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# Self-dual skew codes and factorization of skew polynomials

D. Boucher and F. Ulmer\*

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## Abstract

In previous work the authors generalized cyclic codes to the noncommutative polynomial setting and used this approach to construct new self-dual codes over  $\mathbb{F}_4$ . According to this previous result, such a self-dual code must be  $\theta$ -constacyclic, i.e. the generator polynomial is a right divisor of some noncommutative polynomial  $X^n - a$ . The first result of the paper is that such a self-dual code must be  $\theta$ -cyclic or  $\theta$ -negacyclic, i.e.  $a = \pm 1$ . For codes of length  $2^s$  the noncommutative polynomial approach produced surprisingly poor results. We give an explanation of the length  $2^s$  phenomena by showing that in this case the generating skew polynomial has some unique factorization properties. We also construct self-dual skew codes using least common left multiples of noncommutative polynomials and use this to obtain a new  $[78, 39, 19]_4$  self-dual code.

## 1 Introduction

In [2], self-dual skew codes with good minimum distances were obtained. However, like for cyclic codes ([7]), there is a phenomena in lengths which are a power of 2. For the lengths 4, 8, 16, 32 and 64 there are only three self-dual skew codes, while otherwise there is a large number of self-dual codes

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which increases with the length (cf. [3]). The authors conjectured in [4] that for any  $s$  there are only three self-dual skew codes of length  $2^s$ .

The aim of this paper is to use the factorization of skew polynomials for studying self-dual skew codes. This allows to prove the above conjecture and to give an iterative construction of self-dual skew codes using least common multiples.

The material is organized as follows.

The section 2 is devoted to some generalities about skew codes ([2]) and self-dual skew codes ([4]).

In section 3 it is proven that for all nonnegative  $s$ , there are  $2^{2^{s-1}+1} - 1$   $\theta$ -cyclic codes of length  $2^s$  and dimension  $2^{s-1}$  over  $\mathbb{F}_4$  but that among them only three are self-dual for  $s > 1$ . This gives an answer to Conjecture 1 of [4].

In section 4, we give a construction of self-dual module  $\theta$ -codes which is based on the factorization and least common right multiples (lcrm) of skew polynomials. An example of a  $[78, 39, 19]_4$  self-dual code is given.

## 2 Self-dual skew codes over a finite field

Starting from the finite field  $\mathbb{F}_q$  and an automorphism  $\theta$  of  $\mathbb{F}_q$ , a ring structure is defined in [8] on the set:

$$R = \mathbb{F}_q[X; \theta] = \{a_n X^n + \dots + a_1 X + a_0 \mid a_i \in \mathbb{F}_q \text{ and } n \in \mathbb{N}\}.$$

The addition in  $R$  is defined to be the usual addition of polynomials and the multiplication is defined by the basic rule  $X \cdot a = \theta(a) X$  ( $a \in \mathbb{F}_q$ ) and extended to all elements of  $R$  by associativity and distributivity. The ring  $R$  is called a skew polynomial ring and its elements are skew polynomials. It is a left and right euclidean ring whose left and right ideals are principal. Left and right gcd (gcd) and lcm (lcm) exist in  $R$  and can be computed using the left and right euclidean algorithm ([5], Section 2).

Following [3] we define linear codes using the skew polynomial ring  $R$ .

**Definition 1** Consider  $R = \mathbb{F}_q[X; \theta]$  and let  $f \in R$  be of degree  $n$ . A **module  $\theta$ -code** (or *module skew code*)  $\mathcal{C}$  is a left  $R$ -submodule  $Rg/Rf \subset R/Rf$  in the basis  $1, X, \dots, X^{n-1}$  where  $g$  is a right divisor of  $f$  in  $R$ . We denote this code  $\mathcal{C} = (g)_n^\theta$ . If there exists an  $a \in \mathbb{F}_q \setminus \{0\}$  such that  $g$  divides  $X^n - a$  on the right then the code  $(g)_n^\theta$  is  **$\theta$ -constacyclic**. We will denote it  $(g)_n^{\theta, a}$ . If  $a = 1$ , the code is  **$\theta$ -cyclic** and if  $a = -1$ , it is  **$\theta$ -negacyclic**.

Let us now recall that the **euclidean dual** or **dual** of a linear code  $C$  of length  $n$  over  $\mathbb{F}_q$  can be defined with the **euclidean scalar product** :

$$\forall x, y \in \mathbb{F}_q^n, \langle x, y \rangle = \sum_{i=1}^n x_i y_i$$

as  $C^\perp = \{x \in \mathbb{F}_q^n, \forall y \in C, \langle x, y \rangle = 0\}$ . A linear code  $C$  over  $\mathbb{F}_q$  is **euclidean self-dual** or **self-dual** if  $C = C^\perp$ .

To characterize self-dual module  $\theta$ -codes, we need to define the skew reciprocal polynomial of a skew polynomial (definition 3 of [4]) and also the left monic skew reciprocal polynomial.

**Definition 2** *The skew reciprocal polynomial of  $h = \sum_{i=0}^m h_i X^i \in R$  of degree  $m$  is  $h^* = \sum_{i=0}^m X^{m-i} \cdot h_i = \sum_{i=0}^m \theta^i(h_{m-i}) X^i$ . The left monic skew reciprocal polynomial of  $h$  is  $h^{*\ell} := (1/\theta^m(h_0)) \cdot h^*$ .*

Since  $\theta$  is an automorphism, the map  $*$ :  $R \rightarrow R$  given by  $h \mapsto h^*$  is a bijection. In particular for any  $g \in R$  there exists a unique  $h \in R$  such that  $g = h^*$  and if  $g$  is monic, then  $g = h^{*\ell}$ .

