Homotopy commutative algebra and 2-nilpotent Lie algebra

Michel Dubois-Violette, Todor Popov

Homotopy commutative algebra and 2-nilpotent Lie algebra

Michel Dubois-Violette and Todor Popov

Abstract The homotopy transfer theorem due to Tornike Kadeishvili induces the structure of a homotopy commutative algebra, or C_∞-algebra, on the cohomology of the free 2-nilpotent Lie algebra. The latter C_∞-algebra is shown to be generated in degree one by the binary and the ternary operations.

1 Introduction

Every Universal Enveloping Algebra (UEA) $U\mathfrak{g}$ of a finite dimensional positively graded Lie algebra $\mathfrak{g}$ belongs to the class of Artin-Schelter regular algebras(see e.g. [4]). As every finitely generated graded connected algebra, $U\mathfrak{g}$ has a free minimal resolution which is canonically built from the data of its Yoneda algebra $\mathscr{E} := \mathrm{Ext}_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})$. By construction the Yoneda algebra $\mathscr{E}$ is isomorphic (as algebra) to the cohomology of the Lie algebra (with coefficients in the trivial representation provided by the ground field $\mathbb{K}$)

$$\mathscr{E} = \mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbb{K},\mathbb{K}) \cong H^\bullet(\mathfrak{g},\mathbb{K}) \tag{1}$$

equipped with wedge product between cohomological classes in $H^\bullet(\mathfrak{g},\mathbb{K})$.

The homotopy transfer theorem of Tornike Kadeishvili [7] implies that the Yoneda algebra $\mathscr{E} = \mathrm{Ext}^\bullet_{U\mathfrak{g}}(\mathbb{K},\mathbb{K})$ has the structure of homotopy associative algebra, or A_∞-algebra. Since $\mathscr{E} \cong H^\bullet(\mathfrak{g},\mathbb{K})$ is the cohomology of the exteriour algebra

Michel Dubois-Violette
Laboratoire de Physique Théorique, UMR 8627, Université Paris XI, Bâtiment 210, F-91 405 Orsay Cedex, France e-mail: Michel.Dubois-Violette@th.u-psud.fr

Todor Popov
Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko chaussée, 1784 Sofia, Bulgaria e-mail: tpopov@inrne.bas.bg

$\bigwedge \mathfrak{g}^*$ which is graded-commutative, it has the structure of homotopy commutative and associative algebra, or C_∞-algebra.

Throughout the text $\mathfrak{g}$ will be the free 2-nilpotent graded Lie algebra, with degree one generators in the finite dimensional vector space V over a field $\mathbb{K}$ of characteristic 0,

$$\mathfrak{g} = V \oplus \bigwedge^2 V\ .$$

The UEA $U(V \oplus \wedge^2 V)$ arises naturally in physics in the universal Fock-like space of the parastatistics algebra introduced by H.S. Green [5](see also [3]). Here we will concentrate on the case when V is an ordinary (even) vector space V, when the algebra $U\mathfrak{g}$ is the parafermionic algebra.

The aim of this note is to describe the Yoneda algebra $\mathscr{E}$ of the UEA $U\mathfrak{g}$, i.e., the cohomology $H^\bullet(\mathfrak{g},\mathbb{K})$ with its C_∞-structure induced by the isomorphism (1) through the homotopy transfer.

The cohomology space $H^\bullet(\mathfrak{g},\mathbb{K})$ has a natural $GL(V)$-action. The decomposition of the $GL(V)$-module $H^\bullet(\mathfrak{g},\mathbb{K})$ into irreducible Schur modules V_λ is known since the work of Józefiak and Weyman [6]; it contains all $GL(V)$-modules with self-conjugated Young diagrams $\lambda = \lambda'$ once and exactly once. The decomposition of $\mathscr{E} = H^\bullet(\mathfrak{g},\mathbb{K})$ into Schur modules provides a powerful tool to handle its C_∞-algebra structure.

2 Artin-Schelter regularity

Let $\mathfrak{g}$ be the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \oplus \bigwedge^2 V$ generated by the finite dimensional vector space V having Lie bracket

$$[x,y] := \begin{cases} x\wedge y & x,y \in V \\ 0 & \text{otherwise} \end{cases}\ . \tag{2}$$

We denote the Universal Enveloping Algebra $U\mathfrak{g}$ by PS and will refer to it as *parastatistics algebra* (by some abuse[1]). The parastatistics algebra $PS(V)$ generated in V is graded

$$PS(V) := U\mathfrak{g} = U(V \oplus \bigwedge^2 V) = T(V)/([[V,V],V])\ .$$

We shall write simply PS when the space of generators V is clear from the context.

