

**Performance of time-marching techniques dedicated to  
nonsmooth systems**  
**A nonlinear modal analysis approach**

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# Outline

## ① Introduction

- Context
- Challenges
- Motivation

## ② Modal analysis

- Linear framework
- Nonlinear framework

## ③ Non-smooth application

- Unilateral contact without friction
- Invariant periodic solution
- Linear rod with contact
- Modal motions
- Invariant manifolds

## ④ Time-stepping techniques

- Central differences – Implicit contact
- $\theta$ -method
- Comparison

## ⑤ Conclusion

- Summary
- Future work

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# Context

## Framework

- *Flexible structures*
  - ▶ finite-element models, multibody
  - ▶ possibly with *many degrees-of-freedom*
- Dynamics and vibration
  - ▶ transient
  - ▶ *steady-state* — periodic, quasi-periodic regimes
- Interface nonlinearities — contact, friction

# Context

## Framework

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## Nonlinear modal analysis

- nonlinear modes: periodic motion of an autonomous system, *invariant manifold* on which oscillations take place
- extension of linear modal analysis, i.e. provides an *essential signature of the dynamics*
- numerical framework
- useful for engineering applications

# Challenges

- **Large scale engineering systems** (e.g. FE models, multibody) call for (efficient) **numerical methods** adapted to generic nonlinearities (i.e. not only polynomial)
  - ▶ Space semi-discretization
  - ▶ Time stepping approaches (incl. *shooting* for BVP)
  - ▶ Galerkin methods in time (e.g. harmonic balance, finite-element, etc.)
  - ▶ Space-time discretization

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  - ▶ Contact interface laws
  - ▶ Optimization strategies

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- **Non-smooth** systems (e.g. featuring interface interactions) require specific treatments and methods
  - ▶ Contact interface laws
  - ▶ Optimization strategies
- Beyond nonlinear modal analysis, the **dynamic regimes** of such systems have to be fully qualified including their **stability**
  - ▶ equilibria
  - ▶ periodic or quasi-periodic solutions
  - ▶ chaos



# Time-integration algorithms

## Validation

- Assessment of performances of time-stepping techniques dedicated to flexible mechanical systems featuring unilateral contact conditions
- How are such techniques capable of finding periodic orbits of autonomous (and conservative) systems with non-smooth nonlinearities?
- Preservation of invariant quantities

# Time-integration algorithms

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## Extension

- Description of the dynamics around these invariant quantities
- Stability analysis

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# Linear framework (1)

- Linear 1<sup>st</sup>-order ODE with constant coefficients

$$\dot{\mathbf{z}} + \mathbf{D}\mathbf{z} = \mathbf{0} \quad (1)$$

- Assumed solution

$$\mathbf{z}(t) = \mathbf{Z}e^{\lambda t} \Rightarrow (\mathbf{D} - \lambda\mathbf{I})\mathbf{Z} = \mathbf{0} \quad (2)$$

- ▶ generally complex eigen elements  $(\lambda_j, Z_j)$
- ▶ uncoupled equations  $\equiv$  **invariant subsets**
- ▶ superposition principle  $\equiv$  eigen modes span the configuration space
- Invariant set formulation
  - ▶ Master coordinates  $(x_1, y_1) = (u_1, v_1)$
  - ▶ Functional dependence

$$\begin{aligned} x_i(t) &= a_{1i}u_1(t) + a_{2i}v_1(t) \\ y_i(t) &= b_{1i}u_1(t) + b_{2i}v_1(t) \end{aligned} \quad i = 1, 2, \dots, N \quad (3)$$

- ▶ Substitution into equation of motion (1) to get  $a_{ik}$ 's and  $b_{ik}$ 's

## Linear framework (2)

- Equations of motion in real modal space

$$\begin{pmatrix} \dot{u}_k \\ \dot{v}_k \end{pmatrix} = \begin{bmatrix} -\xi_k \omega_k & \omega_k \sqrt{1 - \xi_k^2} \\ \omega_k \sqrt{1 - \xi_k^2} & -\xi_k \omega_k \end{bmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} \quad k = 1, 2, \dots, N \quad (4)$$

- Reconstruction: superposition principle
- System with two degrees-of-freedom: first linear mode

# Nonlinear framework (1)

- Theoretical tools
  - ▶ Central manifold theorem
  - ▶ Lyapunov theorem
  - ▶ Normal form theorem

- Assumed solution

$$\mathbf{z}(t) = \sum_n \mathbf{Z}_n e^{n\lambda t} \Rightarrow \mathbf{F}(\lambda, \mathbf{Z}) = 0 \quad (5)$$