In order to describe some properties of the skew reciprocal polynomial we need the following morphism of rings already used in [4]:

$$\begin{aligned} \Theta: R &\rightarrow R \\ \sum_{i=0}^n a_i X^i &\mapsto \sum_{i=0}^n \theta(a_i) X^i \end{aligned}$$

**Lemma 1** ([4]) *Let  $f \in R$  be a skew polynomial of degree  $n$  such that  $f = hg$ , where  $h$  and  $g$  are skew polynomials of degrees  $k$  and  $n - k$ . Then*

1.  $f^* = \Theta^k(g^*)h^*$
2.  $(f^*)^* = \Theta^n(f)$ .

In [4], it is established that a module  $\theta$ -code which is self-dual is necessarily  $\theta$ -constacyclic (Corollary 1 of [4]). In this case its generator polynomial  $g$  divides on the right  $X^n - a$  for some  $a$  in  $\mathbb{F}_q \setminus \{0\}$  where  $n = 2k$  is the length of the code. Furthermore the dual of  $(g)_n^{\theta, a}$  is generated by the polynomial  $h^{*\ell}$  where  $h$  is defined by the two equivalent following equalities :

$$\Theta^n(h) \cdot g = X^n - a \Leftrightarrow g \cdot h = X^n - \theta^{-k}(a).$$

The following proposition improves Corollary 1 of [4] :

**Proposition 1** *A self-dual module  $\theta$ -code is either  $\theta$ -cyclic or  $\theta$ -negacyclic.*

**Proof:** If  $C = (g)_n^\theta$  is a self-dual module  $\theta$ -code, then  $C$  is necessarily  $\theta$ -constacyclic ([4], Corollary 1). Let  $a$  be in  $\mathbb{F}_q \setminus \{0\}$  such that  $g$  divides  $X^n - a$  on the right in  $R$ . Consider  $h \in R$  such that  $\Theta^n(h) \cdot g = X^n - a$ . From Lemma 1, we obtain  $\frac{-1}{a} \Theta^{k-n}(g^*)h^* = X^n - \frac{1}{a}$ , showing that  $h^{*\ell} = \frac{1}{\theta^k(h_0)} h^*$  divides  $X^n - \frac{1}{a}$  on the right. Since  $C = (g)_n^\theta$  is a self-dual module  $\theta$ -code we must have  $g = h^{*\ell}$  ([4], Theorem 1). So  $g$  divides on the right the polynomial  $(X^n - a) - (X^n - \frac{1}{a}) = a - \frac{1}{a}$  of degree less than  $g$ . Therefore  $a^2 = 1$ . □

Combining this result with ([4], Theorem 1) we obtain:

**Corollary 1** *A module  $\theta$ -code  $(g)_{2k}^\theta$  with  $g \in \mathbb{F}_q[X; \theta]$  of degree  $k$  is self-dual if and only if  $g = h^{*\ell}$  and  $h$  satisfies*

$$h^{*\ell}h = X^{2k} - \varepsilon \text{ with } \varepsilon \in \{-1, 1\}. \quad (1)$$

### 3 Self-dual module $\theta$ -codes of length $2^s$ over $\mathbb{F}_4$ .

We keep the notation  $R = \mathbb{F}_q[X; \theta]$  and we denote  $(\mathbb{F}_q)^\theta$  the fixed field of  $\theta$ . The properties of the ring  $R$  used in this paper can be found in [5, 6]. The **center**  $Z(R)$  of  $R$  is the commutative polynomial subring  $(\mathbb{F}_q)^\theta[X^{|\theta|}]$  in the variable  $Y = X^{|\theta|}$  where  $|\theta|$  is the order of  $\theta$ . We denote  $Z(R)$  also  $(\mathbb{F}_q)^\theta[Y]$ . Following [6] we call an element of a ring bounded if the left ideal it generates contains a two-sided ideal. In the ring  $R$  all elements are bounded. The monic generator  $f$  of the maximal two-sided ideal contained in  $Rf$  is **the bound of  $f$** . The generators of two-sided ideals in  $R$  are the elements of the form  $X^m f$  where  $f \in Z(R)$ . The two-sided ideals are closed under multiplication, a bound  $f$  is an **irreducible bound** if the two-sided ideal  $(f)$  is maximal. A bound  $f$  with a nonzero constant term belongs to  $Z(R) = (\mathbb{F}_q)^\theta[Y]$  and is an irreducible bound if and only if  $f(Y) \in (\mathbb{F}_q)^\theta[Y]$  is an irreducible (commutative) polynomial ([6], Chap. 3, Th. 12).

**Definition 3** ([6], Chap. 3)  $h \in R$  is **lclm-decomposable**<sup>1</sup> if  $h$  is the least common left multiple of skew polynomials of degree strictly less than  $h$ , i.e.

<sup>1</sup>In [6] the term decomposable is used

$h = \text{lclm}(h_1, h_2)$  where  $h_i \in R$  and  $\deg(h_i) < \deg(h)$ . The polynomial  $h \in R$  is **lclm-indecomposable** if  $h$  is not lclm-decomposable.

**Theorem 1** ([6], Chap. 3, Th. 5 or [5], Th. 1.3) Let  $R = \mathbb{F}_q[X; \theta]$ . If  $h_1 h_2 \cdots h_n$  and  $g_1 g_2 \cdots g_m$  are two decompositions into irreducible factors of  $h \in R$ , then  $m = n$  and there exists a permutation  $\sigma \in S_n$  such that the  $R$ -modules  $R/h_i R$  and  $R/g_{\sigma(i)} R$  are isomorphic. In particular the degrees of the irreducible factors of  $h$  are unique up to permutation.

