{F}_K .

□

Our result extend [21, Th. 2.1], which was stated in the simply connected case. As well as [12, Th. 3.2].

Remark 3.2 Notice that all the computations in the preliminaries remain true when (\bar{M}, \bar{g}) is a CS spacetime and K is closed. Those spaces are also static when K is Killing. Even more, any CS spacetime with K closed is locally a GRW spacetime. But it fails to be globally a GCS spacetime. For instance, the Misner cylinder spacetime $(\mathbb{S}^1 \times \mathbb{R}, \bar{g} = -(d\theta \otimes ds + ds \otimes d\theta))$ admits a timelike parallel vector field $K = \partial_\theta + \partial_s$ which is not globally a gradient, because the Misner cylinder is not causal. From Theorem 3.1 we know that in order to get a global splitting, we need K to be globally a gradient. If we only ask K to be closed, we get that the spacetime is isometric to the quotient of $I \times \mathcal{L}$ by the group of deck transformations of φ , where \mathcal{L} is a leaf of \mathcal{F}_K and φ is the flow of Z .

Remark 3.3 The class of GCS spacetimes is bigger than the class of GRW spacetimes. For instance, any open subset of a GRW spacetime is a GCS spacetime but clearly it does not inherit the warped product structure, in general. Let us point out that the techniques previously used in [1, 2, 4] to obtain uniqueness results for CMC spacelike hypersurfaces in GRW spacetimes, do not work for general GCS spacetimes.

Notice that the assumption on the flow of Z trivially holds when Z is complete. We will see that this assumption also holds when the leaves of \mathcal{F}_K are compact.

Consider a GCS spacetime, K a timelike gradient conformal vector field and let $\varphi : \mathcal{D} \rightarrow \bar{M}$ be the maximal local flow of the normalized vector field Z . If we suppose that the leaves of \mathcal{F}_K are compact, then $\varphi_t : \mathcal{L}_p \rightarrow \mathcal{L}_{\varphi_t(p)}$ is onto for any $p \in M$ and any t such that φ_t is well defined, and therefore, a diffeomorphism. Moreover, we are going to see that φ is well defined in a domain $I \times \mathcal{L}$, I being an interval (a, b) and \mathcal{L} being a leaf of \mathcal{F}_K .

Suppose that I is the maximal interval where $\varphi : I \times \mathcal{L} \rightarrow \bar{M}$ is defined, let us see that I is also the maximal definition interval of each integral curve with initial value on \mathcal{L} . Assume that there exists $p_0 \in \mathcal{L}$ such that $\varphi(t, p_0)$ is defined in $(a, b + \epsilon)$. Consider the leaf $\mathcal{L}_{\varphi(b, p_0)}$ and take $\delta > 0$ such that $(-\delta, \delta) \times \mathcal{L}_{\varphi(b, p_0)} \subset \mathcal{D}$. Now, if $p \in \mathcal{L}$ is an arbitrary point, we can define the flow extension

$$\bar{\varphi}(t, p) = \begin{cases} \varphi(t, p) & \text{if } a < t < b \\ \varphi\left(t - b + \frac{\delta}{2}, \varphi\left(b - \frac{\delta}{2}, p\right)\right) & \text{if } b - \delta < t < b + \delta. \end{cases}$$

We have arrived to a contradiction.

On the other hand, the set $\varphi(I \times \mathcal{L})$ is open in \overline{M} . Consider $q \notin \varphi(I \times \mathcal{L})$, then if we take the maximal interval J where $\varphi : J \times \mathcal{L}_q \rightarrow$ is defined, $\varphi(J \times \mathcal{L}_q) \cap \varphi(I \times \mathcal{L}) = \emptyset$. In consequence $\varphi(I \times \mathcal{L}) = \overline{M}$. Thanks to Theorem 3.1, we get the following result.