- ▶  $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n)$
  - ▶ continuation technique:  $\mathbf{Z}(\lambda)$
  - ▶ geometry of the invariant manifold
- Invariant set formulation
    - ▶ Master coordinates  $(x_1, y_1) = (u, v)$
    - ▶ (Nonlinear) functional dependence

$$\begin{aligned} x_i &= X_i(u, v) \\ y_i &= Y_i(u, v) \end{aligned} \quad i = 1, 2, \dots, N \quad (6)$$

## Nonlinear framework (2)

- ▶ Time differentiation

$$\begin{aligned}\dot{x}_i &= \frac{\partial X_i}{\partial u} \dot{u} + \frac{\partial X_i}{\partial v} \dot{v} \\ \dot{y}_i &= \frac{\partial Y_i}{\partial u} \dot{u} + \frac{\partial Y_i}{\partial v} \dot{v}\end{aligned}\quad i = 1, 2, \dots, N \quad (7)$$

- ▶ Substitution into equation of motion (1) —  $i=1, 2, \dots, N$

$$\begin{aligned}Y_i &= \frac{\partial X_i}{\partial u} v + \frac{\partial X_i}{\partial v} f_1(u, X_1, \dots, X_n, v, Y_2, \dots, Y_N) \\ f_i(u, X_1, \dots, X_n, v, Y_2, \dots, Y_N) &= \frac{\partial Y_i}{\partial u} v + \frac{\partial Y_i}{\partial v} f_1(u, X_1, \dots, X_n, v, Y_2, \dots, Y_N)\end{aligned}\quad (8)$$

- ▶ Power series expansions, nonlinear Galerkin technique...
- ▶ Reduced-dynamics on the invariant manifold

$$\begin{aligned}\dot{u} &= v \\ \dot{v} &= f_1(u, X_1, \dots, X_n, v, Y_2, \dots, Y_N)\end{aligned}\quad (9)$$

## Nonlinear framework (3)

- ▶ energy-dependent frequency
  - ▶ perturbation techniques
  - ▶ stability analysis
- ▶ Cubic nonlinearity: first nonlinear mode



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# Unilateral contact without friction

- Autonomous and conservative system
  - ▶ contact interface :  $\Gamma_c$
  - ▶ no external forcing
  - ▶ no friction, no structural damping

- Gap function on  $\Gamma_c$

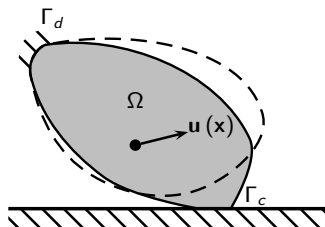
$$g(\mathbf{u}) = \mathbf{u}(\mathbf{x}) \cdot \mathbf{n} - g_0(\mathbf{x})$$

- ▶  $\mathbf{n}$ : outward normal
- ▶  $g_0$ : initial gap

- Contact conditions —  $\forall \mathbf{x} \in \Gamma_c$

$$\tau_N \geq 0, \quad g(\mathbf{u}) \leq 0, \quad g(\mathbf{u}) \cdot \tau_N = 0$$

- ▶  $\tau_N = \boldsymbol{\sigma} \cdot \mathbf{n}$ : contact pressure



# Invariant periodic solution

- Boundary value problem (Dirichlet and Signorini conditions)

$$\rho \ddot{\mathbf{u}} - \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) = 0 \quad \text{on } \Omega \times \mathbb{R}_*^+$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_d \times \mathbb{R}_*^+$$

$$\mathbf{g}(\mathbf{u}) \leq 0, \quad \tau_N \leq 0 \quad \text{and} \quad \mathbf{g}(\mathbf{u}) \cdot \tau_N = 0 \quad \text{on } \Gamma_c \times \mathbb{R}_*^+$$

- Frequency-domain displacement field

$$\mathbf{u}(t) = \sum_{n \in \mathbb{Z}} \hat{\mathbf{u}}_n e^{jn\omega t} \quad \text{with} \quad \hat{\mathbf{u}}_n = \frac{1}{T} \int_T \mathbf{u}(t) e^{-jn\omega t} dt$$