Artin and Schelter [1] introduced a class of regular algebras sharing some "good" homological properties with the polynomial algebra $\mathbb{K}[V]$. These algebras were dubbed Artin-Schelter regular algebras (AS-regular algebra for short).

Definition 1. (AS-regular algebras) A connected graded algebra $\mathscr{A} = \mathbb{K} \oplus \mathscr{A}_1 \oplus \mathscr{A}_2 \oplus \dots$ is called Artin-Schelter regular of dimension d if

(i) $\mathscr{A}$ has finite global dimension d,

(ii) $\mathscr{A}$ has finite Gelfand-Kirillov dimension,

[1] Strictly speaking $PS(V)$ is the creation parastatistics algebra, closed by creation operators alone.

(iii) $\mathscr{A}$ is Gorenstein, i.e., $\mathrm{Ext}^i_{\mathscr{A}}(\mathbb{K},\mathscr{A}) = \delta^{i,d}\mathbb{K}$.

A general theorem claims that the UEA of a finite dimensional positively graded Lie algebra is an AS-regular algebra of global dimension equal to the dimension of the Lie algebra [4]. Hence the parastatistics algebra PS is AS-regular of global dimension $d = \frac{\dim V(\dim V+1)}{2}$. In particular the finite global dimension of PS implies that the ground field $\mathbb{K}$ has a minimal resolution $P_\bullet$ by projective left PS-modules P_n

$$P_\bullet : \qquad 0 \to P_d \to \cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} \mathbb{K} \to 0 \; . \tag{3}$$

Here $\mathbb{K}$ is a trivial left PS-module, the action being defined by the projection ε onto $PS_0 = \mathbb{K}$. Since PS is positively graded and, in the category of positively graded modules over connected locally finite graded algebras, projective module is the same as free module [2], we have $P_n \cong PS \otimes E_n$ where E_n are finite dimensional vector spaces.

The minimal projective resolution is unique (up to an isomorphism). Minimality implies that the complex $\mathbb{K} \otimes_{PS} P_\bullet$ has "zero differentials" hence

$$H_\bullet(\mathbb{K} \otimes_{PS} P_\bullet) = \mathbb{K} \otimes_{PS} P_\bullet = E_n \; .$$

One can calculate the derived functor $\mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K})$ using the resolution $P_\bullet$, it yields

$$\mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K}) = E_n \; . \tag{4}$$

The data of a minimal resolution of $\mathbb{K}$ by free PS-modules provides an easy way to find $\mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K})$. Conversely if the spaces $\mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K})$ are known, then one can constuct a minimal free resolution of $\mathbb{K}$.

The Gorenstein property guarantees that when applying the functor $\mathrm{Hom}_{PS}(-,PS)$ to the minimal free resolution $P_\bullet$ we get another minimal free resolution $P^\bullet := \mathrm{Hom}_{PS}(P_\bullet, PS)$ of $\mathbb{K}$ by right PS-modules

$$P^\bullet : \quad 0 \leftarrow \mathbb{K} \leftarrow P_d^{'} \leftarrow \cdots \leftarrow P_n^{'} \leftarrow \cdots \leftarrow P_2^{'} \leftarrow P_1^{'} \leftarrow P_0^{'} \leftarrow 0 \tag{5}$$

with $P_n^{'} \cong E_n^* \otimes PS$. Note that by construction $E_n^* = \mathrm{Ext}^n_{PS}(\mathbb{K},\mathbb{K})$, thus one has vector space isomorphisms [2]

$$E_n \cong E_n^* \cong \mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K}) \cong \mathrm{Ext}^n_{PS}(\mathbb{K},\mathbb{K}) \; . \tag{6}$$

The Gorenstein property is an analog of the Poincaré duality, it implies $E^*_{d-n} \cong E_n$. The finite global dimension d of PS and the Gorenstein condition imply that its Yoneda algebra

$$\mathscr{E}^\bullet := \mathrm{Ext}^\bullet_{PS}(\mathbb{K},\mathbb{K}) \cong \bigoplus_{n=0}^{d} E_n^*$$

is Frobenius [10]. More on Gorenstein property you can find in the first autor's lecture "Poincaré duality for Koszul algebras" in the present volume.