The noncommutative ring  $R$  is not a unique factorization ring.

**Example 1** For  $\mathbb{F}_4 = \mathbb{F}_2(a)$  where  $a^2 + a + 1 = 0$  and  $\theta$  the Frobenius automorphism  $a \mapsto a^2$  we have in  $\mathbb{F}_4[X; \theta]$  that  $(X + a)(X + 1) = X^2 + a^2 X + a \neq X^2 + aX + a = (X + 1)(X + a)$ . Two factors of a central polynomial always commute:

$$X^2 + 1 = (X + a^2)(X + a) = (X + a)(X + a^2) = (X + 1)(X + 1). \quad (2)$$

In a nonunique factorization ring  $R$  some factorizations can still be unique. As we will see, in  $\mathbb{F}_4[X; \theta]$  the polynomials  $(X + a)(X + a)$ ,  $(X + a)(X + 1)$ ,  $(X + 1)(X + a)$ ,  $(X + a^2)(X + a^2)$ ,  $(X + a^2)(X + 1)$  and  $(X + 1)(X + a^2)$  are lclm-indecomposable, i.e. their factorization into monic irreducible polynomials is unique. In  $\mathbb{F}_4[X; \theta]$  the bounds  $X^2$  and  $X^2 + 1$  are irreducible bounds.

The next theorem characterizes the skew polynomials of  $R$  which do have a unique factorization into irreducible monic skew polynomials :

**Theorem 2** ([6], Chap. 3, Th. 21 and 24) Let  $R = \mathbb{F}_q[X; \theta]$  and  $m \in \mathbb{N}^*$ .

1. If  $h_1, h_2, \dots, h_m$  are monic irreducible polynomials of  $R$  having the same irreducible bound  $f \in R$ , then the product  $h = h_1 h_2 \cdots h_m$  is an lclm-indecomposable monic polynomial in  $R$  if and only if the bound of  $h$  is  $f^m$ .
2. An lclm-indecomposable monic polynomial in  $R$  has a unique factorization into irreducible monic polynomials.

**Lemma 2** Let  $R = \mathbb{F}_q[X; \theta]$ . The lclm-decomposable product  $h_1 h_2$  of two irreducible polynomials  $h_1$  and  $h_2$  having both the same irreducible bound  $f \in R$  is equal to  $f \in R$ .

**Proof:** Otherwise, since the bound of a product divides the products of the bounds ([6], Chap. 3, Th. 12), the bound of  $h_1h_2$  is  $f^2$  and  $h_1h_2$  would be lclm-indecomposable by the above theorem.  $\square$

**Definition 4** Consider  $R = \mathbb{F}_q[X; \theta]$  and let  $f \in Z(R)$  be an irreducible bound which is reducible in  $R$ . To each right factor  $g$  of  $f$  corresponds a unique  $\bar{g} \in \mathbb{F}_q[X; \theta]$  such that  $\bar{g}g = f$  called the **complement** of  $g$  (for  $f$ ). The number of distinct irreducible factors  $g \in \mathbb{F}_q[X; \theta]$  of  $f$  is the **capacity**  $\kappa$  of  $f$ .

**Example 2** Consider  $R = \mathbb{F}_q[X; \theta]$  with  $\theta^2 = id$ . If the central bound  $f = X^2 + \lambda \in Z(R)$  is reducible in  $R$ , then its irreducible monic factors are of the form  $X + a \in R$ . The skew polynomial  $X + \tilde{a} \in R$  is the complement of  $X + a$  if and only if  $(X + \tilde{a})(X + a) = X^2 + (\tilde{a} + \theta(a))X + \tilde{a}a = X^2 + \lambda$ , which is the case if and only if

$$\tilde{a} = \lambda/a \quad \text{and} \quad \theta(a) = -\lambda/a \quad (3)$$

In particular  $\theta^2(a) = a$ . From the previous example we see that the capacity of the central polynomial  $X^2 + 1 \in \mathbb{F}_4[X; \theta]$  is  $\kappa = 3$ .

In the following we will be interested in the case where, as in the previous example, the irreducible factors in  $R$  of an irreducible bound  $f$ , which is reducible in  $R$ , are of degree  $\deg(f)/2$ .

**Lemma 3** Consider  $R = \mathbb{F}_q[X; \theta]$  with  $\theta^2 = id$ .

1. For  $g = \sum_{i=0}^m a_i X^i \in R$  and for  $\bar{g} = \sum_{i=0}^m (-1)^i \theta^{i+1}(a_i) X^i$  we have  $g\bar{g} \in Z(R)$ . In particular the bound of  $g$  is of degree  $\leq 2 \deg(g)$ .
2. Consider a product  $h = h_1 \cdots h_m$  of irreducible monic polynomials having all the same irreducible bound  $f \in Z(R)$  which is reducible in  $R$ . The following assertions are equivalent :
  - (i)  $h$  is lclm-decomposable;
  - (ii)  $h$  is divisible by  $f$ ;
  - (iii) there exists  $i$  in  $\{1, \dots, m-1\}$  such that  $h_{i+1}$  is the complement of  $h_i$  (for  $f$ ).

**Proof:**

1. For  $l \in \{0, \dots, 2m\}$ , the  $l$ -th coefficient of  $G = g\bar{g}$  is given by  $G_l = \sum_{i+j=l} a_i(-1)^j \theta^{l+1}(a_j)$ .

If  $l$  is even, then  $G_l = \sum_{i+j=l} a_i(-1)^{l-i} \theta(a_j) = \sum_{i+j=l} a_i(-1)^i \theta(a_j)$ . As  $\theta^2 = id$ ,  $\theta(G_l) = \sum_{i+j=l} \theta(a_i)(-1)^i a_j = G_l$ .

