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Game semantics and uniqueness of type inhabitation in the simply-typed λ -calculus

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Abstract. The problem of characterizing sequents for which there is a unique proof in intuitionistic logic was first raised by Mints [Min77], initially studied in [BS82] and later in [Aot99]. We address this problem through game semantics and give a new and concise proof of [Aot99]. We also fully characterize a family of λ -terms for Aoto's theorem. The use of games also leads to a new characterization of principal typings for simply-typed λ -terms. These results show that game models can help proving strong structural properties in the simply-typed λ -calculus.

Keywords: games semantics, simply-typed λ -calculus, principal typing, coherence theorem, uniqueness of type inhabitation

1 Introduction

Coherence theorems in category theory are used to ensure the equality of the composition of certain morphisms. In particular such conditions have been studied for cartesian closed categories - models of the simply-typed λ -calculus - in [BS82], a result which was later extended in [Aot99]. These results imply that a unique λ -term inhabits a given typing that verifies certain syntactic constraints. In [BS82], the exhibited typings are *balanced* while in [Aot99] they are *negatively non-duplicated*. A question that arises from these results, is that of the consequences of these constraints on types on their inhabitants. It can easily be observed that balanced typings are exactly inhabited by *affine λ -terms*. It was showed in [Kan07] that the family of *almost linear λ -terms* was included in that of the terms inhabiting negatively non-duplicated typings. One of the aims of this paper is to completely characterize the family of λ -terms that can be typed with negatively non-duplicated typings.

More precisely, we show that using game semantics leads to a concise proof of Aoto's theorem, and in general, gives an accurate method to address structural properties of simply-typed terms. The dialogic-game representation of proofs originates in [Lor59,Lor68,Bla92] and while game semantics has been widely used to study programming language semantics [AM99,HO00] and λ -calculus semantics [Hug00,GFH99,KNO02], to our knowledge it has never been used to

address issues on proofs/terms structures, while it offers two main advantages: first, it brings closer representations of typings and λ -terms which helps associating families of λ -terms with families of typings; second, it provides the analysis of proofs with a fine grained and natural access to the interplay of atomic types occurring in sequents. Our study leads indeed to a rather simple proof of Aoto's theorem, and to a syntactic characterization of the inhabitants of negatively non-duplicating types as *first-order copying λ -terms*, which extends both notions of linear and almost linear terms. These λ -terms could be depicted in Kanazawa's vocabulary as *almost affine*¹. From a more general perspective, game semantics offers a simple way of investigating the relationship between typings and their inhabitants. As an example, we give a new characterization of the principal typings of β -normal terms.

In this paper, we will first recall basic notions on the simply-typed λ -calculus and introduce typing games and strategies; in the third section the correspondence between strategies and simply-typed terms will be presented; finally, we give a new characterization of principal typings, a concise proof of [Aot99] and a full characterization of the terms for [Aot99].

2 Preliminaries

2.1 Simply-typed λ -calculus

Let \mathcal{A} be a countable set of atomic types. The set $\mathcal{T}(\mathcal{A})$ of simple types built upon \mathcal{A} is the smallest set built as the closure of \mathcal{A} under the right-associative connector \rightarrow . We call a *type substitution* σ an endomorphism of $\mathcal{T}(\mathcal{A})$ i.e. a function that verifies $\sigma(\alpha \rightarrow \beta) = \sigma(\alpha) \rightarrow \sigma(\beta)$, for α, β in $\mathcal{T}(\mathcal{A})$. Note that a type substitution is completely defined by the values it takes on \mathcal{A} . A *type relabelling* σ is a type substitution such that $\sigma(\mathcal{A})$ is included in \mathcal{A} . Finally, a *type renaming* denotes a bijective type substitution.

Let us consider a countable set of variables \mathcal{V} . The set Λ of λ -terms built on \mathcal{V} is inductively defined with the following syntactic rules:

$$\Lambda ::= \mathcal{V} \mid \lambda \mathcal{V} . \Lambda \mid (\Lambda \Lambda)$$

We write λ -terms with the usual conventions, omitting sequences of λ 's and unnecessary parentheses. For a term M , the notions of free variables (noted $FV(M)$), bound variables ($BV(M)$) and variables ($V(M) = FV(M) \cup BV(M)$) are defined as usual. We also take for granted the notions of α -conversion, β -reduction and η -conversion. A precise definition of all these notions can be found in [Bar84]. A context is a λ -term with a hole, built according to the following rules:

$$\Lambda_{\square} ::= \square \mid \lambda \mathcal{V} . \Lambda_{\square} \mid \Lambda_{\square} \Lambda \mid \Lambda \Lambda_{\square}$$

¹ This result was independently proved by Kanazawa in a yet unpublished work. Nevertheless, we believe the use of game semantics allows to express more directly the relation between syntactic properties of types and their inhabitants, giving a simpler proof.

The notation adopted for contexts will be $C[], C_1[], \dots$ and grafting a term N in a context $C[]$ is noted $C[N]$. An *occurrence of a subterm* N in M is given by a context $C[]$ such that $M = C[N]$. We say that an occurrence of N in M is *characterized by* $C[]$ when $C[N] = M$. In general, we simply speak about an occurrence N of a subterm of M without mentioning the context that characterizes it. A term N is called a *subterm* of a term M when it has at least an occurrence in M , *i.e.* when there is a context $C[]$ such that $C[N] = M$.

A *typing environment* Γ is a finite subset of $\mathcal{V} \times \mathcal{T}(\mathcal{A})$ such that if (x, α) and (x, β) are in Γ , $\alpha = \beta$ holds. We write such an environment as a sequence of *type assignments* of the form $x : \alpha$. A *typing* $\langle \Gamma; \gamma \rangle$ is a pair made of a typing environment Γ and a type γ . A *typing judgement* of a term M is written $\Gamma \vdash M : \gamma$ and is said *valid* (we also say $\langle \Gamma; \gamma \rangle$ is a typing of M , or M is an inhabitant of $\langle \Gamma; \gamma \rangle$) if it can be obtained with the following inference system:

$$\frac{}{\Gamma, x : \alpha \vdash x : \alpha} \quad \frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x.M : \alpha \rightarrow \beta}$$

$$\frac{\Gamma \vdash M : \alpha \rightarrow \beta \quad \Delta \vdash N : \alpha}{\Gamma \cup \Delta \vdash MN : \beta}$$

As a derivation of a typing judgement for a term M is always constructed by closely following the syntactic structure of M , for a given derivation \mathcal{D} of the judgement $\Gamma \vdash M : \gamma$, there is a unique subderivation (where subderivation is taken in the obvious sense) \mathcal{D}' of \mathcal{D} that corresponds to an occurrence of a subterm N of M . Thus a derivation \mathcal{D} of $\Gamma \vdash M : \gamma$ assigns to each occurrence N of a subterm of M a unique judgement of the form $\Delta \vdash N : \delta$. Given a derivation \mathcal{D} of $\Gamma \vdash M : \gamma$, M is said in *η -long form relative to \mathcal{D}* when for each occurrence of any subterm N of M which is assigned a judgement of the form $\Delta \vdash N : \delta_1 \rightarrow \delta_2$ relative to \mathcal{D} , either N is of the form $\lambda x.N'$ or the context $C[]$ which characterizes this occurrence of N is of the form $C'[[N']]$. Terms in η -long form present several advantages (see [JG95]) and every simply-typed term M can be put in η -long form with respect to a derivation of a judgement [Hue76]. Furthermore, whenever M is in normal form, there is a unique way of deriving any typing judgement of M . In such a case, we say that to an occurrence of N in M is *associated the pair* $\langle \Delta; \delta \rangle$ *relative to* $\langle \Gamma; \gamma \rangle$ when the judgement associated to this occurrence of N relative to the unique derivation of $\Gamma \vdash M : \gamma$ is $\Delta \vdash N : \delta$. A term is said in *long normal form* when it is β -normal and η -long for some typing. A typing $\langle \Gamma; \gamma \rangle$ is a *principal typing* (or *most general typing*) of a term M , if $\Gamma \vdash M : \gamma$ is a valid judgement and if for every typing $\langle \Delta; \delta \rangle$ of M :

- $\Delta = \Delta_1 \cup \Delta_2$, where a type assignment $x : \alpha$ appears in Δ_1 if and only if x is a free variable of M
- there is a type substitution σ such that $\langle \Delta_1; \delta \rangle = \langle \Gamma; \gamma \rangle \cdot \sigma$.

An important result is that if a term M has a typing, then it has a principal typing which is unique up to renaming (see [Hin97]). This is the reason why from now on we will speak of the principal typing of a simply-typed term M . Given any typing $\langle \Gamma; \gamma \rangle$, polarity of types in $\langle \Gamma; \gamma \rangle$ is defined as follows:

- γ has a positive occurrence in $\langle \Gamma; \gamma \rangle$.
- for every type assignment $x : \delta \in \Gamma$, δ has a negative occurrence in $\langle \Gamma; \gamma \rangle$.
- if $\alpha \rightarrow \beta$ has a positive (*resp.* negative) occurrence in $\langle \Gamma; \gamma \rangle$, then α has a negative (*resp.* positive) occurrence and β a positive (*resp.* negative) one in this typing.

A typing $\langle \Gamma; \gamma \rangle$ is said *balanced* if each of its atomic types has at most one positive and at most one negative occurrence in it. It is said *negatively non-duplicating* if each of its atomic types has at most one negative occurrence in it.

Theorem 1. [BS82] *If a term M inhabits a balanced typing, then M is the unique inhabitant of this typing modulo $=_{\beta\eta}$.*
 [Aot99] *If a term M inhabits a negatively non-duplicating typing, then M is the unique inhabitant of this typing modulo $=_{\beta\eta}$.*

2.2 Arenas and typings

In what follows, arenas and games are presented as a restriction of Hyland-Ong games (HO-games) [HO00] and Nickau games [Nic94] as we do not use notions of question and answer. Following [KNO02], arenas associated to types are considered as ordered labelled tree; arenas associated to typings are considered as unordered labelled trees. Given a type γ , a move in the arena of γ is a finite sequence of natural numbers (we write \mathbb{N}^* for the set of such sequences, ϵ for the empty sequence, and $s_1 \cdot s_2$ for the concatenation of such sequences). Elements of \mathbb{N} will be written i, j, i_1, \dots , and elements of \mathbb{N}^* s, s_1, \dots . Given a subset N of \mathbb{N}^* and $i \in \mathbb{N}$, we denote by $i \cdot N$ the set $\{i \cdot s \mid s \in N\}$. Finally, given $s \in \mathbb{N}^*$, we write $|s|$ for the length of s defined as $|\epsilon| = 0$, $|i| = 1$ for $i \in \mathbb{N}$ and $|s_1 \cdot s_2| = |s_1| + |s_2|$.