3 Homology and cohomology of $\mathfrak{g}$

A non-minimal projective(in fact free) resolution of $\mathbb{K}$, $C(\mathfrak{g}) \stackrel{\varepsilon}{\to} \mathbb{K}$ is given by the standard Chevalley-Eilenberg chain complex $C_\bullet(\mathfrak{g}) = (U\mathfrak{g} \otimes_{\mathbb{K}} \wedge^p \mathfrak{g}, d_p)$ with differential maps

$$d_p(u \otimes x_1 \wedge \ldots \wedge x_p) = \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_p$$
$$+ \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_p \qquad (7)$$

The resolution $C_\bullet(\mathfrak{g})$ calculates the homologies of the derived complex $\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})$

$$E_n = \mathrm{Tor}_n^{PS}(\mathbb{K}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})) = H_n(\mathfrak{g}, \mathbb{K}) \ ,$$

coinciding with the homologies $H_n(\mathfrak{g}, \mathbb{K})$ of the Lie algebra $\mathfrak{g}$ with trivial coefficients. The derived complex $\mathbb{K} \otimes_{PS} C_\bullet(\mathfrak{g})$ is the chain complex with degrees $\bigwedge^\bullet \mathfrak{g} = \mathbb{K} \otimes_{PS} PS \otimes \bigwedge^\bullet \mathfrak{g}$ and differentials $\partial_p := id \otimes_{PS} d_p : \bigwedge^p \mathfrak{g} \to \bigwedge^{p-1} \mathfrak{g}$. One has

$$\bigwedge^p \mathfrak{g} = \bigwedge^p (V \oplus \bigwedge^2 V) = \bigoplus_{s+r=p} \bigwedge^s (\bigwedge^2 V) \otimes \bigwedge^r (V) \qquad (8)$$

and differentials $\partial_{p=r+s} : \bigwedge^s(\bigwedge^2 V) \otimes \bigwedge^r(V) \to \bigwedge^{s+1}(\bigwedge^2 V) \otimes \bigwedge^{r-2}(V)$ are given by

$$\partial_p : \ e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \ \otimes \ e_1 \wedge \ldots \wedge e_r \mapsto$$
$$\sum_{i<j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \ \otimes \ e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_r \ .$$

The differential ∂ is induced by the Lie bracket $[\cdot,\cdot] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$, it identifies a pair of degree 1 generators $e_i, e_j \in$ with one degree 2 generator $e_{ij} := (e_i \wedge e_j) = [e_i, e_j]$. The differential ∂_p is the extension of $\partial_2 := -[\cdot,\cdot]$ as coderivation on $\bigwedge^p \mathfrak{g}$.

The dual cochain complex $\mathrm{Hom}_{PS}(C(\mathfrak{g}), \mathbb{K}) = (\bigwedge^\bullet \mathfrak{g}^*, \delta)$ calculates cohomology[2]

$$E_n^* = \mathrm{Ext}^n_{PS}(\mathbb{K}, \mathbb{K}) \cong H^n(\mathrm{Hom}_{PS}(C(\mathfrak{g}), \mathbb{K})) = H^n(\mathfrak{g}, \mathbb{K}) \ . \qquad (9)$$

The coboundary map $\delta^p : \bigwedge^p \mathfrak{g}^* \to \bigwedge^{p+1} \mathfrak{g}^*$ is transposed to the differential ∂_{p+1}

$$\delta^p : \ e^*_{i_1 j_1} \wedge \ldots \wedge e^*_{i_s j_s} \ \otimes \ e^*_{l_1} \wedge \ldots \wedge e^*_{l_r} \mapsto \qquad (10)$$
$$\sum_{k=1}^{s} \sum_{i_k < j_k} (-1)^{i+j} e^*_{i_1 j_1} \wedge \ldots \wedge \hat{e}^*_{i_k j_k} \wedge \ldots \wedge e^*_{i_s j_s} \ \otimes \ e^*_{i_k} \wedge e^*_{j_k} \wedge e^*_{l_1} \wedge \ldots \wedge \ldots \wedge e^*_{l_r} \ ,$$

it is (up to a conventional sign) the extension as derivation of the dualization of the Lie bracket $\delta^1 := [\cdot,\cdot]^* : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$.Thus the algebra $(\bigwedge^\bullet \mathfrak{g}^*, \delta)$ equipped with δ is a *(graded-)commutative* DGA.

