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# Global existence of solutions to a parabolic-elliptic chemotaxis system with critical degenerate diffusion

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**Abstract** This paper is devoted to the analysis of non-negative solutions for a degenerate parabolic-elliptic Patlak-Keller-Segel system with critical nonlinear diffusion in a bounded domain with homogeneous Neumann boundary conditions. Our aim is to prove the existence of a global weak solution under a smallness condition on the mass of the initial data, there by completing previous results on finite blow-up for large masses. Under some higher regularity condition on solutions, the uniqueness of solutions is proved by using a classical duality technique.

*Keywords:* Chemotaxis; Keller-Segel model; Parabolic equation; Elliptic equation; Global existence; Uniqueness.

## 1 Introduction

Chemotaxis is the movement of biological organisms oriented towards the gradient of some substance, called the chemoattractant. The Patlak-Keller-Segel (PKS) model (see [13], [12] and [17]) has been introduced in order to explain chemotaxis cell aggregation by means of a coupled system of two equations: a drift-diffusion type equation for the cell density  $u$ , and a reaction diffusion equation for the chemoattractant concentration  $\varphi$ . It reads

$$(PKS) \left\{ \begin{array}{ll} \partial_t u = \operatorname{div}(\nabla u^m - u \cdot \nabla \varphi) & x \in \Omega, t > 0, \\ -\Delta \varphi = u - \langle u \rangle & x \in \Omega, t > 0, \\ \langle \varphi(t) \rangle = 0 & t > 0, \\ \partial_\nu u = \partial_\nu \varphi = 0 & x \in \partial\Omega, t > 0, \\ u(0, x) = u_0(x) & x \in \Omega, \end{array} \right. \quad (1)$$

where  $\Omega \subset \mathbb{R}^N$  is an open bounded domain,  $\nu$  the outward unit normal vector to the boundary  $\partial\Omega$  and  $m \geq 1$ . An important parameter in this model is the total mass  $M$  of cells, which is formally conserved through the evolution:

$$M = \langle u \rangle = \frac{1}{|\Omega|} \int_{\Omega} u(t, x) \, dx = \frac{1}{|\Omega|} \int_{\Omega} u_0(x) \, dx. \quad (2)$$

Several studies have revealed that the dynamics of (1) depend sensitively on the parameters  $N$ ,  $m$  and  $M$ . More precisely, if  $N = 2$  and  $m = 1$ , it is well-known that the solutions of (1) may blow up in finite time if  $M$  is sufficiently large (see [17, 16]) while solutions are global in time for  $M$  sufficiently small [17], see also the survey articles [4, 10].

The situation is very different when  $m = 1$  and  $N \neq 2$ . In fact, if  $N = 1$ , there is global existence of solutions of (1) whatever the value of the mass of initial data  $M$ , see [8] and the references therein. If  $N \geq 3$ , for all  $M > 0$ , there are initial data  $u_0$  with mass  $M$  for which the corresponding solutions of (1) explode in finite time (see [16]). Thus,

in dimension  $N \geq 3$  and  $m = 1$ , the threshold phenomenon does not take place as in dimension 2, but we expect the same phenomenon when  $N \geq 3$  and  $m$  is equal to the *critical value*  $m = m_c = \frac{2(N-1)}{N}$ . More precisely, we consider a more general version of (1) where the first equation of (1) is replaced by

$$\partial_t u = \operatorname{div}(\phi(u) \nabla u - u \nabla \varphi), \quad t > 0, \quad x \in \Omega,$$

and the diffusivity  $\phi$  is a positive function in  $C^1([0, \infty[)$  which does not grow too fast at infinity. In [8], the authors proved that there is a critical exponent such that, if the diffusion has a faster growth than the one given by this exponent, solutions to (1) (with  $\phi(u)$  instead of  $mu^{m-1}$ ) exist globally and are uniformly bounded, see also [6, 14] for  $N = 2$ . More precisely, the main results in [8] read as follows:

- If  $\phi(u) \geq c(1+u)^p$  for all  $u \geq 0$  and some  $c > 0$  and  $p > 1 - \frac{2}{N}$  then all solutions of (1) are global and bounded.
- If  $\phi(u) \leq c(1+u)^p$  for all  $u \geq 0$  and some  $c > 0$  and  $p < 1 - \frac{2}{N}$  then there exist initial data  $u_0$  such that

$$\lim_{t \rightarrow T} \|u(\cdot, t)\|_\infty = \infty, \text{ for some finite } T > 0.$$

Except for  $N = 2$ , the critical case  $m = \frac{2(N-1)}{N}$  is not covered by the analysis of [8]. Recently, Cieřlak and Laurençot in [7] show that if  $\phi(u) \leq c(1+u)^{1-\frac{2}{N}}$  and  $N \geq 3$ , there are solutions of (1) blowing up in finite time when  $M$  exceeds an explicit threshold. In order to prove that, when  $N \geq 3$  and  $m = \frac{2(N-1)}{N}$ , we have a threshold phenomenon similar to dimension  $N = 2$  with  $m = 1$ , it remains to show that solutions of (1) are global when  $M$  is small enough. The goal of this paper is to show that this is indeed true, see Theorem 2.2 below.

By combining Theorem 2.2 with the blow-up result obtained in [7], we conclude that, for  $N \geq 3$  and  $m = \frac{2(N-1)}{N}$ , there exists  $0 < M_1 \leq M_2 < \infty$  such that the solutions of (1) are global if the mass  $M$  of the initial data  $u_0$  is in  $[0, M_1)$ , and may explode in finite time if  $M > M_2$ . An important open question is whether  $M_1 = M_2$  when  $\Omega$  is a ball in  $\mathbb{R}^N$  and  $u_0$  is a radially symmetric function. Notice that, in the radial case, this result is true when  $N = 2$  and  $m = 1$ , and the threshold value of the mass for blow-up is  $M_1 = M_2 = 8\pi$ , see [6, 16, 15, 18]. Again, for  $N = 2$  and  $m = 1$ , but for regular, connected and bounded domain, it has been shown that  $M_1 = 4\pi = \frac{M_2}{2}$  (see [15, 16] and the references therein). Such a result does not seem to be known for  $N \geq 3$  and  $m = \frac{2(N-1)}{N}$ .

Still, in the whole space  $\Omega = \mathbb{R}^N$  when the equation for  $\varphi$  in (1) is replaced by the Poisson equation  $\varphi = E_N * u$ , with  $E_N$  being the Poisson kernel, it has been shown in [9, 5, 2, 20, 21, 3] that:

- When  $N \geq 3$  and  $1 \leq m < 2 - \frac{2}{N}$ , this modified version of (1) has a global weak solution if  $M = \|u_0\|_1$  is sufficiently small, while finite time blow-up occurs for some initial data with sufficiently large mass.
- When  $N \geq 2$  and  $m > 2 - \frac{2}{N}$ , this modified version of (1) has a global weak solution whatever the value of  $M$ .

- When  $N \geq 2$  and  $m = 2 - \frac{2}{N}$ , there is a threshold mass  $M_c > 0$  such that solutions to this modified version of (1) exist globally if  $M = \|u_0\|_1 \leq M_c$ , and might blow up in finite time if  $M > M_c$ .

From now on, we assume that

$$N \geq 3 \quad \text{and} \quad m = \frac{2(N-1)}{N}.$$

## 2 Main Theorem

Throughout this paper, we deal with weak solutions of (1). Our definition of weak solutions now reads:

**Definition 2.1.** *Let  $T \in (0; \infty]$ . A pair  $(u, \varphi)$  of functions  $u : \Omega \times [0, T) \rightarrow [0, \infty)$ ,  $\varphi : \Omega \times [0, T) \rightarrow \mathbb{R}$  is called a weak solution of (1) in  $\Omega \times [0, T)$  if*

- $u \in L^\infty((0, T); L^\infty(\Omega))$ ;  $u^m \in L^2((0, T); H^1(\Omega))$  and  $\langle u \rangle = M$ .
- $\varphi \in L^2((0, T); H^1(\Omega))$  and  $\langle \varphi \rangle = 0$ .
- $(u, \varphi)$  satisfies the equation in the sense of distributions; i.e.,

$$-\int_0^T \int_\Omega (\nabla u^m \cdot \nabla \psi - u \nabla \varphi \cdot \nabla \psi - u \partial_t \psi) \, dx dt = \int_\Omega u_0(x) \psi(0, x) \, dx,$$

$$\int_0^T \int_\Omega \nabla \varphi \cdot \nabla \psi \, dx dt = \int_0^T \int_\Omega (u - M) \psi \, dx dt,$$

for any continuously differentiable function  $\psi \in C^1([0, T] \times \bar{\Omega})$  with  $\psi(T) = 0$  and  $T > 0$ .