Let us consider $\gamma = \gamma_1 \rightarrow \dots \rightarrow \gamma_n \rightarrow a \in \mathcal{T}(\mathcal{A})$, where a is an atomic type. We inductively define the arena $A_\gamma = (M_\gamma, \tau_\gamma)$ (where M_γ is a finite set of moves and the typing function τ_γ is a function from M_γ to \mathcal{A}) from the arenas (M_{γ_i}, τ_i) associated to γ_i for every $i \in [1, n]$:

1. $M_\gamma = \{\epsilon\} \cup \bigcup_{i=1 \dots n} i \cdot M_{\gamma_i}$,
2. $\tau_\gamma : M_\gamma \mapsto \mathcal{A}$ is defined by:
 - (a) $\tau_\gamma(\epsilon) = a$
 - (b) $\tau_\gamma(i \cdot s) = \tau_{\gamma_i}(s)$, for $i \in [1 \dots n]$ and $s \in \mathbb{N}^*$

The arena associated to a type $\gamma \in \mathcal{T}(\mathcal{A})$ is therefore a finite prefix-closed subset of \mathbb{N}^* whose elements are labelled with atomic types. The parent relation expresses the *enabling* relation [HO00]: given s_1 and s_2 in \mathbb{N}^* , s_1 enables s_2 (written $s_1 \vdash s_2$) when there is i in \mathbb{N} such that $s_2 = s_1 \cdot i$. Given an arena (M, τ) , we define the function $pl : M \mapsto \{O, P\}$ which associates moves to players (P stands for the proponent and O for the opponent), and $\overline{pl} : M \mapsto \{O, P\}$ its inverse function (*i.e.* $pl(m) = O$ iff $\overline{pl}(m) = P$ for every $m \in M$, and $\overline{\overline{pl}} = pl$), by $pl(\epsilon) = O$ and $pl(s_2) = \overline{pl}(s_1)$ for $s_1 \vdash s_2$. In the rest of the document, we

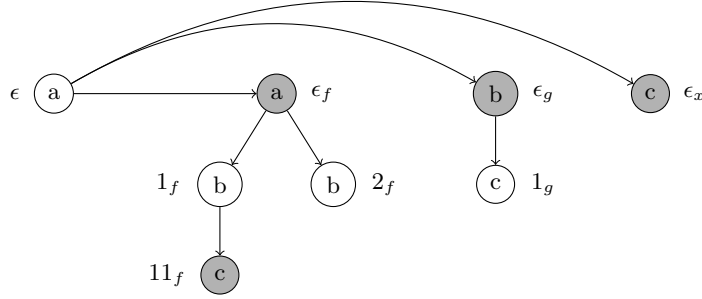


Fig. 1. Example of an arena

use the partition $M^O = \{s \in M \mid pl(s) = O\}$ and $M^P = \{s \in M \mid pl(s) = P\}$ of M which separates P -moves (*i.e.* negative atoms) and O -moves (*i.e.* positive atoms) in M . Remark the correspondence between, atomic types and moves, polarity and pl .

Given a type assignment $x : \gamma$, the associated arena $A_{x:\gamma} = (M_{x:\gamma}, \tau_{x:\gamma})$ is defined similarly to A_γ but moves in $M_{x:\gamma}$ are paired with the variable x , and written $\epsilon_x, i_x, s_x, \dots$ for the pairs $(\epsilon, x), (i, x), (s, x) \dots$ respectively. In the rest of the document, we will use m, n, m_1, \dots for moves. For a variable x , a sequence s in \mathbb{N}^* and $i \in \mathbb{N}$, given the move $m = s_x$, we write $i \cdot m$ (*resp.* $m \cdot i$) for the move $(i \cdot s)_x$ (*resp.* $(s \cdot i)_x$). For a set of moves M , the *preceding* relation $m_1 \prec m_2$ stands for $m_1, m_2 \in M$ iff there exist $i, j \in \mathbb{N}$ and $m \in M$ such that $m_1 = m \cdot i$, $m_2 = m \cdot j$ and $i < j$.

Let us consider a typing $\langle \Gamma; \gamma \rangle$, where $\Gamma = x_1 : \gamma_1, \dots, x_n : \gamma_n$. Its associated arena $A = (M, \tau)$ is defined as:

1. $M = M_\gamma \cup \bigcup_{i \in [1, n]} M_{x_i : \gamma_i}$
2. $\tau(m) = \begin{cases} \tau_\gamma(m) & \text{if } m \in M_\gamma \\ \tau_{x_i : \gamma_i}(m) & \text{if } m \in M_{x_i : \gamma_i} \text{ for some } i \in [1, n] \end{cases}$

Moreover, the enabling relation is extended so that $m_1 \vdash m_2$ holds iff

- $m_2 = m_1 \cdot j$ for some j in \mathbb{N} or
- $m_1 = \epsilon \in M_\gamma$ and $m_2 = \epsilon_{x_i}$ for some $i \in [1, n]$

The arena associated to a typing $\langle \Gamma; \gamma \rangle$ is therefore represented as a forest made of the labelled trees associated to the type γ and to the type assignments in Γ . The enabling relation gives an arborescent structure to this forest, but while the arena associated to a type defines an ordered tree (with the left-to-right relation given by \prec), the arborescent structure of an arena associated to a typing is not ordered (two subtrees which roots are ϵ_x, ϵ_y can be permuted). Note that the notation m_x for a move in the arena associated to some typing $\langle \Gamma; \gamma \rangle$ allows to differentiate this arena from the arena of $\langle \Gamma - \{x : \delta\}; \delta \rightarrow \gamma \rangle$. Figure 1

presents the arena for the sequent $\langle f : (c \rightarrow b) \rightarrow b \rightarrow a, g : c \rightarrow b, x : c; a \rangle$, where O -moves are represented by white nodes, P -moves by dark nodes and oriented black edges represent the enabling relation.

2.3 Games and strategies

In the rest of the document, we use S, S_1, \dots to denote finite sequences of moves $m_1.m_2 \dots m_n$ of a given arena. We write $S_1 \sqsubseteq S_2$ when the sequence of moves S_1 is a prefix of the sequence of moves S_2 . A finite sequence $S = m_1 \dots m_n$ of moves in an arena A is said *justified* if to each occurrence of a move m in S (except the initial occurrence m_1), there is an occurrence of a move n which precedes m in S and which enables m (the occurrence of n is said to *justify* the occurrence of m). Formally, a justified sequence S is written $(m_1, 1, 0) \dots (m_n, n, l)$ where (m, i, j) denotes the move in position i in S , justified by (m', j, k) where $j < i$. This notation will be simplified for the particular case of strategies we are interested in.

Given a finite sequence of moves S , the P -view $\ulcorner S \urcorner$ of S removes moves which are irrelevant for player P in S and is inductively defined as:

$$\begin{aligned} \ulcorner \epsilon \urcorner &= \epsilon && \text{for the initial move } \epsilon \\ \ulcorner S.(m, i, j) \urcorner &= \ulcorner S \urcorner.(m, i, j) && \text{for a } P\text{-move } m \\ \ulcorner S_1.(m_1, j, k).S_2.(m_2, i, j) \urcorner &= \ulcorner S_1 \urcorner.(m_1, j, k).(m_2, i, j) && \text{for an } O\text{-move } m_2. \end{aligned}$$

The O -view of a sequence S can be defined similarly, but is not needed for our purpose. We call a justified sequence $S = m_1 \dots m_n$ a *legal position* if it verifies:

1. $m_1 = \epsilon$ (Initial move condition)
2. for $S = S_1.(m, i, j).(n, i + 1, k).S_2$, $pl(m) = \overline{pl}(n)$ (Alternation)
3. $\ulcorner S \urcorner = S$ (P -view condition)

For a given arena A , we write L_A the set of its legal positions.

In general, a game is defined as an arena to which we associate a set of positions, in the present case, a set of prefix-closed legal positions. Because our study only requires the particular notion of games where available positions are precisely all legal positions in the given arena, we confuse the usual notion of game with that of an arena. Note that, given a legal position L , because L is P -views, an occurrence of an O -move $M \neq \epsilon$ in L is justified by the immediately preceding move (which is a P -move). The notation for occurrences of moves in a legal position S can therefore be simplified as follows:

- an occurrence of an O -move n in S will be denoted by (n, i) if it is in position $2i + 1$ in S in a left-to-right order;
- an occurrence of a P -move m in S will be written (m, i) if this occurrence is justified by a preceding occurrence (n, i) of the O -move n .

Remark that this notation confuses occurrences of a P -move justified by the same occurrence of an O -move (n, i) . In the next section, we will see that these occurrences correspond to the same variable in the term associated to the strategy. When unnecessary, information about justification will be omitted from our notation and an occurrence of a move m will simply be noted m .

Definition 1. Let $A = (M, \tau)$ be an arena and $\Sigma \subseteq L_A$ a finite non-empty and prefix-closed set of legal positions. Σ is a (P -)strategy (or typing strategy) if:

1. the length $|S|$ of any sequence $S \in \Sigma$ is even and different from 0.
2. if $S.m.n_1, S.m.n_2 \in \Sigma$, then $n_1 = n_2$
3. if $S.m.n \in \Sigma$ then $\tau(m) = \tau(n)$

According to the second property of this definition, a strategy Σ is uniquely determined by the set $\max(\Sigma) \subseteq \Sigma$ made of the sequences which are maximal for the partial order \sqsubseteq .

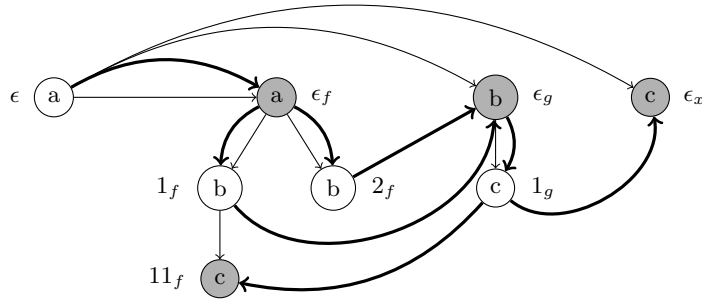


Fig. 2. Example of a typing strategy

As an example, Figure 2 represents the arena for the sequent $\langle f : ((c \rightarrow b) \rightarrow b \rightarrow a), g : c \rightarrow b, x : c; a \rangle$. In this Figure, paths made of thick arrows starting from ϵ denote sequences of moves which are legal positions in the arena if the justification constraint is respected. An example of strategy Σ is given by $\max(\Sigma) = \{(\epsilon, 0).(\epsilon_f, 0).(1_f, 1).(\epsilon_g, 0), (\epsilon, 0).(\epsilon_f, 0).(2_f, 1).(\epsilon_g, 0).(1_g, 2).(\epsilon_x, 0)\}$.

Definition 2. Given an arena $A = (M, \tau)$, a typing strategy Σ on A is called a winning strategy if

- for all $S.m$ in $\max(\Sigma)$, there is no move $n \in M^O$, such that $m \vdash n$.
- for every sequences $S.m_1 \in \Sigma$ and $S.m_1.n \in L_A$, there is a P -move m_2 such that $S.m_1.n.m_2 \in \Sigma$.

The first condition ensures P wins when O can play no move while the second one ensures that P wins with this strategy independently from O 's strategy. In the rest of the document, if not specified otherwise, only winning strategies will be considered.