[2] In the presence of metric one has $\delta := \partial^*$(see below)

4 Homology of $\mathfrak{g}$ as a $GL(V)$-module

An irreducible polynomial $GL(V)$-module V_λ is called Schur module, it has a basis labelled by semistandard Young tableaux which are fillings of the Young diagram λ with the numbers of the set $\{1,\ldots,\dim V\}$. The action of the linear group $GL(V)$ on the space V of the generators of the Lie algebra $\mathfrak{g}$ induces a $GL(V)$-action on the UEA $PS = U\mathfrak{g} \cong S(V \oplus \Lambda^2 V)$ and on the space $\bigwedge^\bullet \mathfrak{g} \cong \bigwedge^\bullet (V \oplus \bigwedge^2 V)$.

In the presence of metric g one has an identification $V \overset{g}{\cong} V^*$, and $\bigwedge^\bullet \mathfrak{g} \overset{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$. The adjoint operator $\partial_p^* : \bigwedge^p \mathfrak{g} \to \bigwedge^{p+1}\mathfrak{g}$ is defined by $g(\partial_p^* v, w) = g(v, \partial_{p+1} w)$. It turns out that the action of ∂_p^* always takes the form (similar to the action of δ^p)

$$\begin{aligned} &\partial_p^* :\ e_{i_1 j_1} \wedge \ldots \wedge e_{i_s j_s} \otimes e_{l_1} \wedge \ldots \wedge e_{l_r} \mapsto \\ &\sum_{k=1}^{s} \sum_{i_k < j_k} (-1)^{i+j} e_{i_1 j_1} \wedge \ldots \wedge \hat{e}_{i_k j_k} \wedge \ldots \wedge e_{i_s j_s} \otimes e_{i_k} \wedge e_{j_k} \wedge e_{l_1} \wedge \ldots \wedge \ldots \wedge e_{l_r}\ , \end{aligned} \tag{11}$$

It is obvious that the maps ∂ and ∂^* both commute with the $GL(V)$-action. The *Laplacian* $\Delta = \oplus_{p\geq 0} \Delta_p$ of the pair $(\mathfrak{g}, g)$ is defined to be the self-adjoint operator

$$\Delta_p = \partial_{p+1}\partial_{p+1}^* + \partial_p^* \partial_p \in \mathrm{End}(\bigwedge^{\mathrm{p}} \mathfrak{g})\ .$$

Its kernel is a complete set of representatives for the homology classes in $H_p(\mathfrak{g}, \mathbb{K})$

$$\ker \Delta_p \cong H_p(\mathfrak{g}, \mathbb{K})\ .$$

The decomposition of the $GL(V)$-module $H_n(\mathfrak{g}, \mathbb{K})$ into irreducible polynomial representations V_λ is given by the following theorem;

Theorem 1 (Józefiak and Weyman [6], Sigg [11]). *The homology $H_\bullet(\mathfrak{g}, \mathbb{K})$ of the 2-nilpotent Lie algebra $\mathfrak{g} = V \oplus \bigwedge^2 V$ decomposes into irreducible $GL(V)$-modules*

$$H_n(\mathfrak{g}, \mathbb{K}) = H_n(\bigwedge^\bullet \mathfrak{g}, \partial) \cong \mathrm{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})(V) \cong \bigoplus_{\lambda:\lambda=\lambda'} V_\lambda \tag{12}$$

where the sum is over self-conjugate Young diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

The data $H_n(\mathfrak{g}, \mathbb{K}) = \mathrm{Tor}_n^{PS}(\mathbb{K}, \mathbb{K})$ encodes the minimal free resolution $P_\bullet$ (cf. (3)).

The Euler characteristics of $P_\bullet$ implies an identity about the $GL(V)$-characters

$$ch\,PS(V)\,.\,ch\left(\bigoplus_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} V_\lambda\right) = 1\ .$$

The character of a Schur module V_λ is the Schur function, $ch\,V_\lambda = s_\lambda(x)$. Due to the Poincaré-Birkhoff-Witt theorem $ch\,PS(V) = ch\,S(V \oplus \bigwedge^2 V)$ thus the identity reads

$$\prod_i \frac{1}{(1-x_i)} \prod_{i<j} \frac{1}{(1-x_i x_j)} \sum_{\lambda:\lambda=\lambda'} (-1)^{\frac{1}{2}(|\lambda|+r(\lambda))} s_\lambda(x) = 1\ . \tag{13}$$

But the latter identity is nothing but rewriting of the Littlewood identity [6]. The moral is that the Littlewood identity reflects a homological property of the algebra PS, namely the above particular structure of the minimal projective (free) resolution of $\mathbb{K}$ by PS-modules.