For  $\varphi \in H^1(\Omega)$  satisfying  $\langle \varphi \rangle = 0$ , we denote by  $C_s$  the Sobolev constant where

$$\|\nabla \varphi\|_2 \geq C_s \|\varphi\|_{2^*}, \quad \text{where } 2^* = \frac{2N}{N-2}. \quad (3)$$

The main theorem gives the existence and uniqueness of a time global weak solution to (1) which corresponds to a degenerate version of the ‘‘Nagai model’’ for the semi-linear Keller-Segel system, when  $u_0 \in L^\infty(\Omega)$  and the initial data is assumed to be small.

**Theorem 2.2.** *Define*

$$M_* := \left( \frac{2 C_s^2}{(m-1) |\Omega|^{\frac{2}{N}}} \right)^{\frac{N}{2}}, \quad (4)$$

where  $C_s$  is the Sobolev constant in (3).

Assume that  $u_0$  is nonnegative function in  $L^\infty(\Omega)$ , which satisfies

$$\|u_0\|_1 < M_*. \quad (5)$$

Then the equation (1) has a global weak solution  $(u, \varphi)$  in the sense of Definition 2.1. Moreover, if we assume that

$$\varphi \in L^\infty((0, T); W^{2, \infty}(\Omega)) \quad (6)$$

for all  $T > 0$  then this solution is unique.

In order to prove the previous theorem, we introduce the following approximated equations

$$(KS)_\delta \begin{cases} \partial_t u_\delta = \operatorname{div}(\nabla(u_\delta + \delta)^m - u_\delta \nabla \varphi_\delta) & x \in \Omega, t > 0, \\ -\Delta \varphi_\delta = u_\delta - \langle u_\delta \rangle & x \in \Omega, t > 0, \\ \partial_\nu u_\delta = \partial_\nu \varphi_\delta = 0 & x \in \partial\Omega, t > 0, \\ u_\delta(0, x) = u_0(x) & x \in \Omega, \end{cases}$$

where  $\delta \in (0, 1)$ , and we show that under a smallness condition on the mass of initial data, the Liapunov function

$$L_\delta(u, \varphi) = \int_\Omega (b_\delta(u) + \frac{1}{2} |\nabla \varphi_\delta|^2 - u_\delta \varphi_\delta) \, dx,$$

yields the  $L^m$  bound of  $u_\delta(t)$  independent of  $\delta$ . Then using Gagliardo-Nirenberg and Poincaré inequalities, we obtain for  $p > m$ , the  $L^p$  bound for  $u_\delta(t)$  independent of  $\delta$ . As a consequence of Sobolev embedding theorem, we improve the regularity of  $\varphi_\delta$ . And thus, under the same assumptions on the initial data, Moser's iteration technique yields the uniform bound of  $u_\delta$ . Then, thanks to the local well-posedness result [8, Theorem 3.1] we obtain the existence of a global solution of  $(KS)_\delta$ . The existence of solutions stated in Theorem 2.2 is then proved using a compactness method; for that purpose we show an additional estimate on  $\partial_t u_\delta^m$  which, together with the already derived estimates, guarantees the compactness in space and time of the family  $(u_\delta)_{\delta \in (0,1)}$ . Finally, in the presence of nonlinear diffusion and under some additional regularity assumption on  $\varphi_\delta$ , we prove the uniqueness using a classical duality technique.

### 3 Approximated Equations

The first equation of (1) is a quasilinear parabolic equation of degenerate type. Therefore, we cannot expect the system (1) to have a classical solution at the point where  $u$  vanishes. In order to prove Theorem 2.2, we use a compactness method and introduce the following approximated equations of (KS):

$$(KS)_\delta \begin{cases} \partial_t u_\delta = \operatorname{div}(\nabla(u_\delta + \delta)^m - u_\delta \nabla \varphi_\delta) & x \in \Omega, t > 0, \\ -\Delta \varphi_\delta = u_\delta - \langle u_\delta \rangle & x \in \Omega, t > 0, \\ \partial_\nu u_\delta = \partial_\nu \varphi_\delta = 0 & x \in \partial\Omega, t > 0, \\ u_\delta(0, x) = u_0(x) & x \in \Omega, \end{cases} \quad (7)$$

where  $\delta \in (0, 1)$ .

The main purpose of this section is to construct the time global strong solution of (7).

#### 3.1 Existence of global strong solution of $(KS)_\delta$

**Theorem 3.1.** *For  $\delta \in (0, 1)$  and  $T > 0$ , we consider an initial condition  $u_0 \in L^\infty(\Omega)$ ,  $u_0 \geq 0$  and such that  $\|u_0\|_1 < M_*$  where  $M_*$  is defined in (4). Then  $(KS)_\delta$  has a global strong solution  $(u_\delta, \varphi_\delta)$  which is bounded in  $L^\infty((0, T) \times \Omega)$  for all  $T > 0$  uniformly with respect to  $\delta \in (0, 1)$ .*

The starting point of the proof of Theorem 3.1 is the following local well-posedness result [8, Theorem 1.3]:

**Lemma 3.2.** *Let the same assumptions as that in Theorem 3.1 hold. There exists a maximal existence time  $T_{max}^\delta \in (0, \infty]$  and a unique solution  $(u_\delta, \varphi_\delta)$  of  $(KS)_\delta$  in  $[0, T_{max}^\delta) \times \Omega$ . Moreover,*

$$\text{if } T_{max}^\delta < \infty \text{ then } \lim_{t \rightarrow T_{max}^\delta} \|u_\delta(t, \cdot)\|_\infty = \infty.$$

*In addition  $\langle u_\delta(t) \rangle = \langle u_0 \rangle = M$  for all  $t \in [0, T_{max}^\delta)$ .*

To prove Theorem 3.1 we need to prove some lemmas which control  $L^m$  norm,  $L^p$  norm and  $L^\infty$  norm of the solution  $u_\delta$  of (7).

### 3.2 $L^p$ -estimates, $1 \leq p \leq \infty$ .

Our goal is to show that if  $\|u_0\|_1$  is small enough then all solutions are global in time and uniformly bounded.

Let us first prove the  $L^m$  bound for  $u_\delta$ .

**Lemma 3.3.** *Let the same assumptions as that in Theorem 3.1 hold and  $(u_\delta, \varphi_\delta)$  be the nonnegative maximal solution of  $(KS)_\delta$ . Then,  $u_\delta$  satisfies the following estimate*

$$\|u_\delta(t)\|_m \leq C_0, \text{ for all } t \in [0, T_{max}^\delta)$$

*and  $\|u_\delta(t)\|_1 = \|u_0\|_1$  where  $C_0$  is a constant independent of  $T_{max}^\delta$  and  $\delta$ .*

*Proof.* In this proof, the solution to equation (7) should be denoted by  $(u_\delta, \varphi_\delta)$  but for simplicity we drop the index.