- Final formulation

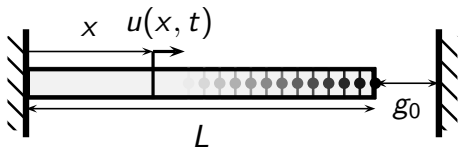
Find  $\{\omega, \hat{\mathbf{u}}\}$ , with  $\hat{\mathbf{u}} = \{\hat{\mathbf{u}}_n, n \in \mathbb{Z}\}$  such as:

$$-\operatorname{div} \boldsymbol{\sigma}(\hat{\mathbf{u}}_n) = (n\omega)^2 \hat{\mathbf{u}}_n \quad \text{on } \Omega \times \mathbb{Z}$$

$$\hat{\mathbf{u}}_n = 0 \quad \text{on } \Gamma_d \times \mathbb{Z}$$

$$\mathbf{g}(\hat{\mathbf{u}}) \leq 0, \quad \tau_N \leq 0 \quad \text{and} \quad \mathbf{g}(\hat{\mathbf{u}}) \cdot \tau_N = 0 \quad \text{on } \Gamma_c \times [0, T]$$

# Linear rod with contact



- Local equation

$$\rho \frac{\partial^2 u}{\partial t^2}(x, t) - E \frac{\partial^2 u}{\partial x^2}(x, t) = 0, \quad x \in ]0, L [$$

- Boundary condition

$$u(0, t) = 0$$

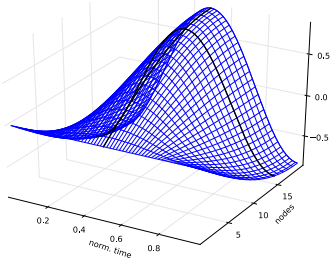
- Unilateral contact conditions

$$u(L, t) - g_0 \leq 0, \quad \sigma_{xx}(L, t) = EA \frac{\partial u}{\partial x}(L, t) \geq 0, \quad (u(L, t) - g_0) \cdot \sigma_{xx}(L, t) = 0$$

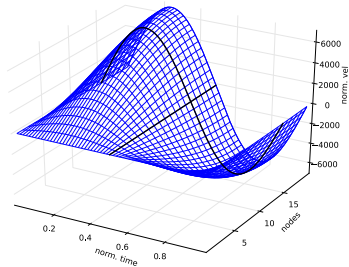
# Modal motions

low energy

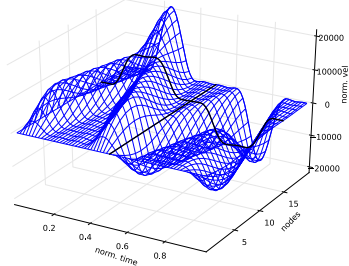
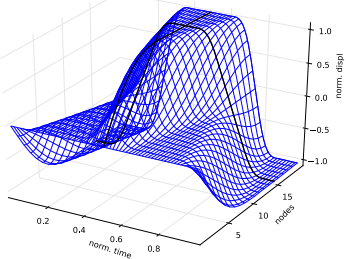
displacement



velocity

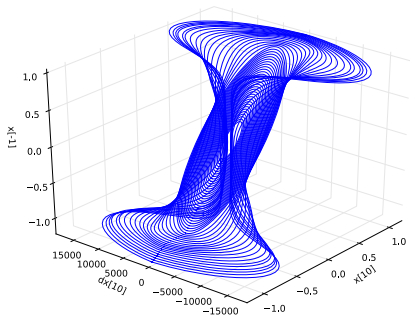


high energy

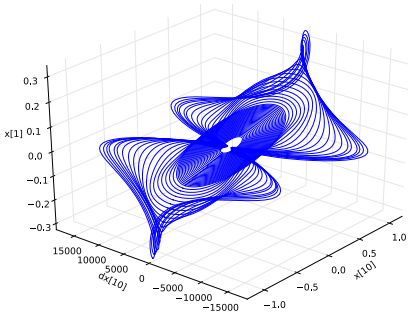


# Invariant manifolds

- Cross section of the invariant manifold
- From low to high energy
- Energy–frequency dependence



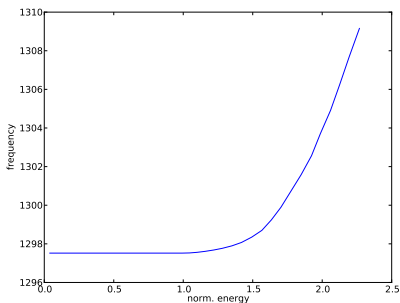
node in contact



node close to the clamped condition

# Invariant manifolds

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# Central differences – Implicit contact

- Problem Find  $\mathbf{u}_{n+1}$  and  $\lambda_{n+1}$  such that:

$$\mathbf{M} \left( \frac{\mathbf{u}_{n+1} - 2\mathbf{u}_n + \mathbf{u}_{n-1}}{\Delta t^2} \right) + \mathbf{K}\mathbf{u}_n - \mathbf{G}^T \lambda_{n+1} = 0$$

with  $\mathbf{G}\mathbf{u}_{n+1} \geq 0$ ,  $\lambda_{n+1} \geq 0$ , and  $\lambda_{n+1}^T \mathbf{G}\mathbf{u}_{n+1} = 0$