5 Homotopy algebras A_∞ and C_∞

Definition 2. (A_∞-algebra) A homotopy associative algebra, or A_∞-algebra, over $\mathbb{K}$ is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{i\in\mathbb{Z}} A^i$ endowed with a family of graded mappings (operations)

$$m_n : A^{\otimes n} \to A, \qquad \deg(m_n) = 2-n \qquad n \geq 1$$

satisfying the Stasheff identities $\mathbf{SI(n)}$ for $n \geq 1$

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \qquad \mathbf{SI(n)}$$

where the sum runs over all decompositions $n = r+s+t$.

Here we assume the Koszul sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$. We define the shuffle product $Sh_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A^{\otimes p+q}$ throughout the expression

$$(a_1 \otimes \ldots \otimes a_p) \sqcup\!\sqcup (a_{p+1} \otimes \ldots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} sgn(\sigma)\, a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all (p,q)-shuffles $Sh_{p,q}$, i.e., over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q)$.

Definition 3. (C_∞-algebra [7]) A homotopy commutative algebra, or C_∞-algebra, is an A_∞-algebra $\{A, m_n\}$ such that each operation m_n vanishes on non-trivial shuffles

$$m_n\left((a_1 \otimes \ldots \otimes a_p) \sqcup\!\sqcup (a_{p+1} \otimes \ldots \otimes a_n)\right) = 0\ , \qquad 1 \leq p \leq n-1\ . \qquad (14)$$

In particular for m_2 we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a C_∞-algebra such that $m_n = 0$ for $n \geq 3$ is a (super-)commutative DGA.

A morphism of two A_∞-algebras A and B is a family of graded maps $f_n : A^{\otimes n} \to B$ for $n \geq 1$ with $\deg f_n = 1-n$ such that the following conditions hold

$$\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes r}) = \sum_{1\leq r\leq n} (-1)^S m_r(f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_r})$$

where the sum is on all decompositions $i_1 + \ldots + i_r = n$ and the sign on RHS is determined by $S = \sum_{k=1}^{r-1}(r-k)(i_k - 1)$. The morphism f is a *quasi-isomorphism of A_∞-algebras* if f_1 is a quasi-isomorphism. It is strict if $f_i = 0$ for all $i \neq 1$. The identity morphism of A is the strict morphism f such that f_1 is the identity of A.

A morphism of C_∞-algebras is a morphism of A_∞-algebras vanishing on non-trivial shuffles $f_n\left((a_1 \otimes \ldots \otimes a_p) \sqcup\!\sqcup (a_{p+1} \otimes \ldots \otimes a_n)\right) = 0\ ,\ 1 \leq p \leq n-1$.

6 Homotopy Transfer Theorem

Lemma 1. *Every cochain complex (A,d) of vector spaces over a field $\mathbb{K}$ has its cohomology $H^\bullet(A)$ as a deformation retract.*

One can always choose a vector space decomposition of the cochain complex (A,d) such that $A^n \cong B^n \oplus H^n \oplus B^{n+1}$ where H^n is the cohomology and B^n is the space of coboundaries, $B^n = dA^{n-1}$. We choose a homotopy $h : A^n \to A^{n-1}$ which identifies B^n with its copy in A^{n-1} and is 0 on $H^n \oplus B^{n+1}$. The projection p to the cohomology and the cocycle-choosing inclusion i given by $A^n \underset{i}{\overset{p}{\rightleftarrows}} H^n$ are chain homomorphisms (satisfying the additional conditions $hh = 0$, $hi = 0$ and $ph = 0$). With these choices done the complex $(H^\bullet(A),0)$ is a deformation retract of (A,d)

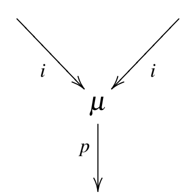
$$h \circlearrowright (A,d) \underset{i}{\overset{p}{\rightleftarrows}} (H^\bullet(A),0) \ , \qquad pi = Id_{H^\bullet(A)} \ , \qquad ip - Id_A = dh + hd \ .$$

Let now (A,d,μ) be a DGA, i.e., A is endowed with an associative product μ compatible with d. The cochain complexes (A,d) and its contraction $H^\bullet(A)$ are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on A can be transferred to an A_∞-structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\bullet(A)$. For a friendly introduction to homotopy transfer theorems in much boarder context we send the reader to the textbook [9], see chapter 9.