Let us define the functional  $L_\delta$  by

$$L_\delta(u, \varphi) = \int_\Omega (b_\delta(u) + \frac{1}{2} |\nabla \varphi|^2 - u \varphi) dx,$$

where

$$b_\delta(u) := \int_1^u \int_1^z \frac{m(\sigma + \delta)^{m-1}}{\sigma} d\sigma dz,$$

such that  $b_\delta(1) = b'_\delta(1) = 0$  and  $b(u) \geq 0$ . According to [11] it is a Liapunov functional for  $(KS)_\delta$ . Indeed,

$$\begin{aligned} \frac{d}{dt} L_\delta(u(t), \varphi(t)) &= \int_\Omega b'_\delta(u) \partial_t u dx - \int_\Omega \Delta \varphi \partial_t \varphi dx - \int_\Omega \partial_t u \varphi dx - \int_\Omega u \partial_t \varphi dx \\ &= \int_\Omega \partial_t u (b'_\delta(u) - \varphi) dx - \int_\Omega (\Delta \varphi + u) \partial_t \varphi dx \\ &= \int_\Omega \operatorname{div} (m (u + \delta)^{m-1} \nabla u - u \nabla \varphi) (b'_\delta(u) - \varphi) dx - \int_\Omega \langle u(t) \rangle \partial_t \varphi dx \\ &= - \int_\Omega (m (u + \delta)^{m-1} \nabla u - u \nabla \varphi) (b''_\delta(u) \nabla u - \nabla \varphi) dx - M \frac{d}{dt} \int_\Omega \varphi dx \\ &= - \int_\Omega u (b''_\delta(u) \nabla u - \nabla \varphi)^2 dx \\ &\leq 0. \end{aligned}$$

Then, we can conclude that for all  $t \in [0, T_{max}^\delta)$  we have  $L_\delta(u(t), \varphi(t)) \leq L_\delta(u_0, \varphi_0)$ . Using Sobolev inequality (3), Hölder inequality, and Young inequality we obtain

$$\int_\Omega u \varphi dx \leq \|\varphi\|_{2^*} \|u\|_{\frac{2N}{N+2}} \leq C_s^{-1} \|\nabla \varphi\|_2 \|u\|_{\frac{2N}{N+2}} \leq \frac{1}{2} \|\nabla \varphi\|_2^2 + \frac{C_s^{-2}}{2} \|u\|_{\frac{2N}{N+2}}^2.$$

Since  $\frac{2}{N+2} < m$ , and using interpolation inequality we get,

$$\|u\|_{\frac{2N}{N+2}} \leq \|u\|_1^{\frac{1}{N}} \|u\|_m^{\frac{N-1}{N}} \leq M^{\frac{1}{N}} |\Omega|^{\frac{1}{N}} \|u\|_m^{\frac{m}{2}}.$$

Then,

$$\int_{\Omega} u \varphi \, dx \leq \frac{1}{2} \|\nabla \varphi\|_2^2 + \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} \|u\|_m^m.$$

Substituting this into the Liapunov functional, we find:

$$\begin{aligned} L_{\delta}(u, \varphi) &\geq \int_{\Omega} (b_{\delta}(u) + \frac{1}{2} |\nabla \varphi|^2) \, dx - \frac{1}{2} \|\nabla \varphi\|_2^2 - \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} \|u\|_m^m \\ &\geq \int_{\Omega} b_{\delta}(u) \, dx - \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} \|u\|_m^m. \end{aligned}$$

We next observe that:

$$\begin{aligned} b_{\delta}(u) &= m \int_1^u \int_1^z \frac{(\delta + s)^{m-1}}{s} \, ds dz \geq m \int_1^u \int_1^z s^{m-2} \, ds dz \\ &\geq \frac{u^m}{m-1} - \frac{m}{m-1} u + 1 \geq \frac{u^m}{m-1} - \frac{m}{m-1} u. \end{aligned}$$

Then:

$$\begin{aligned} L_{\delta}(u, \varphi) &\geq \frac{1}{m-1} \|u\|_m^m - \frac{C_s^{-2}}{2} |\Omega|^{\frac{2}{N}} M^{\frac{2}{N}} \|u\|_m^m - \frac{m}{m-1} M |\Omega| \\ &= \left( \frac{1}{m-1} - \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} \right) \|u\|_m^m - \frac{m}{m-1} M |\Omega|. \end{aligned}$$

Let us define  $\omega_M$  by

$$\omega_M := \frac{1}{m-1} - \frac{C_s^{-2}}{2} M^{\frac{2}{N}} |\Omega|^{\frac{2}{N}} = \frac{|\Omega|^{\frac{2}{N}}}{2 C_s^2} (M_*^{\frac{2}{N}} - M^{\frac{2}{N}}).$$

Since  $M = \|u_0\|_1 < M_*$ , then  $\omega_M$  is positive. Finally we get,

$$L_{\delta}(u_0, \varphi_0) + \frac{m}{m-1} M |\Omega| \geq L_{\delta}(u(t), \varphi(t)) + \frac{m}{m-1} M |\Omega| \geq \omega_M \|u(t)\|_m^m \text{ for } t \in [0, T_{max}^{\delta}).$$

In addition, we can see that  $L_{\delta}(u_0, \varphi_0) \leq C$  where  $C$  is independent of  $\delta \in (0, 1)$ . In fact,

$$L_{\delta}(u_0, \varphi_0) = \int_{\Omega} (b_{\delta}(u_0) + \frac{1}{2} |\nabla \varphi_0|^2 - u_0 \varphi_0) \, dx,$$

and, since  $(\delta + s)^{m-1} \leq \delta^{m-1} + s^{m-1} \leq 1 + s^{m-1}$  we obtain

$$\begin{aligned} b_{\delta}(u_0) &= m \int_1^{u_0} \int_1^z \frac{(\delta + s)^{m-1}}{s} \, ds dz \leq m \int_1^{u_0} \int_1^z \frac{1 + s^{m-1}}{s} \, ds dz \\ &\leq m(u_0 \ln u_0 - u_0 + 1) + \frac{m}{m-1} \left( \frac{u_0^m}{m} - u_0 + 1 \right). \end{aligned}$$

Using Young inequality we get

$$L_{\delta}(u_0, \varphi_0) \leq m \|u_0\|_2^2 + m |\Omega| + \frac{\|u_0\|_m^m}{m-1} + \frac{m |\Omega|}{m-1} + \frac{1}{2} \|\nabla \varphi_0\|_2^2 + \frac{1}{2} \|u_0\|_2^2 + \frac{1}{2} \|\varphi_0\|_2^2.$$

since  $u_0 \in L^{\infty}(\Omega)$  and  $\varphi_0 \in H^1(\Omega)$  we get  $L_{\delta}(u_0, \varphi_0) \leq C$  where  $C$  is independent of  $\delta$  and the proof of the lemma is complete.  $\square$

Thanks to Lemma 3.3, let us now show that for all  $p > m$  the  $L^p$  bound for  $u_\delta$ .

**Lemma 3.4.** *Let the same assumptions as that in Theorem 3.1 hold. Then for all  $T > 0$  and all  $p \in (1, \infty)$  there exists  $C(p, T)$  independent on  $\delta$  such that, for all  $t \in [0, T_{\max}^\delta) \cap [0, T]$ , the solution  $(u_\delta, \varphi_\delta)$  to  $(KS)_\delta$  satisfies*

$$\|u_\delta(t)\|_p \leq C(p, T), \quad (8)$$

and

$$\int_0^t \int_\Omega (\delta + u_\delta)^{m-1} u_\delta^{p-2} |\nabla u_\delta|^2 dx ds \leq C(p, T). \quad (9)$$

To prove the previous lemma we need the following preliminary result [20].

**Lemma 3.5.** *Consider  $0 < q_1 < q_2 \leq 2^*$ . There is  $C_1$  depending only on  $N$  such that*

$$\|u\|_{q_2} \leq C_1^\theta \|u\|_{H^1(\Omega)}^\theta \|u\|_{q_1}^{1-\theta}, \text{ for } u \in H^1(\Omega), \quad (10)$$

with

$$\theta = \frac{2N(q_2 - q_1)}{q_2[(N+2)q_1 + 2N(1 - q_1)]} \in [0, 1].$$

*Proof.* For  $u \in H^1(\Omega)$  we have by Sobolev inequality

$$\|u\|_{2^*} \leq C_N \|u\|_{H^1}. \quad (11)$$

By interpolation inequality we have for  $0 < q_1 < q_2 \leq 2^*$

$$\|u\|_{q_2} \leq \|u\|_{2^*}^\theta \|u\|_{q_1}^{1-\theta}, \quad (12)$$

where  $\frac{1}{q_2} = \frac{\theta(N-2)}{2N} + \frac{1-\theta}{q_1}$ . Hence, substitute (11) into (12) and the lemma is proved.  $\square$

Now, we recall the following generalized Poincaré inequality.

**Lemma 3.6.** *For  $u \in H^1(\Omega)$  we have for  $0 < q_1 \leq 1$  the following inequality*

$$\|u\|_{H^1}^2 \leq C_2(q_1) (\|\nabla u\|_2^2 + \|u\|_{q_1}^2),$$

where  $C_2$  depends only on  $\Omega$  and  $q_1$ .

Now using the last two lemmas, let us prove Lemma 3.4.

*Proof.* In this proof, the solution to equation (7) should be denoted by  $(u_\delta, \varphi_\delta)$  but for simplicity we drop the index.