Theorem 3.4 *Let $(\overline{M}, \overline{g})$ be a GCS spacetime and let K be a timelike gradient conformal vector field on \overline{M} . Suppose that the leaves of \mathcal{F}_K are compact. Then $(\overline{M}, \overline{g})$ admits a global decomposition as a GRW spacetime.*

4. Set up

Consider a (connected) spacelike hypersurface $x : S \rightarrow \overline{M}$ in a GCS spacetime $(\overline{M}, \overline{g})$ of dimension $n + 1$. Let us denote by g the induced metric on S (when \overline{M} is a GRW spacetime, we will agree to represent both \overline{g} and g by the same symbol as the one used in (9)) and by $\phi_S := \phi \circ x$ the restriction of a potential function ϕ of K on S . Note that ϕ_S is constant if and only if K_p is orthogonal to S , for all $p \in S$.

The time orientability of \overline{M} allows us to consider $N \in \mathfrak{X}^\perp(S)$ as the only, globally defined, unit timelike normal vector field on S in the same time-orientation of $-K$, i.e., such that $\overline{g}(K, N) > 0$ holds on all S . Thus, for Z defined in (2), the wrong-way Schwarz inequality (see [17, Prop. 5.30], for instance) gives $\overline{g}(N, Z) \geq 1$ and the equality holds at a point p if and only if $N(p) = -Z(p)$. Recall that, when \overline{M} is a GRW spacetime, a spacelike hypersurface is called a slice if $N = -\partial_t$ everywhere on S , i.e., $t := \pi_I \circ x$ is a constant or equivalently ϕ_S is constant.

Put $K^T := K + \overline{g}(K, N)N$, the tangential component of K along x . It is easy to see

$$\nabla \phi_S = K^T, \quad (10)$$

where ∇ denotes the gradient operator of g . A standard computation allows us to write

$$\Delta \phi_S = n\rho + nH\overline{g}(K, N), \quad (11)$$

where H denotes the mean curvature of S with respect to N and Δ the Laplacian of the induced metric (compare with [3, (2.1)]).

5. Mean curvature of a compact spacelike hypersurface in a GCS spacetime

Proposition 5.1 (Compare with [3, Prop. 2.1]) *In a GCS spacetime, there is no compact spacelike hypersurface whose mean curvature function H satisfies $H\rho > 0$.*

Proof. If S is such a spacelike hypersurface, from (11) the function ϕ must be constant and S is a leaf of the foliation \mathcal{F}_K with $H = 0$, which is a contradiction. \square

We know that a GCS spacetime with a Killing gradient vector field is a static spacetime. For those static GCS spacetimes we have the following two results.

Proposition 5.2 *In a GCS spacetime such that its timelike gradient conformal vector field, K , is Killing, the only compact spacelike hypersurfaces whose mean curvature function is signed (≤ 0 or ≥ 0) are the totally geodesic leaves of the foliation \mathcal{F}_K .*

In particular, there is no compact spacelike hypersurface with strictly positive or strictly negative mean curvature. And so, we get the following result for CMC spacelike hypersurfaces.

Corollary 5.3 *The only compact CMC spacelike hypersurfaces in a GCS spacetime whose timelike gradient conformal vector field, K , is Killing, are the totally geodesic leaves of the foliation \mathcal{F}_K .*

We come back to the general case in which K is a timelike gradient conformal vector field. And we consider the function $h : S \rightarrow \mathbb{R}$ defined by $h := \bar{h}|_S$. From the wrong-way Schwarz inequality [17, Prop. 5.30] we have $\bar{g}(K, N) \geq h$, everywhere on S . This inequality together with (11) provides the following wide extension of [1, Prop. 4.2].

Theorem 5.4 *Let S be a compact CMC spacelike hypersurface in a GCS spacetime. Let p_0 and p^0 be two points of S where ϕ_S attains local minimum and local maximum values, respectively. The mean curvature H of S must satisfy*

$$\frac{-\rho(p_0)}{h(p_0)} \leq H \leq \frac{-\rho(p^0)}{h(p^0)} \quad (12)$$

Proof. Note that $K^T(p_0) = K^T(p^0) = 0$ from (10), and, as consequence, we get the equality $\bar{g}(K, N) = h$ at the points p_0 and p^0 . On the other hand, from (11) we have

$$\rho(p_0) + h(p_0)H \geq 0 \quad \text{and} \quad \rho(p^0) + h(p^0)H \leq 0,$$

which give the announced inequalities. \square

Corollary 5.5 *Let S be a compact CMC spacelike hypersurface in a GCS spacetime. If $\rho \leq 0$ (resp. $\rho \geq 0$) on S , then the mean curvature H of S satisfies $H \geq 0$ (resp. $H \leq 0$).*

In the particular case in which \bar{h} is constant, from (4) we get that K is Killing, and so

Corollary 5.6 *The only compact CMC spacelike hypersurfaces in a GCS spacetime whose timelike gradient conformal vector field, K , has constant length, are the totally geodesic leaves of the foliation \mathcal{F}_K .*

Finally, if we only ask \bar{h} to be constant on a compact CMC spacelike hypersurface we get.