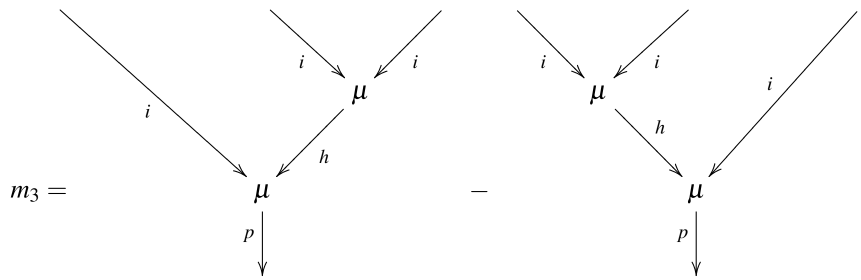
Theorem 2 (Kadeishvili [7]). *Let (A,d,μ) be a (commutative) DGA over a field $\mathbb{K}$. There exists a A_∞-algebra (C_∞-algebra) structure on the cohomology $H^\bullet(A)$ and a $A_\infty(C_\infty)$-quasi-isomorphism $f_i : (\otimes^i H^\bullet(A), \{m_i\}) \to (A, \{d,\mu,0,0,\ldots\})$ such that the inclusion $f_1 = i : H^\bullet(A) \to A$ is a cocycle-choosing homomorphism of cochain complexes. The differential m_1 on $H^\bullet(A)$ is zero ($m_1 = 0$) and m_2 is strictly associative operation induced by the multiplication on A. The resulting structure is unique up to quasi-isomorphism.*

Kontsevich and Soibelman [12] gave an explicit expressions for the higher operations of the induced A_∞-structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion i, the root by the projection p the vertices by the product μ of the (commutative) DGA (A,d,μ) and the internal edges by the homotopy h. The C_∞-structure implies additional symmetries on trees. We will make use of the graphic representation for the binary operation on $H^\bullet(A)$

$$m_2(x,y) := p\mu(i(x),i(y)) \quad \text{or} \quad m_2 =$$



and the ternary one $m_3(x,y,z) = p\mu(i(x), h\mu(i(y),i(z))) - p\mu(h\mu(i(x),i(y)),i(z))$ being the sum of two planar binary trees with three leaves



Theorem 3. *The cohomology $H^\bullet(\mathfrak{g},\mathbb{K}) \cong \mathrm{Ext}^\bullet_{PS}(\mathbb{K},\mathbb{K})$ of the 2-nilpotent graded Lie algebra $\mathfrak{g} = V \otimes \bigwedge^2 V$ is a homotopy commutative algebra which is generated in degree* 1 *(i.e., in $H^1(\mathfrak{g},\mathbb{K})$) by the operations m_2 and m_3.*

Sketch of the proof. Let us choose a metric $g(\,\cdot\,,\,\cdot\,) = \langle\,\cdot\,,\,\cdot\,\rangle$ on the vector space V and an orthonormal basis $\langle e_i, e_j\rangle = \delta_{ij}$. The choice induces a metric on $\bigwedge^\bullet \mathfrak{g} \stackrel{g}{\cong} \bigwedge^\bullet \mathfrak{g}^*$.

Due to the isomorphisms $\mathrm{Tor}^{PS}_n(\mathbb{K},\mathbb{K}) \cong \mathrm{Ext}^n_{PS}(\mathbb{K},\mathbb{K})$ (see eq. 6) and $V \cong V^*$ the theorem 1 implies the decomposition of $H^\bullet(\mathfrak{g},\mathbb{K})$ into irreducible $GL(V)$-modules

$$H^n(\mathfrak{g},\mathbb{K}) \cong H^n(\bigwedge \mathfrak{g}^*, \delta) \cong \mathrm{Ext}^n_{PS}(\mathbb{K},\mathbb{K})(V^*) \cong \bigoplus_{\lambda:\lambda=\lambda'} V_\lambda$$

where the sum is over self-conjugate diagrams λ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$.

In the presence of metric g the differential δ is identified with the adjoint of ∂, $\delta \stackrel{g}{:=} \partial^*$while ∂ plays the role of a homotopy. In view of lemma 1 we have the cohomology $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$ as deformation retract of the complex $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$,

$$pi = Id_{H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)}\ , \qquad ip - Id_{\bigwedge^\bullet \mathfrak{g}^*} = \delta\delta^* + \delta^*\delta\ , \qquad \delta^* \stackrel{g}{=} \partial\ .$$

Here the projection p identifies the subspace $\ker\delta \cap \ker\delta^*$ with $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$, which is the orthogonal complement of the space of the coboundaries $\mathrm{im}\delta$. The cocycle-choosing homomorphism i is Id on $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*)$ and zero on coboundaries.