If  $l$  is odd, then

$$G_l = \sum_{i+j=l} a_i(-1)^j a_j = \sum_{i+j=l, j \text{ even}, i \text{ odd}} a_i a_j - \sum_{i+j=l, j \text{ odd}, i \text{ even}} a_i a_j = 0.$$

So  $G$  belongs to  $(\mathbb{F}_q)^\theta[X^2] = Z(R)$ .

2. The irreducible factors of an irreducible bound  $f$  are all similar and therefore of the same degree  $d$  ([6], Chap. 3, Corollary of Th. 20 or Theorem 4.3 of [5]). The first assertion shows that in our situation this degree  $d$  is equal to  $\deg(f)/2$ .

The implications  $(iii) \Rightarrow (ii) \Rightarrow (i)$  are straightforward.

To prove  $(i) \Rightarrow (iii)$ , we proceed by induction on  $m$ . According to Lemma 2, it is true for  $m = 2$ . Suppose  $m > 2$  and that the result holds for  $i < m$ . Let  $h = h_1 \cdots h_m$  be lclm-decomposable where  $h_i$  are irreducible polynomials with bound  $f$ . Then, there exist  $g_1, \dots, g_m \in R$  such that  $h = g_1 \cdots g_m$  where  $(g_1, \dots, g_m) \neq (h_1, \dots, h_m)$ . If  $g_m = h_m$  then  $g_1 \cdots g_{m-1} = h_1 \cdots h_{m-1}$  is lclm-decomposable and one concludes using the induction hypothesis. Otherwise  $\text{lclm}(g_m, h_m) = \tilde{h}_{m-1} h_m$  divides on the right  $h = h_1 \cdots h_m = \tilde{h}_1 \cdots \tilde{h}_{m-1} h_m$ . So  $h_1 \cdots h_{m-1} = \tilde{h}_1 \cdots \tilde{h}_{m-1}$ . If there exist  $i$  such that  $\tilde{h}_i \neq h_i$ , then  $h_1 \cdots h_{m-1}$  is lclm-decomposable and one concludes using the induction hypothesis; otherwise  $h_{m-1} = \tilde{h}_{m-1}$  and  $\text{lclm}(g_m, h_m) = h_{m-1} h_m$  is lclm-decomposable; so according to Lemma 2,  $h_{m-1} h_m = f$ .

□

**Proposition 2** *Let  $R = \mathbb{F}_q[X; \theta]$  with  $\theta^2 = id$ ,  $1 \leq m \in \mathbb{N}$  and  $f \in Z(R)$  an irreducible bound of degree  $2m$  and capacity  $\kappa$  which is reducible in  $R$ . The number  $A(m)$  of distinct monic right factors  $g \in R$  of degree  $m \deg(f)/2$  of  $f^m$  is  $((\kappa - 1)^{m+1} - 1) / (\kappa - 2)$ .*



**Proof:** The irreducible factors of  $f^m$  and therefore also of its right factor  $g$  are all of degree  $\deg(f)/2$ . If  $g = g_1 g_2 \cdots g_m$  is a factorization into irreducible, and  $\bar{g}_i$  the complement of  $g_i$ , then  $f^m = \bar{g}_m \cdots \bar{g}_2 \bar{g}_1 g_1 g_2 \cdots g_m$ . Therefore  $g$  is always a divisor of  $f^m$  and that we only need to count the different polynomials  $g = g_1 g_2 \cdots g_m$ .

1. If  $g$  is divisible by the central bound  $f \in Z(R)$ , then  $g = g' f$  where  $g' = g'_1 \cdots g'_{m-2}$  and  $g'_i$  of degree  $\deg(f)/2$  : there are  $A(m-2)$  such polynomials.
2. If  $g$  is not divisible by  $f$  then according to Lemma 3, for all  $i$ ,  $g_{i+1}$  is not the complement of  $g_i$ . There are  $\kappa$  choices for  $g_1$  and  $\kappa - 1$  choices for each factor  $g_2, g_3 \dots, g_m$ .

From  $A(0) = 1$ ,  $A(1) = \kappa$  and  $A(m) = A(m-2) + \kappa(\kappa-1)^{m-1}$  we get the result. □

**Corollary 2** *Let  $s$  be a nonnegative integer. The number of module  $\theta$ -cyclic codes over  $\mathbb{F}_4$  with length  $2^s$  and dimension  $2^{s-1}$  is  $2^{2^{s-1}+1} - 1$ .*

**Proof:**  $X^{2^s} - 1 = (X^2 + 1)^{2^{s-1}}$  in  $\mathbb{F}_4[X; \theta]$ , and the capacity of  $X^2 + 1$  is  $\kappa = 3$ , so according to the previous proposition applied with  $m = 2^{s-1}$ , the skew polynomial  $X^{2^s} - 1$  has  $2^{2^{s-1}+1} - 1$  monic right factors of degree  $2^{s-1}$  in  $\mathbb{F}_4[X; \theta]$ . □

**Proposition 3** *Let  $R = \mathbb{F}_q[X; \theta]$  with  $\theta^2 = id$ ,  $1 \leq m \in \mathbb{N}$ ,  $f \in Z(R)$  an irreducible bound which is reducible in  $R$  and  $h_i \in R$  and  $g_j \in R$  monic irreducible polynomials having all the same bound  $f$ . If  $g_m g_{m-1} \cdots g_1$  is lclm-indecomposable and  $f^m = h_1 \cdots h_{m-1} h_m g_m g_{m-1} \cdots g_1$ , then  $h_i$  is the complement of  $g_i$ .*

**Proof:** We proceed by induction on  $m$ . If  $m = 1$  the result is trivial. Suppose that the result holds for  $i < m$ .