Corollary 5.7 *Let (\bar{M}, \bar{g}) be a GCS spacetime with a timelike gradient homothetic vector field K . The only compact CMC spacelike hypersurfaces in \bar{M} on which the length of K is constant are the leaves of the foliation \mathcal{F}_K .*

Proof. Let S be a compact CMC spacelike hypersurface. Again from Theorem 5.4, we have

$$H = -\frac{\rho}{h}.$$

Now, from (11) we get $\Delta\phi_S \geq 0$, and, therefore, ϕ_S must be constant. \square

Remark 5.8 In a more general setting, under the same assumptions as in Theorem 5.4, if the function $-\frac{\rho}{h}$ on S attains its minimum (resp. maximum) at the point where ϕ_S attains a local maximum (resp. a local minimum), then S is necessarily a leaf of the foliation \mathcal{F}_K . In order to prove this assertion, note that we have $H = \frac{-\rho}{h}$ on S from (12). Formula (11) reduces then to

$$\frac{\Delta\phi_S}{h} = nH(\bar{g}(Z, N) - 1).$$

Hence, the function ϕ_S is either subharmonic or superharmonic on a compact Riemannian manifold, and thus it must be constant.

Next, we will see that the technical assumption on the critical points of ϕ_S and $\frac{-\rho}{h}$ in the previous remark, follow from a reasonable geometric assumption.

Let (\bar{M}, \bar{g}) be a GCS spacetime, let K be a timelike gradient conformal vector field on \bar{M} and let Z be its normalized vector field.

Consider $p \in \bar{M}$ and an integral curve $\gamma(t) = \varphi(t, p)$ of Z passing through p . Note that we have

$$\frac{d}{dt} \left(\frac{\rho(\gamma(t))}{h(\gamma(t))} \right) = (\log h(\gamma(t)))'' ,$$

from (4). Therefore, if we suppose that the function $\frac{\rho}{h}$ is decreasing on γ , then

$$(\log h(\gamma(t)))'' \leq 0.$$

On the other hand, ϕ is a global time function. Therefore, ϕ is strictly decreasing along any integral curve of Z .

Thus, we can state,

Theorem 5.9 *Let (\bar{M}, \bar{g}) be a GCS spacetime and let K be a timelike gradient conformal vector field on \bar{M} . Suppose that $(\log h(\gamma(t)))'' \leq 0$ for each integral curve γ of the normalized vector field Z , then the only compact CMC spacelike hypersurfaces are the leaves of the foliation \mathcal{F}_K , which are totally umbilical with mean curvature $H = \frac{-\rho}{h}$.*

Proof. It is enough to see that the function $-\frac{\rho}{h}|_S$ attains its maximum (resp. minimum) at a local minimum (resp. maximum) of the function ϕ_S and, then, we apply Remark 5.8.

Let $q^0 \in S$ such that $\frac{-\rho}{h}(q^0)$ is maximum on S . Consider the closed set

$$Q^0 = \left\{ q \in S / \frac{-\rho}{h}(q) = \frac{-\rho}{h}(q^0) \right\}$$

and let $\bar{q}^0 \in Q^0$ such that $\phi_S(\bar{q}^0)$ is the minimum of ϕ_S on the compact set Q^0 . We are going to prove that \bar{q}^0 is a local minimum of ϕ_S . Indeed, observe that $\nabla\phi_S(\bar{q}^0) = 0$. Otherwise, S is transverse to the leaf $\mathcal{L}_{\bar{q}^0}$. Assume that \bar{q}^0 is not a local minimum of ϕ_S . Take an open neighborhood, $U(\bar{q}^0)$, where the local flow $\varphi(t, p)$ is well defined. The function $\phi(\varphi(t, p))$ is strictly decreasing on t , for each fixed $p \in U$, and it is constant on each leaf of \mathcal{F}_K . The function $\frac{-\rho}{h}(\varphi(t, p))$ is increasing on t , for each fixed $p \in U$, and constant on each leaf of \mathcal{F}_K . From those facts and the choice of \bar{q}^0 , we arrive to a contradiction. An analogous reasoning asserts that \bar{q}^0 must be a local minimum of ϕ_S . \square

In particular, we widely extend [4, Th. 1] (and consequently [1, Theors. 5.1, 5.2]).