Example 1. In the arena of Figure 2, there is a winning strategy Σ such that $\max(\Sigma) = \{S_1, S_2\}$ with:

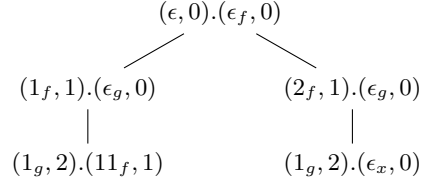


Fig. 3. Example of an arborescent reading

- $S_1 = (\epsilon, 0).(\epsilon_f, 0).(1_f, 1).(\epsilon_g, 0).(1_g, 2).(11_f, 1)$
- $S_2 = (\epsilon, 0).(\epsilon_f, 0).(2_f, 1).(\epsilon_g, 0).(1_g, 2).(\epsilon_x, 0)$

Remark that there is another typing strategy Σ' such that $\max(\Sigma') = \{S'_1, S'_2\}$ where $S'_1 = (\epsilon, 0).(\epsilon_f, 0).(1_f, 1).(\epsilon_g, 0).(1_g, 2).(\epsilon_x, 0)$.

3 Strategies as terms

3.1 Interpreting λ -terms

Given an arena $A = (M, \tau)$, the prefix-closure property of a strategy Σ on A can be used to represent $\max(\Sigma)$ as a tree denoted \mathbb{T}_Σ . We call this tree *the arborescent reading of Σ* and given a strategy Σ it is inductively defined on prefix-closed set of alternating sequences as $\mathbb{T}_\Sigma = m_1.m_2[\mathbb{T}_{\Sigma_1}, \dots, \mathbb{T}_{\Sigma_p}]$ if:

- for every S in $\max(\Sigma)$, $S = m_1.m_2.S'$ for some S' ,
- $\{n_1, \dots, n_p\} = \{m' \mid m_2 \vdash m'\}$ and $n_i < n_j$ iff $i < j$,
- $\Sigma_i = \{n_i.S' \mid m_1.m_2.n_i.S' \in \Sigma\}$.

For example the strategy defined by $\max(\Sigma) = \{S_1, S_2\}$ in Example 1 has the arborescent reading (also pictured on Figure 3)

$$\mathbb{T}_\Sigma = (\epsilon, 0).(\epsilon_f, 0)[(1_f, 1).(\epsilon_g, 0)[(1_g, 2).(11_f, 1)], (2_f, 1).(\epsilon_g, 0)[(1_g, 2).(\epsilon_x, 0)]]$$

Given an arena A , let us note $FV_A = \{((\epsilon_x, 0), x) \mid \text{there is a variable } x, \epsilon_x \in M\}$

Definition 3. Let the interpretation of a strategy Σ on an arena $A = (M, \tau)$ be $\llbracket \Sigma \rrbracket = \llbracket \mathbb{T}_\Sigma, FV_A \rrbracket$ which is inductively defined on \mathbb{T}_Σ as:

$$\llbracket (m, i).(n, j)[\mathbb{T}_{\Sigma_1}, \dots, \mathbb{T}_{\Sigma_q}], V \rrbracket = \lambda x_1 \dots x_p. x [\llbracket \mathbb{T}_{\Sigma_1}, W \rrbracket] \dots [\llbracket \mathbb{T}_{\Sigma_q}, W \rrbracket]$$

where $m \in M^O$, $n \in M^P$, the set W is equal to $V \cup \{((m \cdot k, i), x_k) \mid m \cdot k \in M\}$, the x_k 's being fresh variables, and $((n, j), x)$ is in W .

The second rule of the interpretation associates variables to occurrences of P -moves. We say that an occurrence of a variable x is a *realization* of a P -move m (or of (m, i)) in $\llbracket \Sigma \rrbracket$ when $\llbracket \Sigma \rrbracket = C[xN_1 \dots N_n] = C[\llbracket (m, i)[\mathbb{T}_{\Sigma_1}, \dots, \mathbb{T}_{\Sigma_n}], V \rrbracket]$.

Note that $((m, i), x)$ is in V . Moreover, if we suppose $\llbracket \Sigma \rrbracket$ respects the syntactic convention of Barendregt, *i.e.* if every variable in $\llbracket \Sigma \rrbracket$ is uniquely identified by its name, then it is easy to see that every occurrences of a variable x are realizations of the same (m, i) . In this case, we simply say that *the variable x is a realization of the P-move m* (or of (m, i)). In the rest of the document, $\llbracket \Sigma \rrbracket$ is supposed to verify Barendregt's convention for every strategy Σ .

This interpretation of winning strategies as λ -terms allows us establishing a bijection between the set of strategies in the arena associated to the pair $\langle \Gamma; \gamma \rangle$ and the set of inhabitants of that pair modulo $\beta\eta$ -conversion.

Lemma 1. *Given the arena A associated to the typing pair $\langle \Gamma; \gamma \rangle$, and a winning strategy Σ on A , $\Gamma \vdash \llbracket \Sigma \rrbracket : \gamma$ is a valid judgement.*

Proof. The proof is done by induction on the structure of \mathbb{T}_Σ .

In the course of the proof of this Lemma, it can be noted that $\llbracket \Sigma \rrbracket$ is β -normal and η -long with respect to $\langle \Gamma; \gamma \rangle$.

Lemma 2. *Given $\Gamma \vdash N : \gamma$, there is a winning strategy Σ on A such that $\llbracket \Sigma \rrbracket =_{\beta\eta} N$.*

Proof. Without loss of generality we suppose that N is in long normal form with respect to the typing pair $\langle \Gamma; \gamma \rangle$. We then construct Σ inductively on the structure of N .

To establish the bijection between terms and strategies, it suffices to complete the picture with the following Lemma.

Lemma 3. *Given an arena A , and two winning strategies Σ_1 and Σ_2 on A , $\llbracket \Sigma_1 \rrbracket = \llbracket \Sigma_2 \rrbracket$ iff $\Sigma_1 = \Sigma_2$.*

3.2 Game-theoretic characterization of Principal Typings

Given a simply-typed λ -term N , its principal typing enjoys two equivalent characterizations:

1. an intentional one, through the Hindley-Milner inference algorithm [DM82].
2. an extensional one, by specifying that every other typing of N can be found by substitution on the principal one.

In this section we propose a third characterization of principal typings of β -normal terms by means of game semantics.

Definition 4. *Given a typing arena $A = (M, \tau)$ and a typing strategy Σ on A , the binary relations \triangleleft and \triangleleft_Σ on $M^O \times M^P$ are defined as follows:*

1. $n \triangleleft m$ iff $\tau(m) = \tau(n)$
2. $n \triangleleft_\Sigma m$ iff there is $S \in \Sigma$, such that $S = S'.n.m$.

For a P -move m , we note $\triangleleft^m = \{n \in M^O \mid n \triangleleft m\}$ the set of its *possible antecedents*, and $\triangleleft_\Sigma^m = \{n \in M^O \mid n \triangleleft_\Sigma m\}$ the set of its antecedents in Σ . For any typing strategy Σ , \triangleleft_Σ^m is a subset of \triangleleft^m . Remark that for any P -moves m_1, m_2 , $\tau(m_1) = \tau(m_2)$ is equivalent to $\triangleleft^{m_1} = \triangleleft^{m_2}$.

Let us define $R_\Sigma \subseteq M^P \times M^P$ as $m_1 R_\Sigma m_2$ iff $\triangleleft_\Sigma^{m_1} \cap \triangleleft_\Sigma^{m_2} \neq \emptyset$. We note R_Σ^* the transitive closure of R_Σ ; it is easy to verify that R_Σ^* is an equivalence relation. Then, all elements (m_1, m_2) in R_Σ , have at least one antecedent $n \in M^O$ in common in Σ ; this implies $\tau(n) = \tau(m_1) = \tau(m_2)$. Thus, whenever $m_1 R_\Sigma^* m_2$, $\tau(m_1) = \tau(m_2)$.

Definition 5. *Let us consider a typing arena $A = (M, \tau)$ and a strategy Σ on A . Σ is a covering typing strategy if*

1. Σ is a winning typing strategy. (winning condition)
2. For all moves $m = \epsilon_x \in M^P$, there is a sequence $S.m \in \Sigma$. (ϵ -completeness)
3. For all O -moves m , there is a P -move n and a sequence $S \in \Sigma$ (or S is empty), such that $S.m.n \in \Sigma$. (O -completeness)
4. For all m_1, m_2 in M^P , $\tau(m_1) = \tau(m_2)$ iff $m_1 R_\Sigma^* m_2$. (least typing constraint)

Theorem 2. *Given a term N , a typing pair $\langle \Gamma; \gamma \rangle$ and the corresponding arena $A = (M, \tau)$. The two following properties are equivalent*

1. the strategy Σ on A such that $\llbracket \Sigma \rrbracket =_{\beta\eta} N$ is a covering strategy
2. $\langle \Gamma; \gamma \rangle$ is the principal typing of the long normal form of N relative to $\langle \Gamma; \gamma \rangle$

Proof. Without loss of generality we assume that N is in long normal form relatively to $\langle \Gamma; \gamma \rangle$. We prove the equivalence between the two following propositions:

- 1'. Σ is not a covering strategy on A
- 2'. $\langle \Gamma; \gamma \rangle$ is not the principal typing of N

The implication $2' \Rightarrow 1'$ is proved using a disjunction of cases. First, we suppose that there is a type assignment $x : \alpha$ in Γ such that x has no occurrence in N ; then Σ violates the condition of ϵ -completeness and thus is not a covering strategy. Secondly, we let $\langle \Gamma'; \gamma' \rangle$ be the principal typing of N and suppose there is type substitution σ such that $\langle \Gamma; \gamma \rangle = \langle \Gamma'; \gamma' \rangle \cdot \sigma$; we associate to a type substitution a transformation on the initial arena (a morphism of arena) and show that if σ is not a relabelling then Σ violates one of the condition in Definition 5. The implication $1' \Rightarrow 2'$ is proved with similar arguments.

4 Expressing structural properties

4.1 First-order copying terms

We are now about to define syntactically the family of λ -terms that correspond to strategies in arenas associated to negatively non-duplicating typing pairs. This definition requires some intricate relation between the variables that occur

in a term. The main technical difficulty for defining this relation comes from α -conversion. Thus, we adopt Barendregt's naming convention of bound variables that consists in giving a different name to each variable introduced by each λ -abstraction. Under this convention, and given a normal λ -term N in η -long form for some typing $\langle \Gamma; \gamma \rangle$ and two of its variables x, y , the binary relation $x \mathbb{I}_{ij}^N y$ on $V(N) \times V(N)$ (where $i, j \in \mathbb{N}$) holds iff

$$N = C[xN_1 \dots N_{i-1}(\lambda x_1 \dots x_{j-1} y x_{j+1} \dots x_n \cdot N) N_{i+1} \dots N_m]$$

Definition 6. *Let us consider a λ -term N in η -long form for some typing and which variables have distinct names. Given two variables x and y in $V(N)$, $x \approx_N y$ is verified iff:*

1. $x = y$.
2. *or there are two variables z_1, z_2 and i, j in \mathbb{N} such that $z_1 \mathbb{I}_{ij}^N x$, $z_2 \mathbb{I}_{ij}^N y$, and $z_1 \approx_N z_2$*

The idea behind the relation $x \approx_N y$ on a term N is to express that the variables x and y are *recursively introduced* by the same variable.

Example 2. Given $N = \lambda f g. g(f(\lambda x_1 y_1. x_1(\lambda z_1. z_1)))(f(\lambda x_2 y_2. y_2(x_2(\lambda z_2. z_2))))$, the relations $x_1 \approx_N x_2$, $y_1 \approx_N y_2$ and $z_1 \approx_N z_2$ hold.