We choose  $p > 1$ ,  $K \geq 0$  and we multiply the first equation in (7) by  $(u - K)_+^{p-1}$  and



integrate by parts using the boundary conditions for  $u$  and  $\varphi$  to see that

$$\begin{aligned}
\frac{1}{p} \frac{d}{dt} \|(u - K)_+\|_p^p &= -m(p-1) \int_{\Omega} (\delta + u)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&+ (p-1) \int_{\Omega} u \nabla \varphi (u - K)_+^{p-2} \cdot \nabla u dx \\
&= -m(p-1) \int_{\Omega} (\delta + u - K + K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&+ (p-1) \int_{\Omega} (u - K + K) \nabla \varphi \cdot (u - K)_+^{p-2} \nabla u dx \\
&\leq -m(p-1) \int_{\Omega} (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&+ (p-1) \int_{\Omega} (u - K)_+^{p-1} \nabla \varphi \cdot \nabla u dx + (p-1)K \int_{\Omega} \nabla \varphi (u - K)_+^{p-2} \cdot \nabla u dx \\
&\leq -m(p-1) \int_{\Omega} (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&- \frac{p-1}{p} \int_{\Omega} (u - K)_+^p \Delta \varphi dx - K \int_{\Omega} (u - K)_+^{p-1} \Delta \varphi dx \\
&\leq -m(p-1) \int_{\Omega} (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx + (I),
\end{aligned}$$

where, thanks to the second equation in (7),

$$\begin{aligned}
(I) &= \frac{p-1}{p} \int_{\Omega} (u - K)_+^p (u - M) dx + K \int_{\Omega} (u - K)_+^{p-1} (u - M) dx \\
&= \frac{p-1}{p} \|(u - K)_+\|_{p+1}^{p+1} + \frac{p-1}{p} (K - M) \|(u - K)_+\|_p^p \\
&+ K \|(u - K)_+\|_p^p + K(K - M) \|(u - K)_+\|_{p-1}^{p-1} \\
&\leq K^2 \|(u - K)_+\|_{p-1}^{p-1} + 2K \|(u - K)_+\|_p^p + \|(u - K)_+\|_{p+1}^{p+1}.
\end{aligned}$$

Since for  $a > 0$  and  $b > 0$  we have  $a^{p-1}b \leq a^{p+1} + b^{\frac{p+1}{2}}$  and  $a^p b \leq a^{p+1} + b^{p+1}$  then,

$$(I) \leq 3 \|(u - K)_+\|_{p+1}^{p+1} + (2K)^{p+1} + K^{p+1}, \quad (13)$$

and we get

$$\begin{aligned}
\frac{d}{dt} \|(u - K)_+\|_p^p &\leq -m(p-1) \int_{\Omega} (u + \delta - K)^{m-1} (u - K)_+^{p-2} |\nabla u|^2 dx \\
&+ 3p \|(u - K)_+\|_{p+1}^{p+1} + C_p K^{p+1},
\end{aligned} \quad (14)$$

for all  $t \in [0, T_{\max}^\delta)$ .

The term  $\|(u - K)_+\|_{p+1}^{p+1}$  can be estimated with the help of Lemma 3.5 and Lemma 3.6.

Assuming now that  $p > 2$  we remark that  $0 < \frac{2}{p+m-1} \leq 1$  and  $1 < \frac{2(p+1)}{p+m-1} = \frac{2N}{N-2} \frac{1+p}{1+\frac{Np}{N-2}} \leq \frac{2N}{N-2}$ , then thanks to Lemma 3.5 and Lemma 3.6 we obtain

$$\begin{aligned}
\|(u - K)_+\|_{\frac{p+m-1}{2}}^{\frac{p+m-1}{2}} \|\frac{2(p+1)}{p+m-1}\| &\leq C(p) \left( \|\nabla(u - K)_+\|_{\frac{p+m-1}{2}}^{\frac{2(p+1)}{p+m-1} \theta} \|(u - K)_+\|_{\frac{p+m-1}{2}}^{\frac{2(p+1)}{p+m-1} (1-\theta)} \right. \\
&+ \left. \|(u - K)_+\|_{\frac{p+m-1}{2}}^{\frac{p+m-1}{2}} \|\frac{2(p+1)}{p+m-1}\| \right),
\end{aligned} \quad (15)$$

where

$$\theta = \frac{p+m-1}{p+1} \in (0, 1). \quad (16)$$

Since

$$\|(u-K)_+\|_{\frac{p+m-1}{2}}^{\frac{2(p+1)}{p+m-1}} \|\frac{2(p+1)}{p+m-1}\| = \int_{\Omega} (u-K)_+^{p+1} dx = \|(u-K)_+\|_{\frac{p+1}{2}}^{p+1}, \quad (17)$$

$$\|(u-K)_+\|_{\frac{p+m-1}{2}}^{\frac{2(p+1)}{p+m-1}(1-\theta)} = \left( \int_{\Omega} (u-K)_+ dx \right)^{(p+1)(1-\theta)} = \|(u-K)_+\|_1^{\frac{2}{N}}, \quad (18)$$

and by Lemma 3.3

$$\|(u-K)_+\|_1 = \int_{u \geq K} (u-K) dx \leq \frac{1}{K^{m-1}} \int_{u \geq K} K^{m-1} u dx \leq \frac{\|u\|_m^m}{K^{m-1}} \leq \frac{C_0^m}{K^{m-1}}, \quad (19)$$

we substitute (17), (18) and (19) into (15) and obtain

$$\|(u-K)_+\|_{\frac{p+1}{2}}^{p+1} \leq C_3(p) \left\{ \|\nabla(u-K)_+\|_2^{\frac{m+p-1}{2}} K^{\frac{-2(m-1)}{N}} + K^{-(m-1)(p+1)} \right\}. \quad (20)$$

We may choose  $K = K_*$  large enough such that

$$3 p C_3(p) K_*^{\frac{-2(m-1)}{N}} \leq \frac{2 p (p-1) m}{(m+p-1)^2},$$

Hence

$$\frac{d}{dt} \|(u-K_*)_+\|_p^p \leq C(p) K_*^{p+1},$$

so that

$$\|(u(t)-K_*)_+\|_p^p \leq C(p) t + \|u_0\|_p^p, \text{ for } t \in [0, T_{\max}^\delta].$$

As

$$\begin{aligned} \int_{\Omega} |u|^p dx &\leq \int_{u < 2K_*} (2K_*)^{p-1} |u| dx + \int_{u \geq 2K_*} |u - K_* + K_*|^p dx \\ &\leq (2K_*)^{p-1} M + \int_{u \geq 2K_*} (2|u - K_*|)^p dx, \\ &\leq (2K_*)^{p-1} M + 2^p \|(u-K_*)_+\|_p^p, \end{aligned}$$

the previous inequality warrants that

$$\|u(t)\|_p \leq C(p, T), \quad t \in [0, T_{\max}] \cap [0, T], \quad (21)$$

where  $C(p, T)$  is a constant independent of  $\delta$ .

We next take  $K = 0$  in (14), integrate with respect to time and use (8) to obtain (9).  $\square$

Thanks to Lemma 3.4, we can improve the regularity of  $\varphi_\delta$ .

**Lemma 3.7.** *Let the same assumptions as that in Theorem 3.1 hold, the solution  $\varphi_\delta$  satisfies*

$$\|\nabla \varphi_\delta(t)\|_\infty \leq L(T), \quad t \in [0, T_{\max}^\delta] \cap [0, T]$$

where  $T > 0$  and  $L$  is a positive constant independent of  $\delta$ .