- Algorithm

1. Prediction of  $\mathbf{u}_{n+1}$  by ignoring contact forces  $\lambda_{n+1}$
2. If  $(\mathbf{G}\mathbf{u}_{n+1})_{(i)} < 0$  then  $\mathbf{u}_{n+1}$  is modified so that  $(\mathbf{G}\mathbf{u}_{n+1})_{(i)} = 0$

- Properties

- ▶ conditionally stable
- ▶ proof of convergence in 1D
- ▶ possibilities of lumped mass
- ▶ energy losses affect the solution after few impacts

# $\theta$ -method (1)

- Integral formulation

$$\int_t^{t+\Delta t} \mathbf{M} d\dot{\mathbf{u}} + \int_t^{t+\Delta t} \mathbf{K} \mathbf{u} dt - \int_t^{t+\Delta t} \boldsymbol{\lambda} dt = 0$$

- Assumptions

- ▶  $\mathbf{u}_{t+\Delta t} - \mathbf{u}_t = \Delta t((1-\theta)\dot{\mathbf{u}}_t + \theta\dot{\mathbf{u}}_{t+\Delta t})$  with  $0 \leq \theta \leq 1$
- ▶  $\int_t^{t+\Delta t} \mathbf{M} d\dot{\mathbf{u}} = \mathbf{M}(\dot{\mathbf{u}}_{t+\Delta t} - \dot{\mathbf{u}}_t)$
- ▶  $\int_t^{t+\Delta t} \mathbf{K} \mathbf{u} dt = \Delta t((1-\xi)\mathbf{K}\mathbf{u}_t + \xi\mathbf{K}\mathbf{u}_{t+\Delta t})$  with  $0 \leq \xi \leq 1$
- ▶  $\int_t^{t+\Delta t} \boldsymbol{\lambda} dt = \Delta t \boldsymbol{\lambda}_{t+\Delta t}$

- Problem Find  $\mathbf{u}_{n+1}$  and  $\boldsymbol{\lambda}_{n+1}$  such that:

$$\left( \frac{\mathbf{M}}{\theta \Delta t^2} + \xi \mathbf{K} \right) \mathbf{u}_{t+\Delta t} + \frac{\mathbf{M}}{\theta \Delta t} \mathbf{v}_t + \left( \frac{\mathbf{M}}{\theta \Delta t^2} - (1-\xi)\mathbf{K} \right) \mathbf{u}_t - \mathbf{G}^T \boldsymbol{\lambda}_{t+\Delta t} = 0$$

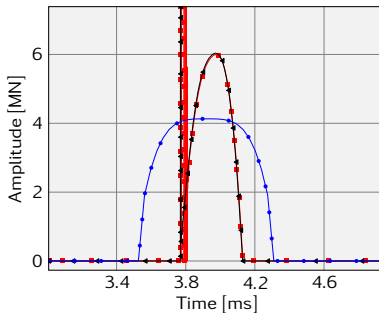
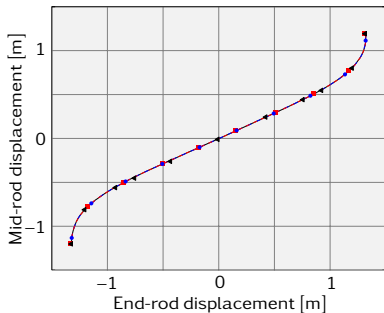
with  $\mathbf{G}\mathbf{u}_{t+\Delta t} \geq 0$ ,  $\boldsymbol{\lambda}_{t+\Delta t} \geq 0$ , and  $\boldsymbol{\lambda}_{t+\Delta t}^T \mathbf{G}\mathbf{u}_{t+\Delta t} = 0$

## $\theta$ -method (2)

- Algorithm
  1. Steepest descent in  $\mathbf{u}_{n+1}$
  2. Uzawa in  $\lambda_{n+1}$
- Properties
  - ▶ Best parameters:  $\theta = \xi = 0.5$
  - ▶ Inconditionally stable in a linear framework
  - ▶ Energy conservation

# Comparison

- Initial conditions *on* the calculated invariant manifold
- Time-stepping integration
- Comparison of the three numerical approaches: implicit — , explicit — , and NLM —



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- Full unilateral and periodic formulation, *ie, no regularization*
- Strategy dedicated to flexible structures
- Numerical proof of the existence of these periodic motions
  - ▶ Agreement between three different numerical techniques
  - ▶ Need for mathematical proof (already existing?)
- Capture the well-known stiffening effects observed for systems undergoing unilateral contact conditions

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- Friction?