This definition is naturally extended to subterms of N :

Definition 7. *Let us consider a term N in long normal form for some typing $\langle \Gamma; \gamma \rangle$, and two of its subterms N_1 and N_2 ; $N_1 \approx_N N_2$ is verified if:*

- $N_1 = x_1$, $N_2 = x_2$, and $x_1 \approx_N x_2$
- $N_1 = \lambda x_1. P_1$, $N_2 = \lambda x_2. P_2$, $x_1 \approx_N x_2$ and $P_1 \approx_N P_2$
- $N_1 = x_1 P_1 \dots P_n$, $N_2 = x_2 Q_1 \dots Q_n$, $x_1 \approx_N x_2$ and $P_i \approx_N Q_i$ for every $i \in [1, n]$.

Example 3. In the previous example, we have $x_1(\lambda z_1. z_1) \approx_N x_2(\lambda z_2. z_2)$. But we do not have $f(\lambda x_1 y_1. x_1(\lambda z_1. z_1)) \approx_N f(\lambda x_2 y_2. y_2(x_2(\lambda z_2. z_2)))$.

Remark that for a term $N = C_1[C_2[P_1][P_2]]$ in long normal form for some typing $\langle \Gamma; \gamma \rangle$ and where $P_1 \approx_N P_2$, the term $N' = C'_1[(\lambda x. C'_2[x][x])P]$ that is β -convertible to N is also an inhabitant of $\langle \Gamma; \gamma \rangle$. In the previous example, $(\lambda F f g. g(f(\lambda x_1 y_1. F x_1))(f(\lambda x_2 y_2. y_2(F x_2))))(\lambda x. x(\lambda z. z))$ is simply typable and has the same most general typing as N .

In the rest of the document, we focus on a particular case for this relation:

Definition 8. *Let us consider a term N in long normal form for its principal typing $\langle \Gamma; \gamma \rangle$. N is said first-order copying (written copy(1)) whenever two subterms N_1 and N_2 are assigned the same type α in $\langle \Gamma; \gamma \rangle$ iff $N_1 \approx_N N_2$.*

Example 4. The term in Example 2 is not copy(1) because $f(\lambda x_1 y_1. x_1(\lambda z_2. z_2))$ and $f(\lambda x_2 y_2. y_2(x_2(\lambda z_2. z_2)))$ have the same type in the principal typing of N but are not in relation with \approx_N . The term $N = f(h(\lambda x_1 y_1. y_1))(h(\lambda x_2 y_2. y_2))$ is copy(1) and can be expanded into $(\lambda z. f z z)(h(\lambda x y. y))$ preserving its principal typing. Note that z is of atomic type in any typing for which N is in η -long form.

The notion is called first-order copying because the maximal subterms of N which verify the relation \approx_N are of atomic type in the principal typing of N just as in the last example above.

4.2 Proof uniqueness for copy(1)-terms

We are now going to see that first-order copying are precisely the inhabitants of negatively non-duplicating typings. But we first prove that negatively non-duplicating typings have at most one inhabitant.

Lemma 4. *Let us consider a negatively non-duplicating typing $\langle \Gamma; \gamma \rangle$ and the associated arena $A = (M, \tau)$. There is at most one occurrence of each O -move m in every sequence of a winning strategy Σ on A .*

Proof. If we suppose there is a strategy Σ which contains a sequence S of the form $S_1.n.m_1.S_2.n.m_2$ where $n \in M^O$ then we can prove Σ must contain an infinite number of sequences which is absurd.

Theorem 3. [Aot99] *Let $\langle \Gamma; \gamma \rangle$ be a negatively non-duplicating typing. There is at most one inhabitant of $\langle \Gamma; \gamma \rangle$.*

Proof. Let us suppose that N is an inhabitant of $\langle \Gamma; \gamma \rangle$ and let $A = (M, \tau)$ be the arena $\langle \Gamma; \gamma \rangle$. Then we let Σ be the strategy on A such that $N = \llbracket \Sigma \rrbracket$. We recall that a P -move m corresponds to a negative occurrence of an atomic type $a = \tau(m)$. Suppose there is another strategy $\Sigma' \neq \Sigma$ in A ; we show by induction on the length of a sequence S in Σ' that $S \in \Sigma$.

1. if $|S| = 2$, then $S = (\epsilon, 0).(m, 0)$ for some P -move m . Because the pair is negatively non-duplicating, every P -move $m' \neq m$ in M verifies $\tau(m') \neq \tau(m)$. Therefore S must also be in Σ .
2. if $|S| = 2p+2$, then S is of the form $S_1.(m, p).(n, i)$. By induction hypothesis, S_1 is in Σ . Moreover, if $S_1.(n, p).(m, i)$ is in Σ' , there exists a P -move m' such that $S_1 = S_2.(m', j)$ (where $m' \vdash n$). But because Σ is a winning strategy there must exist a P -move m'' in M such that there is a sequence of the form $S_1.(n, p).(m'', k)$ in Σ . Again, the typing being negatively non-duplicating, m is the only P -move such that $\tau(m) = \tau(n)$ and therefore we must have $m = m''$. Moreover, according to Lemma 4, $k = j$.

We now turn to proving that negatively non-duplicating typings are inhabited by first-order copying λ -terms and that conversely the principal typing of first-order copying λ -terms is negatively non-duplicating. This correspondence is similar to the one between balanced typings and affine λ -terms [Bel76, Hir91].

Lemma 5. *Let us consider an arena $A = (M, \tau)$, a strategy Σ on it and the term $N = \llbracket \Sigma \rrbracket$. Two variables x_1, x_2 in $V(N)$ verify $x_1 \approx_N x_2$ iff there is $m \in M^P$ such that x_1 and x_2 are realizations of m in $\llbracket \Sigma \rrbracket$.*

Proof. First, the right implication is shown by induction on the definition of the \approx_N relation on variables.

For the left implication, let us first define $J(S)$ for a sequence S in a strategy Σ as follows:

- $J((\epsilon, 0)) = 0$.
- $J(S_1.(m, i).S_2.(n, i)) = 1 + J(S_1.(m, i))$ if n is a P -move.
- $J(S.(m, i)) = J(S)$ if m is a O -move.

Let us suppose the two variables x_1 and x_2 are realizations of the same P -move m in $\llbracket \Sigma \rrbracket$, which is formally written as:

$$N = \begin{cases} C_1[\llbracket (m, i_1).[\mathbb{T}_1, \dots, \mathbb{T}_n], V_1 \rrbracket] = C_1[x_1[\llbracket \mathbb{T}_1, V_1 \rrbracket] \dots \llbracket \mathbb{T}_n, V_1 \rrbracket] \\ C_2[\llbracket (m, i_2).[\mathbb{T}'_1, \dots, \mathbb{T}'_n], V_2 \rrbracket] = C_2[x_2[\llbracket \mathbb{T}'_1, V_2 \rrbracket] \dots \llbracket \mathbb{T}'_n, V_2 \rrbracket] \end{cases}$$

Let us consider $S_1.(n, i_1)$ and $S_2.(n, i_2)$, the sequences in Σ for the respective occurrences of n in the equation above. We prove the rest of the lemma by induction on $p = \max(J(S_1.n), J(S_2.n))$.

Theorem 4. *If a negatively non-duplicating typing $\langle \Gamma; \gamma \rangle$ types a term N then N is copy(1).*

Proof. Consider the strategy Σ on the arena A associated to $\langle \Gamma; \gamma \rangle$ such that $\llbracket \Sigma \rrbracket = N$. According to Theorem 3, Σ is the unique winning strategy in A . Suppose $N = C_1[N_1] = C_2[N_2]$, such that $N_1 = \lambda x_1 \dots x_q. z_1 P_1 \dots P_{p_1}$ and $N_2 = \lambda y_1 \dots y_q. z_2 Q_1 \dots Q_{p_2}$ are of the same type δ in $\langle \Gamma; \gamma \rangle$. We can write $\llbracket \Sigma \rrbracket$ as:

$$\llbracket \Sigma \rrbracket = \begin{cases} C_1[\llbracket \mathbb{T}_1, V_1 \rrbracket] = C_1[\llbracket n_1.m_1.[\mathbb{T}_{1,1}, \dots, \mathbb{T}_{1,p_1}], V_1 \rrbracket] = C_1[N_1] \\ C_2[\llbracket \mathbb{T}_2, V_2 \rrbracket] = C_2[\llbracket n_2.m_2.[\mathbb{T}_{2,1}, \dots, \mathbb{T}_{2,p_2}], V_2 \rrbracket] = C_2[N_2] \end{cases}$$

where n_1, n_2 are O -moves, and m_1, m_2 P -moves. Because N_1 and N_2 are of the same type, $\tau(m_1) = \tau(m_2)$ and because $\langle \Gamma; \gamma \rangle$ is negatively non-duplicating $m_1 = m_2 = m$. This implies $p_1 = p_2 = p$. By induction on the trees \mathbb{T}_1 and \mathbb{T}_2 , we prove $N_1 \approx_N N_2$.

Theorem 5. *Given a first-order copying term N in long normal form for its principal typing $\langle \Gamma; \gamma \rangle$, $\langle \Gamma; \gamma \rangle$ is negatively non-duplicating.*

Proof. Given the arena $A = (M, \tau)$ associated to $\langle \Gamma; \gamma \rangle$, and the strategy Σ on A , such that $N = \llbracket \Sigma \rrbracket$, we know Σ is a covering strategy (Theorem 2). Suppose there exists $n_1 \neq n_2$ in M^P , such that $\tau(n_1) = \tau(n_2)$. By Theorem 2, this leads to $n_1 R_\Sigma^* n_2$, which implies the existence of two sequences $S_1.n_1$ and $S_2.n_2$ in Σ such that $N = C_i[\llbracket n_i.[\mathbb{T}_{i1}, \dots, \mathbb{T}_{im}], V_i \rrbracket] = C_i[N_i]$ for $i \in \{1, 2\}$. N_1 and N_2 are of the same atomic type $\tau(n_2) = \tau(n_1)$ then $N_1 = x_1 P_1 \dots P_n$ and $N_2 = x_2 Q_1 \dots Q_m$. According to Lemma 5, $x_1 \approx_N x_2$ does not hold which implies that $N_1 \approx_N N_2$ does not hold either: we obtain a contradiction with the hypothesis that N is first-order copying.

5 Conclusion

Games are widely used in logic for the study of programming language semantics [HO00], logic semantics [Bla92], and λ -calculus semantics [Hug00]. In this article, we show that game semantics brings a new point of view on the structural properties of the simply-typed λ -calculus. It was already known that the balanced types used in [BS82] are exactly the principal typings of affine λ -terms *i.e.* terms M for which each subterm $\lambda x.M'$ in it has at most one free occurrence of x in M' and each free variable x has at most one occurrence in M [Bel76,Hir91]. Using games, we obtain a full correspondence between negatively non-duplicating typings and the newly introduced class of *first-order copying λ -terms*.