*Proof.* Using standard elliptic regularity estimates for  $\varphi_\delta$ , we infer from Lemma 3.4 that given  $T > 0$ , and  $p \in (1, \infty)$ , there is  $C(p, T)$  such that

$$\|\varphi_\delta(t)\|_{W^{2,p}} \leq C(p) \|u_\delta(t)\|_p \leq C(p, T), \text{ for } t \in [0, T_{\max}) \cap [0, T].$$

Lemma 3.7 then readily follows from Sobolev embedding theorem upon choosing  $p > N$ .  $\square$

**Lemma 3.8.** *Let  $N \geq 3$ ,  $r \geq 4$ ,  $u \in L^{\frac{r}{4}}(\Omega)$ , and  $u^{\frac{r+m-1}{2}} \in H^1(\Omega)$ . Then it holds that*

$$\|u\|_r \leq C_1^{\frac{2\theta}{r+m-1}} \|u\|_{\frac{r}{4}}^{1-\theta} \|u^{\frac{r+m-1}{2}}\|_{H^1}^{\frac{2\theta}{r+m-1}} \quad (22)$$

with

$$\theta = \frac{3N(r+m-1)}{(3N+2)r + 4N(m-1)} \in (0, 1). \quad (23)$$

*Proof.* For  $r \geq 4$ , we can see that

$$\|u\|_r = \left( \int_{\Omega} (u^{\frac{r+m-1}{2}})^{\frac{2r}{r+m-1}} dx \right)^{\frac{1}{r}} = \|u^{\frac{r+m-1}{2}}\|_{\frac{\frac{2}{r+m-1}}{\frac{2r}{r+m-1}}},$$

and

$$\frac{r}{2(r+m-1)} < 1 < \frac{2r}{r+m-1} < 2 < \frac{2N}{N-2}.$$

By Lemma 3.5,

$$\|u\|_r = \|u^{\frac{r+m-1}{2}}\|_{\frac{\frac{2}{r+m-1}}{\frac{2r}{r+m-1}}} \leq \left( C_1^\theta \|u^{\frac{r+m-1}{2}}\|_{H^1(\Omega)}^\theta \|u^{\frac{r+m-1}{2}}\|_{\frac{r}{2(r+m-1)}}^{1-\theta} \right)^{\frac{2}{r+m-1}}$$

and

$$\begin{aligned} \theta &= \frac{2N \left( \frac{2r}{r+m-1} - \frac{r}{2(r+m-1)} \right)}{\frac{2r}{r+m-1} \left( 2N \left( 1 - \frac{r}{2(r+m-1)} \right) + (N+2) \frac{r}{2(r+m-1)} \right)} \\ &= \frac{3N(r+m-1)}{(3N+2)r + 4N(m-1)} \in (0, 1). \end{aligned}$$

In addition, we have

$$\|u^{\frac{r+m-1}{2}}\|_{\frac{r}{2(r+m-1)}} = \left( \int_{\Omega} |u|^{\frac{r+m-1}{2} \frac{r}{2(r+m-1)}} dx \right)^{\frac{2(r+m-1)}{r}} = \|u\|_{\frac{r}{4}}^{\frac{r+m-1}{2}},$$

and we obtain (22).  $\square$

We are now in a position to prove the uniform  $L^\infty(\Omega)$  bound for  $u_\delta$ .

**Lemma 3.9.** *Let the same assumptions as that in Theorem 3.1 hold, and  $(u_\delta, \varphi_\delta)$  be the nonnegative maximal solution of (7). For all  $T > 0$ , there is  $C_\infty(T)$  such that*

$$\|u_\delta(t)\|_\infty \leq C_\infty(T), \text{ for all } t \in [0, T_{\max}^\delta) \cap [0, T],$$

where  $C_\infty(T)$  is a positive constant independent on  $\delta$ .

*Proof.* In this proof we omit the index  $\delta$ , and we employ Moser's iteration technique developed in [1, 21] to show the uniform norm bound for  $u$ .

We multiply the first equation in (7) by  $u^{r-1}$ , where  $r \geq 4$ , and integrate it over  $\Omega$ . Then, we have

$$\begin{aligned} \frac{d}{dt} \frac{\|u\|_r^r}{r} &= - \int_{\Omega} (\nabla(u + \delta)^m - u \nabla \varphi) \cdot \nabla u^{r-1} \, dx \\ &= -m(r-1) \int_{\Omega} (u + \delta)^{m-1} u^{r-2} |\nabla u|^2 \, dx + (r-1) \int_{\Omega} u^{r-1} \nabla \varphi \cdot \nabla u \, dx \\ &\leq -m(r-1) \int_{\Omega} u^{m+r-3} |\nabla u|^2 \, dx + (r-1) \int_{\Omega} u^{r-1} \nabla \varphi \cdot \nabla u \, dx. \end{aligned}$$

By Young's inequality and Lemma 3.7,

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \|u\|_r^r &\leq \frac{-4m(r-1)}{(r+m-1)^2} \int_{\Omega} |\nabla u^{\frac{r+m-1}{2}}|^2 \, dx + \frac{2(r-1) \|\nabla \varphi\|_{\infty}}{(r+m-1)} \int_{\Omega} u^{\frac{r-m+1}{2}} |\nabla u^{\frac{r+m-1}{2}}| \, dx \\ &\leq \frac{-2m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + \frac{r-1}{2m} \|\nabla \varphi\|_{\infty}^2 \int_{\Omega} u^{r-m+1} \, dx \\ &\leq \frac{-2m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + C(T) r \int_{\Omega} u^{r-m+1} \, dx. \end{aligned}$$

Using Hölder and Young inequalities and Lemma 3.3 we obtain

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \|u\|_r^r &\leq \frac{-2m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + r C(T) \|u\|_1^{\frac{m-1}{r-1}} \|u\|_r^{\frac{r(r-m)}{r-1}} \\ &\leq \frac{-2m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + C^r + r^2 \|u\|_r^r, \end{aligned} \quad (24)$$

where we have used that  $r^{\frac{r-1}{r-m}} \leq r^2$  for  $r \geq 4$ .

By Lemma 3.8, we have for  $r \geq 4$

$$\|u\|_r^r \leq C_1^{\frac{2r\theta}{r+m-1}} \|u\|_{\frac{r}{4}}^{r(1-\theta)} \|u^{\frac{r+m-1}{2}}\|_{H^1}^{\frac{2r\theta}{r+m-1}}, \quad (25)$$

where

$$\theta = \frac{3N(r+m-1)}{(3N+2)r + 4N(m-1)} < 1.$$

Therefore, Young inequality and (25) yield that

$$\begin{aligned} 2r^2 \|u\|_r^r &\leq 2r^2 C_1^{\frac{2r\theta}{r+m-1}} \|u\|_{\frac{r}{4}}^{r(1-\theta)} \|u^{\frac{r+m-1}{2}}\|_{H^1}^{\frac{2r\theta}{r+m-1}} \\ &\leq \frac{\theta r}{r+m-1} \frac{m(r-1)}{(r+m-1)^2} \frac{r+m-1}{\theta r C_2(1)} \|u^{\frac{r+m-1}{2}}\|_{H^1}^2 \\ &\quad + \frac{r+m-1-\theta r}{r+m-1} \left( C_2(1) \frac{\theta(r+m-1)r}{m(r-1)} \right)^{\frac{\theta r}{r(1-\theta)+m-1}} \\ &\quad \times (2r^2)^{\frac{(r+m-1)}{r(1-\theta)+m-1}} C_1^{\frac{2\theta r}{r(1-\theta)+m-1}} \|u\|_{\frac{r}{4}}^{(1-\theta)r \frac{(r+m-1)}{r(1-\theta)+m-1}}, \end{aligned}$$

where  $C_2(1)$  is the Poincaré constant defined in Lemma 3.6. Then we obtain

$$\begin{aligned} 2r^2 \|u\|_r^r &\leq \frac{m(r-1)}{C_2(1)(r+m-1)^2} \|u^{\frac{r+m-1}{2}}\|_{H^1}^2 \\ &\quad + C_1^{\frac{\theta r}{r(1-\theta)+m-1}} 2^{\frac{(r+m-1)}{r(1-\theta)+m-1}} r^{\frac{2(r+m-1)+\theta r}{r(1-\theta)+m-1}} \|u\|_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}}. \end{aligned}$$

Now, since  $N > 2$ , which gives  $4N \geq 3N + 2$ , we find the following upper bound for  $\theta$

$$\theta \leq \frac{3N}{3N+2} \quad (26)$$

In addition,

$$\frac{\theta r}{r(1-\theta) + m - 1} \leq \frac{\theta}{1-\theta} = -1 + \frac{1}{1-\theta} \leq \frac{3N}{2}, \quad (27)$$

$$\frac{r+m-1}{r(1-\theta) + m - 1} \leq \frac{r+m-1}{(1-\theta)(r+m-1)} \leq \frac{1}{1-\theta} \leq \frac{3N+2}{2}, \quad (28)$$

and

$$\frac{2(r+m-1) + \theta r}{r(1-\theta) + m - 1} \leq \frac{2+\theta}{1-\theta} \leq 9N+4. \quad (29)$$

As  $C_1 \geq 1$  and  $r \geq 1$ , we get

$$2 r^2 \|u\|_r^r \leq \frac{m(r-1)}{C_2(1)(r+m-1)^2} \|u^{\frac{r+m-1}{2}}\|_{H^1}^2 + C r^{9N+4} \|u\|_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}}. \quad (30)$$