We apply the Kadeishvili homotopy transfer Theorem 2 for the commutative DGA $(\bigwedge^\bullet \mathfrak{g}^*, \mu, \delta^\bullet)$ and its deformation retract $H^\bullet(\bigwedge^\bullet \mathfrak{g}^*) \cong H^\bullet(\mathfrak{g},\mathbb{K})$ and conclude that the cohomology $H^\bullet(\mathfrak{g},\mathbb{K})$ is a C_∞-algebra.

The Kontsevich and Soibelman tree representations of the operations m_n provide explicit expressions. Let us take μ to be the super-commutative product $\wedge$ on the DGA $(\bigwedge^\bullet \mathfrak{g}^*, \delta^\bullet)$. The projection p maps onto the Schur modules V_λ with $\lambda = \lambda'$.

The binary operation on the degree 1 generators $e_i \in H^1(\mathfrak{g},\mathbb{K})$ is trivial, one gets

$$m_2(e_i, e_j) = p(e_i \wedge e_j) = 0 \qquad p(V_{(1^2)}) = 0.$$

Hence $H^\bullet(\mathfrak{g},\mathbb{K})$ could not be generated in $H^1(\mathfrak{g},\mathbb{K})$ as algebra with product m_2.

The ternary operation m_3 restricted to $H^1(\mathfrak{g},\mathbb{K})$ is nontrivial, indeed one has

$$\begin{aligned} m_3(e_i,e_j,e_k) &= p\left\{e_i\wedge\partial(e_j\wedge e_k)-\partial(e_i\wedge e_j)\wedge e_k\right\} = p\left\{e_{ij}\wedge e_k - e_i\wedge e_{jk}\right\} \\ &= p\left\{(e_{ij}\wedge e_k + e_{jk}\wedge e_i + e_{ki}\wedge e_j) - e_{ki}\wedge e_j\right\} = e_{ik}\wedge e_j \in H^2(\mathfrak{g},\mathbb{K}) \end{aligned}$$

The completely antisymmetric combination in the brackets $(\ldots)$ spans the Schur module $V_{(1^3)}$, $p(e_{ij}\wedge e_k + e_{jk}\wedge e_i + e_{ki}\wedge e_j) = 0$ yields a Jacobi-type identity. The monomials $e_{ij}\wedge e_k$ modulo $V_{(1^3)}$ span a Schur module $V_{(2,1)}\in H^2(\mathfrak{g},\mathbb{K})$ with basis in bijection with the semistandard Young tableaux $e_{ik}\wedge e_j \leftrightarrow \begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array}$ and $e_{ij}\wedge e_k \leftrightarrow \begin{array}{|c|c|}\hline i & k \\ \hline j \\ \cline{1-1}\end{array}$.

We check the symmetry condition on ternary operation m_3 in C_∞-algebra; indeed m_3 vanishes on the (signed) shuffles $Sh_{1,2}$ and $Sh_{2,1}$

$$m_3(e_i\sqcup e_j\otimes e_k) = m_3(e_i,e_j,e_k) - m_3(e_j,e_i,e_k) + m_3(e_j,e_k,e_i) = 0 = m_3(e_i\otimes e_j\sqcup e_k).$$

It is important that in the complexes $(\bigwedge^p\mathfrak{g},\partial_p)$ and $(\bigwedge^p\mathfrak{g}^*,\delta^p)$ two different degees are involved; one is the homological degree $p := r+s$ counting the number of $\mathfrak{g}$-generators, while the second is the tensor degree $t := 2s+r$(also called weight). The differentials ∂ and δ preserve the tensor degree t but the spaces $H_n(\mathfrak{g},\mathbb{K})$ and $H^n(\mathfrak{g},\mathbb{K})$ are not homogeneous in t. The operation m_n is bigraded by homological and tensor gradings of bidegree $(p,t) = (2-n,0)$. The bi-grading impose the vanishing of many higher products.