Even though the problems of coherence for the simply-typed λ -calculus seem quite theoretical, they have recently proved to be useful in designing parsing algorithms for grammars of simply-typed λ -terms [dG01,Mus01]. Determining a λ -term only with its typing properties allows indeed to partially characterize it up to $\beta\eta$ -conversion. This idea has been used in [Kan07] to propose a very general parsing technique using [Aot99] coherence theorem for *almost linear* λ -terms and in [Sal10] to prove that parsing grammars of simply-typed λ -terms is decidable. Based on the result presented in this paper, we expect to extend Kanazawa's algorithm to first-order copying terms.

The question of extending coherence theorems to a wider class of terms remains an open-question. A family of typings has been given in [BD05] under the name of *deterministic typings*, with a method similar to the one given in this article, but it seems hard to give a syntactic characterization of the terms which inhabits such typings. Nevertheless, the use of game semantics gives an easy framework to study proof structures and while we focused on normal proofs, we shall try to extend our results to non-normal ones. For example, it would be interesting to investigate a generalization of the game-theoretical characterization of principal typings of β -normal terms to non β -normal ones will also be addressed.

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A Isomorphic arenas

Arenas are a particular kind of relational structures. Original games defined by Hyland and Ong and Nickau mainly focus on β -reductions, through the composition of strategies; this led to the construction of categories of arenas and games. Because we consider terms in β -reduced form, this is not needed and we study formal properties of arenas described as forest of labelled trees. The following notion of homomorphism of arena is a transformation which allows to describe two notions: the notion of type substitution of a typing $\langle \Gamma; \gamma \rangle$ and the notion of curryfication on the term or typing.

In order to do so, let us first remark that given a typing $\langle \Gamma; \gamma \rangle$ and its associated arena $A = (M, \tau)$, to an occurrence of a formula or type α in $\langle \Gamma; \gamma \rangle$, there is an associated prefix-closed set of moves M' for which there is a least upper bound for the transitive closure of the enabling relation (written $\widehat{M'}$); moreover (M', τ) is equal to A_α , the arena associated to α , modulo renaming of the moves; we call (M', τ) a *sub-arena* of (M, τ) , and we note $\sigma_A(M') = \alpha$ the formula in $\langle \Gamma; \gamma \rangle$ represented by this sub-arena. Finally, given an arena $A = (M, \tau)$, we note $\mathcal{F}(A)$ the set of all its sub-arenas.

Given two arenas $A_1 = (M_1, \tau_1)$ and $A_2 = (M_2, \tau_2)$, we define a homomorphism of arena as a mapping of the moves of the initial arena A_1 with sets of moves in sub-arenas of A_2 ; this mapping preserves certain conditions on the typing function and on the enabling relation. Formally, a homomorphism of arena $F : A_1 \mapsto \mathcal{F}(A_2)$ is defined as:

1. $F(\widehat{\epsilon}) = \epsilon$.
2. for m, n in M_1 , if $m \vdash n$ then $\widehat{F(m)} \vdash \widehat{F(n)}$.
3. for m_1, m_2 in M_1 , if $\tau_1(m_1) = \tau_2(m_2)$ then $\sigma_{A_2}(F(m_1)) = \sigma_{A_2}(F(m_2))$

Remark that 1. and 2. imply pl is preserved by homomorphism as follows: for every move m in M_1 , $pl(m) = pl(\widehat{F(m)})$.

A homomorphism of arena F is an isomorphism if F defines a bijection between M_1 and M_2 which respects 3. Two isomorphic arenas are noted $A_1 \cong A_2$. Because $F(m)$ is the singleton $\{\widehat{F(m)}\}$ if F is an isomorphism, $F(m)$ will denote $\{\widehat{F(m)}\}$. Remark that the identity id defines an isomorphism between the arenas associated to a typing $\langle x_1 : \gamma_1, \dots, x_n : \gamma_n; \gamma \rangle$ and the ones associated to any typing found by permutation of the type assignments in the context Γ . A type substitution can also be seen as a homomorphism of arena: let us consider a typing $\langle \Gamma; \gamma \rangle$ and its arena $A = (M, \tau)$. Given the arena $A' = (M', \tau')$ associated to $\langle \Gamma; \gamma \rangle \cdot [a \mapsto \alpha]$, there is an associated homomorphism $F : A \mapsto \mathcal{F}(A')$ such that $\widehat{F(m)} = m$ and:

- if $\tau(m) \neq a$, $F(m) = \{m\}$ and $\tau(m) = \tau'(\widehat{F(m)})$
- otherwise, $\sigma_{A'}(F(m)) = \alpha$

An example is graphically illustrated in Figure 4.

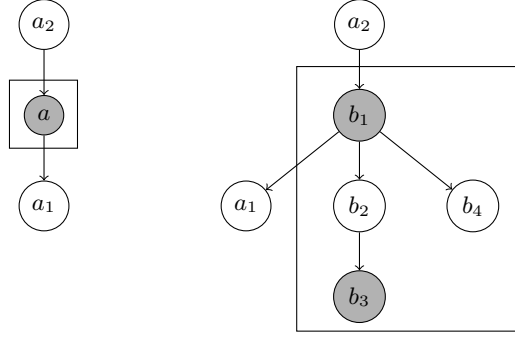


Fig. 4. Type substitution $[a \mapsto (b_3 \rightarrow b_2) \rightarrow b_4 \rightarrow b_1]$ in $(a_1 \rightarrow a) \rightarrow a_2$

Lemma 6. *The arenas $A_1 = (M_1, \tau_1)$ and $A_2 = (M_2, \tau_2)$ associated respectively to $\langle x_1 : \gamma_1, x_2 : \gamma_2 \dots, x_n : \gamma_n; \gamma \rangle$ and $\langle x_2 : \gamma_2 \dots, x_n : \gamma_n; \gamma_1 \rightarrow \gamma \rangle$ are isomorphic.*

Proof. Let us consider the arenas A_1 and A_2 as enunciated, and define a renaming $\mathbf{abs} : M_1 \mapsto M_2$ such that:

- $\mathbf{abs}(s_{x_1}) = \{1.s\}$ for every move $s_{x_1} \in M_1$
- $\mathbf{abs}(i \cdot s) = \{(i+1) \cdot s\}$ for every move $i \cdot s$ in M_1 , where $i \in \mathbb{N}$ and $s \in \mathbb{N}^*$
- $\mathbf{abs}(m) = \{m\}$ for every other move m in M_1

Moreover, we define $\tau_2(\mathbf{abs}(m)) = \tau_1(m)$ for every move m in M_1 . The function \mathbf{abs} defines an isomorphism between A_1 and A_2 . According to the renaming of the moves, it is easy to check that the enabling relation in A_1 is verified in A_2 and vice versa.

Given an isomorphism F between the arenas A and B , and a sequence $S = m_1 \dots m_n$ of moves in A , we write $F(S) = F(m_1) \dots F(m_n)$.

Lemma 7. *Let us consider two isomorphic arenas $A_1 = (M_1, \tau_1)$ and $A_2 = (M_2, \tau_2)$ for an isomorphism $F : A_1 \mapsto A_2$. For every legal position S in A_1 , $F(S)$ is a legal position in A_2 .*

Proof. Let us consider the legal position S in A_1 and show $F(S)$ is a justified sequence. S being legal in A_1 , we know that every prefix S' of S of length superior to 2 verifies the justification condition: $S' = S_1.m.S_2.n$ and m justifies n . But the sequence $F(S) = F(S_1).F(m).F(S_2).F(n)$ remains justified in A_2 as $F(m) \vdash F(n)$.

The sequence $F(S)$ verifies the initial move condition as $F(\epsilon) = \{\epsilon\}$ and the alternation condition as $p\ell(F(m)) = p\ell(m)$. Finally, the P -view condition is also verified; indeed, if $S = \epsilon$, $P(F(S)) = P(\epsilon) = \epsilon$; if $S = S'.m$ where m is in M_1^P , $P(S.m) = P(S).m$ and $P(F(S.m)) = P(F(S).F(m)) = P(F(S)).F(m)$ as $F(m) \in M_2^P$; if $S = S_1.(m_1, j).S_2.(m_2, j)$ and $m_2 \in M_1^O$ the property is verified by stability of $p\ell$ and of the enabling relation.

Lemma 8. *Given two isomorphic arenas A_1 and A_2 , there is a winning strategy on A_1 if and only if there is a winning strategy on A_2*

Proof. Let us consider $A_1 = (M_1, \tau_1)$ and $A_2 = (M_2, \tau_2)$ and the isomorphism F between A_1 and A_2 . Suppose there is a winning strategy Σ on A_1 ; according to Lemma 7, for every sequence $S \in \Sigma$, $F(S)$ is a legal position in (M_2, τ_2) . Moreover, by definition $m \vdash n$ iff $\widehat{F(m)} \vdash \widehat{F(n)}$, which implies the strategy $F(\Sigma) = \{F(S) \mid S \in \Sigma\}$ to be winning in A_2 .

B Proof of the Term-Strategy correspondence

Lemma 1 *Given the arena A associated to the typing pair $\langle \Gamma; \gamma \rangle$, and a winning strategy Σ on A , $\Gamma \vdash \llbracket \Sigma \rrbracket : \gamma$ is a valid judgement.*

Proof. Let us write $A = (M, \tau)$. The proof is done by induction on the structure of \mathbb{T}_Σ :

- if $\mathbb{T}_\Sigma = (\epsilon, 0)(m, 0)$, then $\llbracket \Sigma \rrbracket = \lambda x_1, \dots, x_n. x$ such that for $i \in \mathbb{N}$ and $m = i \in M$ holds iff $i \in [1, n]$. Let us write $\tau(m) = a$. If there is a variable x such that $m = \epsilon_x$, then A is the arena of a typing $\langle \Gamma, x : a; \gamma_1 \rightarrow \dots \rightarrow \alpha_n \rightarrow a \rangle$, which types $\llbracket \Sigma \rrbracket$. Otherwise, if $m = i$, for some i in $[1, n]$, then $x = x_i$ and A is the arena of a typing of the form $\langle \Gamma; \gamma_1 \rightarrow \dots \rightarrow \gamma_{i-1} \rightarrow a \rightarrow \dots \rightarrow \gamma_n \rightarrow a \rangle$ which types $\llbracket \Sigma \rrbracket$.
- if $\mathbb{T}_\Sigma = (\epsilon, 0)(m, 0)[\mathbb{T}_{\Sigma_1}, \dots, \mathbb{T}_{\Sigma_p}]$, then

$$\llbracket \mathbb{T}_\Sigma, FV_A \rrbracket = \lambda x_1 \dots x_n. x [\llbracket \mathbb{T}_{\Sigma_1}, FV_A \cup V \rrbracket \dots \llbracket \mathbb{T}_{\Sigma_p}, FV_A \cup V \rrbracket]$$

where $V = \{(i, 0), x_i \mid i \in \mathbb{N} \text{ and } m = i \in M\}$. Given $\langle \Gamma; \gamma_1 \rightarrow \dots \rightarrow \gamma_n \rightarrow a \rangle$ the typing associated to A , $FV_A \cup V$ corresponds to the variables in the typing context $\Gamma, x_1 : \gamma_1, \dots, x_n : \gamma_n$. This typing context must contain a type assignment $x : \alpha_1 \rightarrow \dots \rightarrow \alpha_n \rightarrow a$, where for every $i \in [1, n]$, α_i is the type associated to the sub-arena of A made of the set of moves which have $m \cdot i$ has common prefix. Then $\llbracket \mathbb{T}_{\Sigma_j}, FV_A \cup V \rrbracket$ corresponds to the interpretation of the winning strategy Σ_j on the arena associated to $\langle \Gamma, x_1 : \gamma_1, \dots, x_n : \gamma_n; \alpha_j \rangle$, for every $j \in [1, p]$; by induction hypothesis, $\Gamma, x_1 : \gamma_1, \dots, x_n : \gamma_n \vdash \llbracket \mathbb{T}_{\Sigma_j}, FV_A \cup V \rrbracket : \alpha_j$. Because the typing environments of these typings contain a type assignment $x : \gamma_1 \rightarrow \dots \rightarrow \gamma_n \rightarrow a$, we finally obtain $\Gamma \vdash \llbracket \Sigma \rrbracket : \gamma_1 \rightarrow \dots \rightarrow \gamma_n \rightarrow a$ as a valid judgement.