Using Lemma 3.6 we have

$$\|u^{\frac{r+m-1}{2}}\|_{H^1}^2 \leq C_2(1) \left( \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + \|u^{\frac{r+m-1}{2}}\|_1^2 \right). \quad (31)$$

Using Hölder inequality, Young inequality and Lemma 3.3, we get

$$\|u^{\frac{r+m-1}{2}}\|_1^2 = \|u\|_{\frac{r+m-1}{2}}^{r+m-1} \leq \|u\|_r^r \frac{r+m-3}{r-1} \|u\|_1^{\frac{r-m+1}{r-1}} \leq \|u\|_r^r \frac{r+m-3}{r-1} \|u_0\|_1^{\frac{r-m+1}{r-1}},$$

then,

$$\begin{aligned} \frac{m(r-1)}{(r+m-1)^2} \|u^{\frac{r+m-1}{2}}\|_1^2 &\leq (r-1)^{\frac{r-1}{r+m-3}} \frac{r+m-3}{r-1} \|u\|_r^r \\ &\quad + \frac{2-m}{r-1} \left( \frac{m}{(r+m-1)^2} \|u_0\|_1^{\frac{r+m-1}{r-1}} \right)^{\frac{r-1}{2-m}} \\ &\leq r^2 \|u\|_r^r + \left( \frac{m}{(r+m-1)^2} \|u_0\|_1^{\frac{r+m-1}{r-1}} \right)^{\frac{r-1}{2-m}} \\ &\leq r^2 \|u\|_r^r + C_4^r. \end{aligned} \quad (32)$$

Now substituting (32) and (31) into (30) we get

$$\begin{aligned} 2 r^2 \|u\|_r^r &\leq \frac{m(r-1)}{(r+m-1)^2} \left( \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + \|u^{\frac{r+m-1}{2}}\|_1^2 \right) + C r^{9N+4} \|u\|_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}} \\ &\leq \frac{m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + r^2 \|u\|_r^r + C_4^r + C r^{9N+4} \|u\|_{\frac{r}{4}}^{\frac{(1-\theta)r(r+m-1)}{r(1-\theta)+m-1}}, \end{aligned}$$

hence

$$r^2 \|u\|_r^r \leq \frac{m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + C_4^r + C r^{9N+4} \|u\|_{\frac{r}{4}}^{\frac{r(1-\theta)(r+m-1)}{r(1-\theta)+m-1}}.$$

We apply Young inequality again to the last term of the above inequality. It is easy to see that

$$\frac{2}{3N+2} \leq 1-\theta \leq \frac{(1-\theta)(r+m-1)}{r(1-\theta)+m-1} = \frac{(1-\theta)r + (1-\theta)(m-1)}{r(1-\theta)+m-1} < 1,$$

so that

$$r^2 \|u\|_r^r \leq \frac{m(r-1)}{(r+m-1)^2} \|\nabla u^{\frac{r+m-1}{2}}\|_2^2 + C_4^r + 1 + (C r^{9N+4})^{3N+1} \|u\|_{\frac{r}{4}}^r, \quad (33)$$

for any  $r \in [4, \infty)$ .

Substituting (33) into (24) we end up with

$$\frac{d}{dt} \|u\|_r^r \leq r C_4^r + r + r (C r^{9N+4})^{3N+1} \|u\|_{\frac{r}{4}}^r \leq C_5^r + C r^\alpha \|u\|_{\frac{r}{4}}^r, \quad (34)$$

for any  $r \in [4, \infty)$ , where  $\alpha = (9N+4)(3N+1) + 1$ . After integrating (34) from 0 to  $t$ , we obtain the  $L^r$  estimate for  $u$  as follows:

$$\sup_{0 < t < T} \|u(t)\|_r^r \leq \|u_0\|_r^r + T C_5^r + C r^\alpha T \sup_{0 < t < T} \|u(t)\|_{\frac{r}{4}}^r. \quad (35)$$

Since

$$\|u_0\|_r \leq \|u_0\|_\infty^{\frac{r-1}{r}} \|u_0\|_1^{\frac{1}{r}} \leq C_6,$$

then

$$\sup_{0 < t < T} \|u(t)\|_r^r \leq C_7(T) r^\alpha \max \left\{ C_6, \sup_{0 < t < T} \|u(t)\|_{\frac{r}{4}} \right\}^r, \quad (36)$$

and we obtain for  $r \geq 4$

$$\sup_{0 < t < T} \|u(t)\|_r \leq C_7(T)^{\frac{1}{r}} r^{\frac{\alpha}{r}} \max \left\{ C_6, \sup_{0 < t < T} \|u(t)\|_{\frac{r}{4}} \right\}. \quad (37)$$

We are now in a position to derive the claimed  $L^\infty$  estimate. To this end, we set

$$\alpha_p := \max \left\{ C_6, \sup_{0 < t < T} \|u(t)\|_{4^p} \right\}$$

for  $p \geq 0$ . Then we take  $r = 4^p$  with  $p \geq 0$  in (37) which reads

$$\begin{aligned} \alpha_p &\leq 4^{\frac{\alpha p}{4^p}} C_7(T)^{\frac{1}{4^p}} \max \left\{ C_6, \sup_{0 < t < T} \|u(t)\|_{4^{p-1}} \right\}, \\ &\leq 4^{\frac{\alpha}{2^p}} C_7(T)^{\frac{1}{4^p}} \alpha_{p-1} \end{aligned}$$

since  $p \leq 2^p$  for  $p \geq 1$ . Arguing by induction we conclude that

$$\alpha_p \leq 4^{\alpha \sum_{k=1}^p 2^{-k}} C_7(T)^{\sum_{k=1}^p 4^{-k}} \alpha_0.$$

Then by using Lemma 3.3 we get

$$\sup_{0 < t < T} \|u(t)\|_{4^p} \leq 4^\alpha C_7(T) \alpha_0 \leq C_8(T).$$

Consequently, by letting  $p$  tend to  $\infty$ , we see that  $u \in L^\infty((0, T) \times \Omega)$  and

$$\sup_{0 < t < T} \|u(t)\|_\infty \leq C_8(T). \quad (38)$$

Since the right hand side is independent of  $\delta$ , we have proved the lemma.  $\square$

**Lemma 3.10.** *Let the same assumptions as that in Theorem 3.1 hold, and  $(u_\delta, \varphi_\delta)$  be the solution to (7). Then for all  $T > 0$  there is  $C_9(T)$  such that the solution  $u_\delta$  satisfies the following derivation estimate*

$$\int_0^T \|\partial_t u_\delta^m\|_{(W^{1,N+1})'} dt \leq C_9(T).$$

*Proof.* Consider  $\psi \in W^{1,N+1}(\Omega)$  and  $t \in (0, T)$ , we have

$$\begin{aligned} & \left| \int_\Omega m u_\delta^{m-1}(t) \partial_t u_\delta(t) \psi dx \right| \\ &= m \left| \int_\Omega \nabla(u_\delta^{m-1} \psi) \cdot (\nabla u_\delta^m - u_\delta \nabla \varphi_\delta) dx \right| \\ &= m \left| \int_\Omega (u_\delta^{m-1} \nabla \psi + \psi \nabla u_\delta^{m-1}) \cdot (\nabla u_\delta^m - u_\delta \nabla \varphi_\delta) dx \right| \\ &\leq m \int_\Omega [u_\delta^{m-1} |\nabla u_\delta^m| |\nabla \psi| + u_\delta^m |\nabla \psi| |\nabla \varphi_\delta| \\ &\quad + |\psi| m(m-1) u_\delta^{2m-3} |\nabla u_\delta|^2 + |\psi|(m-1) u_\delta^{m-1} |\nabla u_\delta| |\nabla \varphi_\delta|] dx \\ &\leq m \left[ \|u_\delta\|_\infty^{m-1} \|\nabla u_\delta^m\|_2 \|\nabla \psi\|_2 + \|\nabla \psi\|_2 \|u_\delta\|_\infty^m \|\nabla \varphi_\delta\|_\infty |\Omega|^{\frac{1}{2}} \right. \\ &\quad \left. + \|\psi\|_\infty \frac{4m(m-1)}{(2m-1)^2} \|\nabla u_\delta^{m-\frac{1}{2}}\|_2^2 + \|\psi\|_2 \frac{m-1}{m} \|\nabla u_\delta^m\|_2 \|\nabla \varphi_\delta\|_\infty \right]. \end{aligned}$$