Corollary 5.10 *In a GRW spacetime \bar{M} whose warping function satisfies $(\log f)'' \leq 0$, the only compact CMC spacelike hypersurfaces are the slices.*

Example 5.11 As an easy application, consider the family of GRW spacetimes $\bar{M}_{a,b}$, $a \in \mathbb{R}^+$, $b \in \mathbb{R} \setminus \{0\}$, with base \mathbb{R} , warping function $f(t) = ae^{bt}$ and arbitrary fiber. Clearly $(\log f)'' = 0$ holds. Therefore, if S is a compact CMC spacelike hypersurface in $\bar{M}_{a,b}$, then S is a slice with mean curvature $H = -b$.

Remark 5.12 Let \bar{M} be a GRW spacetime, whose warping function satisfies $(\log f)'' \geq 0$. As a consequence of the Theorem 5.4, if S is a compact CMC spacelike hypersurface in \bar{M} with mean curvature H , then

$$\frac{-f'(t(p^0))}{f(t(p^0))} \leq H \leq \frac{-f'(t(p_0))}{f(t(p_0))}.$$

Observe that $\max \phi(t(S)) = \max \frac{-f'(t(S))}{f(t(S))}$ and $\min \phi(t(S)) = \min \frac{-f'(t(S))}{f(t(S))}$. This may be interpreted as that the mean curvature of the hypersurface is bounded by the minimum and maximum values of the mean curvatures of the slices intersecting S .

Proposition 5.13 *In a GRW spacetime \bar{M} such that $(\log f)'' \leq 0$ and $f' \leq 0$ (resp. $f' > 0$), there exists no compact CMC spacelike hypersurface in M such that its mean curvature satisfies $H < 0$ (resp. $H > 0$).*

Proof. Otherwise, from Theorem 5.4 there are points $p_0, p^0 \in S$ such that

$$H(p_0) \leq \frac{-f'(t(p_0))}{f(t(p_0))} \leq \frac{-f'(t(p^0))}{f(t(p^0))} \leq H(p^0)$$

\square

Proposition 5.14 *Let \bar{M} be a GRW spacetime whose warping function satisfies $(\log f)'' \leq 0$. If the mean curvature function H of a compact spacelike hypersurface S in \bar{M} satisfies $H < 0$ or $H > 0$, then S does not intersect any maximal slice.*

Proof. From the inequality in the proof of the previous result, it follows that f' has no zero on S . \square

6. CMC spacelike hypersurfaces in a Einstein GRW spacetime

In this section we deal with GRW spacetimes which are Einstein, i.e., its Ricci tensor is proportional to its metric. If \overline{M} has base I , fiber F and warping function f , it is well-know that \overline{M} is Einstein with $\overline{\text{Ric}} = \bar{c}\langle \cdot, \cdot \rangle$, if and only if (F, g) has constant Ricci curvature c and f satisfies the following differential equations

$$\frac{f''}{f} = \frac{\bar{c}}{n} \quad \text{and} \quad \frac{\bar{c}(n-1)}{n} = \frac{c + (n-1)(f')^2}{f^2} \tag{13}$$

Moreover, \overline{M} has constant sectional curvature \bar{C} if and only if F has constant sectional curvature C (that is, \overline{M} is a classical Robertson-Walker spacetime) and f satisfies (13) with $c = (n-1)C$ and $\bar{c} = n\bar{C}$.

All the positive solutions to (13) were given in [2] and collected in the following table (in each case, the interval of definition I of f is the maximal one where f is positive.)