Lemma 2 *Given a typing pair $\langle \Gamma; \gamma \rangle$, its associated arena A , and a term N such that $\Gamma \vdash N : \gamma$ is a valid judgement, then there is a winning strategy Σ on A such that $\llbracket \Sigma \rrbracket =_{\beta\eta} N$.*

Proof. Without loss of generality we suppose that N is in long-normal form with respect to the pair $\langle \Gamma; \alpha \rangle$. We then construct Σ inductively on the structure of N .

- if $N = x$: then every typing for which N is η -long must be of the form $\langle \Gamma, x : a; a \rangle$, where a is atomic. The corresponding arena $A = (M, \tau)$ verifies the existence of two moves $m_1 = \epsilon$ and $m_2 = \epsilon_x$ such that no move $m_1 \cdot i$ and $m_2 \cdot i$ (for $i \in \mathbb{N}^*$) exist in M . Moreover, $\tau(\epsilon) = \tau(\epsilon_x)$. This is equivalent to the existence of a strategy $\Sigma = \{\epsilon, \epsilon_x\}$ on the arena A ; this strategy verifies $\llbracket \Sigma \rrbracket = x$.
- if $N = \lambda x.N'$: let us consider $A = (M, \tau)$ the arena associated to $\langle \Gamma; \alpha \rightarrow \beta \rangle$, a typing for which N is η -long. We consider \mathbf{abs} the isomorphism defined in the previous section; then $\mathbf{abs}^{-1}(A) = A'$ is the arena associated to $\langle \Gamma, x : \alpha; \beta \rangle$ which implies A' is a typing arena for N' . By induction hypothesis, the associated game has a winning strategy Σ' such that $\llbracket \Sigma' \rrbracket = N'$. But $\mathbf{abs}(\Sigma') = \Sigma$ is a winning strategy on A according to Lemma 8. Finally, for $\mathbb{T}_{\Sigma'} = (\epsilon, 0).(m, 0)[\mathbb{T}_{\Sigma'_1}, \dots, \mathbb{T}_{\Sigma'_n}]$, we obtain

$$\llbracket \mathbb{T}_{\Sigma'}, FV_{A'} \rrbracket = \lambda x_1 \dots \lambda x_n. y [\llbracket \mathbb{T}_{\Sigma'_1}, FV_{A'} \cup V' \rrbracket \dots \llbracket \mathbb{T}_{\Sigma'_n}, FV_{A'} \cup V' \rrbracket] = N'$$

where $V' = \{((m, 0), i) \mid i \in \mathbb{N} \text{ and } m = i \in M'\}$, and the following interpretation for Σ :

$$\llbracket \mathbb{T}_{\Sigma}, FV_A \rrbracket = \lambda x \lambda x_1 \dots \lambda x_n. y [\llbracket \mathbb{T}_{\mathbf{abs}(\Sigma'_1)}, FV_A \cup V \rrbracket \dots \llbracket \mathbb{T}_{\mathbf{abs}(\Sigma'_n)}, FV_A \cup V \rrbracket]$$

where V is the analog of V' in A . Then $\llbracket \Sigma \rrbracket$ is syntactically equal to $\lambda x.N'$ for $FV_A \cup V = FV_{A'} \cup V'$, an hypothesis we can suppose (otherwise, the equality is verified modulo α -equivalence).

- if $N = xN_1 \dots N_n$: let $\langle \Gamma, x : \alpha_1 \rightarrow \dots \rightarrow \alpha_n \rightarrow a; a \rangle$ be a typing for which N is η -long. Then for every $i \in [1, n]$, $\langle \Gamma; \alpha_i \rangle$ is a typing for the term N_i , which remains in η -long form for it. By considering A_i the arena for $\langle \Gamma, x : \alpha_1 \rightarrow \dots \rightarrow \alpha_n \rightarrow a; \alpha_i \rangle$, the induction hypothesis leads to the existence of a strategy Σ_i on A_i , such that $N_i = \llbracket \Sigma_i \rrbracket$. Let us build a renaming on moves $\mathbf{app} : M_1 + \dots + M_n \rightarrow M$ as follows:
 - if $m \in M_i$ for some $i \in [1, n]$ and there is $s \in \mathbb{N}^*$ such that $m = s$, then $\mathbf{app}(m) = (i \cdot s)_x$
 - otherwise $\mathbf{app}(m) = m$ otherwise.

We define a new arena made of the set of moves $M = \mathbf{app}(M_1 + \dots + M_n) \cup \{\epsilon\}$ and the typing function τ such that $\tau(\epsilon) = \tau(\epsilon_x)$ and $\tau(\mathbf{app}(m)) = \tau_i(m)$, for $m \in M_i$ and $i \in [1, n]$. Then, $A = (M, \tau)$ is the arena associated to $\langle \Gamma, x : \alpha_1 \rightarrow \dots \rightarrow \alpha_n \rightarrow a; a \rangle$ because every element of the form $s_x \in M_i$ is preserved in M (Γ is unchanged) and $\tau(\epsilon) = \tau(\epsilon_x)$. We now consider the typing strategy Σ , such that $\mathbb{T}_{\Sigma} = \epsilon. \epsilon_x [\mathbf{app}(\llbracket \Sigma_1 \rrbracket), \dots, \mathbf{app}(\llbracket \Sigma_n \rrbracket)]$ on A . Then

$$\llbracket \mathbb{T}_{\Sigma}, FV_A \rrbracket = x [\llbracket \mathbf{app}(\mathbb{T}_{\Sigma_1}), FV_A \rrbracket \dots \llbracket \mathbf{app}(\mathbb{T}_{\Sigma_n}), FV_A \rrbracket]$$

Because \mathbf{app} does not rename moves of the form ϵ_y , every term N_i is equal to $\llbracket \mathbf{app}(\mathbb{T}_{\Sigma_i}), FV_{A_i} \rrbracket = N_i$. This leads to $\llbracket \Sigma \rrbracket = xN_1 \dots N_n$.

Lemma 3 *Given an arena A , and two winning strategies Σ_1 and Σ_2 on A , if $\llbracket \Sigma_1 \rrbracket = \llbracket \Sigma_2 \rrbracket$ then $\Sigma_1 = \Sigma_2$.*

Proof. Let us proceed by induction on \mathbb{T}_{Σ_1} :

- if $\mathbb{T}_{\Sigma_1} = (\epsilon, 0)(m_1, 0)$, then $\llbracket \Sigma_1 \rrbracket = \lambda x_1 \dots x_n. z_1$. The interpretation of Σ_2 is equal to such a term iff $\Sigma_2 = \{(\epsilon, 0), (m_2, 0)\}$. Moreover, $\llbracket \Sigma_1 \rrbracket = \llbracket \Sigma_2 \rrbracket$ iff m_1 and m_2 refer to the same variable modulo α -equivalence, *i.e.* $m_1 = m_2$.
- if $\mathbb{T}_{\Sigma_1} = (\epsilon, 0)(m_1, 0)[\mathbb{T}_{\Sigma_{11}}, \dots, \mathbb{T}_{\Sigma_{1m}}]$, then

$$\llbracket \mathbb{T}_{\Sigma_1}, FV_A \rrbracket = \lambda x_1 \dots x_n. z_1 \llbracket \mathbb{T}_{\Sigma_{11}}, FV_A \cup V \rrbracket \dots \llbracket \mathbb{T}_{\Sigma_{1m}}, FV_A \cup V \rrbracket$$

where $FV_A = \{(\epsilon_x, 0), x \mid \text{there is a variable } x \text{ such that } \epsilon_x \in M\}$ and $V = \{(i, 0), x_i \mid \epsilon \vdash m \text{ and } m = i \in \mathbb{N}\}$. Moreover, \mathbb{T}_{Σ_2} must be of the form $(\epsilon, 0)(m_2, 0)[\mathbb{T}_{\Sigma_{21}}, \dots, \mathbb{T}_{\Sigma_{2m}}]$ so that $\llbracket \Sigma_1 \rrbracket = \llbracket \Sigma_2 \rrbracket$. This implies

$$\llbracket \mathbb{T}_{\Sigma_2}, FV_A \rrbracket = \lambda x_1 \dots x_n. z_2 \llbracket \mathbb{T}_{\Sigma_{21}}, FV_A \cup V \rrbracket \dots \llbracket \mathbb{T}_{\Sigma_{2m}}, FV_A \cup V \rrbracket$$

Then $\llbracket \mathbb{T}_{\Sigma_{1j}}, FV_A \cup V \rrbracket = \llbracket \mathbb{T}_{\Sigma_{2j}}, FV_A \cup V \rrbracket$ must stand for every $j \in [1, m]$, which implies by induction hypothesis, $\mathbb{T}_{\Sigma_{1j}} = \mathbb{T}_{\Sigma_{2j}}$. Moreover, z_1 and z_2 must again refer to the same variable, which implies $m_1 = m_2$, so finally, $\Sigma_1 = \Sigma_2$.

C Proof of Theorem 2

Lemma 9. *Let us consider a term N in long-normal form for $\langle \Gamma; \gamma \rangle$ and $\langle \Gamma'; \gamma' \rangle$. Given A and A' the respective arenas of these typings, and Σ, Σ' the strategies in A and A' respectively, such that $\llbracket \Sigma \rrbracket = \llbracket \Sigma' \rrbracket = N$, then $\Sigma = \Sigma'$*

Proof. We prove it by induction on Σ the arborescent reading of which is of the general form $(\epsilon, 0).(m, 0)[\mathbb{T}_{\Sigma_1}, \dots, \mathbb{T}_{\Sigma_{n_1}}]$. Similarly the arborescent reading of Σ' is of the form $(\epsilon, 0).(m', 0)[\mathbb{T}_{\Sigma'_1}, \dots, \mathbb{T}_{\Sigma'_{n_2}}]$ and

$$N = \begin{cases} \llbracket \Sigma \rrbracket = \lambda x_1 \dots x_{p_1}. x \llbracket \mathbb{T}_{\Sigma_1}, FV_A \cup V \rrbracket \dots \llbracket \mathbb{T}_{\Sigma_{n_1}}, FV_A \cup V \rrbracket \\ \llbracket \Sigma' \rrbracket = \lambda x_1 \dots x_{p_2}. x' \llbracket \mathbb{T}_{\Sigma'_1}, FV_A \cup V \rrbracket \dots \llbracket \mathbb{T}_{\Sigma'_{n_2}}, FV_A \cup V \rrbracket \end{cases}$$

and N is of the form $\lambda x_1 \dots x_1 \dots x_p. y N_1 \dots N_n$. Then $p_1 = p_2 = p$, which implies we can force $V = V'$. Moreover, $m = m'$ and $n_1 = n_2 = n$. By induction hypothesis, $\Sigma_i = \Sigma'_i$ for every $[1, n]$ which finally leads to $\Sigma = \Sigma'$.