Using Lemma 3.8, Lemma 3.9, and the embedding of  $W^{1,N+1}(\Omega)$  in  $L^\infty(\Omega)$ , we end up with

$$|\langle \partial_t u_\delta^m(t), \psi \rangle| \leq C(T) \left( \|\nabla u_\delta(t)^m\|_2 + \|\nabla u_\delta^{m-\frac{1}{2}}(t)\|_2^2 + 1 \right) \|\psi\|_{W^{1,N+1}},$$

and a duality argument gives

$$\|\partial_t u_\delta^m(t)\|_{(W^{1,N+1})'} \leq C(T) \left( \|\nabla u_\delta^m(t)\|_2 + \|\nabla u_\delta^{m-\frac{1}{2}}(t)\|_2^2 + 1 \right).$$

Integrating the above inequality over  $(0, T)$  and using Lemma 3.4 with  $p = 2$  and  $p = m$  give Lemma 3.10.  $\square$

## 4 Proof of Theorem 2.2

### 4.1 Existence

In this section, we assume that  $u_0$  is a nonnegative function in  $L^\infty(\Omega)$  satisfying (5). For  $\delta \in (0, 1)$ ,  $(u_\delta, \varphi_\delta)$  denotes the solution to  $(KS)_\delta$  constructed in Section 3. To prove existence of a weak solution, we use a compactness method. For that purpose, we first study the compactness properties of  $(u_\delta, \varphi_\delta)_\delta$ .

**Lemma 4.1.** *There are functions  $u$  and  $\varphi$  and a sequence  $(\delta_n)_{n \geq 1}$ ,  $\delta_n \rightarrow 0$ , such that, for all  $T > 0$  and  $p \in (1, \infty)$ ,*

$$u_{\delta_n} \longrightarrow u, \text{ in } L^p((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0, \quad (39)$$

$$\varphi_{\delta_n} \longrightarrow \varphi, \text{ in } L^p((0, T); W^{2,p}(\Omega)) \text{ as } \delta_n \rightarrow 0. \quad (40)$$

*In addition,  $u \in L^\infty((0, T) \times \Omega)$  for all  $T > 0$  and is nonnegative.*

*Proof.* Thanks to Lemma 3.4 and Lemma 3.9,  $(u_\delta^m)_\delta$  is bounded in  $L^2((0, T); H^1(\Omega))$  while  $(\partial_t u_\delta^m)_\delta$  is bounded in  $L^1((0, T); (W^{1, N+1})'(\Omega))$  by Lemma 3.10.

Since  $H^1(\Omega)$  is compactly embedded in  $L^2(\Omega)$  and  $L^2(\Omega)$  is continuously embedded in  $(W^{1, N+1})'(\Omega)$ , it follows from [19, corollary 4] that  $(u_\delta^m)$  is compact in  $L^2((0, T) \times \Omega)$  for all  $T > 0$ . Since  $r \mapsto r^{\frac{1}{m}}$  is  $\frac{1}{m}$ -Hölder continuous, it is easy to check that the previous compactness property implies that  $(u_\delta)$  is compact in  $L^{2m}((0, T) \times \Omega)$  for all  $T > 0$ . There are thus a function  $u \in L^{2m}((0, T) \times \Omega)$  for all  $T > 0$  and a sequence  $(\delta_n)_{n \geq 1}$  such that

$$u_{\delta_n} \longrightarrow u \text{ in } L^{2m}((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0, \quad (41)$$

for all  $T > 0$ , owing to Lemma 3.9, we may also assume that

$$u_{\delta_n} \xrightarrow{*} u \text{ in } L^\infty((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0. \quad (42)$$

for all  $T > 0$ . It readily follows from (41) and (42), and Hölder inequality that (39) holds true. Since elliptic regularity ensure that

$$\|\varphi_{\delta_k} - \varphi_{\delta_n}\|_{W^{2,p}} \leq C(p) \|u_{\delta_k} - u_{\delta_n}\|_p,$$

for all  $k \geq 1, n \geq 1$ , and  $p \in (1, \infty)$ , a straightforward consequence of (39) is that  $(\varphi_{\delta_n})_{n \geq 1}$  is a Cauchy sequence in  $L^p((0, T); W^{2,p}(\Omega))$  and thus converges to some function  $\varphi$  in that space. Finally, the nonnegativity of  $u$  follows easily from that of  $u_{\delta_n}$  by (39).  $\square$

*Proof of Theorem 2.2 (existence).* It remains to identify the equations solved by the limit  $(u, \varphi)$  of  $(u_{\delta_n}, \varphi_{\delta_n})_{n \geq 1}$  constructed in Lemma 4.1. To this end we first note that, owing to (39) and the boundedness of  $(u_{\delta_n})_n$  and  $u$  in  $L^\infty((0, T) \times \Omega)$ , we have

$$u_{\delta_n}^m \longrightarrow u^m \text{ in } L^p((0, T) \times \Omega) \text{ as } \delta_n \rightarrow 0, \quad (43)$$

for all  $T > 0$ . Since  $(\nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}})_{n \geq 1}$  and  $(\nabla u_{\delta_n}^m)_{n \geq 1}$  are bounded in  $L^2((0, T) \times \Omega)$  for all  $T > 0$  by Lemma 3.4 with  $p = 2$  and  $p = m + 1$ , we may extract a further subsequence (not relabeled) such that

$$\nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} \rightharpoonup \nabla u^{\frac{m+1}{2}} \text{ in } L^2((0, T) \times \Omega), \quad (44)$$

$$\nabla u_{\delta_n}^m \rightharpoonup \nabla u^m \text{ in } L^2((0, T) \times \Omega), \quad (45)$$

for all  $T > 0$ . Then if  $\psi \in L^4((0, T) \times \Omega; \mathbb{R}^N)$ ,

$$\begin{aligned} & \left| \int_0^T \int_\Omega \psi \cdot [\nabla(u_{\delta_n} + \delta_n)^m - \nabla u^m] \, dx ds \right| \\ &= \frac{2}{m+1} \left| \int_0^T \int_\Omega \psi \cdot \left[ (u_{\delta_n} + \delta_n)^{\frac{m-1}{2}} \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} - u^{\frac{m-1}{2}} \nabla u^{\frac{m+1}{2}} \right] \, dx ds \right| \\ &\leq \frac{2}{m+1} \left| \int_0^T \int_\Omega \psi \cdot \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} \left( (u_{\delta_n} + \delta_n)^{\frac{m-1}{2}} - u^{\frac{m-1}{2}} \right) \, dx ds \right| \\ &+ \frac{2}{m+1} \left| \int_0^T \int_\Omega u^{\frac{m-1}{2}} \psi \cdot \left( \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} - \nabla u^{\frac{m+1}{2}} \right) \, dx ds \right| \\ &\leq \frac{2}{m+1} \|\psi\|_4 \|\nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}}\|_2 \|(u_{\delta_n} + \delta_n)^{\frac{m-1}{2}} - u^{\frac{m-1}{2}}\|_4 \\ &+ \frac{2}{m+1} \left| \int_0^T \int_\Omega u^{\frac{m-1}{2}} \psi \cdot \left( \nabla(u_{\delta_n} + \delta_n)^{\frac{m+1}{2}} - \nabla u^{\frac{m+1}{2}} \right) \, dx ds \right|. \end{aligned}$$

Since  $u^{\frac{m-1}{2}} \psi \in L^2((0, T) \times \Omega)$ , we deduce from (39) and (44) that the right-hand side of the above inequality converges to zero as  $n \rightarrow \infty$ . In other words,

$$\nabla(u_{\delta_n} + \delta_n)^m \rightharpoonup \nabla u^m \text{ in } L^{\frac{4}{3}}((0, T) \times \Omega), \quad (46)$$



for all  $T > 0$ .