TABLE

1	$\bar{c} > 0$	$c > 0$	$f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)}e^{-bt}, \quad a > 0, \quad b = \sqrt{\bar{c}/n}$
2	$\bar{c} > 0$	$c = 0$	$f(t) = ae^{\varepsilon bt}, \quad a > 0, \quad \varepsilon = \pm 1, \quad b = \sqrt{\bar{c}/n}$
3	$\bar{c} > 0$	$c < 0$	$f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)}e^{-bt} \quad a \neq 0, \quad b = \sqrt{\bar{c}/n}$
4	$\bar{c} = 0$	$c = 0$	$f(t) = a, \quad a > 0$
5	$\bar{c} = 0$	$c < 0$	$f(t) = \varepsilon \sqrt{\frac{-c}{(n-1)}}t + a, \quad \varepsilon = \pm 1$
6	$\bar{c} < 0$	$c < 0$	$f(t) = a_1 \cos(bt) + a_2 \sin(bt), \quad a_1^2 + a_2^2 = cn/\bar{c}(n-1), \quad b = \sqrt{-\frac{\bar{c}}{n}}$

Note that, from (13), we have

$$(n-1)(\log f)'' = \frac{c}{f^2}.$$

Therefore, as a direct application of Corollary 5.10, we get the following wide extension on [2, Cor. 5],

Theorem 6.1 *Every compact CMC spacelike hypersurface in an Einstein GRW spacetime, whose fiber has Ricci curvature $c \leq 0$ (cases 2 to 6 of previous table), must be a slice.*

Now we will deal with the remaining case in the table. Consider a GRW spacetime \overline{M} , with fiber a Riemannian manifold with constant positive Ricci curvature c , (F, g) , and warping function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ given by

$$f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)}e^{-bt},$$

where \bar{c} and a are positive constants and $b = \sqrt{\frac{\bar{c}}{n}}$. In this case, $(\log f)''(t) = \frac{c}{(n-1)f^2(t)} > 0$ and we know from Remark 5.12, that any compact CMC spacelike hypersurface S in \bar{M} with mean curvature H satisfies

$$\frac{-f'(t(p^0))}{f(t(p^0))} \leq H \leq \frac{-f'(t(p_0))}{f(t(p_0))}$$

where $p^0 = \max t(S)$ and $p_0 = \min t(S)$.

Since, we have

$$\frac{f'(t)}{f(t)} = b \left(1 - \frac{2cn}{4a^2c(n-1)e^{2bt} + cn} \right),$$

we get

$$-b \leq \frac{f'(t)}{f(t)} \leq b,$$

and we are now in position to state the following result,

Theorem 6.2 *Let \bar{M} be a GRW spacetime whose fiber is a Riemannian manifold with constant positive Ricci curvature c and warping function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ given by*

$$f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)}e^{-bt},$$

where \bar{c} and a are positive constants, and $b = \sqrt{\frac{\bar{c}}{n}}$. There is no compact CMC spacelike hypersurface in \bar{M} with mean curvature $H \geq b$ or $H \leq -b$. Even more, any compact CMC spacelike hypersurface in \bar{M} with mean curvature $H \in]-b, b[$, intersects the only slice in \bar{M} with mean curvature H .

Remark 6.3 In the setting of the previous result, the only slice with mean curvature H is the one given by $t = \frac{1}{2b} \log \left(\frac{cn(b+H)}{4a^2\bar{c}(n-1)(b-H)} \right)$.

Remark 6.4 For the De Sitter spacetime, $I = \mathbb{R}$, $F = \mathbb{S}^n$ (the unit n -sphere), and $f(t) = \cosh t$, it is well-known that there exists a compact CMC spacelike hypersurface with mean curvature $H^2 \leq 1$, which is not a slice. Therefore, Theorem 6.2 is the best possible result in this direction for the case 1 of the table.