Lemma 10. *Let us consider a typing $\langle \Gamma; \gamma \rangle$ and its arena $A = (M, \tau)$. Suppose there is a strategy Σ on A such that $\llbracket \Sigma \rrbracket = N$. If $\langle \Gamma; \gamma \rangle$ is not the principal typing of N , then Σ is not a covering strategy on A*

Proof. Consider A the arena of $\langle \Gamma; \gamma \rangle$ and Σ the strategy on A such that $N = \llbracket \Sigma \rrbracket$. Let us also consider $A' = (M', \tau')$ the arena associated to the principal typing $\langle \Gamma'; \gamma' \rangle$ of N and Σ' the typing strategy on A' such that $N = \llbracket \Sigma' \rrbracket$. According to Lemma 9,

$$\Sigma = \Sigma' \quad (*)$$

For convenience, we note $\llbracket \Sigma \rrbracket_A$ and $\llbracket \Sigma \rrbracket_{A'}$ the interpretation of Σ in A and A' respectively.

If $\langle \Gamma; \gamma \rangle$ is not the principal typing of N , then one of the three following cases is verified:

1. there is a type assignment $x : \alpha$ in Γ such that x is not free in N . Then there is some move of the kind ϵ_x which does not appear in any of the sequences of Σ . By definition Σ is not a covering strategy (the ϵ -completeness condition is not verified).
2. We now suppose the existence of a type substitution σ , such that $\langle \Gamma; \gamma \rangle = \langle \Gamma'; \gamma' \rangle \cdot \sigma$ and which contains a substitution of the kind $[a \mapsto \alpha]$, where α is not an atomic type. Let us write $\mathbf{sub} : A' \mapsto \mathcal{F}(A)$ the corresponding homomorphism of arena. Let us therefore consider a move $m \in M'$ such that $\tau(m) = a$; because α is not atomic there is at least one move $n \in \mathbf{sub}(m)$ such that $\widehat{\mathbf{sub}(m)} \vdash n$:

- if $\widehat{\mathbf{sub}(m)}$ is an O -move. First, if there is no sequence S in Σ in which m appears, the O -move completeness condition is violated and Σ cannot be a covering strategy in A . So let' suppose

$$\llbracket \Sigma \rrbracket_A = C[\llbracket (\widehat{\mathbf{sub}(m)}, j) \cdot (\widehat{\mathbf{sub}(m')}, k) [\mathbb{T}_{\Sigma_1}, \dots, \mathbb{T}_{\Sigma_n}], V \rrbracket_A]$$

$$\llbracket \Sigma \rrbracket_A = C[\lambda x_1 \dots x_p. y \llbracket \mathbb{T}_{\Sigma_1}, V' \rrbracket_A \dots \llbracket \mathbb{T}_{\Sigma_n}, V' \rrbracket_A]$$

for some $j \in \mathbb{N}$. Because $\widehat{\mathbf{sub}(m)}$ has at least one child, $p \geq 1$. By definition of the interpretation of Σ , there exists $i \in [1, p]$ and a move $n \in \mathbf{sub}(m)$ such that $((n, j), x_i)$ in V' . But n is not a move in A' and $\llbracket (m, j) \cdot (m', k) [\mathbb{T}_{\Sigma_1}, \dots, \mathbb{T}_{\Sigma_n}] \rrbracket_{A'} = y \llbracket \mathbb{T}_{\Sigma_1}, V' \rrbracket_{A'} \dots \llbracket \mathbb{T}_{\Sigma_n}, V' \rrbracket_{A'}$. It implies $\llbracket \Sigma \rrbracket_{A'} \neq N$ and we obtain a contradiction with (*).

- if $\widehat{\mathbf{sub}(m)}$ is a P -move. First, if there is a sequence of the shape $S.m$ in Σ , then

$$\llbracket \Sigma \rrbracket_A = C[\llbracket (\widehat{\mathbf{sub}(m)}, j) [\mathbb{T}_{\Sigma_1} \dots \mathbb{T}_{\Sigma_p}], V \rrbracket_A] = C[xN_1 \dots N_p]$$

where $p > 0$. But $\llbracket (\widehat{\mathbf{sub}(m)}, j) \mathbb{T}_{\Sigma_1} \dots \mathbb{T}_{\Sigma_p}, V \rrbracket_{A'}$ is equal to x and again there is a contradiction with (*). If m appears in no sequence of Σ , then there is no sequence $S.m.n.S'$ in Σ on A . Again, the strategy cannot be a covering strategy on A , as n would be an O -move not appearing in any sequence of Σ .

3. so let us suppose σ is only made of substitutions of the form $[a \mapsto b]$, where a and b are atomic types. Then the homomorphism associated to σ is an isomorphism. Let us suppose σ is not a renaming: there exist two distinct atomic types a_1 and a_2 such that $\sigma(a_1) = \sigma(a_2) = b$. We consider two moves $m_1, m_2 \in M'$ such that $\tau'(m_1) = a_1$ and $\tau'(m_2) = a_2$
 - if m_1, m_2 are O -moves in A' . Suppose the first three conditions of the definition of a covering strategy are verified for the strategy Σ on A . It again ensures the existence of some sequences $S_1.\mathbf{sub}(m_1).n_1$ and $S_2.\mathbf{sub}(m_2).n_2$ in Σ . By definition of a typing strategy, all P -moves n such that $\mathbf{sub}(m_i)$ is in \triangleleft_{Σ}^n (for $i = \{1, 2\}$) must verify $\tau(n) = \tau(\mathbf{sub}(m_i)) = b$. If the least typing constraint condition is verified we obtain $n_1 R_{\Sigma}^* n_2$. Let us suppose $n_1 R_{\Sigma} n_2$ for simplicity; then there is a move m in $\triangleleft_{\Sigma}^{n_1} \cap \triangleleft_{\Sigma}^{n_2}$ and n_1 and n_2 must verify $\tau'(\mathbf{sub}^{-1}(n_1)) = \tau'(\mathbf{sub}^{-1}(n_2))$ in A' , as $\llbracket \Sigma \rrbracket_{A'} = N$. This contradicts the hypothesis $a_1 \neq a_2$.

- suppose m_1 is a P -move in A' ; if $\text{sub}(m_1)$ and $\text{sub}(m_2)$ appear in some sequences of Σ , the proof is identical to the previous case. Otherwise, let us again suppose the first three conditions of a covering strategy are verified on Σ and the absence of a sequence $S_1.\text{sub}(m_1)$ in Σ . Suppose the least typing constraint condition is also verified. Then $\triangleleft_{\Sigma}^{\text{sub}(m_1)} = \emptyset$ which would imply $\tau(\text{sub}(m_1)) \neq \tau(m)$, for every move $m \neq m_1$ in M , and in particular for m_2 , if m_2 is a P -move; otherwise there is such a P -move m such that $m_2 \in \triangleleft_{\Sigma}^m$.

Lemma 11. *Let us consider a typing $\langle \Gamma; \gamma \rangle$ and its arena A . Suppose there is a strategy Σ on A such that $\llbracket \Sigma \rrbracket = N$. If Σ is not a covering strategy on A , then $\langle \Gamma; \gamma \rangle$ is not the principal typing of N*

Proof. Let us suppose Σ is not a covering strategy on A :

- if the winning condition is not verified, according to Lemma 2, $\langle \Gamma; \gamma \rangle$ is not a typing for N .
- if the ϵ -completeness condition is not verified, the domain of Γ extends $FV(N)$.
- if the O -completeness condition is not verified, there is an O -move n which does not appear in any of the sequences S of Σ . Also, given the P -move m such that $m \vdash n$, this move cannot appear in any sequence of Σ (otherwise n and any move enabled by m would appear in some $S \in \Sigma$ by definition of a winning strategy). Let us consider the prefix-closed set of moves $E = \{m \cdot s \in M \mid s \in \mathbb{N}^*\}$. Then (E, τ) is a sub-arena of A ; given $\sigma_A((E, \tau)) = \alpha$, we can build a new typing $\langle \Delta; \delta \rangle$ which arena is obtained by replacing E by the move m labelled with a fresh atomic type a , and Σ is still a strategy which is interpreted as N . The newly obtained typing $\langle \Delta; \delta \rangle$ verifies $\langle \Delta; \delta \rangle \cdot [a \mapsto \alpha] = \langle \Gamma; \gamma \rangle$, and the latest is not the principal typing of N .
- if the least typing constraint condition is not verified, there exist m_1 and m_2 in M^P such that $\tau(m_1) = \tau(m_2) = a$, but $m_1 R_{\Sigma}^* m_2$ is not verified. For $i \in \{1, 2\}$, we consider the sets $E_i = \{m \mid m_i R_{\Sigma}^* m\} \cup \{n \mid m_i R_{\Sigma}^* m \text{ and } Snm \in \Sigma\}$; then $E_1 \cap E_2 = \emptyset$ and we can build a new arena (M, τ') which only differs from A by labeling of the moves in E_1 with a fresh atomic type a_1 , and the moves in E_2 with a fresh atomic type a_2 . The typing $\langle \Delta; \delta \rangle$ of N associated to this new arena verifies $\langle \Gamma; \gamma \rangle = \langle \Delta; \delta \rangle \cdot [a_1 \mapsto a, a_2 \mapsto a]$ and again $\langle \Gamma; \gamma \rangle$ cannot be the principal typing of N .

Theorem 2 *Given a term N in normal form, a typing pair $\langle \Gamma; \gamma \rangle$ and the corresponding arena $A = (M, \tau)$. The two following properties are equivalent*

1. *there is a covering strategy Σ on A such that $N = \llbracket \Sigma \rrbracket$*
2. *$\langle \Gamma; \gamma \rangle$ is the principal typing of $N = \llbracket \Sigma \rrbracket$ and N is η -long in $\langle \Gamma; \gamma \rangle$*

Proof. The equivalence between the two following propositions:

- 1'. Σ is not a covering strategy on A
- 2'. $\langle \Gamma; \gamma \rangle$ is not the principal typing of N

is given by Lemma 10 and 11

D Proofs of Theorems in section 4.1

Lemma 4 *Let us consider a negatively non-duplicating typing $\langle \Gamma; \gamma \rangle$ and the associated arena $A = (M, \tau)$. Then every sequence in a winning strategy Σ on A has at most one occurrence of each O -move m .*

Proof. Let us suppose this is wrong: there is a strategy Σ which contains a sequence S of the form $S_1.m.n_1.S_2.m.n_2$ where m is a O -move. Because the typing is negatively non-duplicating, there is at most one P -move n of type $\tau(m)$ in the arena. This implies $n_1 = n_2$. Hence, we can consider a sequence of moves S in a winning strategy as a sequence of pairs of moves $M^O \times M^P$ written $(m, f(m))$, f being a function which associates to a move $m \in M^O$ the unique move $n \in M^P$ such that $\tau(n) = \tau(m)$ if it exists. Then for every O -move m' such that there exists $i \in \mathbb{N}$ and $f(m) \cdot i = m' \in M$, the sequence $S_1.(m, f(m)).S_2.(m, f(m)).(m', f(m'))$ belongs to Σ by definition of a winning strategy. There is therefore an infinite number of sequences of the form $S_1.((m, f(m)).S_2)^*$ (where $((m, f(m)).S_2)^*$ denotes a sequence made of an arbitrary number of concatenations of $(m, f(m)).S_2$) in Σ , which is absurd.