1. If  $h_m$  is the complement of  $g_m$ , then we can divide both sides by the central polynomial  $h_m g_m = f$  to obtain  $h_1 \cdots h_{m-1} g_{m-1} \cdots g_1 = f^{m-1}$ . Since  $g_{m-1} \cdots g_1$  is lclm-indecomposable, we obtain the result by induction.

2. Otherwise  $h_1 \cdots h_{m-1} h_m$  is divisible by  $f$ . After simplification we obtain

$(h'_1 \cdots h'_{m-2} h'_{m-3} g_m)(g_{m-1} \cdots g_1) = f^{m-1}$ . Applying the induction hypothesis we obtain the contradiction that  $g_m$  is the complement of  $g_{m-1}$  and therefore that  $f$  divides  $g_m \cdots g_1$ . Since  $g_{m-1} \cdots g_1$  is lclm-indecomposable, the above lemma shows that this case is not possible.  $\square$

In the following we want to decide in some special cases if a product of linear polynomials  $(X + \alpha_1)(X + \alpha_2) \cdots (X + \alpha_m)$  generates a self-dual code. The main difficulty is that the skew reciprocal polynomial of a monic polynomial is not always monic:  $(X + \alpha)^* = \theta(\alpha)X + 1 = \theta(\alpha)(X + 1/\theta(\alpha))$ .

**Lemma 4** *Consider  $0 < m \in \mathbb{N}$  and  $\alpha_1, \alpha_2, \dots, \alpha_m$  in  $\mathbb{F}_q \setminus \{0\}$ . For  $g = (X + \alpha_1)(X + \alpha_2) \cdots (X + \alpha_m) \in \mathbb{F}_q[X; \theta]$  we have  $g^* =$*

$$\theta^m(\alpha_1 \cdots \alpha_m) \left( X + \frac{\theta^{m-1}(\alpha_1 \cdots \alpha_{m-1})}{\theta^m(\alpha_1 \cdots \alpha_m)} \right) \cdots \left( X + \frac{\theta(\alpha_1)}{\theta^2(\alpha_1 \alpha_2)} \right) \left( X + \frac{1}{\theta(\alpha_1)} \right)$$

from which we can deduce  $g^{*\ell}$  by dividing on the left by  $\theta^m(\alpha_1 \cdots \alpha_m)$ .

**Proof:** We proceed by induction on  $m$ . For  $m = 1$  the result holds. Assume that the result holds for  $k < m$ . Lemma 1 shows

$$((X + \alpha_1) \cdots (X + \alpha_m))^* = \theta^{m-1}((X + \alpha_m)^*)((X + \alpha_1) \cdots (X + \alpha_{m-1}))^*.$$

By induction we only need to express  $h = \theta^m((X + \alpha_m)^*) \theta^{m-1}(\alpha_1 \cdots \alpha_{m-1})$  as a product of a constant times a monic linear polynomial. By direct computation we obtain

$$\begin{aligned} h &= \theta^m(\alpha_m) \left( X + \frac{1}{\theta(\alpha_m)} \right) \theta^{m-1}(\alpha_1 \cdots \alpha_{m-1}) \\ &= \theta^m(\alpha_1 \cdots \alpha_m) \left( X + \frac{\theta^{m-1}(\alpha_1 \cdots \alpha_{m-1})}{\theta^m(\alpha_1 \cdots \alpha_m)} \right). \end{aligned}$$

The claim now follows by induction.  $\square$

**Proposition 4** *Consider  $\mathbb{F}_4 = \mathbb{F}_2(a)$ ,  $\theta$  the Frobenius automorphism  $a \mapsto a^2$  and  $h \in \mathbb{F}_4[X; \theta]$  to be monic of degree  $m \in \mathbb{N}$ . Then  $h^{*\ell}h = (X^2 + 1)^m$  if and only if*

$$h = \begin{cases} (X + 1)^m & \text{if } m \text{ is odd} \\ (X + 1)^{m-1}(X + u), u \in \{1, a, a^2\} & \text{if } m \text{ is even.} \end{cases}$$

**Proof:** ( $\Leftarrow$ ): If  $m$  is odd, the previous Lemma shows that the skew polynomial  $h = (X + 1)^m$  satisfies  $h^{*\ell}h = (X^2 + 1)^m$ . Let us assume that  $m$  is even and consider  $h = (X + 1)^{m-1}(X + u)$  with  $u \in \mathbb{F}_4 \setminus \{0\}$ . Then  $h^* = \Theta((X + u)^*)(X + 1)^{m-1} = u(X + u^2)(X + 1)^{m-1}$ , which gives  $h^{*\ell} = (X + u^2)(X + 1)^{m-1}$  and  $hh^{*\ell} = (X^2 + 1)^m$ . Furthermore this product commutes because  $(X^2 + 1)^m$  is central.

( $\Rightarrow$ ): The polynomial  $h$  is of the form  $(X + \alpha_1) \cdots (X + \alpha_m)$ . We will show by induction that we can choose  $\alpha_1 = \cdots = \alpha_m = 1$  for  $m$  odd and  $\alpha_1 = \cdots = \alpha_{m-1} = 1$  for  $m$  even. This will show that  $h$  must be one of the above polynomials and prove the claim.