7. Calabi-Bernstein's type results

An important family of spacelike hypersurfaces in a GRW spacetime are the spacelike graphs. Recall that any spacelike hypersurface is locally a spacelike graph, globally under additional assumptions [1, Sect. 3]. Let us briefly recall the notion of spacelike graph. Consider a GRW spacetime (\bar{M}, \bar{g}) , with fiber a Riemannian manifold (F, g) and warping function $f : I \rightarrow \mathbb{R}^+$. Let $u \in C^\infty(F)$ be a function such that $u(F) \subset I$. Consider the graph of u in \bar{M}

$$F_u := \{(u(p), p) : p \in F\} \subset \bar{M}.$$

The metric induced by (9) on F_u is written on F as follows

$$g_u = -du^2 + f(u)^2 g, \quad (14)$$

where $f(u) := f \circ u$. Therefore, F_u is spacelike if and only if $|Du| < f(u)$ everywhere on F , where Du denotes the gradient of u with respect to g . The unit timelike normal vector field to F_u in the same time orientation of $-\partial_t$ is

$$N = \frac{-f(u)}{\sqrt{f(u)^2 - |Du|^2}} \left(\partial_t + \frac{1}{f(u)^2} Du \right).$$

It is not difficult to see that F_u is spacelike and has constant mean curvature H (relative to N) if and only if u satisfies

$$\operatorname{div} \left(\frac{Du}{f(u)\sqrt{f(u)^2 - |Du|^2}} \right) = -nH - \frac{f'(u)}{\sqrt{f(u)^2 - |Du|^2}} \left(n + \frac{|Du|^2}{f(u)^2} \right) \quad (E.1)$$

$$|Du| < f(u), \quad (E.2)$$

where div denotes the divergence operator of (F, g) , $f(u) := f \circ u$ and $f'(u) := f' \circ u$. Equation (E) is elliptic and it is the Euler-Lagrange equation of a relevant variational problem, see [3, 8]. In fact, let (F, g) be a (connected) compact Riemannian manifold, with dimension $n \geq 2$ and let f be a positive smooth function defined on an open interval I of \mathbb{R} . Consider the class of smooth real valued functions u on F such that $u(F) \subset I$ and $|Du| < f(u)$ on all F . Over this class, consider the functional

$$\mathcal{A}(u) := \int_F f(u)^{n-1} \sqrt{f(u)^2 - |Du|^2} dV_g, \quad (15)$$

where dV_g is the canonical measure given by g . The Euler-Lagrange equation for critical points of this functional, under the constraint

$$\int_F \left(\int_{t_0}^u f(t)^n dt \right) dV_g = \text{constant}, \quad (16)$$

is precisely (E). Notice that $\mathcal{A}(u)$, given in (15), is the n -area of (F, g_u) and (16) is a volume constraint for the graph. Therefore, u is a critical point of (15) under the constraint (16) if and only if F_u has constant mean curvature H . Equation (E) is then called the constant mean curvature equation for spacelike graphs in \overline{M} .

We end this paper with uniqueness and non-existence results for the equation (E). The following two theorems widely extend the results [1, Th. 5.9] and [2, Th. 7], respectively.

Theorem 7.1 *Let (F, g) be a compact Riemannian manifold and let $f : I \rightarrow \mathbb{R}^+$ be a smooth function such that $(\log f)'' \leq 0$. If H is a real constant, the only entire solutions $u \in C^\infty(F)$, $u(F) \subset I$, to the CMC spacelike hypersurface equation (E) are the constant functions $u = u_0$ such that $-\frac{f'(u_0)}{f(u_0)} = H$.*

Theorem 7.2 *Let (F, g) be a compact Riemannian manifold whose Ricci curvature is a non positive constant c . Let H be a real constant and let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be one of the functions in the cases from two to six of the previous table. The only entire solutions $u \in C^\infty(F)$, $u(F) \subset I$, to the constant mean curvature differential equation (E) are the constant functions $u = u_0$ such that $-\frac{f'(u_0)}{f(u_0)} = H$.*

Finally for the remaining case in the previous table, we get,

Theorem 7.3 *Let (F, g) be a compact Riemannian manifold with positive constant Ricci curvature c , and let f be the function defined by*

$$f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)}e^{-bt},$$

where \bar{c} and a are positive constants and $b = \sqrt{\frac{\bar{c}}{n}}$. Let H be a real number. Then

i) *If $H \in \mathbb{R} \setminus]-b, b[$, there exists no solution to the CMC spacelike hypersurface equation (E).*

ii) *If $H \in]-b, b[$ and $u \in C^\infty(F)$, $u(F) \subset I$, is a solution to the CMC spacelike hypersurface equation (E), then*

$$\frac{1}{2b} \log \left(\frac{cn(b+H)}{4a^2\bar{c}(n-1)(b-H)} \right) \in u(F).$$

Acknowledgments

The first two authors are partially supported by the Spanish MEC-FEDER Grant MTM2010-18099 and the Junta de Andalucía Regional Grant P09-FQM-4496.