Now, we are going to show that  $(u, \varphi)$  in Lemma 4.1 is the desired weak solution in Theorem 2.2. Let  $T > 0$  and  $\psi \in C^1([0, T] \times \overline{\Omega})$  with  $\psi(T) = 0$ . The solution of (7) satisfies

$$\int_0^T \int_{\Omega} [\nabla(u_{\delta_n} + \delta_n)^m \cdot \nabla\psi - u_{\delta_n} \nabla\varphi_{\delta_n} \cdot \nabla\psi - u_{\delta_n} \partial_t\psi] \, dxdt = \int_{\Omega} u_0 \psi(0, x) \, dx, \quad (47)$$

and,

$$\int_0^T \int_{\Omega} [\nabla\varphi_{\delta_n} \cdot \nabla\psi + M \psi - u_{\delta_n} \psi] \, dxdt = 0. \quad (48)$$

From (46) we see that

$$\int_0^T \int_{\Omega} \nabla(u_{\delta_n} + \delta_n)^m \cdot \nabla\psi \, dxdt \longrightarrow \int_0^T \int_{\Omega} \nabla u^m \cdot \nabla\psi \, dxdt \text{ as } \delta_n \rightarrow 0.$$

From (39) we get

$$\int_0^T \int_{\Omega} u_{\delta_n} \partial_t\psi \, dxdt \longrightarrow \int_0^T \int_{\Omega} u \partial_t\psi \, dxdt \text{ as } \delta_n \rightarrow 0.$$

From (39) and (40) we get

$$\int_0^T \int_{\Omega} u_{\delta_n} \nabla\varphi_{\delta_n} \cdot \nabla\psi \, dxdt \longrightarrow \int_0^T \int_{\Omega} u \nabla\varphi \cdot \nabla\psi \, dxdt \text{ as } \delta_n \rightarrow 0.$$

Thus we conclude that  $u$  satisfies

$$\int_0^T \int_{\Omega} (\nabla u^m \cdot \nabla\psi - u \nabla\varphi \cdot \nabla\psi - u \cdot \partial_t\psi) \, dxdt = \int_{\Omega} u_0(x) \cdot \psi(0, x) \, dx.$$

Similarly, from (40) we see that

$$\int_0^T \int_{\Omega} \nabla\varphi_{\delta_n} \cdot \nabla\psi \, dxdt \longrightarrow \int_0^T \int_{\Omega} \nabla\varphi \cdot \nabla\psi \, dxdt \text{ as } \delta_n \rightarrow 0,$$

and from (39) we see that

$$\int_0^T \int_{\Omega} u_{\delta_n} \psi \, dxdt \longrightarrow \int_0^T \int_{\Omega} u \psi \, dxdt \text{ as } \delta_n \rightarrow 0.$$

Thus, we have constructed a weak solution  $(u, \varphi)$  of (KS).  $\square$

## 4.2 Uniqueness

In this section, we prove the uniqueness statement of Theorem 2.2 under the additional assumption (6) on  $\varphi$ . The proof relies on a classical duality technique, and on the method presented in [2]

*Proof.* The proof estimates the difference of weak solutions in dual space  $H^1(\Omega)'$  of  $H^1(\Omega)$ , motivated by the fact that the nonlinear diffusion is monotone in this norm.

Assume that we have two different weak solutions  $(u_1, \varphi_1)$  and  $(u_2, \varphi_2)$  to equations (1) corresponding to the same initial conditions, and fix  $T > 0$ . We put

$$(u, \varphi) = (u_1 - u_2, \varphi_1 - \varphi_2) \text{ in } [0, T] \times \Omega.$$

Then  $\varphi$  is the strong solution of

$$\begin{cases} -\Delta \varphi &= u & \text{in } \Omega, \\ \partial_\nu \varphi &= 0 & \text{on } \partial\Omega, \\ \langle \varphi \rangle &= 0. \end{cases} \quad (49)$$

Since  $\partial_t u \in L^2((0, T); H^1(\Omega)')$ , we have

$$-\Delta \partial_t \varphi = \partial_t u_1 - \partial_t u_2 = \partial_t u \text{ in } H^1(\Omega)',$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|_2^2 &= \int_{\Omega} \nabla \varphi \cdot \nabla \partial_t \varphi \, dx \\ &= - \langle \Delta \partial_t \varphi, \varphi \rangle_{(H^1)', H^1} = \langle \partial_t u, \varphi \rangle_{(H^1)', H^1}. \end{aligned} \quad (50)$$

Now it follows from (1) that  $u$  satisfies the equation

$$\begin{cases} \partial_t u &= \operatorname{div}(\nabla(u_1^m - u_2^m)) - \operatorname{div}(u_1 \nabla \varphi + u \nabla \varphi_2) \\ \partial_\nu u &= 0 \\ u(0, x) &= 0. \end{cases} \quad (51)$$

Substituting (51) in (50), we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla \varphi\|_2^2 = \int_{\Omega} (u_1^m - u_2^m) \Delta \varphi \, dx + \int_{\Omega} u_1 |\nabla \varphi|^2 \, dx + \int_{\Omega} u \nabla \varphi_2 \cdot \nabla \varphi \, dx. \quad (52)$$

The first integral on the right-hand side of (52) is nonnegative due to the fact that  $z \mapsto z^m$  is an increasing function. The second integral on the right-hand side of (52) can be estimated by

$$\left| \int_{\Omega} u_1 |\nabla \varphi|^2 \, dx \right| \leq \|u_1\|_{\infty} \int_{\Omega} |\nabla \varphi|^2 \, dx.$$

For the last integral, using an integration by parts we obtain

$$\begin{aligned} \int_{\Omega} u \nabla \varphi_2 \cdot \nabla \varphi \, dx &= - \int_{\Omega} \Delta \varphi \nabla \varphi_2 \cdot \nabla \varphi \, dx \\ &= \int_{\Omega} \nabla \varphi \cdot \nabla (\nabla \varphi_2 \cdot \nabla \varphi) \, dx \\ &= \sum_{i,j} \int_{\Omega} \partial_i \varphi \partial_{ij}^2 \varphi_2 \partial_j \varphi \, dx + \sum_{i,j} \int_{\Omega} \partial_i \varphi \partial_j \varphi_2 \partial_{ij}^2 \varphi \, dx. \end{aligned} \quad (53)$$

integrating by parts the second integral on the right-hand side of (53),

$$\begin{aligned} \sum_{i,j} \int_{\Omega} \partial_i \varphi \partial_j \varphi_2 \partial_{ij}^2 \varphi \, dx &= \sum_{i,j} \frac{1}{2} \int_{\Omega} \partial_j \varphi_2 \partial_j |\partial_i \varphi|^2 \, dx \\ &= \frac{1}{2} \int_{\Omega} \nabla \varphi_2 \cdot \nabla (|\nabla \varphi|^2) \, dx \\ &= -\frac{1}{2} \int_{\Omega} \Delta \varphi_2 |\nabla \varphi|^2 \, dx \\ &\leq C(T) \|\nabla \varphi\|_2^2, \end{aligned}$$

since  $-\Delta\varphi_2 = u_2 - \langle u_2 \rangle \in L^\infty((0, T) \times \Omega)$ . Together with (53) the previous inequality implies

$$\begin{aligned} \left| \int_{\Omega} u \nabla\varphi_2 \cdot \nabla\varphi \, dx \right| &\leq C(T) \int_{\Omega} (|D^2\varphi_2| + 1) |\nabla\varphi|^2 \, dx. \\ &\leq C(T) (\|\varphi_2\|_{L^\infty((0, T); W^{2, \infty}(\Omega))} + 1) \int_{\Omega} |\nabla\varphi|^2 \, dx, \end{aligned}$$

provided that the  $L^\infty((0, T); W^{2, \infty}(\Omega))$  norm of the function  $\varphi_2$  is bounded. Thus, substituting the above estimates in (52), one finally obtains

$$\frac{d}{dt} \int_{\Omega} |\nabla\varphi|^2 \, dx \leq C(T) \int_{\Omega} |\nabla\varphi|^2 \, dx. \quad (54)$$

Notice that  $\|\nabla\varphi(0)\|_2 = 0$  which follows from (49) and the property  $u(0) = 0$ . Thus, inequality (54) implies

$$\|\nabla\varphi(t)\|_2^2 \leq e^{C(T)t} \|\nabla\varphi(0)\|_2^2 = 0.$$

Consequently,  $\nabla\varphi(t) = 0$  for all  $t \in [0, T]$  and, since  $\langle \varphi(t) \rangle = 0$ , we have  $\varphi(t) = 0$  for all  $t \in [0, T]$ . Using (49), we conclude that  $u(t) = 0$  for all  $t \in [0, T]$ . Consequently  $(u_1, \varphi_1) = (u_2, \varphi_2)$ .  $\square$

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