1. For  $m = 1$  we get from the lemma that  $(X + 1/\theta(\alpha_1))(X + \alpha_1) = X^2 + 1$  or that  $X + 1/\theta(\alpha_1)$  is the complement of  $X + \alpha_1$ . Using formula (3) we obtain  $1/\theta(\alpha_1) = 1/\alpha_1$  and  $\theta(\alpha_1) = 1/\alpha_1$ . Therefore  $\alpha_1^2 + 1 = (\alpha_1 + 1)^2 = 0$  and  $\alpha_1 = 1$ .
2. For  $m = 2$ . If  $h = (X + \alpha_1)(X + \alpha_2)$  is lcm-decomposable, then  $h$  is divisible and therefore equal to  $X^2 + 1 = (X + 1)(X + 1)$ . If  $h$  is lcm-indecomposable, then combining proposition 3 and the previous lemma we obtain that  $X + 1/\theta(\alpha_1)$  is the complement of  $X + \alpha_1$ . Like in the case  $m = 1$  this implies that  $\alpha_{m-1} = 1$ .
3. Suppose  $m > 2$  and that the result holds for  $i < m$ . If  $X^2 + 1$  divides  $h$ , then  $h = q(X^2 + 1)$ . Lemma 1 shows that  $h^{*\ell}h = (X^2 + 1)q^{*\ell}q(X^2 + 1)$ . Therefore  $q^{*\ell}q = (X^2 + 1)^{m-2}$  and we obtain the result for  $q$  by induction, which gives also the result for  $h = (X^2 + 1)q$ . Otherwise  $h = (X + \alpha_1) \cdots (X + \alpha_m)$  is lcm-indecomposable (Lemma 3). Combining proposition 3 and the previous lemma we obtain that  $X + \theta^{i-1}(\alpha_1 \cdots \alpha_{i-1})/\theta^i(\alpha_1 \cdots \alpha_i)$  is the complement of  $X + \alpha_i$ . Dividing  $(X^2 + 1)^m$  on the right by  $X + \alpha_m$  and on the left by its complement, we obtain the result by induction for  $\alpha_1, \dots, \alpha_{m-1}$ . If  $m$  is even we get  $\alpha_1 = \dots = \alpha_{m-1} = 1$  and the result follows. If  $m$  is odd we get  $\alpha_1 = \dots = \alpha_{m-2} = 1$ . Using  $\alpha_1 \alpha_2 \cdots \alpha_{m-2} = 1$  we get from the above formula that  $X + 1/\theta(\alpha_{m-1})$  is the complement of  $X + \alpha_{m-1}$ . Like in the case  $m = 1$  this implies that  $\alpha_{m-1} = 1$  and completes the proof.

□

This shows that for any integer  $s \geq 1$ , from the  $2^{2^{s-1}+1} - 1$   $\theta$ -cyclic codes over  $\mathbb{F}_4$  of length  $2^s$ , only 3 are self-dual and proves Conjecture 1 of [4] :

**Corollary 3** Consider  $\mathbb{F}_4 = \mathbb{F}_2(a)$ ,  $\theta$  the Frobenius automorphism  $a \mapsto a^2$ ,  $s > 1$  an integer and  $g \in \mathbb{F}_4[X; \theta]$  monic of degree  $2^{s-1}$ . The code  $(g)_{2^s}^\theta$  is self-dual if and only if,

$$g = (X + u)(X + 1)^{2^{s-1}-1}, \quad u \in \{1, a, a^2\}.$$

**Proof:** The code  $(g)_{2^s}^\theta$  is self-dual if and only if  $g = h^{*\ell}$  with  $h^{*\ell}h = X^{2^s} - 1$ . According to the previous proposition applied with  $m = 2^{s-1}$ , one gets  $h = (X + 1)^{2^{s-1}-1}(X + u)$  with  $u \in \{1, a, a^2\}$  so according to Lemma 1,  $h^* = \Theta(1 + u^2X)(X + 1)^{2^{s-1}-1}$  and  $g = h^{*\ell} = (X + u^2)(X + 1)^{2^{s-1}-1}$ .  $\square$

## 4 Construction of self-dual $\theta$ -codes with $\theta^2 = id$

In [7] a characterization of the generator polynomials of (classical) self-dual cyclic codes of length  $n$  over  $\mathbb{F}_{2^m}$  is given using the factorization of  $X^n - 1$  in  $\mathbb{F}_{2^m}[X]$ . In [2], self-dual  $\theta$ -cyclic codes over  $\mathbb{F}_4$  are constructed by solving polynomial systems satisfied by the coefficients of their generator polynomials. Since the polynomial system becomes increasingly difficult to solve, we propose in this section a construction that allows to construct self-dual codes from suitable smaller degree polynomials.

**Proposition 5** Consider a finite field  $\mathbb{F}_q$  of characteristic  $p$ ,  $R = \mathbb{F}_q[X; \theta]$  with  $\theta^2 = id$  and  $k = p^s \times t$  with  $s \in \mathbb{N}^*$  and  $t$  an integer not multiple of  $p$ . For  $h \in R$  of degree  $k$  the polynomial  $g = h^{*\ell}$  generates a self-dual  $\theta$ -code over  $\mathbb{F}_q$  of length  $n = 2k$  if and only if

1.  $Y^t - \varepsilon = f_1(Y)f_2(Y)\cdots f_m(Y) \in (\mathbb{F}_q)^\theta[Y] = Z(R)$  ( $\varepsilon = \pm 1$  and  $Y = X^2$ ), where  $f_i(X^2) \in R$  are monic polynomials that are pairwise coprime with the property that  $f_i^{*\ell} = f_i$ .
2.  $h_i^{*\ell}h_i = f_i^{p^s}$
3.  $h = \text{lcrm}(h_1, \dots, h_m)$