Given a justified sequence $S.(n, i)$, we note $J(S.(n, i))$ the value inductively defined as:

- $J((\epsilon, 0)) = 0$.
- $J(S_1.(m, i).S_2.(n, i)) = 1 + J(S_1.(m, i))$ if n is a P -move.
- $J(S.(m, i)) = J(S)$ if m is a O -move.

Then $J(S.(n, i))$ corresponds to the number of enabling $m \vdash m'$ (where $m \in M^O$ and $m' \in M^P$ from (n, i) to $(\epsilon, 0)$ in $S.(n, i)$).

Lemma 12. *Let us consider an arena $A = (M, \tau)$, a strategy Σ on A and the term $N = \llbracket \Sigma \rrbracket$.*

If there is a P -move $n \in M^P$ such that x_1 and x_2 are realizations of n in $\llbracket \Sigma \rrbracket$, then $x_1 \approx_N x_2$.

Proof. Let us suppose the two variables x_1 and x_2 are realizations of the same P -move n in $\llbracket \Sigma \rrbracket$; this is written as:

$$N = \begin{cases} C_1[\llbracket (n, i_1).T_1 \dots T_n, V_1 \rrbracket] = C_1[x_1[\llbracket T_1, V_1 \rrbracket] \dots \llbracket T_n, V_1 \rrbracket] \\ C_2[\llbracket (n, i_2).T_1' \dots T_n', V_2 \rrbracket] = C_2[x_2[\llbracket T_1', V_2 \rrbracket] \dots \llbracket T_n', V_2 \rrbracket] \end{cases}$$

Let us consider $S_1.(n, i_1)$ and $S_2.(n, i_2)$ the sequences in Σ which contain the occurrences of n in the equation above. Let us prove $x_1 \approx_N x_2$ by induction on $p = \max(J(S_1.(n, i_1)), J(S_2.(n, i_2)))$:

- If $p = 1$, then $S_1 = S_2 = \epsilon, i_1 = i_2 = 0$ and $x_1 = x_2$ are occurrences of the same variable.

- We suppose the property to be true for $p > 1$. Because there is a unique O -move m such that $m \vdash n$, $S'_1.(m, i_1) \sqsubseteq S_1$ and $S'_2.(m, i_2) \sqsubseteq S_2$ are verified. Now, because $p > 1$, there is a P -move n' which enables m , and integers j_1 and j_2 such that $S'_1 = S''_1.(n', j_1)$ and $S'_2 = S''_2.(n', j_2)$. By definition $\max(J(S''_1.(n', j_1)), J(S''_2.(n', j_2))) = \max(J(S_1.(n, i_1)), J(S_2.(n, i_2))) - 1$. Hence, given y_1 and y_2 the variables which correspond to the realizations of these two occurrences of n' in $[\Sigma]$, we obtain by induction hypothesis $y_1 \approx y_2$. Finally

$$\begin{aligned} N &= C'_k[(n, j_k)\mathbb{T}_{k,1} \dots \mathbb{T}_{k,j} \dots \mathbb{T}_{k,q}, W_k] \\ &= C'_k[y_k[\mathbb{T}_{k,1}, W_k] \dots [\mathbb{T}_{k,j}, W_k] \dots [\mathbb{T}_{k,q}, W_k]] \\ &= C'_k[y_k[\mathbb{T}_{k,1}, W_k] \dots \lambda x_{1,k}^j \dots x_{i-1,k}^j x_k x_{i+1,k}^j \dots x_{r,k}^j \cdot N_{k,j} \dots [\mathbb{T}_{k,q}, W_k]] \end{aligned}$$

for $k \in \{1, 2\}$. Therefore $y_k \mathbb{I}_{j_i}^N x_k$ which leads to $x_1 \approx_N x_2$.

Lemma 13. *Let us consider an arena $A = (M, \tau)$, a strategy Σ on A and the term $N = [\Sigma]$.*

If two variables $x_1, x_2 \in V(N)$ verify $x_1 \approx_N x_2$, then there is a P -move $n \in M^P$ which realizes both x_1 and x_2 in $[\Sigma]$

Proof. Let us consider two variables x_1 and x_2 in N , such that $x_1 \approx_N x_2$. We first recall that variables have distinct name in N by definition of the interpretation of a strategy. We proceed by induction on the definition of \approx_N :

- if $x_1 = x_2 = x$. According to the definition of the interpretation of Σ ,

$$N = \begin{cases} C_1[[(n_1, i_1). \mathbb{T}_1 \dots \mathbb{T}_n, V_1]] = C_1[x_1 N_1 \dots N_n] \\ C_2[[(n_2, i_2). \mathbb{T}'_1 \dots \mathbb{T}'_n, V_2]] = C_2[x_2 P_1 \dots P_n] \end{cases}$$

and $n_1 = n_2 = n$ in M^P where n is the unique move such that $((n, i), x)$ appears in $V_1 \cap V_2$, for some $i \in \mathbb{N}$.

- if there are two variables z_1 and z_2 such that $z_1 \approx_N z_2$ and naturals i, j such that $z_1 \mathbb{I}_{i_j}^N x_1$ and $z_2 \mathbb{I}_{i_j}^N x_2$. By induction hypothesis, there is a P -move n which realizes z_1 and z_2 in Σ :

$$N = \begin{cases} C_1[[(n, i_1). \mathbb{T}_1 \dots \mathbb{T}_n, V_1]] = C_1[z_1 N_1 \dots N_n] \\ C_2[[(n, i_2). \mathbb{T}'_1 \dots \mathbb{T}'_n, V_2]] = C_2[z_2 P_1 \dots P_n] \end{cases}$$

Because $z_1 \mathbb{I}_{i_j}^N x_1$ and $z_2 \mathbb{I}_{i_j}^N x_2$, we know that $N_i = \lambda y_1 \dots y_{j-1} x_1 y_{j+1} \dots y_p \cdot N'$ and $P_i = \lambda y'_1 \dots y'_{j-1} x_2 y'_{j+1} \dots y'_p \cdot P'$. But by definition of the interpretation, x_1 and x_2 must be realizations of the move $(n \cdot i) \cdot j$.

Lemma 14. *Let us consider two sequences $S_1.m_1.n$ and $S_2.m_2.n$ in a strategy Σ in an arena A . Suppose that A is associated to a negatively non-duplicating typing. Then for every sequence S , the sequence $S_1.m_1.n.S$ is in Σ iff $S_2.m_2.n.S$ is in Σ .*

Proof. We proceed by induction on $|S|/2$:

- if $|S| = 0$ the result is already given.
- suppose the lemma to be true for $|S|/2 \in \mathbb{N}$ and the existence of a sequence $S_i.m_i.n.S.m'.n' \in \Sigma$, for some i in $\{1, 2\}$. Given $j \neq i \in \{1, 2\}$, there must be a sequence of the form $S_j.m_j.n.S.m'.n''$ in Σ by the definition of a winning strategy. Because the typing is negatively non-duplicating, $n' = n''$.

Theorem 4 *If a negatively non-duplicating pair $\langle \Gamma; \gamma \rangle$ types a term N then N is copy(1).*

Proof. Let us consider the arena $A = (M, \tau)$ associated to $\langle \Gamma; \gamma \rangle$, a negatively non-duplicating typing. By hypothesis, there is a strategy Σ on A , such that $N = \llbracket \Sigma \rrbracket$. According to Theorem 3, Σ is the unique winning strategy in A . Suppose $N = C_1[N_1] = C_2[N_2]$, such that N_1 and N_2 are of the same type δ in $\langle \Gamma; \gamma \rangle$ and such that there is no subterms $N'_1 = C'_1[N_1]$, $N'_2 = C'_2[N_2]$ of the same type. The general forms of N_1 and N_2 are respectively $\lambda x_1 \dots x_q.z_1 P_1 \dots P_{p_1}$, and $\lambda y_1 \dots y_q.z_2 Q_1 \dots Q_{p_2}$, because N is η -long for this typing. Moreover, we can write $\llbracket \Sigma \rrbracket$ as:

$$\llbracket \Sigma \rrbracket = \begin{cases} C_1[\llbracket \mathbb{T}_1, V_1 \rrbracket] = C_1[\llbracket m_1.n_1.\mathbb{T}_{1,1} \dots \mathbb{T}_{1,p_1}, V_1 \rrbracket] = C_1[N_1] \\ C_2[\llbracket \mathbb{T}_2, V_2 \rrbracket] = C_2[\llbracket m_2.n_2.\mathbb{T}_{2,1} \dots \mathbb{T}_{2,p_2}, V_2 \rrbracket] = C_2[N_2] \end{cases}$$

where m_1, m_2 are O -moves, and n_1, n_2 are P -moves. Because N_1 and N_2 are of the same type, we obtain $\tau(m_1) = \tau(m_2)$. Moreover, $\langle \Gamma; \gamma \rangle$ is negatively non-duplicating which implies $n_1 = n_2 = n$; considering $p \in \mathbb{N}$ the number of moves enabled by the move n in the arena A , we obtain $p_1 = p_2 = p$. Let us proceed by induction on the trees \mathbb{T}_1 and \mathbb{T}_2

- if $\mathbb{T}_1 = m_1.n$ and $\mathbb{T}_2 = m_2.n$, then $N_1 = \lambda x_1 \dots x_q.z_1$ and $N_2 = \lambda y_1 \dots y_q.z_2$, for $q \geq 0$. Then z_1 and z_2 come from the interpretation of two occurrences of n in Σ and according to Lemma 5, $z_1 \approx_N z_2$. Moreover, the subterms N_1 and N_2 are of the same type γ . If $q > 0$, for every i in $[1, q]$ the variables x_i and y_i are of the same type; these variables correspond to the P -moves $m_1 \cdot i$ and $m_2 \cdot i$ in the interpretation of Σ . Because they are of the same type, $\tau(m_1 \cdot i) = \tau(m_2 \cdot i)$; but the typing being negatively non-duplicating, $m_1 \cdot i = m_2 \cdot i$ must stand for each $i \in [1, n]$ which implies $m_1 = m_2 = m$. This forces $x_i \approx_N y_i$ for every $i \in [1, n]$; so finally $N_1 \approx_N N_2$.
- otherwise, according to Lemma 14, $\mathbb{T}_{1,i} = \mathbb{T}_{2,i}$ for every $i \in [1, p]$. By induction hypothesis on the trees \mathbb{T}_1 and \mathbb{T}_2 , for every $i \in [1, p]$, the interpretation of the trees $\mathbb{T}_{1,i}$ and $\mathbb{T}_{2,i}$ leads to the terms P_i and Q_i respectively, and $P_i \approx_N Q_i$. Because $z_1 \approx_N z_2$ and $x_j \approx_N y_j$ for every $j \in [1, q]$ by hypothesis of induction, $N_1 \approx_N N_2$.