References

- [1] Alías L J, Romero A and Sánchez M 1995 Uniqueness of complete spacelike of constant mean curvature in Generalized Robertson-Walker spacetimes *Gen. Relat. Gravit.* **27** 71–84.
- [2] Alías L J, Romero A and Sánchez M 1997 Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems *Tôhoku Math. J.* **49** 337–345.
- [3] Alías L J, Romero A and Sánchez M 1997 Spacelike hypersurfaces of constant mean curvature in certain spacetimes *Nonlinear Anal-Theor* **30** 655–661.
- [4] Alías L J and Montiel S 2002 Uniqueness of spacelike hypersurfaces with constant mean curvature in generalized Robertson-Walker spacetimes *Differential Geometry, Valencia, 2001* World Sci. Publ., River Edge, N.J. 59–69.
- [5] Barbosa J M L and Olikier V 1993 Stable spacelike hypersurfaces of constant mean curvature in Lorentz space *Geometry and Global Analysis*, First MSJ Internat. Res. Inst., Tôhoku Univ. 161–164.
- [6] Beem J K, Ehrlich P E and Easley K L 1996 *Global Lorentzian Geometry, second edition* Pure and Applied Mathematics vol. 202 (Marcel Dekker New York)
- [7] Choquet-Bruhat Y 1976 The problem of constraints in General Relativity: solution of the Lichnerowicz equation *Differential Geometry and Relativity* Math. Phys. Appl. Math. **3** D. Reidel, Dordrecht, 225–235.

- [8] Colares A G and Brasil A 1999 On constant mean curvature spacelike hypersurfaces in Lorentz manifolds *Math. Cotemp.* **17** 99–135.
- [9] Daftardar V and Dadhich N 1994 Gradient conformal Killing Vectors and Exact solutions *Gen. Relat. Gravit.* **26** 859–868.
- [10] Duggal K L and Sharma R 1999 *Symmetries of Spacetimes and Riemannian Manifolds* Mathematics and Its Applications vol. 487 (Kluwer Academic Publishers).
- [11] Earley D, Isenberg J, Marsden J and Moncrief V 1986 Homothetic and conformal symmetries of solutions to Einstein equations *Commun. Math. Phys.* **106** 106–137.
- [12] Gutiérrez M and Olea B 2009 Global decomposition of a Lorentzian manifold as a GRW spacetime *Diff. Geom. Appl.* **27** 146–156.
- [13] Harris S G 1985 A characterization of Robertson-Walker spaces by lightlike sectional curvature *Gen. Relat. Gravit.* **17** 493–498.
- [14] Hawking S W 1968 The existence of cosmic time functions *P. Roy. Soc. Lond. A. Mat.* **308** 433–435.
- [15] Latorre J M and Romero A 2002 Uniqueness of noncompact spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes *Geom. Dedicata* **93** 1–10.
- [16] Marsden J E and Tipler F J 1980 Maximal hypersurfaces and foliations of constant mean curvature in General Relativity *Phys. Rep.* **66** 109–139.
- [17] O’Neill B 1983 *Semi-Riemannian Geometry with applications to Relativity* (Academic Press).
- [18] Rainer M and Schmidt H-J 1995 Inhomogeneous cosmological models with homogeneous inner hypersurface geometry *Gen. Relat. Gravit.* **27** 1265-1293.
- [19] Reboucas M J, Skea J E F and Tavakol R K 1996 Cosmological models expressible as gradient vector fields *J. Math. Phys.* **37** 858–873.
- [20] Romero A and Sánchez M 1994 On Completeness of Certain Families of Semi-Riemannian Manifolds *Geom. Dedicata* **53** 103–117.
- [21] Sánchez M 1998 On the Geometry of Generalized Robertson-Walker spacetimes: geodesics *Gen. Relat. Gravit.* **30** 915–932.
- [22] Zafiris E 1998 Irreducible decomposition of Einstein’s equations in spacetimes with symmetries *Ann. Phys.* **263** 155–178.