**Proof:**

1. ( $\Leftarrow$ ): We have to prove that  $h^{*\ell}h = X^{2tp^s} - \varepsilon$  (Corollary 1). From  $h = \text{lcrm}(h_1, \dots, h_m)$  we obtain that  $h = h_i q_i$  with  $q_i \in R$ . Lemma 1 shows that  $h^{*\ell} = \tilde{q}_i h_i^{*\ell}$  where  $\tilde{q}_i \in R$ . Therefore  $h^{*\ell}h = \tilde{q}_i (h_i^{*\ell} h_i) q_i = \tilde{q}_i (f_i)^{p^s} q_i = \tilde{q}_i q_i (f_i)^{p^s}$  (because  $(f_i)^{p^s} \in (\mathbb{F}_q)^\theta[X^2]$  is central), showing that  $\text{lcm}((f_1)^{p^s}, \dots, (f_m)^{p^s})$  is a right divisor of  $h^{*\ell}h$  in  $R$ . To prove the claim it remains to show that

$$\text{lcm}((f_1)^{p^s}, \dots, (f_m)^{p^s}) = (f_1)^{p^s} \cdots (f_m)^{p^s} = X^n - \varepsilon. \quad (4)$$

Comparing degrees we obtain from relation (4) that  $h^{*\ell}h = X^n - \varepsilon$ . In order to prove the first equality of relation (4) we first show that the least common right multiple of polynomials in  $Z(R) \subset R$  coincide when viewed as polynomials either in  $R$  or in the commutative polynomial ring  $Z(R)$ . Both  $R$  and  $Z(R)$  are euclidean rings and the (left and right for  $R$ ) euclidean division has a unique quotient and unique remainder. Therefore a division in  $Z(R)$  is also a (left and right) division in  $R$ . Since the lcm can be computed in both cases using the extended euclidean algorithm ([5], Section 2), they coincide in both rings. In the commutative ring  $Z(R) = (\mathbb{F}_q)^\theta[Y]$  the first equality of relation (4) is a consequence of Gauss Lemma and the claim follows.

2. ( $\Rightarrow$ ): Corollary 1 shows that if  $g = h^{*\ell}$  generates a self-dual  $\theta$ -code over  $\mathbb{F}_q$  of length  $n = 2k$ , then  $h^{*\ell}h = X^n - \varepsilon = ((X^2)^t - \varepsilon)^{p^s} = (Y^t - \varepsilon)^{p^s}$  (where  $Y = X^2$ ). Since  $(Y^t - \varepsilon)^{*\ell} = Y^t - \varepsilon$ , a decomposition  $Y^t - \varepsilon = f_1(Y)f_2(Y) \cdots f_m(Y) \in (\mathbb{F}_q)^\theta[Y] = Z(R)$  into pairwise coprime  $f_i(Y)$  with  $f_i^{*\ell} = f_i$  must exist in  $\mathbb{F}_q[Y]$ . We noted above that the division in  $Z(R)$  and  $R$  coincide in  $Z(R)$ , so that  $(f_i^{p^s})^{*\ell} = f_i^{p^s}$  are pairwise coprime in  $Z(R)$  and  $R$ . According to ([5], Theorem 4.1), we have  $h^{*\ell} = \text{lcm}(h_1^{*\ell}, \dots, h_m^{*\ell})$  where  $h_i^{*\ell} = \text{gcd}(f_i^{p^s}, h^{*\ell})$  are pairwise coprime in  $R$ . In particular, according to [8],  $\deg(\text{lcm}(h_i^{*\ell}, h_j^{*\ell})) = \deg(h_i^{*\ell}) + \deg(h_j^{*\ell})$  for  $i \neq j$  and  $\deg(h^{*\ell}) = \deg(\text{lcm}(h_i^{*\ell})) = \sum \deg(h_i^{*\ell})$ .

We now show that  $h_i$  divides  $f_i^{p^s}$  and  $h$  on the left :

- Let  $\delta_i$  be the degree of  $f_i^{p^s}$  and  $d_i$  be the degree of  $h_i$ . Since  $f_i \in Z(R)$ ,  $\delta_i$  is even. Applying Lemma 1 to  $f_i^{p^s} = q_i h_i^*$  we obtain  $(f_i^{p^s})^* = \Theta^{\delta_i - d_i}(h_i^{**}) q_i^* = \Theta^{\delta_i - d_i}(\Theta^{d_i}(h_i)) q_i^* = \Theta^{\delta_i}(h_i) q_i^* = h_i q_i^*$  ( $\delta_i$  is even and  $\theta^2 = id$ ). So  $h_i$  divides on the left  $(f_i^{p^s})^*$ . As  $(f_i)^{p^s}$  is

central, it is equal to  $(f_i^{p^s})^*$  times a constant, so  $h_i$  divides on the left  $(f_i^{p^s})^{*\ell} = f_i^{p^s}$ .

- Since  $h_i^{*\ell}$  divides  $h^{*\ell}$  on the right, we also have  $h^* = p_i h_i^*$ . Using Lemma 1, we obtain  $\Theta^k(h) = h^{**} = \Theta^{k-d_i}(h_i^{**})p_i^*$ . Therefore  $\Theta^k(h) = \Theta^{k-d_i}(\Theta^{d_i}(h_i))p_i^* = \Theta^k(h_i)p_i^*$ . Since  $\Theta$  is a morphism of rings,  $h_i$  divides  $h$  on the left.