On the level of Schur modules the ternary operation glues three fundamental $GL(V)$-representations $V_\square$ into a Schur module $V_{(2,1)}$. By iteration of the process of gluing boxes we generate all elementary hooks $V_k := V_{(k+1,1^k)}$,

$$m_3(V_\square,V_\square,V_\square) = V_{\tiny\yng(2,1)}\,, \quad m_3\left(V_{\tiny\yng(2,1)},V_\square,V_\square\right) = V_{\tiny\yng(3,1,1)}\,,\ldots, m_3(V_k,V_0,V_0) = V_{k+1}\ .$$

In our context the more convenient notation for Young diagrams is due to Frobenius: $\lambda := (a_1,\ldots,a_r|b_1,\ldots b_r)$ stands for a diagram λ with a_i boxes in the i-th row on the right of the diagonal, and with b_i boxes in the i-th column below the diagonal and the rank $r = r(\lambda)$ is the number of boxes on the diagonal.

For self-dual diagrams $\lambda = \lambda'$, i.e., $a_i = b_i$ we set $V_{a_1,\ldots,a_r} := V_{(a_1,\ldots,a_r|a_1,\ldots a_r)}$ when $a_1 > a_2 > \ldots > a_r \geq 0$ (and set the convention $V_{a_1,\ldots,a_r} := 0$ otherwise). Any two elementary hooks V_{a_1} and V_{a_2} can be glued together by the binary operation m_2, the decomposition of $m_2(V_{a_1},V_{a_2}) \cong m_2(V_{a_2},V_{a_1})$ is given by

$$m_2(V_{a_1},V_{a_2}) = V_{a_1,a_2} \oplus \left(\bigoplus_{i=1}^{a_2} V_{a_1+i,a_2-i}\right) \qquad a_1 \geq a_2$$

where the "leading" term V_{a_1,a_2} has the diagram with minimal height. Hence any m_2-bracketing of the hooks $V_{a_1},V_{a_2}\ldots,V_{a_r}$ yields[3] a sum of $GL(V)$-modules

[3] The operation m_2 is associative thus the result does not depend on the choice of the bracketing.

$$m_2(\dots m_2(m_2(V_{a_1},V_{a_2}),V_{a_3}),\dots,V_{a_r}) = V_{a_1,\dots,a_r} \oplus \dots$$

whose module with minimal height is precisely $V_{a_1,\dots,a_r}$. We conclude that all elements in the C_∞-algebra $H^\bullet(\mathfrak{g},\mathbb{K})$ can be generated in $H^1(\mathfrak{g},\mathbb{K})$ by m_2 and m_3. □

Acknowledgements We are grateful to Jean-Louis Loday for many enlightening discussions and his encouraging interest. The work was supported by the French-Bulgarian Project Rila under the contract Egide-Rila N112.

References

1. M. Artin, W.F. Schelter, *Graded algebras of global dimension 3.* Adv. Math. **66**(1987), 171-216.
2. H. Cartan. *Homologie et cohomologie d' une algèbre graduée.* Séminaire Henri Cartan, **11**(1958), 1-20.
3. M. Dubois-Violette, T. Popov, *Homogeneous Algebras, Statistics and Combinatorics*, LMP **61**(2002), 159-170.
4. G. Floystad, J.E. Vayne, *Artin-Schelter regular algebras of dimension five*, Algebra, Geometry, and Mathematical Physics, Banach Center Publications, **93**, (2011), 19-39.
5. H. S. Green, *A Generalized Method of Field Quantization*, Phys. Rev. **90**(1953), 270-273.
6. T. Józefiak, J. Weyman, *Representation-theoretic interpretation of a formula of D. E. Littlewood*, Math. Proc. Cambridge Phil. Soc., **103**(1988), 193-196.
7. T. Kadeishvili, *The A_∞-algebra Structure and Cohomology of Hochschild and Harrison*, Proc. of Tbil. Math. Inst. **91**(1988), 19-27.
8. J.-L. Loday and T. Popov, *Parastatistics Algebra, Young Tableaux and the Super Plactic Monoid.* Int. J. Geom. Meth. in Mod. Phys. **5** (2008), 1295-1314.
9. J.-L. Loday, B. Vallette, *Algebraic Operads*, Grundlehren Math. Wiss. **346**, Springer, Heidelberg, 2012.
10. D.-M. Lu, J. H. Palmieri, Q.-S. Wu, J. J. Zhang, *Koszul Equivalences in A_∞-Algebras*, arXiv[math.RA]0710.5492.
11. S. Sigg, *Laplacian and homology of free 2-step nilpotent Lie algebras*, J. Algebra **185**(1996), 144-161.
12. M. Kontsevich, Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, Math. Phys. Stud., **21**(2000), Kluwer Acad. Publ., Dordrecht, 255-307.