Since  $h_i^{*\ell}$  divides  $h^{*\ell}$  on the right and  $h_i$  divides  $h$  on the left, we obtain  $h^{*l}h = \tilde{g}_i h_i^{*l} h_i g_i$ . Since two factors of a decomposition of the central polynomial  $h^{*l}h = \tilde{g}_i h_i^{*l} h_i g_i$  into two factors commute,  $h_i^{*l} h_i$  divides  $h^{*l}h = X^n - \varepsilon$  on the right. According to Theorem 4.1 of [5],  $h_i^{*\ell} h_i = \text{lcm}(\text{gcd}(h_i^{*\ell} h_i, f_j^{p^s}), j = 1, \dots, m)$ . We now note that both  $h_i^{*\ell}$  and  $h_i$  divide the central polynomial  $f_i^{p^s}$ , so that the product  $h_i^{*\ell} h_i$  divides  $(f_i^{p^s})^2$ . For  $j \neq i$  we obtain  $\text{gcd}(h_i^{*\ell} h_i, f_j^{p^s}) = 1$  and  $h_i^{*\ell} h_i = \text{gcd}(h_i^{*\ell} h_i, f_i^{p^s})$ . In particular,  $h_i^{*\ell} h_i$  divides  $f_i^{p^s}$ .

For  $i \in \{1, \dots, m\}$  the polynomials  $f_i^{p^s}$  are pairwise coprime, showing that their divisors  $h_i^{*\ell} h_i$  are also pairwise coprime. Therefore

$$\begin{aligned} \deg(\text{lcm}(h_i^{*\ell} h_i)) &= \sum_{i=0}^m \deg(h_i^{*\ell} h_i) = 2 \sum_{i=0}^m \deg(h_i^{*\ell}) \\ &= 2 \deg(h^{*\ell}) = \sum_{i=0}^m \deg(f_i^{p^s}). \end{aligned}$$

From  $\sum_{i=0}^m \deg(h_i^{*\ell} h_i) = \sum_{i=0}^m \deg(f_i^{p^s})$  and the fact that  $h_i^{*\ell} h_i$  divides  $f_i^{p^s}$ , we obtain  $h_i^{*\ell} h_i = f_i^{p^s}$ .

As  $h_i$  divides  $h$  on the left,  $\text{lcm}(h_i, i = 1, \dots, m)$  also divides  $h$  on the left. Since  $\text{gcd}(h_i^{*\ell}, h_j^{*\ell}) = 1$  implies  $\text{gcd}(h_i, h_j) = 1$  we have  $\deg(\text{lcm}(h_i, i = 1, \dots, m)) = \sum \deg(h_i) = \deg(h)$ . Therefore  $h = \text{lcm}(h_i, i = 1, \dots, m)$ .

□

**Example 3** Let  $\mathbb{F}_4 = \mathbb{F}_2(a)$ ,  $\theta$  the Frobenius automorphism  $a \mapsto a^2$  and  $R = \mathbb{F}_4[X; \theta]$ . In  $\mathbb{F}_2[Y] = Z(R)$  (where  $Y = X^2$ ), we have  $Y^{39} - 1 =$

$f_1(Y)f_2(Y)f_3(Y)f_4(Y)$  where:

$$\begin{aligned}
f_1(Y) &= Y + 1 \\
f_2(Y) &= Y^2 + Y + 1 \\
f_3(Y) &= Y^{12} + Y^{11} + Y^{10} + Y^9 + Y^8 + Y^7 + Y^6 + Y^5 + Y^4 + Y^3 + Y^2 + Y + 1 \\
f_4(Y) &= (Y^{12} + Y^{11} + Y^{10} + Y^9 + Y^5 + Y^4 + Y^3 + Y^2 + 1) \\
&\quad (Y^{12} + Y^{11} + Y^{10} + Y^9 + Y^8 + Y^7 + Y^6 + Y^5 + Y^4 + Y^3 + Y^2 + Y + 1)
\end{aligned}$$

The polynomials  $f_i$  are pairwise coprime polynomials satisfying  $f_i^{*\ell} = f_i$  ( $i \in \{1, \dots, 4\}$ ).

The skew polynomials

$$\begin{aligned}
h_1 &= X + 1 \\
h_2 &= X^2 + X + 1 \\
h_3 &= X^{12} + aX^{11} + X^{10} + X^8 + aX^6 + a^2X^4 + a^2X^2 + X + a^2 \\
h_4 &= X^{24} + a^2X^{23} + X^{22} + a^2X^{20} + X^{19} + a^2X^{18} + X^{17} + aX^{15} + X^{13} + a^2X^{12} \\
&\quad + a^2X^{11} + aX^9 + a^2X^7 + a^2X^6 + a^2X^5 + a^2X^4 + aX^2 + X + a
\end{aligned}$$

satisfy  $h_i^{*\ell}h_i = f_i (= f_i(X^2))$  ( $i \in \{1, 2, 3, 4\}$ ).

According to Proposition 5 for  $h = \text{lcrm}(h_1, h_2, h_3, h_4)$  the skew polynomial  $g = h^{*\ell} = \text{lclm}(h_1^{*\ell}, h_2^{*\ell}, h_3^{*\ell}, h_4^{*\ell})$  below

$$\begin{aligned}
&X^{39} + a^2X^{38} + a^2X^{37} + X^{36} + a^2X^{34} + aX^{33} + aX^{32} + a^2X^{31} + aX^{30} + a^2X^{29} + a^2X^{28} + \\
&aX^{27} + a^2X^{26} + a^2X^{25} + X^{24} + a^2X^{22} + X^{20} + X^{19} + a^2X^{17} + X^{15} + a^2X^{14} + a^2X^{13} \\
&+ aX^{12} + a^2X^{11} + a^2X^{10} + aX^9 + a^2X^8 + aX^7 + aX^6 + a^2X^5 + X^3 + a^2X^2 + a^2X + 1
\end{aligned}$$

generates a [78, 39] self-dual code over  $\mathbb{F}_4$ . Using the generator matrix of the code (cf. [2, 3]) one can verify in MAGMA (cf. [1]) that its minimum distance is equal to 19. Therefore  $g$  generates a  $[78, 39, 19]_4$  self-dual code.

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